CHAPTER 19

Inner Product on Λ

19.1. First Properties of the Inner Product

19.1.1. In this chapter, we preserve the setup of the previous chapter. In particular, we write $\Lambda = \Lambda_\lambda$ where $\lambda \in X^+$ is fixed, except in subsections 19.2.3, 19.3.6 and 19.3.7.

Let $\rho_1 : \mathbf{U} \to \mathbf{U}$ be the algebra isomorphism given by

$$\rho_1(E_i) = -v_i F_i, \quad \rho_1(F_i) = -v_i^{-1} E_i, \quad \rho_1(K_\mu) = K_{-\mu}.$$

Let $\rho : \mathbf{U} \to \mathbf{U}^{opp}$ be the algebra isomorphism given by the composition $S\rho_1$ where $S : \mathbf{U} \to \mathbf{U}^{opp}$ is the antipode. We have

$$\rho(E_i) = v_i \tilde{K}_i F_i, \quad \rho(F_i) = v_i \tilde{K}_{-i} E_i, \quad \rho(K_\mu) = K_\mu.$$

It is clear that $\rho^2 = 1$.

Proposition 19.1.2. *There is a unique bilinear form* $(,) : \Lambda \times \Lambda \to \mathbf{Q}(v)$ *such that*

(a) $(\eta, \eta) = 1$;

(b) $(ux, y) = (x, \rho(u)y)$ *for all* $x, y \in \Lambda$ *and* $u \in \mathbf{U}$.

This bilinear form is symmetric. If $x \in (\Lambda)_\nu, y \in (\Lambda)_{\nu'}$ *with* $\nu \neq \nu'$, *then* $(x, y) = 0$.

For any $u \in \mathbf{U}$, we consider the linear map of the dual space $\Lambda^* = \mathrm{Hom}(\Lambda, \mathbf{Q}(v))$ into itself, given by $\xi \mapsto u(\xi)$ where $u(\xi)(x) = \xi(\rho(u)x)$ for all $x \in \Lambda$. This defines a $\mathbf{U}$-module structure on Λ^*, since $\rho : \mathbf{U} \to \mathbf{U}^{opp}$ is an algebra homomorphism. Let $\xi_0 \in \Lambda^*$ be the unique linear form such that $\xi_0(\eta) = 1$ and ξ_0 is zero on $(\Lambda)_\nu$ for $\nu \neq 0$. It is clear that $E_i \xi_0 = 0$ for all $i \in I$ and $K_\mu \xi_0 = v^{\langle \mu, \lambda \rangle} \xi_0$ for all $\mu \in Y$. We show that $F_i^{\langle i, \lambda \rangle + 1} \xi_0 = 0$. It is enough to show that, for any x in a weight space of Λ, the vector $E_i^{\langle i, \lambda \rangle + 1} x$ cannot be a non-zero multiple of η. This follows from Lemma 5.1.6, since $E_i \eta = 0$ and $F_i^{\langle i, \lambda \rangle + 1} \eta = 0$. From the description 18.1.1(a) of

Λ, we now see that there is a unique homomorphism of **U**-modules $\Lambda \to \Lambda^*$ which takes η to ξ_0.

Now it is clear that there is a 1-1 correspondence between homomorphisms of **U**-modules $\Lambda \to \Lambda^*$ which take η to ξ_0 and bilinear forms $(,)$ on Λ which satisfy (a) and (b). The existence and uniqueness of the form $x, y \mapsto (x, y)$ follow. The form $x, y \mapsto (y, x)$ satisfies the defining properties of the form $x, y \mapsto (x, y)$, since $\rho^2 = 1$. By the uniqueness, we see that these two forms coincide; hence $(,)$ is symmetric.

Proposition 19.1.3. *Let $\nu \in \mathbf{N}[I]$.*

(a) *We have $(L(\Lambda)_\nu, L(\Lambda)_\nu) \in \mathbf{A}$.*

(b) *For any $i \in I$ such that $\nu_i > 0$, and any $x \in L(\Lambda)_{\nu-i}, x' \in L(\Lambda)_\nu$ we have $(\tilde{F}_i x, x') = (x, \tilde{E}_i x')$ modulo $v^{-1}\mathbf{A}$.*

When $\operatorname{tr} \nu = 0$, (a) and (b) are trivial. We may therefore assume that $\operatorname{tr} \nu = N > 0$ and that both (a),(b) are already known for ν' with $\operatorname{tr} \nu' < N$. Since $L(\Lambda)_\nu = \sum_{i;\nu_i>0} \tilde{F}_i L(\Lambda)_{\nu-i}$, and $\tilde{E}_i(L(\Lambda)_\nu) \subset L(\Lambda)_{\nu-i}$ whenever $\nu_i > 0$, we see that (a) for ν follows from the induction hypothesis. (We use (a) and (b) for $\operatorname{tr} \nu' = N - 1$.)

We now prove (b) for ν. Let i, x, x' be as in (b). By 18.2.2, we may assume that $x = F_i^{(s)} y$ and $x' = F_i^{(s')} y'$ where $y \in L(\Lambda)_{\nu-i-si}$, $y' \in L(\Lambda)_{\nu-s'i}, E_i y = E_i y' = 0$ and $s \ge 0, s' \ge 0$, $s + \langle i, \lambda - \nu + i' \rangle \ge 0, s' + \langle i, \lambda - \nu \rangle \ge 0$. We must show that

(c) $(F_i^{(s+1)} y, F_i^{(s')} y') = (F_i^{(s)} y, F_i^{(s'-1)} y') \mod v^{-1}\mathbf{A}$.

For any r, r' we have from the definitions

$$(F_i^{(r)} y, F_i^{(r')} y') = (y, v_i^{r^2} \tilde{K}_{-ri} E_i^{(r)} F_i^{(r')} y')$$
$$= v_i^{r^2} \begin{bmatrix} r - r' + \langle i, \lambda - \nu + s'i' \rangle \\ r \end{bmatrix}_i (y, K_{-ri} F_i^{(r'-r)} y').$$

This is zero unless $r' \ge r$. By symmetry, it is also zero unless $r \ge r'$. Thus

$$(F_i^{(r)} y, F_i^{(r')} y') = \delta_{r,r'} v_i^{r^2} \begin{bmatrix} \langle i, \lambda - \nu + s'i' \rangle \\ r \end{bmatrix}_i (y, K_{-ri} y')$$
$$= \delta_{r,r'} v_i^{r^2 - r\langle i, \lambda - nu + s'i' \rangle} \begin{bmatrix} \langle i, \lambda - \nu + s'i' \rangle \\ r \end{bmatrix}_i (y, y').$$

Hence (c) is equivalent to

(d)
$$\delta_{s+1,s'} v_i^{-(s+1)((s+1)+\langle i,\lambda-\nu\rangle)} \begin{bmatrix} 2(s+1)+\langle i,\lambda-\nu\rangle \\ s+1 \end{bmatrix}_i (y,y') = \delta_{s,s'-1} v_i^{-s(s+2+\langle i,\lambda-\nu\rangle)} \begin{bmatrix} 2(s+1)+\langle i,\lambda-\nu\rangle \\ s \end{bmatrix}_i (y,y') \mod v^{-1}\mathbf{A}.$$

We may therefore assume that $s+1=s'$. Now y, y' are contained in $L(\Lambda)_{\nu-i-si}$ and $\mathrm{tr}\,(\nu-i-si)<N$; hence, by the induction hypothesis, we have $(y,y')\in\mathbf{A}$. We see that to prove (d) it suffices to prove

(e)
$$v_i^{-(s+1)((s+1)+\langle i,\lambda-\nu\rangle)} \begin{bmatrix} 2(s+1)+\langle i,\lambda-\nu\rangle \\ s+1 \end{bmatrix}_i = v_i^{-s(s+2+\langle i,\lambda-\nu\rangle)} \begin{bmatrix} 2(s+1)+\langle i,\lambda-\nu\rangle \\ s \end{bmatrix}_i \mod v^{-1}\mathbf{Z}[v^{-1}].$$

From the inequality $s+1+\langle i,\lambda-\nu\rangle\geq 0$, we deduce $2(s+1)+\langle i,\lambda-\nu\rangle\geq s+1$ and $2(s+1)+\langle i,\lambda-\nu\rangle\geq s$. Since $v_i^{-pq}\begin{bmatrix} p+q \\ p \end{bmatrix}_i\in 1+v^{-1}\mathbf{Z}[v^{-1}]$ for $p\geq 0, q\geq 0$, it follows that we have

$$v_i^{-(s+1)((s+1)+\langle i,\lambda-\nu\rangle)} \begin{bmatrix} 2(s+1)+\langle i,\lambda-\nu\rangle \\ s+1 \end{bmatrix}_i \in 1+v^{-1}\mathbf{Z}[v^{-1}]$$

and

$$v_i^{-s(s+2+\langle i,\lambda-\nu\rangle)} \begin{bmatrix} 2(s+1)+\langle i,\lambda-\nu\rangle \\ s \end{bmatrix}_i \in 1+v^{-1}\mathbf{Z}[v^{-1}]$$

and (e) follows. The proposition is proved.

Lemma 19.1.4. *Let $b, b'\in\mathcal{B}(\lambda)$.*

(a) *If $b'\neq\pm b$ then $(b^-\eta, b'^-\eta)\in v^{-1}\mathbf{A}$.*

(b) *We have $(b^-\eta, b^-\eta)\in 1+v^{-1}\mathbf{A}$.*

We may assume that $b, b'\in\mathcal{B}_\nu$ for some ν. We argue by induction on $N=\mathrm{tr}\,\nu$. The case where $N=0$ is trivial. Assume now that $N\geq 1$. We can find $i\in I$ such that $\nu_i>0$ and $b_1\in\mathcal{B}_{\nu-i}$ such that $\tilde{\phi}_i b_1=b$ and $\tilde{\epsilon}_i b=b_1$ (both equalities are modulo $v^{-1}\mathcal{L}(\mathbf{f})$). By 18.3.8, we have $\tilde{E}_i(b^-\eta)=b_1^-\eta$ and $\tilde{F}_i(b_1^-\eta)=b^-\eta$ (both equalities are modulo $v^{-1}L(\Lambda)$). It follows that $b_1^-\eta\neq 0$. (From $b_1^-\eta=0$ we could deduce by applying $\tilde{F}_i$ that $b^-\eta\in v^{-1}L(\Lambda)$ which contradicts $b\in\mathcal{B}(\lambda)$.) Thus $b_1\in\mathcal{B}(\lambda)$. Using again 18.3.8, we have $\tilde{E}_i(b'^-\eta)=(\tilde{\epsilon}_i b')^-\eta \mod v^{-1}L(\Lambda)$.

Using the previous proposition, we have

(c) $(b^-\eta, b'^-\eta) = (\tilde{F}_i(b_1^-\eta), b'^-\eta) = (b_1^-\eta, \tilde{E}_i(b'^-\eta)) = (b_1^-\eta, (\tilde{\epsilon}_i b')^-\eta)$

equalities modulo $v^{-1}\mathbf{A}$. Assume first that $b = b'$. Then $(\tilde{\epsilon}_i b')^-\eta = b_1^-\eta$ mod $v^{-1}L(\Lambda)$ and (c) becomes $(b^-\eta, b^-\eta) = (b_1^-\eta, b_1^-\eta)$ mod $v^{-1}L(\Lambda)$; by the induction hypothesis, we have $(b_1^-\eta, b_1^-\eta) \in 1 + v^{-1}\mathbf{A}$ so that $(b^-\eta, b^-\eta) \in 1 + v^{-1}\mathbf{A}$, as required.

Assume next that $b' \neq \pm b$. There are two cases: we have either $\tilde{\epsilon}_i b' = b_2$ mod $v^{-1}\mathcal{L}(\mathbf{f})$ for some $b_2 \in \mathcal{B}$ or $\tilde{\epsilon}_i b' \in v^{-1}\mathcal{L}(\mathbf{f})$. If the second alternative occurs, then $(b_1^-\eta, (\tilde{\epsilon}_i b')^-\eta) \in v^{-1}\mathbf{A}$ by 19.1.3(a); hence $(b^-\eta, b'^-\eta) \in v^{-1}\mathbf{A}$, by (c). If the first alternative occurs, then $b_2 \in \mathcal{B}(\lambda)$ (by the same argument as the one showing that $b_1 \in \mathcal{B}(\lambda)$) and we have $b_2 \neq \pm b_1$ (if we had $b_2 = \pm b_1$, then by applying $\tilde{\phi}_i$ we would deduce $b' = \pm b$ mod $v^{-1}\mathcal{L}(\mathbf{f})$; hence $b' = \pm b$).

From (c) we have $(b^-\eta, b'^-\eta) = (b_1^-\eta, b_2^-\eta)$ mod $v^{-1}\mathbf{A}$ and from the induction hypothesis we have $(b_1^-\eta, b_2^-\eta) \in v^{-1}\mathbf{A}$. It follows that $(b^-\eta, b'^-\eta) \in v^{-1}\mathbf{A}$. The lemma is proved.

19.2. Normalization of Signs

19.2.1. Let $\mathbf{B}_\nu$ be as in 14.4.2. From 17.3.7 and 18.1.7, we see that the following two conditions for an element $x \in \mathbf{f}_\nu$ are equivalent:

(a) $x \in \mathbf{B}_\nu + v^{-1}\mathcal{L}(\mathbf{f})_\nu$

(b) $x = \tilde{\phi}_{i_1}\tilde{\phi}_{i_2}\cdots\tilde{\phi}_{i_t}1$ mod $v^{-1}\mathcal{L}(\mathbf{f})_\nu$,

for some sequence $i_1, i_2, \dots, i_t$ in I such that $i_1 + i_2 + \cdots + i_t = \nu$.

For the proof of Theorem 14.4.3, we shall need the following result. We regard $\mathbf{f} \otimes \Lambda$ as an $\mathbf{f}$-module with θ_i acting as ϕ_i (see 18.1.3.)

Lemma 19.2.2. (a) *Let $b \in \mathbf{B}_\nu$ and $b' \in \mathbf{B}_{\nu'}$. The vector $\tilde{\phi}_i(b \otimes b'^-\eta)$ is equal modulo $v^{-1}\mathcal{L}(\mathbf{f}) \odot L(\Lambda)$ to $\tilde{\phi}_i(b) \otimes b'^-\eta$ or to $b \otimes \tilde{F}_i(b'^-\eta)$.*

(b) *Let $b_0 \in \mathbf{B}_\nu$. The vector $b_0(1 \otimes \eta)$ is equal modulo $v^{-1}\mathcal{L}(\mathbf{f}) \odot L(\Lambda)$ to $b_1 \otimes b_2^-\eta$ for some $b_1 \in \mathbf{B}_{\nu_1}$, $b_2 \in \mathbf{B}_{\nu_2}$ with $\nu_1 + \nu_2 = \nu$.*

We prove (a). By 18.2.2, (which is now known to be valid unconditionally) there exists $r_0 \geq 0$ such that $\nu_i' \geq r_0$ and $b'^-\eta = F_i^{(r_0)}x'$ mod $v^{-1}L(\Lambda)$ where $x' \in L(\Lambda)_{\nu'-r_0 i}$, $E_i x' = 0$, $x' \neq 0$.

By 16.2.7(b), there exists $r_1 \geq 0$ such that $\nu_i \geq r_1$ and $b = \phi_i^{(r_1)}x$ mod $v^{-1}\mathcal{L}(\mathbf{f})$ where $x \in \mathcal{L}(\mathbf{f})_{\nu-r_1 i}, \epsilon_i x = 0, x \neq 0$.

By 18.2.5 (which is now known to hold unconditionally), we have $\tilde{\phi}_i(b \otimes b'^-\eta) = \tilde{\phi}_i(\phi_i^{(r_1)}x \otimes F_i^{(r_0)}x')$ mod $v^{-1}\mathcal{L}(\mathbf{f}) \odot L(\Lambda)$.

By 17.1.15, $\tilde{\phi}_i(\phi_i^{(r_1)}x \otimes F_i^{(r_0)}x')$ is equal modulo $v^{-1}\mathcal{L}(\mathbf{f}) \odot L(\Lambda)$ to $\tilde{\phi}_i^{r_1+1}x \otimes \tilde{F}_i^{r_0}x'$ or to $\tilde{\phi}_i^{r_1}x \otimes \tilde{F}_i^{r_0+1}x'$ or equivalently to $\tilde{\phi}_i(b) \otimes b'^{-}\eta$ or $b \otimes \tilde{F}_i(b'^{-}\eta)$. This proves (a).

We prove (b). From 19.2.1, it follows that

$$b_0 = \tilde{\phi}_{i_1}\tilde{\phi}_{i_2}\cdots\tilde{\phi}_{i_t}1 \mod v^{-1}\mathcal{L}(\mathbf{f}),$$

for some sequence $i_1, i_2, \dots, i_t$ in I such that $i_1+i_2+\cdots+i_t = \nu$. We have

$$\begin{aligned} b_0(1\otimes\eta) &= b_0\Xi(1) = \Xi(b_0) = \Xi(\tilde{\phi}_{i_1}\tilde{\phi}_{i_2}\cdots\tilde{\phi}_{i_t}1)\\ &= \tilde{\phi}_{i_1}\tilde{\phi}_{i_2}\cdots\tilde{\phi}_{i_t}(\Xi(1)) = \tilde{\phi}_{i_1}\tilde{\phi}_{i_2}\cdots\tilde{\phi}_{i_t}(1\otimes\eta). \end{aligned}$$

The third equality is modulo $v^{-1}\mathcal{L}(\mathbf{f}) \odot L(\Lambda)$.

It remains to show that $\tilde{\phi}_{i_1}\tilde{\phi}_{i_2}\cdots\tilde{\phi}_{i_t}(1 \otimes \eta)$ is equal to $b_1 \otimes b_2^{-}\eta$ mod $v^{-1}\mathcal{L}(\mathbf{f}) \odot L(\Lambda)$ for some $b_1 \in \mathbf{B}_{\nu_1}$ and $b_2 \in \mathbf{B}_{\nu_2}$ with $\nu_1 + \nu_2 = \nu$. We show that this holds for any sequence $i_1, i_2, \dots, i_t$, by induction on t. The case where $t = 0$ is trivial. We assume that $t \geq 1$. Using the induction hypothesis and (a), we have that $\tilde{\phi}_{i_1}\tilde{\phi}_{i_2}\cdots\tilde{\phi}_{i_t}(1\otimes\eta)$ is equal to $\tilde{\phi}_{i_1}b\otimes b'^{-}\eta$ or to $b\otimes\tilde{F}_{i_1}(b'^{-}\eta)$ modulo $v^{-1}\mathcal{L}(\mathbf{f})\odot L(\Lambda)$, for some $b \in \mathbf{B}_{\nu_1}, b' \in \mathbf{B}_{\nu_2}$ such that $\nu_1+\nu_2 = \nu-i_1$. We have $\tilde{\phi}_{i_1}b = b_1 \mod v^{-1}\mathcal{L}(\mathbf{f})$ for some $b_1 \in \mathbf{B}_{\nu_1+i_1}$ and $\tilde{F}_{i_1}(b'^{-}\eta) = (\tilde{\phi}_{i_1}b')^{-}\eta = b_2^{-}\eta \mod v^{-1}\mathcal{L}(\mathbf{f})$ for some $b_2 \in \mathbf{B}_{\nu_2+i_1}$; (b) follows.

19.2.3. Proof of Theorem 14.4.3. The theorem is obvious when $\nu = 0$. Thus we may assume that $\operatorname{tr}\nu = N > 0$ and that the result is true when N is replaced by $N' \in [0, N-1]$.

We first prove part (b) of the theorem. Recall that $\sigma(\mathcal{B}_\nu) = \mathcal{B}_\nu$. (See 14.2.5(c).) Assume that $b, b' \in \mathbf{B}_\nu$ satisfy $\sigma(b) = -b'$. We will show that this leads to a contradiction.

We can find $i \in I, n > 0$ such that $\nu_i \geq n$, and $b'' \in \mathbf{B}_{\nu-ni}\cap\mathcal{B}_{i;0}$ such that $b = \pi_{i,n}b''$. By 17.3.7, we have $\tilde{\phi}_i^n b'' = b \mod v^{-1}\mathcal{L}(\mathbf{f})$. Since $b'' \in \mathcal{B}_{i;0}$, we can find $\beta \in \mathcal{L}(\mathbf{f})$ such that $\epsilon_i(\beta) = 0$ and $b'' = \beta \mod v^{-1}\mathcal{L}(\mathbf{f})$. This is a special case of the equality $B_N = B(N)$ in 16.3.5(a). By definition, we have $\tilde{\phi}_i^n\beta = \phi_i^{(n)}\beta$. Since $\tilde{\phi}_i^n$ preserves $v^{-1}\mathcal{L}(\mathbf{f})$, we have $\tilde{\phi}_i^n\beta = \tilde{\phi}_i^n b''$ mod $v^{-1}\mathcal{L}(\mathbf{f})$; hence $b = \phi_i^{(n)}\beta \mod v^{-1}\mathcal{L}(\mathbf{f})$.

For any $j \in I$, we define $c_j \in \mathbf{N}$ by $b'' \in \mathcal{B}_{j;c_j}$. Thus, we have $c_i = 0$. Since the root datum is assumed to be Y-regular, we can find $\lambda \in X$ such that $\langle i, \lambda\rangle = 0$ and $\langle j, \lambda\rangle \geq c_j$ for all $j \in I - \{i\}$. These inequalities show, using the definition of the c_j, that $\sigma(b'')\eta \neq 0$, where $\eta \in \Lambda = \Lambda_\lambda$ is as in 3.5.7.

In the $\mathbf{f}$-module $\mathbf{f} \otimes \Lambda$, we have $\theta_i^{(n)}(1 \otimes \eta) = \theta_i^{(n)} \otimes \eta$ since $F_i\eta = 0$ and $\tilde{K}_{-i}\eta = 0$ (recalling that $\langle i, \lambda \rangle = 0$). By definition of the $\mathbf{f}$-module structure on $\mathbf{f} \otimes \Lambda$, we have

$$\sigma(\beta)\theta_i^{(n)}(1 \otimes \eta) = \sigma(\beta)(\theta_i^{(n)} \otimes \eta) = \theta_i^{(n)} \otimes \sigma(\beta)(\eta) + z$$

where z is in the kernel of the obvious projection $pr_n : \mathbf{f} \otimes \Lambda \to \mathbf{f}_{ni} \otimes \Lambda$. Since $b = \theta_i^{(n)}\beta \mod v^{-1}\mathcal{L}(\mathbf{f})$, we have $-b' = \sigma(b) = \sigma(\beta)\theta_i^{(n)} \mod v^{-1}\mathcal{L}(\mathbf{f})$; hence

(a) $$-b'(1 \otimes \eta) = \theta_i^{(n)} \otimes \sigma(\beta)(\eta) + z \mod v^{-1}\mathcal{L}(\mathbf{f}) \odot L(\Lambda).$$

We have used that $x \in \mathcal{L}(\mathbf{f}) \implies x(1 \otimes \eta) \in \mathcal{L}(\mathbf{f}) \odot L(\Lambda)$; since $x(1 \otimes \eta) = \Xi(x)$, this follows from Lemma 18.2.7, which is now known unconditionally.

By 19.2.2(b), we have $b'(1 \otimes \eta) = b_1 \otimes b_2^- \eta \mod v^{-1}\mathcal{L}(\mathbf{f}) \odot L(\Lambda)$ where $b_1 \in \mathbf{B}_{\nu_1}, B_2 \in \mathbf{B}_{\nu_2}$ and $\nu_1 + \nu_2 = \nu$. Comparing with (a), we deduce that

$$\theta_i^{(n)} \otimes \sigma(\beta)^-(\eta) + b_1 \otimes b_2^- \eta + z \in v^{-1}\mathcal{L}(\mathbf{f}) \odot L(\Lambda).$$

Since $\beta = b'' \mod v^{-1}\mathcal{L}(\mathbf{f})$, we have $\sigma(\beta) = \sigma(b'') \mod v^{-1}\mathcal{L}(\mathbf{f})$. By the induction hypothesis, we have $\sigma(b'') \in \mathbf{B}_{\nu - ni}$. Recall also that $\sigma(b'')\eta \neq 0$. Thus we have

$$\theta_i^{(n)} \otimes \sigma(b'')\eta + b_1 \otimes b_2^- \eta + z \in v^{-1}\mathcal{L}(\mathbf{f}) \odot L(\Lambda).$$

By the definition of z, this implies that

$$\sigma(b'')^- \eta + b_2^- \eta \in v^{-1}L(\Lambda)$$

if $b_2 \in \mathbf{B}_{\nu - ni}$ and $b_2^- \eta \neq 0$ and

$$\sigma(b'')^- \eta \in v^{-1}L(\Lambda)$$

if $b_2 \notin \mathbf{B}_{\nu - ni}$ or $b_2^- \eta = 0$.

Both alternatives are impossible, since, by the induction hypothesis, $\sigma(b'')^- \eta, b_2^- \eta$ (in the first case) and $\sigma(b'')^- \eta$ (in the second case) are a part of an $\mathbf{A}$-basis of $L(\Lambda)$; by the induction hypothesis, we cannot have $\sigma(b'') + b_2 = 0$. This proves part (b) of the theorem.

We now prove part (a) of the theorem. Assume that $b, b' \in \mathbf{B}_\nu$ satisfy $b' = -b$. Since $\sigma(b) \in \mathcal{B}_\nu = \mathbf{B}_\nu \cup (-\mathbf{B}_\nu)$, we have either $\sigma(b) = b_1$ with $b_1 \in \mathbf{B}_\nu$ or $\sigma(b) = -b_2$ with $b_2 \in \mathbf{B}_\nu$. The second alternative cannot occur, by part (b). Thus the first alternative holds. But then $b_1 = -\sigma(b')$ and this again contradicts part (b). This proves part (a).

We prove part (c). Let $b \in \mathbf{B}_\nu$. We have $\sigma(b) \in \mathbf{B}_\nu \cup (-\mathbf{B}_\nu)$ and $\sigma(b) \notin (-\mathbf{B}_\nu)$ by (b), hence $\sigma(b) \in \mathbf{B}_\nu$. This proves part (c). Clearly, parts (d) and (e) follow from part (a) since $\mathcal{B}_\nu$ is a signed basis of $\mathbf{f}_\nu$. The theorem is proved.

19.3. Further Properties of the Inner Product

19.3.1. We shall denote by ${}_\mathcal{A}\Lambda_\lambda$ the image of the canonical map ${}_\mathcal{A}\mathbf{f} \to \Lambda_\lambda$. The canonical basis $\mathbf{B}(\Lambda_\lambda)$ of Λ_λ (see 14.4.11) is clearly an $\mathcal{A}$-basis of ${}_\mathcal{A}\Lambda_\lambda$. For any $\nu \in \mathbf{N}[I]$, let $({}_\mathcal{A}\Lambda_\lambda)_\nu$ be the image of ${}_\mathcal{A}\mathbf{f}_\nu$ under the canonical map $\mathbf{f} \to \Lambda_\lambda$. We have a direct sum decomposition ${}_\mathcal{A}\Lambda_\lambda = \oplus_\nu ({}_\mathcal{A}\Lambda_\lambda)_\nu$.

Proposition 19.3.2. *${}_\mathcal{A}\Lambda_\lambda$ is stable under the operators $x^-, x^+ : \Lambda_\lambda \to \Lambda_\lambda$, for any $x \in {}_\mathcal{A}\mathbf{f}$.*

For x^-, this is obvious. To prove the assertion about x^+, we may assume that $x = \theta_i^{(n)}$ for some i, n. Let $y \in ({}_\mathcal{A}\Lambda_\lambda)_\nu$. We show that $E_i^{(n)} y \in {}_\mathcal{A}\Lambda_\lambda$ by induction on $N = \text{tr}\,\nu$. If $N = 0$, the result is obvious. Assume that $N \geq 1$. We may assume that $y = F_j^{(t)} y'$ where $1 \leq t \leq \nu_j$ and $y' \in ({}_\mathcal{A}\Lambda_\lambda)_{\nu - tj}$. By 3.4.2(b), the operator $E_i^{(n)} F_j^{(t)}$ on Λ_λ is an $\mathcal{A}$-linear combination of operators $F_j^{(t')} E_i^{(n')}$. By the induction hypothesis, we have $E_i^{(n')} y' \in {}_\mathcal{A}\Lambda_\lambda$; hence $F_j^{(t')} E_i^{(n')} y' \in {}_\mathcal{A}\Lambda_\lambda$ so that $E_i^{(n)} y = E_i^{(n)} F_j^{(t)} y' \in {}_\mathcal{A}\Lambda_\lambda$. This completes the proof.

The following result is a strengthening of Lemma 19.1.4.

Proposition 19.3.3. *Let $b, b' \in \mathcal{B}(\lambda)$.*

(a) *If $b' \neq \pm b$ then $(b^-\eta, b'^-\eta) \in v^{-1}\mathbf{Z}[v^{-1}]$.*

(b) *We have $(b^-\eta, b^-\eta) \in 1 + v^{-1}\mathbf{Z}[v^{-1}]$.*

We shall prove by induction on $\text{tr}\,\nu$ that

(c) $(x, y) \in \mathcal{A}$

for any $x, y \in ({}_\mathcal{A}\Lambda_\lambda)_\nu$. When $\text{tr}\,\nu = 0$, (c) is trivial. We may therefore assume that $\text{tr}\,\nu = N > 0$ and that (c) is already known for ν' with $\text{tr}\,\nu' < N$. We may assume that $x = F_i^{(r)} x'$ where $0 < r \leq \nu_i$ and $x' \in ({}_\mathcal{A}\Lambda_\lambda)_{\nu - ri}$. From the definitions we have

(d) $(F_i^{(r)} x', y) = (x', v_i^{r^2} \tilde{K}_{-ri} E_i^{(r)} y)$.

By 19.3.2, we have $\tilde{K}_{-ri} E_i^{(r)} y \in ({}_\mathcal{A}\Lambda_\lambda)_{\nu - ri}$; hence the right hand side of (d) is in $\mathcal{A}$, by the induction hypothesis. This proves (c). The proposition follows by combining (c) with Lemma 19.1.4, since $\mathbf{A} \cap \mathcal{A} = \mathbf{Z}[v^{-1}]$.

19.3.4. From the description 18.1.1(a) of Λ_λ, we see that there is a unique **Q**-linear isomorphism $^- : \Lambda_\lambda \to \Lambda_\lambda$ such that $\overline{u\eta_\lambda} = \bar{u}\eta_\lambda$ for all $u \in \mathbf{U}$. It has square equal to 1.

Theorem 19.3.5. *Let $b \in \Lambda_\lambda$. We have $b \in \mathbf{B}(\Lambda_\lambda)$ if and only if*

(1) $b \in {}_{\mathcal{A}}\Lambda_\lambda$, $\bar{b} = b$ and

(2) there exists a sequence $i_1, i_2, \ldots, i_p$ in I such that $b = \tilde{F}_{i_1}\tilde{F}_{i_2}\cdots\tilde{F}_{i_p}\eta_\lambda$ mod $v^{-1}L(\Lambda_\lambda)$.

We have $b \in \pm\mathbf{B}(\Lambda_\lambda)$ if and only if b satisfies (1) and

(3) $(b, b) = 1 \mod v^{-1}\mathbf{Z}[v^{-1}]$.

If $b \in \pm\mathbf{B}(\Lambda_\lambda)$, then b obviously satisfies (1); it satisfies (3) by 19.3.3.

Assume now that b satisfies (1) and (3). Since the canonical basis is almost orthonormal, from Lemma 14.2.2 it follows that there exists $b' \in \mathbf{B}(\Lambda_\lambda)$ such that $b = \pm b' \mod v^{-1}L(\Lambda_\lambda)$. Since $\overline{b - (\pm b')} = b - (\pm b')$, it follows that $b - (\pm b') = 0$.

Assume now that $b \in \mathbf{B}(\Lambda_\lambda)$. We show that b satisfies (2). We have $b = \beta^-\eta_\lambda$ for some $\beta \in \mathbf{B}$. We can find a sequence $i_1, i_2, \ldots, i_p$ in I such that $\beta = \tilde{\phi}_{i_1}\tilde{\phi}_{i_2}\cdots\tilde{\phi}_{i_p}1 \mod v^{-1}\mathcal{L}(\mathbf{f})$. Using 18.3.8, it follows that $b = \tilde{F}_{i_1}\tilde{F}_{i_2}\cdots\tilde{F}_{i_p}\eta_\lambda \mod v^{-1}L(\Lambda_\lambda)$, as required.

Finally, assume that b satisfies (1),(2). Let $i_1, i_2, \ldots, i_p$ in I be as in (2). Using again 18.3.8, we see that $b = (\tilde{\phi}_{i_1}\tilde{\phi}_{i_2}\cdots\tilde{\phi}_{i_p}1)^-\eta_\lambda \mod v^{-1}L(\Lambda_\lambda)$. Let β be the unique element of $\mathbf{B}$ such that

$$\beta = \tilde{\phi}_{i_1}\tilde{\phi}_{i_2}\cdots\tilde{\phi}_{i_p}1 \mod v^{-1}\mathcal{L}(\mathbf{f}).$$

We have $b = \beta^-\eta_\lambda \mod v^{-1}L(\Lambda_\lambda)$. Let $b' = \beta^-\eta_\lambda$. Then $b - b' \in {}_{\mathcal{A}}\Lambda_\lambda$, $\overline{b - b'} = b - b'$ and $b - b' \in v^{-1}L(\Lambda_\lambda)$. It follows that $b - b' = 0$. Thus, $b \in \mathbf{B}(\Lambda_\lambda)$. The theorem is proved.

19.3.6. We will now investigate the relation between the inner product $(,)$ on $\mathbf{f}$ (see 1.2.5) and the inner product $(,)$ on Λ_λ, which we now denote by $(,)_\lambda$ since $\lambda \in X^+$ will vary.

Proposition 19.3.7. *Let $x, y \in \mathbf{f}$. When $\lambda \in X^+$ tends to ∞ (in the sense that $\langle i, \lambda\rangle$ tends to ∞ for all i), then the inner product $(x^-\eta_\lambda, y^-\eta_\lambda)_\lambda \in \mathbf{Q}(v)$ converges in $\mathbf{Q}((v^{-1}))$ to (x, y).*

We may assume that both x and y belong to $\mathbf{f}_\nu$ for some ν. We prove the proposition by induction on $N = \operatorname{tr} \nu$. When $N = 0$, the result is trivial. Assume now that $N \geq 1$. We may assume that $\nu_i > 0$ and $x = \theta_i x'$ for some i and some $x' \in \mathbf{f}_{\nu-i}$. We have

$$(x^- \eta_\lambda, y^- \eta_\lambda)_\lambda = (F_i x'^- \eta_\lambda, y^- \eta_\lambda)_\lambda = (x'^- \eta_\lambda, v_i \tilde{K}_{-i} E_i y^- \eta_\lambda)_\lambda.$$

Using the commutation formula 3.1.6(b) and the equality $E_i \eta_\lambda = 0$ we see that the last inner product is equal to

$$\begin{aligned}
&(v_i - v_i^{-1})^{-1}(x'^- \eta_\lambda, v_i \tilde{K}_{-i}(-r_i(y)^- \tilde{K}_{-i} + \tilde{K}_i({}_i r(y)^-))\eta_\lambda)_\lambda \\
&= -(v_i - v_i^{-1})^{-1} v_i^{-2\langle i,\lambda\rangle + \langle i,|y|\rangle - 1}(x'^- \eta_\lambda, r_i(y)^- \eta_\lambda)_\lambda \\
&+ (1 - v_i^{-2})^{-1}(x'^- \eta_\lambda, {}_i r(y)^- \eta_\lambda)_\lambda.
\end{aligned}$$

Using the induction hypothesis, we see that in the last expression, the first term converges to 0 for $\lambda \to \infty$ (note that $v_i^{-\langle i,\lambda\rangle}$ converges to 0) and the second term converges to $(1 - v_i^{-2})^{-1}(x', {}_i r(y))$ which by 1.2.13(a), is equal to (x, y). The proposition is proved.