CHAPTER 18

Study of the Operators $\tilde{F}_i, \tilde{E}_i$ on Λ_λ

18.1. Preliminaries

18.1.1. In this chapter we assume that the root datum is Y-regular. Let $\lambda \in X^+$. As in 3.5.6, we set $\Lambda_\lambda = \mathbf{f}/\sum_i \mathbf{f}\theta_i^{\langle i,\lambda\rangle+1}$. Since λ will be fixed in this chapter, we shall write Λ instead of Λ_λ. As in 3.5.7, we denote the image of $1 \in \mathbf{f}$ by $\eta \in \Lambda$.

Recall that there is a unique $\mathbf{U}$-module structure on Λ such that $E_i\eta = 0$ for all $i \in I$, $K_\mu\eta = v^{\langle\mu,\lambda\rangle}\eta$ for all $\mu \in Y$, and F_i acts by the map obtained from left multiplication by θ_i on $\mathbf{f}$. From the triangular decomposition for $\mathbf{U}$, we see that Λ can be naturally identified with the $\mathbf{U}$-module

(a) $$\mathbf{U}/(\sum_i \mathbf{U}E_i + \sum_\mu \mathbf{U}(K_\mu - v^{\langle\mu,\lambda\rangle}) + \sum_i \mathbf{U}F_i^{\langle i,\lambda\rangle+1})$$

by the unique isomorphism which makes η correspond to the image of $1 \in \mathbf{U}$.

For any $\nu \in \mathbf{N}[I]$, we denote by $(\Lambda)_\nu$ the image of $\mathbf{f}_\nu$ under the canonical map $\mathbf{f} \to \Lambda$. We have a direct sum decomposition $\Lambda = \oplus_\nu(\Lambda)_\nu$. Note that $(\Lambda)_\nu$ is contained in the $(\lambda-\nu)$-weight space $\Lambda^{\lambda-\nu}$ (the containment may be strict if the root datum is not X-regular).

18.1.2. By Theorem 14.3.2(b), the subset $\cup_{i,n;n\ge\langle i,\lambda\rangle+1}{}^\sigma\mathcal{B}_{i;n}$ of $\mathcal{B}$ is a signed basis of the $\mathbf{Q}(v)$-subspace $\sum_i \mathbf{f}\theta_i^{\langle i,\lambda\rangle+1}$ of $\mathbf{f}$. Hence the natural projection $\mathbf{f} \to \Lambda$ maps this subset to zero and maps its complement $\mathcal{B}(\lambda) = \cap_{i\in I}(\cup_{n;0\le n\le\langle i,\lambda\rangle}{}^\sigma\mathcal{B}_{i,n})$ bijectively onto a signed basis of the $\mathbf{Q}(v)$-vector space Λ. Thus $\{b^-\eta | b \in \mathcal{B}(\lambda)\}$ is a signed basis of Λ.

18.1.3. We shall regard $\mathbf{f}$ as an object of $\mathcal{D}_i$, for any $i \in I$ as in 17.3.1. Since $\mathbf{f}$ is a $\mathfrak{U}$-module (see 15.1.4), the tensor product $\mathbf{f}\otimes\Lambda$ is a $\mathfrak{U}$-module with

$$\phi_i(x\otimes y) = \phi_i(x)\otimes\tilde{K}_i^{-1}y + x\otimes F_i(y)$$

and

$$\epsilon_i(x\otimes y) = \epsilon_i(x)\otimes\tilde{K}_i^{-1}y + (v_i - v_i^{-1})x\otimes\tilde{K}_i^{-1}E_i(y)$$

for all $x \in \mathbf{f}$ and $y \in \Lambda$. (See 15.1.5.) Hence for each $i \in I$, we have $\mathbf{f}\otimes\Lambda \in \mathcal{D}_i$.

Lemma 18.1.4. *There is a unique* $\mathbf{Q}(v)$*-linear map* $\Xi : \mathbf{f} \to \mathbf{f} \otimes \Lambda$ *such that*

(a) $\Xi(1) = 1 \otimes \eta$;

(b) $\Xi(\phi_i x) = \phi_i(\Xi(x))$ *for all* $x \in \mathbf{f}$ *and all* $i \in I$;

(c) $\Xi(\epsilon_i x) = \epsilon_i(\Xi(x))$ *for all* $x \in \mathbf{f}$ *and all* $i \in I$.

By 3.1.4, there is a unique algebra homomorphism $\mathbf{f} \to \mathbf{f} \otimes \mathbf{U}$ such that $\theta_i \mapsto \theta_i \otimes \tilde{K}_{-i} + 1 \otimes F_i$ for all $i \in I$. Composing this with the linear map $\mathbf{f} \otimes \mathbf{U} \to \mathbf{f} \otimes \Lambda$ (identity on the first factor, the map $u \mapsto u\eta$ on the second factor) we obtain a linear map $\Xi : \mathbf{f} \to \mathbf{f} \otimes \Lambda$ which clearly satisfies (a) and (b). We show that it satisfies (c). For $x = 1$, (c) is trivial. Since the algebra $\mathbf{f}$ is generated by the various θ_j, it is enough to show that (c) holds for $x = \theta_j x'$, assuming that it holds for x'. We have

$$\Xi(\epsilon_i x) = \Xi(\epsilon_i \phi_j x') = \Xi(v^{i \cdot j} \phi_j \epsilon_i x' + \delta_{i,j} x') = v^{i \cdot j} \phi_j \epsilon_i \Xi(x') + \delta_{i,j} \Xi(x')$$

and

$$\epsilon_i(\Xi(x)) = \epsilon_i(\Xi(\phi_j x')) = \epsilon_i \phi_j(\Xi(x'));$$

hence (c) holds for x. This proves the existence of Ξ. The uniqueness of Ξ (assuming only (a),(b)) is clear since $\mathbf{f}$ is generated by the θ_j as an algebra.

18.1.5. Let $\mathcal{L}(\mathbf{f})$ be as in 17.3.3. We have $\mathcal{L}(\mathbf{f}) = \oplus_\nu \mathcal{L}(\mathbf{f})_\nu$ (sum over all $\nu \in \mathbf{N}[I]$) where $\mathcal{L}(\mathbf{f})_\nu$ is the $\mathbf{Z}[v^{-1}]$-submodule of $\mathbf{f}$ generated by $\mathcal{B}_\nu$.

Lemma 18.1.6. (a) *If* $b \in \mathcal{B}$ *is not equal to* ± 1, *then there exist* $i \in I$ *and* $b'' \in \mathcal{B}$ *such that* $b - \tilde{\phi}_i b'' \in v^{-1}\mathcal{L}(\mathbf{f})$.

(b) *If* $\nu \in \mathbf{N}[I]$ *is non-zero, then*

$$\mathcal{L}(\mathbf{f})_\nu = \sum_{i;\nu_i > 0} \tilde{\phi}_i(\mathcal{L}(\mathbf{f})_{\nu - i}).$$

We prove (a). According to 14.3.3, if b is as in (a), then there exist $i \in I$ and $n > 0$ such that $b \in \mathcal{B}_{i;n}$. By 17.3.7, we then have $\tilde{\phi}_i b'' - b \in v^{-1}\mathcal{L}(\mathbf{f})$ for some $b'' \in \mathcal{B}$.

We prove (b). The sum $\sum_{i;\nu_i>0} \tilde{\phi}_i(\mathcal{L}(\mathbf{f})_{\nu-i})$ is a $\mathbf{Z}[v^{-1}]$-submodule of $\mathcal{L}(\mathbf{f})_\nu$ (by 17.3.4) and the corresponding quotient module is annihilated by v^{-1} (by (a)). By Nakayama's lemma, this quotient is zero; therefore (b) holds. The lemma is proved.

Proposition 18.1.7. *Let $\nu \in \mathbf{N}[I]$.*

(a) *$\mathcal{L}(\mathbf{f})_\nu$ coincides with the $\mathbf{Z}[v^{-1}]$-submodule of $\mathbf{f}$ generated by the elements $\tilde{\phi}_{i_1}\tilde{\phi}_{i_2}\cdots\tilde{\phi}_{i_t}1$ for various sequences $i_1, i_2, \dots, i_t$ in I in which i appears exactly ν_i times for each $i \in I$.*

(b) *The subset of $\mathcal{L}(\mathbf{f})_\nu/v^{-1}\mathcal{L}(\mathbf{f})_\nu$ consisting of the images of the elements $\pm\tilde{\phi}_{i_1}\tilde{\phi}_{i_2}\cdots\tilde{\phi}_{i_t}1$ for various $i_1, i_2, \dots, i_t$, as above, coincides with the image of $\mathcal{B}_\nu$ in $\mathcal{L}(\mathbf{f})_\nu/v^{-1}\mathcal{L}(\mathbf{f})_\nu$.*

This follows immediately from the previous lemma.

18.1.8. We will denote by $L(\Lambda)$ the **A**-submodule of Λ generated by the signed basis $\{b^-\eta | b \in \mathcal{B}(\lambda)\}$ of Λ. We have a direct sum decomposition $L(\Lambda) = \oplus_\nu L(\Lambda)_\nu$ (ν runs over $\mathbf{N}[I]$) where $L(\Lambda)_\nu$ is the **A**-submodule of Λ generated by the elements $\{b^-\eta | b \in \mathcal{B}(\lambda) \cap \mathcal{B}_\nu\}$. We have $L(\Lambda)_\nu \subset (\Lambda)_\nu$.

Since Λ is integrable (see 3.5.6), Λ belongs to the category $\mathcal{C}'_i$ for any $i \in I$; hence the operators $\tilde{E}_i, \tilde{F}_i : \Lambda \to \Lambda$ (see 16.1.4) are well-defined. For any $\nu \in \mathbf{N}[I]$, we will denote by $L'(\Lambda)_\nu$ the **A**-submodule of Λ generated by the elements $\tilde{F}_{i_1}\tilde{F}_{i_2}\cdots\tilde{F}_{i_t}\eta$ for various sequences $i_1, i_2, \dots, i_t$ in I in which i appears exactly ν_i times for each $i \in I$.

Let $L'(\Lambda) = \sum_\nu L'(\Lambda)_\nu \subset \Lambda$. We have $L'(\Lambda)_\nu \subset (\Lambda)_\nu$.

18.2. A General Hypothesis and Some Consequences

Until the end of 18.3.6, we shall make the following

General hypothesis 18.2.1. *N is a fixed integer ≥ 1 such that, for any $\nu \in \mathbf{N}[I]$ with $tr\,\nu < N$, we have*

(a) *$L(\Lambda)_\nu = L'(\Lambda)_\nu$;*

(b) *if i is such that $\nu_i > 0$, then $\tilde{F}_i(x^-\eta) = (\tilde{\phi}_i x)^-\eta \mod v^{-1}L(\Lambda)$, for all $x \in \mathcal{L}(\mathbf{f})_{\nu-i}$;*

(c) *if i is such that $\nu_i > 0$, then $\tilde{E}_i(b^-\eta) = (\tilde{\epsilon}_i b)^-\eta \mod v^{-1}L(\Lambda)$, for all $b \in \mathcal{B}_\nu$ such that $b^-\eta \neq 0$; in particular, $\tilde{E}_i(L(\Lambda)_\nu) \subset L(\Lambda)_{\nu-i}$.*

In this section and the next we will derive various consequences of the general hypothesis; we will eventually show that this is not only a hypothesis, but a theorem (see 18.3.8).

Lemma 18.2.2. *Let ν be such that $tr\,\nu < N$, let $i \in I$ and let $x \in L'(\Lambda)_\nu$. Write $x = \sum_{r=0}^{\nu_i} F_i^{(r)} x_r$ where the $x_r \in (\Lambda)_{\nu-ri}$ satisfy $E_i x_r = 0$ for all r and $x_r = 0$ unless $r + \langle i, \lambda - \nu\rangle \geq 0$ (see 16.1.4). Then*

(a) *$x_r \in L'(\Lambda)_{\nu-ri}$ for all r.*

(b) *If, in addition,* $x = b^-\eta \mod v^{-1}L(\Lambda)$ *for some* $b \in \mathcal{B}_\nu \cap \mathcal{B}(\lambda)$, *then there exist* $r_0 \in [0, \nu_i]$ *and* $b_0 \in \mathcal{B}_{\nu - r_0 i} \cap \mathcal{B}(\lambda)$ *such that* $x_{r_0} = b_0^- \eta$ mod $v^{-1}L'(\Lambda)_{\nu - r_0 i}$, *and* $x_r \in v^{-1}L'(\Lambda)_{\nu - ri}$ *for all* $r \neq r_0$.

Let t be an integer such that $0 \leq t \leq \nu_i$ and $x_r = 0$ for $r > t$. We prove (a) by induction on t. If $t = 0$, then (a) is obvious. Assume now that $t \geq 1$. We have $\tilde{E}_i x = \sum_{r=0}^{t-1} F_i^{(r)} x_{r+1}$ and $x_{r+1} = 0$ unless $r + \langle i, \lambda - \nu + i' \rangle \geq 0$. (If we had simultaneously $x_{r+1} \neq 0$ and $r + \langle i, \lambda - \nu + i' \rangle < 0$ then $r + 1 + \langle i, \lambda - \nu \rangle < 0$, a contradiction.) By the general hypothesis, we have $\tilde{E}_i x \in L(\Lambda)_{\nu - i}$. By the induction hypothesis applied to $\tilde{E}_i x$, we have $x_r \in L'(\Lambda)_{\nu - ri}$ for all $r > 0$. Hence $F_i^{(r)} x_r = \tilde{F}_i^r x_r \in L'(\Lambda)_\nu$ for all $r > 0$. Since $x \in L'(\Lambda)_\nu$, it follows that $x_0 \in L'(\Lambda)_\nu$. This proves (a).

We prove (b) by induction on t as above. If $t = 0$, then (b) is obvious. Assume now that $t \geq 1$. By the general hypothesis, we have $\tilde{E}_i x = (\tilde{\epsilon}_i b)^- \eta$ mod $v^{-1}L(\Lambda)$. By 17.3.7, we have that $\tilde{\epsilon}_i b$ is equal modulo $v^{-1}\mathcal{L}(\mathbf{f})$ to either 0 or to b' for some $b' \in \mathcal{B}_{\nu - i}$.

If the first alternative occurs, or if the second alternative occurs with $(b')^- \eta = 0$, then $\tilde{E}_i(b^- \eta) \in v^{-1}L(\Lambda)$; applying (a) to $v\tilde{E}_i(b^-\eta)$ we see that $x_r \in v^{-1}L(\Lambda)$ for all $r > 0$. We then have $x_0 = b^- \eta \mod v^{-1}L(\Lambda)$, as required.

Hence we may assume that

(c) $\tilde{\epsilon}_i b = b' \mod v^{-1}\mathcal{L}(\mathbf{f})$, where $b' \in \mathcal{B}_{\nu - i} \cap \mathcal{B}(\lambda)$.

We have, by assumption,

(d) $\tilde{E}_i x = \tilde{E}_i(b^- \eta) \mod v^{-1}\tilde{E}_i \mathcal{L}(\mathbf{f})_\nu$.

By the general hypothesis, we have $\tilde{E}_i \mathcal{L}(\mathbf{f})_\nu \subset \mathcal{L}(\mathbf{f})_{\nu - i}$ and $\tilde{E}_i(b^- \eta) = (\tilde{\epsilon}_i b)^- \eta \mod v^{-1}L(\Lambda)_{\nu - i}$ (we have $b^- \eta \neq 0$, by assumption). Introducing this in (d), and using (c), we obtain

$$\tilde{E}_i x = b'^- \eta \mod v^{-1}\mathcal{L}(\mathbf{f})_{\nu - i}.$$

By the induction hypothesis applied to $\tilde{E}_i x$, we see that there exist $r_0 \in [1, \nu_i]$ and $b_0 \in \mathcal{B}_{\nu - r_0 i} \cap \mathcal{B}(\lambda)$ such that $x_{r_0} = b_0^- \eta \mod v^{-1}L'(\Lambda)_{\nu - r_0 i}$, and $x_r \in v^{-1}L'(\Lambda)_{\nu - ri}$ for all r such that $r > 0$ and $r \neq r_0$. It follows that

$$\tilde{E}_i x = \tilde{F}_i^{r_0 - 1} x_{r_0} \mod v^{-1}L(\Lambda).$$

By the general hypothesis, we have

$$\tilde{F}_i \tilde{E}_i x = \tilde{F}_i \tilde{E}_i(b^- \eta) = \tilde{F}_i((\tilde{\epsilon}_i b)^- \eta) = (\tilde{\phi}_i \tilde{\epsilon}_i b)^- \eta$$

(equalities modulo $v^{-1}L(\Lambda)$.) Since $\tilde{\epsilon}_i b = b'$ mod $v^{-1}\mathcal{L}(\mathbf{f})$, we see from 17.3.7 that $\tilde{\phi}_i \tilde{\epsilon}_i b = b$ mod $v^{-1}\mathcal{L}(\mathbf{f})$. It follows that $\tilde{F}_i \tilde{E}_i x = b^- \eta = x$ (equalities modulo $v^{-1}L(\Lambda)$). We deduce that $x = \tilde{F}_i(\tilde{F}_i^{r_0 - 1} x_{r_0}) = \tilde{F}_i^{r_0} x_{r_0}$ (equalities modulo $v^{-1}L(\Lambda)$.) Since $\tilde{F}_i^r x_r \in v^{-1}L(\Lambda)$ for all $r > 0$, $r \neq r_0$ and $x = \sum_r \tilde{F}_i^r x_r$, we deduce that $x_0 \in v^{-1}L(\Lambda)$. This completes the proof.

Lemma 18.2.3. *Let $i \in I$ and let $x \in \mathcal{L}(\mathbf{f})$. Write $x = \sum_{r \geq 0} \phi_i^{(r)} x_r$ where $x_r \in \mathbf{f}$ are 0 for all but finitely many r and $\epsilon_i x_r = 0$ for all r (see 16.1.2(c)). Then $x_r \in \mathcal{L}(\mathbf{f})$ for all r.*

This is a special case of Lemma 16.2.7(b).

18.2.4. If H, H' are two subsets of $\mathbf{f}, \Lambda$ respectively, we denote by $H \odot H'$ the subgroup of $\mathbf{f} \otimes \Lambda$ generated by the vectors $h \otimes h'$ with $h \in H$, $h' \in H'$.

Lemma 18.2.5. *Assume that $\mathrm{tr}\, \nu < N$ and let $i \in I$. Then*

$$\tilde{\phi}_i(\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)_\nu) \subset \mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$$

and

$$\tilde{\epsilon}_i(\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)_\nu) \subset \mathcal{L}(\mathbf{f}) \odot L'(\Lambda).$$

By Lemmas 18.2.2(a) and 18.2.3, the $\mathbf{A}$-module $\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)_\nu$ is generated by elements $\phi^{(a)} x \otimes F_i^{(a')} x'$ where $x \in \mathcal{L}(\mathbf{f})$ and $x' \in L'(\Lambda)_{\nu - a'i}$ satisfy $\epsilon_i(x) = 0$ and $E_i(x') = 0$. The image of such elements under $\tilde{\phi}_i$ or $\tilde{\epsilon}_i$ is contained in $\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$ by Corollary 17.1.15. The lemma follows.

Lemma 18.2.6. *Let $x \in L'(\Lambda)_\nu$ where $\mathrm{tr}\, \nu < N$. Assume that there exists $b \in \mathcal{B}_\nu \cap \mathcal{B}(\lambda)$ such that $x = b^- \eta$ mod $v^{-1}L'(\Lambda)$. Assume also that $\tilde{F}_i x \notin v^{-1}L'(\Lambda)$. Then $\tilde{\phi}_i(1 \otimes x) = 1 \otimes \tilde{F}_i x$ mod $v^{-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$.*

By 18.2.2, we may assume that $x = F_i^{(s)} x'$ where $x' \in L'(\Lambda)_{\nu - si}$ satisfies $E_i x' = 0$. Since $x' \neq 0$ and $E_i x' = 0$, we have $n = \langle i, \lambda - \nu + si' \rangle \in \mathbf{N}$; moreover, $F_i^{(n+1)} x' = 0$. By Corollary 17.1.15, $\tilde{\phi}_i(1 \otimes F_i^{(s)} x')$ is equal modulo $v^{-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$ to $1 \otimes F_i^{(s+1)} x'$ (if $s < n$) or to $\theta_i \otimes F_i^{(s)} x'$ (if $s \geq n$). If the second alternative occurs, then $\tilde{F}_i x = \tilde{F}_i^{s+1} x' = 0$, contradicting our assumptions. Thus the first alternative occurs and the lemma is proved.

Lemma 18.2.7. *For any ν such that $tr\,\nu \leq N$, we have $\Xi(\mathcal{L}(\mathbf{f})_\nu) \subset \mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$.*

We argue by induction on $\text{tr}\,\nu$. If $\nu = 0$, the result is obvious. Assume that $\nu \neq 0$ and that the result is known when ν is replaced by ν' with $\text{tr}\,\nu' < \text{tr}\,\nu$. By 18.1.6(b), the $\mathbf{Z}[v^{-1}]$-module $\mathcal{L}(\mathbf{f})_\nu$ is spanned by vectors $\tilde{\phi}_i x$ with $i \in I$ and $x \in \mathcal{L}(\mathbf{f})_{\nu-i}$, so it suffices to show that for such i, x, we have $\Xi(\tilde{\phi}_i x) \in \mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$. Since Ξ is a morphism in $\mathcal{D}_i$, we have $\Xi(\tilde{\phi}_i x) = \tilde{\phi}_i(\Xi(x))$. By the induction hypothesis, we have $\Xi(x) \in \mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$.

From the definition of Ξ we see immediately that

$$\Xi(\mathbf{f}_{\nu'}) \subset \sum_{\nu'';\ \text{tr}\,\nu'' \leq\ \text{tr}\,\nu'} \mathbf{f} \otimes \mathbf{U}^-_{\nu''}\eta.$$

Combining this with the previous inclusion, we see that

$$\Xi(x) \in \sum_{\nu'';\ \text{tr}\,\nu'' < N} \mathcal{L}(\mathbf{f}) \odot L'(\Lambda)_{\nu''}.$$

Hence it is enough to show that

$$\tilde{\phi}_i(\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)_{\nu''}) \subset \mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$$

whenever $\text{tr}\,\nu'' < N$.

Now the $\mathbf{A}$-module $\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)_{\nu''}$ is spanned by vectors of the form $\phi^{(a)}x \otimes F_i^{(c)}y$ where $x \in \mathcal{L}(\mathbf{f}), y \in L'(\Lambda)_{\nu''-ci}$ satisfy $\epsilon_i x = 0, E_i y = 0$ (see Lemmas 18.2.2, 18.2.3). Hence it suffices to show that $\tilde{\phi}_i(\phi_i^{(a)}x \otimes F_i^{(c)}y)$ belongs to the $\mathbf{Z}[v^{-1}]$-submodule generated by the vectors $\phi_i^{(a')}x \otimes F_i^{(c')}y$ for various $a', c' \geq 0$. But this follows from Corollary 17.1.15. The lemma is proved.

18.2.8. Consider the linear form $\mathbf{f} \to \mathbf{Q}(v)$ which takes $\mathbf{f}_\nu$ to zero for all $\nu \neq 0$ and takes 1 to 1; tensoring it with the identity map of Λ, we obtain a $\mathbf{Q}(v)$-linear map $pr : \mathbf{f} \otimes \Lambda \to \Lambda$.

From the definitions, we see easily that

(a) $pr(\Xi(x)) = x^-\eta$ for all $x \in \mathbf{f}$.

Lemma 18.2.9. (a) *We have $pr(\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)) \subset L'(\Lambda)$.*

(b) *Let $i \in I$. Let $y \in \mathcal{L}(\mathbf{f}) \odot L'(\Lambda)_\nu$ where $tr\,\nu < N$. We have $pr(\tilde{\phi}_i(y)) = \tilde{F}_i(pr(y)) \mod v^{-1}L'(\Lambda)_{\nu+i}$.*

Let $x \in \mathcal{L}(\mathbf{f})_\nu$ and let $x' \in L'(\Lambda)_{\nu'}$. If $\nu \neq 0$, we have $pr(x \otimes x') = 0$; if $\nu = 0$, we have $x = f1$ where $f \in \mathbf{Z}[v^{-1}]$ and $pr(x \otimes x') = fx'$. Thus (a) holds.

We prove (b). By Lemmas 18.2.2, 18.2.3, we may assume that $y = \phi_i^{(a)} z \otimes F_i^{(a')} z'$ where $z \in \mathcal{L}(\mathbf{f})$ (homogeneous) and $z' \in L'(\Lambda)_{\nu'}$ satisfy $\epsilon_i(z) = 0$, $E_i(z') = 0$ and $a, a' \in \mathbf{N}$. We may assume that $z' \neq 0$. Let n be the smallest integer ≥ 0 such that $F_i^{n+1} z' = 0$. By Corollary 17.1.15, we have that $\tilde{\phi}_i(y)$ is equal to

(c) $\phi_i^{(a+1)} z \otimes F_i^{(a')} z'$ modulo $v^{-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$, if $a + a' \geq n$, and to

(d) $\phi_i^{(a)} z \otimes F_i^{(a'+1)} z'$ modulo $v^{-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$, if $a + a' < n$.

If $a > 0$ or $z \notin \mathbf{f}_0$, then y and both vectors (c),(d) are in the kernel of pr, by the definition of pr; on the other hand, by (a), we have $pr(v^{-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)) \subset v^{-1}L'(\Lambda)$. Hence in this case the lemma holds for y. Hence we may assume that $a = 0$ and $z = 1$. We then have $pr(y) = F_i^{(a')} z'$; moreover, by the previous argument:

$$pr(\tilde{\phi}_i(y)) = F_i^{(a'+1)}(z') \mod v^{-1}L'(\Lambda)$$

if $a' < n$ and

$$pr(\tilde{\phi}_i(y)) = 0 \mod v^{-1}L'(\Lambda)$$

if $a' \geq n$.

On the other hand, by the definition of $\tilde{F}_i$, we have

$$\tilde{F}_i(pr(y)) = \tilde{F}_i(F_i^{(a')}(z')) = F_i^{(a'+1)}(z').$$

It remains to observe that $F_i^{(a'+1)}(z') = 0$ if $a' \geq n$ (by the definition of n). The lemma is proved.

Lemma 18.2.10. *Let $x \in \mathcal{L}(\mathbf{f})_\nu$ with $\operatorname{tr} \nu < N$. We have $(\tilde{\phi}_i x)^- \eta = \tilde{F}_i(x^- \eta) \mod v^{-1}L'(\Lambda)$.*

Using 18.2.8(a) and the commutation of Ξ with $\tilde{\phi}_i$, we have

$$(\tilde{\phi}_i x)^- \eta = pr(\Xi(\tilde{\phi}_i x)) = pr(\tilde{\phi}_i(\Xi(x))).$$

Using again 18.2.8(a), we have

$$\tilde{F}_i(x^- \eta) = \tilde{F}_i(pr(\Xi(x))).$$

It remains to show that

$$pr(\tilde{\phi}_i(\Xi(x))) = \tilde{F}_i(pr(\Xi(x))) \mod v^{-1}L'(\Lambda).$$

This follows from Lemma 18.2.9(b) applied to $y = \Xi(x)$. (We have $\Xi(x) \in \mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$ by 18.2.7, and $\Xi(x) \in \sum_{\nu''; \operatorname{tr} \nu'' \leq \operatorname{tr} \nu} \mathbf{f} \otimes \mathbf{U}^-_{\nu''}\eta$; hence

$$\Xi(x) \in \sum_{\nu''; \operatorname{tr} \nu'' < N} \mathcal{L}(\mathbf{f}) \odot L'(\Lambda)_{\nu''}$$

so that Lemma 18.2.9(b) is applicable.) The lemma is proved.

Lemma 18.2.11. *If* $tr\ \nu = N$, *we have* $L(\Lambda)_\nu \subset L'(\Lambda)_\nu$.

By definition, $L(\Lambda)_\nu$ consists of the vectors of form $x^-\eta$ where $x \in \mathcal{L}(\mathbf{f})_\nu$. Since $\nu \neq 0$, the $\mathbf{Z}[v^{-1}]$-module $\mathcal{L}(\mathbf{f})_\nu$ is equal to $\sum_{i;\nu_i>0} \tilde{\phi}_i(\mathcal{L}(\mathbf{f})_{\nu-i})$ (see 18.1.6). Hence it suffices to show that $(\tilde{\phi}_i(x))^-\eta \in L'(\Lambda)_\nu$ for any $i \in I$ such that $\nu_i > 0$ and any $x \in \mathcal{L}(\mathbf{f})_{\nu-i}$. By 18.2.10, we have $(\tilde{\phi}_i x)^-\eta = \tilde{F}_i(x^-\eta)$ mod $v^{-1}L'(\Lambda)_\nu$. Hence it suffices to show that $\tilde{F}_i(x^-\eta) \in L'(\Lambda)_\nu$.

By the definition of $L(\Lambda)_{\nu-i}$, we have $x^-\eta \in L(\Lambda)_{\nu-i}$. Using our general hypothesis, we deduce that $x^-\eta \in L'(\Lambda)_{\nu-i}$. It remains to observe that $\tilde{F}_i(L'(\Lambda)_{\nu-i}) \subset L'(\Lambda)_\nu$ (from the definitions). The lemma is proved.

Lemma 18.2.12. *If* $tr\ \nu = N$, *we have* $L'(\Lambda)_\nu \subset L(\Lambda)_\nu + v^{-1}L'(\Lambda)_\nu$.

We have

$$L'(\Lambda)_\nu = \sum_{i;\nu_i>0} \tilde{F}_i L'(\Lambda)_{\nu-i} = \sum_{i;\nu_i>0} \tilde{F}_i L(\Lambda)_{\nu-i}$$
$$= \sum_{i;\nu_i>0} \mathbf{A}\tilde{F}_i(\mathcal{L}(\mathbf{f})^-_{\nu-i}\eta).$$

The first and third equalities are by definition; the second one follows from our general hypothesis. Hence it suffices to show that

$$\tilde{F}_i(x^-\eta) \in L(\Lambda)_\nu + v^{-1}L'(\Lambda)_\nu$$

for all $x \in \mathcal{L}(\mathbf{f})_{\nu-i}$ (where $\nu_i > 0$).

By 18.2.10, we have $\tilde{F}_i(x^-\eta) = (\tilde{\phi}_i x)^-\eta \mod v^{-1}L'(\Lambda)_\nu$. On the other hand, we have $\tilde{\phi}_i x \in \mathcal{L}(\mathbf{f})_\nu$ (see 17.3.4); hence we have $(\tilde{\phi}_i x)^-\eta \in L(\Lambda)_\nu$. The lemma is proved.

Lemma 18.2.13. *If* $tr\ \nu = N$, *we have* $L(\Lambda)_\nu = L'(\Lambda)_\nu$.

By 18.2.11, $L(\Lambda)_\nu$ is an $\mathbf{A}$-submodule of $L'(\Lambda)_\nu$. The corresponding quotient module is annihilated by v^{-1}, see Lemma 18.2.12. This quotient is then zero by Nakayama's lemma. The lemma is proved.

18.3. Further Consequences of the General Hypothesis

Lemma 18.3.1. *Let* $y = \tilde{F}_{i_1}\tilde{F}_{i_2}\cdots\tilde{F}_{i_t}\eta \in \Lambda$ *where* $i_1 + i_2 + \cdots + i_t = \nu$ *and* $t = tr\ \nu \leq N$. *Let* $x = \tilde{\phi}_{i_1}\tilde{\phi}_{i_2}\cdots\tilde{\phi}_{i_t}(1 \otimes \eta) \in \mathbf{f} \otimes \Lambda$. *Assume that* $y \notin v^{-1}L'(\Lambda)$. *Then* $x = 1 \otimes y \mod v^{-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$.

We argue by induction on t. If $t = 0$, there is nothing to prove. Assume now that $t > 0$ and that the result is known for $t - 1$.

Let $y' = \tilde{F}_{i_2} \cdots \tilde{F}_{i_t}\eta \in \Lambda$ and let $x' = \tilde{\phi}_{i_2} \cdots \tilde{\phi}_{i_t}(1 \otimes \eta) \in \mathbf{f} \otimes \Lambda$. Since $\tilde{F}_{i_1}(v^{-1}L'(\Lambda)) \subset v^{-1}L'(\Lambda)$, and $y \notin v^{-1}L'(\Lambda)$, we have $y' \notin v^{-1}L'(\Lambda)$. By the induction hypothesis, we have $x' = 1 \otimes y' \mod v^{-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$. Applying $\tilde{\phi}_{i_1}$ and using 18.2.5, we deduce that $x = \tilde{\phi}_{i_1}x' = \tilde{\phi}_{i_1}(1 \otimes y') \mod v^{-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$.

By our general hypothesis, we have $y' = (\tilde{\phi}_{i_2} \cdots \tilde{\phi}_{i_t}1)^{-}\eta \mod v^{-1}L(\Lambda)$; hence, by 18.1.7(b), we have $y' = b^{-}\eta \mod v^{-1}L(\Lambda)_{\nu-i_1}$ for some $b \in \mathcal{B}_{\nu-i_1}$. Since $y' \notin v^{-1}L'(\Lambda)_{\nu-i_1}$, we have $b \in \mathcal{B}(\lambda)$. Applying Lemma 18.2.6, we see that $\tilde{\phi}_{i_1}(1 \otimes y') = 1 \otimes \tilde{F}_{i_1}y' = 1 \otimes y \mod v^{-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$. (That lemma is applicable since $\tilde{F}_{i_1}y' = y \notin v^{-1}L'(\Lambda)$.) Hence $x = 1 \otimes y \mod v^{-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$. The lemma is proved.

Lemma 18.3.2. *If* $\operatorname{tr} \nu \leq N$, *then* $\tilde{E}_i(L(\Lambda)_\nu) \subset L'(\Lambda)$.

We will prove, for any $n \geq 0$, that $\tilde{E}_i(L(\Lambda)_\nu) \subset v^n L'(\Lambda)$, by descending induction on n. This is obvious for n large since $L(\Lambda)_\nu$ is a finitely generated $\mathbf{A}$-module. Hence it is enough to prove the following statement.

(a) Assume that $n \geq 1$ and $\tilde{E}_i(L(\Lambda)_\nu) \subset v^n L'(\Lambda)$; then $\tilde{E}_i(L(\Lambda)_\nu) \subset v^{n-1}L'(\Lambda)$.

We first show that

(b) $\tilde{\epsilon}_i(\mathcal{L}(\mathbf{f})_{\nu'} \odot L(\Lambda)_{\nu''}) \subset v^n\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$

provided that $\nu' + \nu'' = \nu$. In the case where $\operatorname{tr} \nu'' < N$, this follows from 18.2.5. Assume now that $\operatorname{tr} \nu'' = N$; then $\nu' = 0$. It suffices to show that $\tilde{\epsilon}_i(1 \otimes x) \in v^n\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$ for any $x \in L(\Lambda)_\nu$. We write $x = \sum_{r \geq 0} F_i^{(r)}x_r$ where $x_r = 0$ unless $r + \langle i, \lambda - \nu\rangle \geq 0$ and $E_i x_r = 0$. By the assumption of (a), we have $\sum_{r\geq 1} F_i^{(r-1)}x_r \in v^n L'(\Lambda)$ and by 18.2.2, we deduce that $F_i^{(r-1)}x_r \in v^n L'(\Lambda)$ for $r \geq 1$, or equivalently, $\tilde{F}_i^{r-1}x_r \in v^n L'(\Lambda)$ for $r \geq 1$. Using the general hypothesis $(r-1)$ times, we have $\tilde{E}_i^{r-1}(\tilde{F}_i^{r-1}x_r) \subset v^n L'(\Lambda)$; hence $x_r \in v^n L'(\Lambda)$ for all $r \geq 1$. We have $\tilde{\epsilon}_i(1 \otimes x) = \sum_{r\geq 1} \tilde{\epsilon}_i(1 \otimes F^{(r)}x_r)$ since $\epsilon_i(1 \otimes x_0) = 0$. By 17.1.15, this belongs to the $\mathbf{Z}[v^{-1}]$-submodule generated by the elements $\theta_i^{(r_1)} \otimes F^{(r_2)}x_r$ with $r \geq 1$ and $r_1 + r_2 = r - 1$ and these elements belong to $v^n\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$. Thus (b) is proved.

To prove (a), it suffices to show that $\tilde{E}_i(y) \in v^{n-1}L'(\Lambda)$ for all y of the form $y = \tilde{F}_{i_1}\tilde{F}_{i_2} \ldots \tilde{F}_{i_t}\eta$ where $i_1 + i_2 + \cdots + i_t = \nu$. If $y \in v^{-1}L'(\Lambda)$, then our inductive assumption shows that $\tilde{E}_i(vy) \in v^n L'(\Lambda)$; hence $\tilde{E}_i(y) \in v^{n-1}L'(\Lambda)$, as desired. Thus we may assume that $y \notin v^{-1}L'(\Lambda)$. Using now Lemma 18.3.1, we see that

(c) $\tilde{\phi}_{i_1}\tilde{\phi}_{i_2} \cdots \tilde{\phi}_{i_t}(1 \otimes \eta) = (1 \otimes y) + v^{-1}z$

where $z \in \mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$. From the definition of the operators $\tilde{\phi}_i$ on $\mathbf{f} \otimes \Lambda$ we see that we have necessarily

$$z \in \sum_{\nu'+\nu''=\nu} \mathcal{L}(\mathbf{f})_{\nu'} \odot L'(\Lambda)_{\nu''}.$$

Hence, by (b), we have

$$\tilde{\epsilon}_i(z) \in v^n \mathcal{L}(\mathbf{f}) \odot L'(\Lambda).$$

Thus applying $\tilde{\epsilon}_i$ to (c), we obtain

(d) $\tilde{\epsilon}_i \tilde{\phi}_{i_1} \tilde{\phi}_{i_2} \cdots \tilde{\phi}_{i_t}(1 \otimes \eta) = \tilde{\epsilon}_i(1 \otimes y) \mod v^{n-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$.

We have (using 18.2.7)

$$\tilde{\epsilon}_i \tilde{\phi}_{i_1} \tilde{\phi}_{i_2} \cdots \tilde{\phi}_{i_t}(1 \otimes \eta) = \tilde{\epsilon}_i \tilde{\phi}_{i_1} \tilde{\phi}_{i_2} \cdots \tilde{\phi}_{i_t}(\Xi(1)) = \Xi(\tilde{\epsilon}_i \tilde{\phi}_{i_1} \tilde{\phi}_{i_2} \cdots \tilde{\phi}_{i_t} 1)$$
$$\subset \Xi(\mathcal{L}(\mathbf{f})_\nu) \subset \mathcal{L}(\mathbf{f}) \odot L'(\Lambda) \subset v^{n-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda).$$

Hence from (d) we deduce that

(e) $\tilde{\epsilon}_i(1 \otimes y) \in v^{n-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$.

We write $y = \sum_{r \geq 0} F_i^{(r)} y_r$, where the $y_r \in (\Lambda)_{\nu - ri}$ satisfy $E_i y_r = 0$ for all r and $y_r = 0$ unless $r + \langle i, \lambda - \nu \rangle \geq 0$.

By our inductive assumption, we have $\tilde{E}_i y = \sum_{r \geq 1} F_i^{(r-1)} y_r \in v^n L(\Lambda)$. Using 18.2.2, we deduce that $y_r \in v^n L(\Lambda)$ for $r \geq 1$. We have $\tilde{\epsilon}_i(1 \otimes y) = \sum_{r \geq 1} \tilde{\epsilon}_i(1 \otimes F_i^{(r)} y_r)$, since $\epsilon_i(1 \otimes y_0) = 0$. By 17.1.15, we have for any $r \geq 1$, $\tilde{\epsilon}_i(1 \otimes F_i^{(r)} y_r) = 1 \otimes F_i^{(r-1)} y_r$ plus a linear combination with coefficients in $v^{-1}\mathbf{Z}[v^{-1}]$ of terms $\theta_i^{(r_1)} \otimes F_i^{(r_2)} y_r$ where $r_1 + r_2 = r - 1$. The last linear combination is in $v^{n-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$ since $y_r \in v^n L(\Lambda)$ for $r \geq 1$. Taking sum over $r \geq 1$ we obtain

$$\tilde{\epsilon}_i(1 \otimes y) = 1 \otimes \tilde{E}_i y \mod v^{n-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda).$$

Using this and (e), we deduce that $1 \otimes \tilde{E}_i y \in v^{n-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$. Applying pr, we obtain

$$pr(1 \otimes \tilde{E}_i y) \in v^{n-1} pr(\mathcal{L}(\mathbf{f}) \odot L'(\Lambda));$$

hence $\tilde{E}_i y \in v^{n-1} L'(\Lambda)$. The lemma is proved.

Lemma 18.3.3. *Let ν be such that $\operatorname{tr} \nu \leq N$, let $i \in I$ and let $x \in L'(\Lambda)_\nu$. Write $x = \sum_{r=0}^{\nu_i} F_i^{(r)} x_r$ where the $x_r \in (\Lambda)_{\nu - ri}$ satisfy $E_i x_r = 0$ for all r and $x_r = 0$ unless $r + \langle i, \lambda - \nu \rangle \geq 0$ (see 16.1.4). Then $x_r \in L'(\Lambda)_{\nu - ri}$ for all r.*

When $\operatorname{tr} \nu < N$, this is just Lemma 18.2.2(a). In the case where $\operatorname{tr} \nu = N$, we can use the same proof since the inclusion $\tilde{\epsilon}_i(L(\Lambda)_\nu) \subset L'(\Lambda)$ is now known for $\operatorname{tr} \nu = N$ by the previous lemma.

Lemma 18.3.4. *Let $x \in L'(\Lambda)_\nu$ where $\mathrm{tr}\,\nu \leq N$. We have*

$$\tilde{\epsilon}_i(1 \otimes x) = 1 \otimes \tilde{E}_i x \mod v^{-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda).$$

Using the previous lemma we can assume that $x = F_i^{(r)} x'$ where $x' \in L'(\Lambda)_{\nu - ri}$ and $E_i x' = 0$. We can assume that $x' \neq 0$. Then

$$n = \langle i, \lambda - \nu + ri' \rangle \in \mathbf{N};$$

we have $F_i^{(n+1)} x' = 0$. By 17.1.15, $\tilde{\epsilon}_i(1 \otimes F_i^{(s)} x') = 1 \otimes F_i^{(s-1)} x'$ mod $v^{-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$. The lemma follows.

Lemma 18.3.5. *Assume that $\mathrm{tr}\,\nu \leq N$. Then*

$$\tilde{\epsilon}_i(\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)_\nu) \subset \mathcal{L}(\mathbf{f}) \odot L'(\Lambda).$$

When $\mathrm{tr}\,\nu < N$ this is shown in 18.2.5. When $\mathrm{tr}\,\nu \leq N$, the same proof applies since Lemma 18.3.3 is now available.

Lemma 18.3.6. *Assume that $\mathrm{tr}\,\nu = N$. Let i be such that $\nu_i > 0$. Then $\tilde{E}_i(b^- \eta) = (\tilde{\epsilon}_i b)^- \eta$ mod $v^{-1}L'(\Lambda)$, for all $b \in \mathcal{B}_\nu$ such that $b^- \eta \neq 0$.*

We can find $i_1, i_2, \ldots, i_N$ with $i_1 + i_2 + \cdots + i_N = \nu$ such that

$$b = \tilde{\phi}_{i_1} \tilde{\phi}_{i_2} \cdots \tilde{\phi}_{i_N} 1 \mod v^{-1}\mathcal{L}(\mathbf{f})$$

(see 18.1.7(b)). Then

$$y = \tilde{F}_{i_1} \tilde{F}_{i_2} \cdots \tilde{F}_{i_N} \eta \in L(\Lambda)_\nu$$

satisfies $b^- \eta = y$ mod $v^{-1}L'(\Lambda)_\nu$ (see Lemma 18.2.10) and $y \notin v^{-1}L'(\Lambda)$. By 18.3.1, we have $\tilde{\phi}_{i_1} \tilde{\phi}_{i_2} \cdots \tilde{\phi}_{i_N}(1 \otimes \eta) = 1 \otimes y = 1 \otimes b^- \eta$ up to elements in $v^{-1} \sum_{\mathrm{tr}\,\nu' \leq N} \mathcal{L}(\mathbf{f}) \odot L'(\Lambda)_{\nu'}$. From this we deduce using Lemma 18.3.5, that

$$\tilde{\epsilon}_i \tilde{\phi}_{i_1} \tilde{\phi}_{i_2} \cdots \tilde{\phi}_{i_N}(1 \otimes \eta) = \tilde{\epsilon}_i(1 \otimes b^- \eta) \mod v^{-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda).$$

We have

$$\begin{aligned} \tilde{\epsilon}_i \tilde{\phi}_{i_1} \tilde{\phi}_{i_2} \cdots \tilde{\phi}_{i_N}(1 \otimes \eta) &= \tilde{\epsilon}_i \tilde{\phi}_{i_1} \tilde{\phi}_{i_2} \cdots \tilde{\phi}_{i_N}(\Xi(1)) \\ &= \Xi(\tilde{\epsilon}_i \tilde{\phi}_{i_1} \tilde{\phi}_{i_2} \cdots \tilde{\phi}_{i_N} 1) \\ &= \Xi(\tilde{\epsilon}_i(b)) \end{aligned}$$

modulo $v^{-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$ (using Lemma 18.2.7). Using Lemma 18.3.4, we have $\tilde{\epsilon}_i(1 \otimes b^- \eta) = 1 \otimes \tilde{E}_i(b^- \eta)$ mod $v^{-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$. We deduce that $\Xi(\tilde{\epsilon}_i(b)) = 1 \otimes \tilde{E}_i(b^- \eta)$ mod $v^{-1}\mathcal{L}(\mathbf{f}) \odot L'(\Lambda)$. Apply pr to this congruence and use $pr(\Xi(x)) = x^- \eta$ for all $x \in \mathbf{f}$. We deduce that $(\tilde{\epsilon}_i(b))^- \eta = \tilde{E}_i(b^- \eta)$ mod $v^{-1}L'(\Lambda)$. The lemma is proved.

18.3.7. From the lemmas above, we see that, if we assume the general hypothesis 18.2.1 for N, then the properties (a),(b),(c) in 18.2.1 also hold when N is replaced by $N+1$. Since they are obvious for $N=1$, we see that we have proved by induction the following result.

Theorem 18.3.8. *Let* $\nu \in \mathbf{N}[I]$. *We have*

(a) $L(\Lambda)_\nu = L'(\Lambda)_\nu$;

(b) *for any* i *we have* $\tilde{F}_i(x^-\eta) = (\tilde{\phi}_i x)^-\eta \mod v^{-1}L(\Lambda)$, *for all* $x \in \mathcal{L}(\mathbf{f})_\nu$;

(c) *if* i *is such that* $\nu_i > 0$, *then* $\tilde{E}_i(b^-\eta) = (\tilde{\epsilon}_i b)^-\eta \mod v^{-1}L(\Lambda)$, *for all* $b \in \mathcal{B}_\nu$ *such that* $b^-\eta \neq 0$; *in particular,* $\tilde{E}_i(L(\Lambda)_\nu) \subset L(\Lambda)_{\nu-i}$.

From now on, we shall not distinguish between $L(\Lambda)$ and $L'(\Lambda)$.