CHAPTER 17

Applications

17.1. First Application to Tensor Products

17.1.1. In this chapter we shall give three applications of Proposition 16.3.5: two to tensor products, and one to $\mathbf{f}$.

17.1.2. Let $\tilde{M}, M \in \mathcal{C}'_i$. Then $\tilde{M} \otimes M$ is an object in $\mathcal{C}'_i$ (see 5.3.1). Now let $P \in \mathcal{D}_i$ and $M \in \mathcal{C}'_i$. We define $\mathbf{Q}(v)$-linear maps $\phi_i, \epsilon_i : P \otimes M \to P \otimes M$ by

$$\phi_i(x \otimes y) = \phi_i(x) \otimes \tilde{K}_i^{-1} y + x \otimes F_i(y)$$
$$\epsilon_i(x \otimes y) = \epsilon_i(x) \otimes \tilde{K}_i^{-1} y + (v_i - v_i^{-1}) x \otimes \tilde{K}_i^{-1} E_i(y)$$

where $\tilde{K}_i : M \to M$ is the linear map given by $\tilde{K}_i y = v_i^n y$ for $y \in M^n$. It is easy to check that $(P \otimes M, \phi_i, \epsilon_i)$ is an object of $\mathcal{D}_i$. (This also follows from 15.1.5.) Hence the linear maps $\tilde{\phi}_i, \tilde{\epsilon}_i : P \otimes M \to P \otimes M$ are well-defined.

From the definitions we deduce (using the quantum binomial formula) that

$$\phi_i^{(t)}(x \otimes y) = \sum v_i^{-t't''} \phi_i^{(t')} x \otimes \tilde{K}_i^{-t'} F_i^{(t'')} y$$

for all $x \in P, y \in M$ and $t \geq 0$; the sum is taken over all $t', t'' \in \mathbf{N}$ such that $t' + t'' = t$.

Lemma 17.1.3. (a) *If* $(,) : P \times P \to \mathbf{Q}(v)$ *and* $(,) : M \times M \to \mathbf{Q}(v)$ *are admissible symmetric bilinear forms in the sense of 16.2.2, then the symmetric bilinear form on* $P \otimes M$ *given by* $(x \otimes y, x' \otimes y') = (x, x')(y, y')$ *is admissible.*

(b) *If* $(,) : \tilde{M} \times \tilde{M} \to \mathbf{Q}(v)$ *and* $(,) : M \times M \to \mathbf{Q}(v)$ *are admissible symmetric bilinear forms in the sense of 16.2.2, then the symmetric bilinear form on* $\tilde{M} \otimes M$ *given by* $(x \otimes y, x' \otimes y') = (x, x')(y, y')$ *is admissible.*

We prove (a). Let $x, x' \in P, y \in M^n, y' \in M^{n'}$. We have

$$(x \otimes y, \epsilon_i(x' \otimes y')) = (x \otimes y, \epsilon_i(x') \otimes \tilde{K}_i^{-1} y' + (v_i - v_i^{-1}) x' \otimes \tilde{K}_i^{-1} E_i(y'))$$
$$= (x, \epsilon_i(x'))(y, \tilde{K}_i^{-1} y') + (v_i - v_i^{-1})(x, x')(y, \tilde{K}_i^{-1} E_i(y'))$$
$$= \delta_{n,n'} v_i^{-n}(1 - v_i^{-2})(\phi_i x, x')(y, y')$$
$$+ \delta_{n-2,n'} v_i^{-n'-2} v_i^{n-1}(v_i - v_i^{-1})(x, x')(F_i(y), y').$$

On the other hand, we have

$$(\phi_i(x \otimes y), x' \otimes y') = (\phi_i(x) \otimes \tilde{K}_i^{-1} y + x \otimes F_i(y), x' \otimes y')$$
$$= \delta_{n,n'} v_i^{-n} (\phi_i(x), x')(y, y') + \delta_{n-2,n'}(x, x')(F_i(y), y').$$

This proves (a). The proof of (b) is entirely similar.

17.1.4. We consider the following example. Let P_0 be the $\mathbf{Q}(v)$-vector space with basis $\beta_0, \beta_1, \beta_2, \ldots$. We define $\mathbf{Q}(v)$-linear maps $\phi_i, \epsilon_i : P_0 \to P_0$ by $\phi_i(\beta_s) = [s+1]_i \beta_{s+1}$ for $s \geq 0$ and $\epsilon_i(\beta_s) = v_i^{s-1} \beta_{s-1}$ for $s \geq 0$ (with the convention $\beta_{-1} = 0$). It is easy to check that this makes P_0 into an object of $\mathcal{D}_i$.

We have $\phi_i^{(t)}(\beta_s) = \left[\begin{smallmatrix} s+t \\ t \end{smallmatrix} \right]_i \beta_{s+t}$ for $s \geq 0, t \geq 0$, and $\epsilon_i^t(\beta_s) = v_i^{-t(t+1)/2+st} \beta_{s-t}$, for $s \geq 0, t \geq 0$, (with the convention $\beta_{-1} = \beta_{-2} = \cdots = 0$). Let $(,)$ be the symmetric bilinear form on P_0 given by

$$(\beta_s, \beta_{s'}) = \delta_{s,s'} \prod_{t=1}^{s} (1 - v_i^{-2t})^{-1}.$$

It is easy to check that this bilinear form is admissible.

17.1.5. We fix an integer $n \geq 0$. Let M_n be the $\mathbf{Q}(v)$-vector space with basis $b_0, b_1, \ldots, b_n$ with $\mathbf{Z}$-grading such that b_m has degree $n - 2m$. It will be convenient to define $b_m = 0$ for $m > n$ and for $m < 0$.

Let $E_i, F_i : M_n \to M_n$ be the linear maps given by $E_i(b_s) = [n-s+1]_i b_{s-1}$ and $F_i(b_s) = [s+1]_i b_{s+1}$ for all s. It is easy to check that in this way M_n is an object of $\mathcal{C}'_i$. Note that for $t \geq 0$, we have $E_i^{(t)}(b_s) = \left[\begin{smallmatrix} n-s+t \\ t \end{smallmatrix} \right]_i b_{s-t}$ and $F_i^{(t)}(b_s) = \left[\begin{smallmatrix} s+t \\ t \end{smallmatrix} \right]_i b_{s+t}$ for all s. Let $(,)$ be the bilinear form on M_n given by $(b_s, b_{s'}) = \delta_{s,s'} v_i^{-s(n-s)} \left[\begin{smallmatrix} n \\ s \end{smallmatrix} \right]_i$ for $0 \leq s \leq n$ and $0 \leq s' \leq n$. It is easy to check that $(,)$ is an admissible form on M_n.

17.1.6. Let P_0 be as in 17.1.4 and let M_n be as in 17.1.5 ($n \geq 0$). Then, as in 17.1.2, $P = P_0 \otimes M_n$ is a well-defined object of $\mathcal{D}_i$. Note that P has a basis $\{b_{s,s'} = \beta_s \otimes b_{s'} | s \geq 0, 0 \leq s' \leq n\}$.

For any $t \geq 0$ we have

$$\phi_i^{(t)} b_{s,s'} = \sum v_i^{-t'(n-2s'-t'')} \begin{bmatrix} t'+s \\ t' \end{bmatrix}_i \begin{bmatrix} t''+s' \\ t'' \end{bmatrix}_i b_{s+t',s'+t''}$$

where the sum is taken over all $t', t'' \in \mathbf{N}$ such that $t' + t'' = t$, and

$$\epsilon_i b_{s,s'} = v_i^{-n+2s'+s-1} b_{s-1,s'} + v_i^{-n+2s'-2}(v_i - v_i^{-1})[n - s' + 1]_i b_{s,s'-1}.$$

By convention, we set $b_{s,s'} = 0$ if either $s < 0$ or $s' < 0$ or $s' > n$.

For any $m \in \mathbf{Z}$, we define P^m to be the $\mathbf{Q}(v)$-subspace of P spanned by the vectors $b_{s,s'}$ with $s + s' = m$. We have $P = \oplus_m P^m$, $\phi_i^{(t)} P^m \subset P^{m+t}$ and $\epsilon_i P^m \subset P^{m-1}$.

By definition, the operator $\tilde{\phi}_i : P \to P$ (resp. $\tilde{\epsilon}_i : P \to P$) is an (infinite) linear combination of operators $\phi_i^{(t+1)} \epsilon_i^t$ (resp. $\phi_i^{(t)} \epsilon_i^{t+1}$); it follows that

(a) $\tilde{\phi}_i(P^m) \subset P^{m+1}$ and $\tilde{\epsilon}_i(P^m) \subset P^{m-1}$.

17.1.7. Let

$$\zeta_{s,s'} = \sum_{t=0}^{s'} v_i^{-t(n+t-s')} \begin{bmatrix} s+t \\ t \end{bmatrix}_i b_{s+t,s'-t}$$

for $s \geq 0, s' \geq 0,\ s + s' \leq n$,

$$\zeta_{s,s'} = \sum_{t=0}^{s'} v_i^{-t(s+t)} \begin{bmatrix} n+t-s' \\ t \end{bmatrix}_i b_{s+t,s'-t}$$

for $s \geq 0,\ 0 \leq s' \leq n,\ s + s' \geq n$; the two definitions agree if $s + s' = n$. Note that $\zeta_{s,s'} \in P^{s+s'}$.

For $s + s' \geq n$, we have

(a) $$b_{s,s'} = \sum_{t'=0}^{s'} v_i^{-t's-t'} \begin{bmatrix} -1-n+s' \\ t' \end{bmatrix}_i \zeta_{s+t',s'-t'}.$$

Indeed, the right hand side of this equality is, by definition,

$$\sum_{t'=0}^{s'} \sum_{t''=0}^{s'-t'} v_i^{-t''(s+t'+t'')-t's-t'} \begin{bmatrix} n+t''-s'+t' \\ t'' \end{bmatrix}_i \begin{bmatrix} -1-n+s' \\ t' \end{bmatrix}_i b_{s+t'+t'',s'-t'-t''}.$$

The coefficient of $b_{s+t,s'-t}$ (where $0 \leq t \leq s'$) is

$$\sum_{t'+t''=t} v_i^{-t''(s+t'+t'')-t's-t'} \begin{bmatrix} n+t-s' \\ t'' \end{bmatrix}_i \begin{bmatrix} -1-n+s' \\ t' \end{bmatrix}_i.$$

We replace the exponent $-t''(s + t' + t'') - t's - t'$ by $(n + t - s')t' - (-1 - n + s')t'' + f$ where $f = t(-n - t - s + s' - 1)$ depends on t', t'' only through their sum. Hence the coefficient of $b_{s+t,s'-t}$ is

$$v_i^f \sum_{t'+t''=t} v_i^{(n+t-s')t'-(-1-n+s')t''} \begin{bmatrix} n+t-s' \\ t'' \end{bmatrix}_i \begin{bmatrix} -1-n+s' \\ t' \end{bmatrix}_i$$

$$= v^f \begin{bmatrix} t-1 \\ t \end{bmatrix} = v^f \delta_{t,0} = \delta_{t,0};$$

(a) is proved.

From the definitions and from (a), we see that the subset B of P consisting of the vectors $\zeta_{s,s'}$ (with $s \geq 0$ and $0 \leq s' \leq n$) is a basis of P.

For $m \in \mathbf{Z}$, let $\mathcal{L}^m$ be the $\mathbf{Z}[v^{-1}]$-submodule of P generated by the vectors $b_{s,s'}$ with $s + s' = m$.

Lemma 17.1.8. (a) *For any $s \geq 0$ and $0 \leq s' \leq n$, we have*

$$\zeta_{s,s'} - b_{s,s'} \in v^{-1}\mathcal{L}^{s+s'}.$$

(b) *For $m \geq 0$, $\mathcal{L}^m$ is the $\mathbf{Z}[v^{-1}]$-submodule of P generated by the vectors $\zeta_{s,s'}$ with $s + s' = m$.*

Assume first that $s + s' \leq n$. The coefficient of $b_{s+t,s'-t}$ in $\zeta_{s,s'}$ is in $v_i^{-t(n+t-s')+st}(1 + v^{-1}\mathbf{Z}[v^{-1}])$. Here $t \geq 0$; hence $-t(n + t - s') + st = t(s + s' - n) - t^2 \leq 0$; the inequality becomes an equality only for $t = 0$.

Assume next that $s + s' \geq n$. The coefficient of $b_{s+t,s'-t}$ in $\zeta_{s,s'}$ is in

$$v_i^{-t(s+t)+t(n-s')}(1 + v^{-1}\mathbf{Z}[v^{-1}]).$$

Here $t \geq 0$; hence $-t(s+t)+t(n-s') = t(n-s-s')-t^2 \leq 0$; the inequality becomes an equality only for $t = 0$. This proves (a).

The previous proof also shows that the matrix expressing the vectors $\zeta_{s,s'}$ in terms of the vectors $b_{s,s'}$ (with $s+s' = m$ fixed) is upper triangular, with diagonal entries equal to 1 and with off-diagonal entries in $v^{-1}\mathbf{Z}[v^{-1}]$. This implies (b). The lemma is proved.

Lemma 17.1.9. *The $\mathcal{A}$-submodule ${}_{\mathcal{A}}P$ of P generated by B is stable under $\epsilon_i, \phi_i^{(t)} : P \to P$ for all $t \geq 0$. (In other words, the basis B of P is integral.)*

The formulas in 17.1.6 show that $\epsilon_i(b_{s,s'}) \in {}_{\mathcal{A}}P$ and $\phi_i^{(t)}(b_{s,s'}) \in {}_{\mathcal{A}}P$ for all $t \geq 0$. The lemma follows.

Lemma 17.1.10. *Assume that $0 \leq s \leq n$ and $t \geq 0$. We have*

$$\phi_i^{(t)} b_{s,0} = \zeta_{s,t}$$

if $s + t \leq n$ and

$$\phi_i^{(t)} b_{s,0} = \sum_{u;u\geq 0;u\geq t-n;u\leq s+t-n} \begin{bmatrix} t+s-n \\ u \end{bmatrix}_i \zeta_{s+u,t-u}$$

if $s+t \geq n$.

Assume first that $s+t \leq n$. We have

$$\phi_i^{(t)} b_{s,0} = \sum_{t'=0}^{t} v_i^{-t'(n-t+t')} \begin{bmatrix} t'+s \\ t' \end{bmatrix}_i b_{s+t',t-t'} = \zeta_{s,t}.$$

Assume now that $s+t \geq n$. We have

$$\phi_i^{(t)} b_{s,0} = \sum_{t';t'\geq 0;t'\leq t;t-t'\leq n} v_i^{-t'(n-t+t')} \begin{bmatrix} t'+s \\ t' \end{bmatrix}_i b_{s+t',t-t'}$$

$$= \sum_{t',t''\in\mathbf{N};t'+t''\leq t;t-t'\leq n} v_i^{-t''(s+t')-t''} v_i^{-t'(n-t+t')}$$

$$\times \begin{bmatrix} -1-n+t-t' \\ t'' \end{bmatrix}_i \begin{bmatrix} t'+s \\ t' \end{bmatrix}_i \zeta_{s+t'+t'',t-t'-t''}$$

$$= \sum_{u=0}^{t} \Big(\sum_{t',t'';t'+t''=u;t'\geq 0;t''\geq 0;t'\geq t-n} v_i^{-t''(s+t')-t''-t'(n-t+t')}$$

$$\times \begin{bmatrix} -1-n+t-t' \\ t'' \end{bmatrix}_i \begin{bmatrix} t'+s \\ t' \end{bmatrix}_i \zeta_{s+u,t-u} \Big).$$

Since the index t' satisfies $t' \geq t-n$ and $u \geq t'$, the index u must satisfy $u \geq t-n$. We substitute $\left[\begin{smallmatrix} -1-n+t-t' \\ t'' \end{smallmatrix}\right]_i = (-1)^{t''} \left[\begin{smallmatrix} n-t+u \\ t'' \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} t'+s \\ t' \end{smallmatrix}\right]_i = (-1)^{t'} \left[\begin{smallmatrix} -s-1 \\ t' \end{smallmatrix}\right]_i$ and $v_i^{-t''(s+t')-t''-t'(n-t+t')} = v_i^{-(n-t+u)t'+(-s-1)t''}$.

The condition on u implies $n-t+u \geq 0$; hence $\left[\begin{smallmatrix} n-t+u \\ t'' \end{smallmatrix}\right]_i$ is automatically zero unless $n-t+u \geq t''$, i.e., if $t' \geq t-n$. Thus the condition $t' \geq t-n$ can be omitted in the summation and we obtain

$$\sum_{\substack{0\leq u\leq t \\ u\geq t-n}} \sum_{\substack{t',t''\geq 0 \\ t'+t''=u}} (-1)^u v^{-(n-t+u)t'+(-s-1)t''}$$

$$\times \begin{bmatrix} n-t+u \\ t'' \end{bmatrix}_i \begin{bmatrix} -s-1 \\ t' \end{bmatrix}_i \zeta_{s+u,t-u}$$

$$= \sum_{u;0\leq u\leq t;u\geq t-n} (-1)^u \begin{bmatrix} n-t+u-s-1 \\ u \end{bmatrix}_i \zeta_{s+u,t-u}$$

$$= \sum_{u;0\leq u\leq t;u\geq t-n} \begin{bmatrix} -n+t+s \\ u \end{bmatrix}_i \zeta_{s+u,t-u}.$$

(We have used 1.3.1(e), 1.3.1(a).) Recall that $s+t \geq n$. It follows that $\left[\begin{smallmatrix} -n+t+s \\ u \end{smallmatrix}\right]_i = 0$ unless $u \leq t+s-n$ and then the condition $u \leq t$ is automatic. Hence our sum becomes $\sum_{u;u\geq 0;u\geq t-n;u\leq s+t-n} \left[\begin{smallmatrix} t+s-n \\ u \end{smallmatrix}\right]_i \zeta_{s+u,t-u}$. The lemma is proved.

Lemma 17.1.11. *We consider the partition of B into the subsets*

$$B(t) = \{\zeta_{s,t} | s + t \leq n\} \cup \{\zeta_{t+2s-n,n-s} | s + t > n\}$$

where $t = 0, 1, 2, \ldots$.

(a) *For any* $t \geq 0$, *the set* $B(t) \cup B(t + 1) \cup B(t + 2) \cup \cdots$ *is a basis of* $\phi_i^{(t)} P$.

(b) *The basis B of P is adapted.*

From 17.1.10, we see that

$$\zeta_{s,t} \in \phi_i^{(t)} P$$

if $s + t \leq n$ and

$$\zeta_{s,t} \in \phi_i^{(2t+s-n)} P$$

if $s + t \geq n$. (The last inclusion is seen by induction on t.) It follows that

(c) $B(t) \cup B(t + 1) \cup B(t + 2) \cup \cdots \subset \phi_i^{(t)} P$.

Hence $X(t) \subset \phi_i^{(t)} P$ where $X(t)$ is the subspace of P spanned by $B(t) \cup B(t + 1) \cup B(t + 2) \cup \cdots$. We now prove the inclusion

(d) $\phi_i^{(t)} b \subset X(t)$

for any $b \in B \cap P^m$, by induction on $m \geq 0$.

Note that $B(0) = \{\zeta_{s,0} | 0 \leq s \leq n\} = \{b_{s,0} | 0 \leq s \leq n\}$. If $b \in B(0)$, then (d) follows from 17.1.10. If $m = 0$, then $b = b_{0,0} \in B(0)$, hence (d) holds. Assume now that $m \geq 1$. If $b \in B(0)$, then (d) holds; hence we may assume that $b \in B - B(0)$. Then $b \in X(1)$ and by (c) we have $b = \phi_i y$ where $y \in P^{m-1}$. By the induction hypothesis we have $\phi_i^{(t+1)} y \in X(t+1) \subset X(t)$, hence $\phi_i^{(t)} b \in X(t)$. This proves (d). Thus (a) is proved.

From (a) we see that $\{b \in B | b \notin \phi_i P\} = B(0)$. Let $\pi_t : B_0 \to B_t$ be the bijection given by

(e) $\pi_t \zeta_{s,0} = \zeta_{s,t}$ if $s + t \leq n$ and $\pi_t \zeta_{s,0} = \zeta_{t+2s-n,n-s}$ if $s + t \geq n$.

From 17.1.10, we see that $\phi_i^{(t)} \zeta_{s,0} - \pi_t \zeta_{s,0} \in X(t + 1)$, hence

(f) $\phi_i^{(t)} \zeta_{s,0} = \pi_t \zeta_{s,0} \bmod \phi_i^{(t+1)} P$.

The lemma is proved.

Lemma 17.1.12. *Consider the admissible form $(,)$ on P defined as in 17.1.3, in terms of the admissible forms 17.1.4, 17.1.5, on M_n and P_0. Then B is almost orthonormal with respect to $(,)$.*

From the definition it is clear that the basis $(b_{s,s'})$ of P is almost orthonormal (actually different elements in this basis are orthogonal to each other). Since B is related to this basis as described in 17.1.8, it follows that B is also almost orthonormal.

We now see that the hypotheses of 16.3.5 are verified in our case. Applying Proposition 16.3.5 to B, and taking into account 17.1.11(e),(f), we obtain the following result.

Proposition 17.1.13. *We have*

$\tilde{\phi}_i(\zeta_{s,s'}) = \zeta_{s,s'+1} \mod v^{-1}\mathcal{L}(P)$ *if* $s+s'<n$,

$\tilde{\phi}_i(\zeta_{s,s'}) = \zeta_{s+1,s'} \mod v^{-1}\mathcal{L}(P)$ *if* $s+s'\geq n$,

$\tilde{\epsilon}_i(\zeta_{s,s'}) = \zeta_{s,s'-1} \mod v^{-1}\mathcal{L}(P)$ *if* $s+s'\leq n$ *and* $s'\geq 1$,

$\tilde{\epsilon}_i(\zeta_{s,s'}) = \zeta_{s-1,s'} \mod v^{-1}\mathcal{L}(P)$ *if* $s+s'>n$,

$\tilde{\epsilon}_i(\zeta_{s,0}) = 0 \mod v^{-1}\mathcal{L}(P)$ *if* $s\leq n$.

Using Lemma 17.1.8, we can restate the proposition as follows.

Corollary 17.1.14. $\tilde{\phi}_i(b_{s,s'}) = b_{s,s'+1} \mod v^{-1}\mathcal{L}(P)\cap P^{s+s'+1}$ *if* $s+s'<n$,

$\tilde{\phi}_i(b_{s,s'}) = b_{s+1,s'} \mod v^{-1}\mathcal{L}(P)\cap P^{s+s'+1}$ *if* $s+s'\geq n$,

$\tilde{\epsilon}_i(b_{s,s'}) = b_{s,s'-1} \mod v^{-1}\mathcal{L}(P)\cap P^{s+s'-1}$ *if* $s+s'\leq n$,

$\tilde{\epsilon}_i(b_{s,s'}) = b_{s-1,s'} \mod v^{-1}\mathcal{L}(P)\cap P^{s+s'-1}$ *if* $s+s'>n$.

What we actually get are the statements of the corollary with $\mathcal{L}(P)\cap P^{s+s'\pm 1}$ replaced by $\mathcal{L}(P)$. But $b_{s,s'}\in P^{s+s'}$; hence from 17.1.6(a), $\tilde{\phi}_i(b_{s,s'})\in P^{s+s'+1}$ and $\tilde{\epsilon}_i(b_{s,s'})\in P^{s+s'-1}$. The corollary follows.

Corollary 17.1.15. *Let* $P\in\mathcal{D}_i$, $M\in\mathcal{C}'_i$ *and let* $(P\otimes M,\phi_i,\epsilon_i)\in\mathcal{D}_i$ *be defined as in 17.1.2. Let* $x\in P$ *and* $y\in M^n$ *be such that* $\epsilon_i x=0, E_i y=0$. *(Then* $n\geq 0$.*) For any* $m\geq 0$, *let* $\mathcal{L}_m$ *be the* $\mathbf{Z}[v^{-1}]$*-submodule of* $P\otimes M$ *generated by the vectors* $\phi_i^{(s)}x\otimes F_i^{(s')}y$ *with* $s+s'=m$. *We set* $\mathcal{L}_{-1}=0$. *We have*

$\tilde{\phi}_i(\phi_i^{(s)}x\otimes F_i^{(s')}y) = \phi_i^{(s)}x\otimes F_i^{(s'+1)}y \mod v^{-1}\mathcal{L}_{s+s'+1}$ *if* $s+s'<n$;

$\tilde{\phi}_i(\phi_i^{(s)}x\otimes F_i^{(s')}y) = \phi_i^{(s+1)}x\otimes F_i^{(s')}y \mod v^{-1}\mathcal{L}_{s+s'+1}$ *if* $s+s'\geq n$;

$\tilde{\epsilon}_i(\phi_i^{(s)}x\otimes F_i^{(s')}y) = \phi_i^{(s)}x\otimes F_i^{(s'-1)}y \mod v^{-1}\mathcal{L}_{s+s'-1}$ *if* $s+s'\leq n$;

$\tilde{\epsilon}_i(\phi_i^{(s)}x\otimes F_i^{(s')}y) = \phi_i^{(s-1)}x\otimes F_i^{(s')}y \mod v^{-1}\mathcal{L}_{s+s'-1}$ *if* $s+s'>n$.

We may identify $P_0\otimes M_n$ with the subspace of $P\otimes M$ spanned by the vectors $\phi_i^{(s)}x\otimes F_i^{(s')}y$ with $s\geq 0$ and $0\leq s'\leq n$. It is in fact a subobject in $\mathcal{D}_i$. Therefore the result follows from the previous corollary.

17.2. Second Application to Tensor Products

17.2.1. We consider two integers $p\geq 0$ and $n\geq 0$ and form the tensor product $M=M_p\otimes M_n$. This is again an object of $\mathcal{C}'_i$; hence the operators

$\tilde{E}_i, \tilde{F}_i : M_p \otimes M_n \to M_p \otimes M_n$ are well-defined. Now M_n has a basis $\{b_{s'}|0 \le s' \le n\}$ as in 17.1.5; similarly, M_p has a basis $\{b_s|0 \le s \le p\}$ as in 17.1.5; Then

$$\{b_{s,s'} = b_s \otimes b_{s'}|0 \le s \le p, 0 \le s' \le n\}$$

is a basis of M. As in 17.1.7, we define

$$\zeta_{s,s'} = \sum_{t=0}^{s'} v_i^{-t(n+t-s')} \begin{bmatrix} s+t \\ t \end{bmatrix}_i b_{s+t,s'-t}$$

for $0 \le s \le p$, $s' \ge 0$, $s + s' \le n$,

$$\zeta_{s,s'} = \sum_{t=0}^{s'} v_i^{-t(s+t)} \begin{bmatrix} n+t-s' \\ t \end{bmatrix}_i b_{s+t,s'-t}$$

for $0 \le s \le p$, $0 \le s' \le n$, $s+s' \ge n$; the two definitions agree if $s+s' = n$.

The vectors $\zeta_{s,s'}$ just described form a basis B of the vector space M, which is related to the basis $(b_{s,s'})$ by a matrix with entries in $\mathbf{Z}[v^{-1}]$ whose constant terms form the identity matrix. (This is seen as in 17.1.8 or can be deduced from that lemma, using the natural surjective map $P \to M$ which takes $b_{s,s'}$ to $b_{s,s'}$ if $s \le p$ and to zero if $s > p$; that map also takes $\zeta_{s,s'}$ to $\zeta_{s,s'}$ if $s \le p$ and to zero if $s > p$.) Hence the $\mathbf{A}(\mathbf{Z})$-submodule of M, generated by the elements $(b_{s,s'})$, coincides with the $\mathbf{A}(\mathbf{Z})$-submodule generated by the elements $(\zeta_{s,s'})$; we denote it by $\mathcal{L}(M)$.

As in 17.1.9, we see that the $\hat{\mathcal{A}}$-submodule of M generated by B is stable under $E_i^{(n)}, F_i^{(n)}$; hence B is an integral basis. As in 17.1.12, we see that B is almost orthonormal with respect to the form $(,)$ on M defined as in 17.1.3 in terms of the forms $(,)$ on M_p, M_n (see 17.1.5). As in 17.1.11, we see that the basis B of M is adapted. (Again, this could be deduced from the corresponding result for P.)

We now see that the hypotheses of 16.3.5 are verified in our case. Applying Proposition 16.3.5 to B, we obtain the following result, analogous to 17.1.13.

Proposition 17.2.2. *We have*

$\tilde{F}_i(\zeta_{s,s'}) = \zeta_{s,s'+1} \mod v^{-1}\mathcal{L}(M)$ *if* $s + s' < n$;

$\tilde{F}_i(\zeta_{s,s'}) = \zeta_{s+1,s'} \mod v^{-1}\mathcal{L}(M)$ *if* $s < p$ *and* $s + s' \ge n$;

$\tilde{F}_i(\zeta_{p,s'}) = 0 \mod v^{-1}\mathcal{L}(M)$ *if* $s + s' \ge n$;

$\tilde{E}_i(\zeta_{s,s'}) = \zeta_{s,s'-1} \mod v^{-1}\mathcal{L}(M)$ *if* $s + s' \le n$ *and* $s' \ge 1$;

$\tilde{E}_i(\zeta_{s,s'}) = \zeta_{s-1,s'} \mod v^{-1}\mathcal{L}(M)$ *if* $s + s' > n$;

$\tilde{E}_i(\zeta_{s,0}) = 0 \mod v^{-1}\mathcal{L}(M)$ *if* $s \le n$.

As in 17.1.4, we can restate the proposition as follows.

Corollary 17.2.3. *$\tilde{F}_i(b_{s,s'}) = b_{s,s'+1} \mod v^{-1}\mathcal{L}(M)$ if $s+s'<n$;*

$\tilde{F}_i(b_{s,s'}) = b_{s+1,s'} \mod v^{-1}\mathcal{L}(M)$ if $s<p$ and $s+s'\geq n$;

$\tilde{F}_i(b_{p,s'}) = 0 \mod v^{-1}\mathcal{L}(M)$ if $s+s'\geq n$;

$\tilde{E}_i(b_{s,s'}) = b_{s,s'-1} \mod v^{-1}\mathcal{L}(M)$ if $s+s'\leq n$ and $s'\geq 1$;

$\tilde{E}_i(b_{s,s'}) = b_{s-1,s'} \mod v^{-1}\mathcal{L}(M)$ if $s+s'>n$;

$\tilde{E}_i(b_{s,0}) = 0 \mod v^{-1}\mathcal{L}(M)$ if $s\leq n$.

Corollary 17.2.4. *Let $\tilde{M}\in \mathcal{C}'_i$, $M\in\mathcal{C}'_i$ and let $\tilde{M}\otimes M\in\mathcal{C}'_i$ be defined as in 5.3.1. Let $x\in\tilde{M}^p$ and $y\in M^n$ be such that $E_ix=0, E_iy=0$. (Then $p\geq 0, n\geq 0$.) For any $m\geq 0$, let $\mathcal{L}$ be the $\mathbf{A}(\mathbf{Z})$-submodule of $\tilde{M}\otimes M$ generated by the vectors $F_i^{(s)}x\otimes F_i^{(s')}y$. We have*

$$\tilde{F}_i(F_i^{(s)}x\otimes F_i^{(s')}y) = F_i^{(s)}x\otimes F_i^{(s'+1)}y \mod v^{-1}\mathcal{L} \text{ if } s+s'<n;$$
$$\tilde{F}_i(F_i^{(s)}x\otimes F_i^{(s')}y) = F_i^{(s+1)}x\otimes F_i^{(s')}y \mod v^{-1}\mathcal{L} \text{ if } s+s'\geq n;$$
$$\tilde{E}_i(F_i^{(s)}x\otimes F_i^{(s')}y) = F_i^{(s)}x\otimes F_i^{(s'-1)}y \mod v^{-1}\mathcal{L} \text{ if } s+s'\leq n;$$
$$\tilde{E}_i(F_i^{(s)}x\otimes F_i^{(s')}y) = F_i^{(s-1)}x\otimes F_i^{(s')}y \mod v^{-1}\mathcal{L} \text{ if } s+s'>n.$$

We may identify $M_p\otimes M_n$ with the subspace of $\tilde{M}\otimes M$ spanned by the vectors $F_i^{(s)}x\otimes F_i^{(s')}y$ with $0\leq s\leq p$ and $0\leq s'\leq n$. It is in fact a subobject in $\mathcal{C}'_i$. Therefore the result follows from the previous corollary.

17.3. The Operators $\tilde{\phi}_i, \tilde{\epsilon}_i : \mathbf{f}\to\mathbf{f}$

17.3.1. We shall regard $\mathbf{f}$ as a $\mathfrak{U}$-module as in 15.1.4. Thus, for each $i\in I$, $\phi_i:\mathbf{f}\to\mathbf{f}$ acts as left multiplication by θ_i and $\epsilon_i:\mathbf{f}\to\mathbf{f}$ is the linear map ${}_ir$ in 1.2.13. For any $i\in I$, $\mathbf{f}$ with the operators $\phi_i,\epsilon_i:\mathbf{f}\to\mathbf{f}$ is then an object of $\mathcal{D}_i$. (See 16.1.1.) Hence the operators $\tilde{\phi}_i,\tilde{\epsilon}_i:\mathbf{f}\to\mathbf{f}$ (see 16.1.3) are well-defined.

Note that the form $(,)$ on $\mathbf{f}$ is admissible in the sense of 16.2.2 for any i.

17.3.2. For a fixed $i\in I$, we define a $\mathbf{Q}(v)$-basis B^i of $\mathbf{f}$ as follows. By definition, $B^i=\sqcup_{t\geq 0}B^i(t)$ where $B^i(0)$ is any subset of $\mathcal{B}_{i;0}$ such that $\mathcal{B}_{i;0}=B^i(0)\sqcup(-B^i(0))$ and, for $t>0$, $B^i(t)$ is the image of $B^i(0)$ under $\pi_{i,t}:\mathcal{B}_{i;0}\cong\mathcal{B}_{i;t}$ (see 14.3.2(c)). By definition, we have $\mathcal{B}=B^i\cup(-B^i)$ and B^i is *adapted* (in the sense of 16.3.1) to $\mathbf{f}\in\mathcal{D}_i$.

By definition of $\mathcal{B}$, we see that B^i is almost orthonormal for $(,)$ and the $\mathcal{A}$-module it generates is ${}_{\mathcal{A}}\mathbf{f}$.

17.3.3. Let $\mathcal{L}(\mathbf{f}) = \{x \in {}_{\mathcal{A}}\mathbf{f} | (x,x) \in \mathbf{A}\}$. From Theorem 14.2.3 and Lemma 14.2.2, it follows that $\mathcal{L}(\mathbf{f})$ is the $\mathbf{Z}[v^{-1}]$-submodule of $\mathbf{f}$ generated by B^i.

Lemma 17.3.4. (a) *${}_{\mathcal{A}}\mathbf{f}$ is stable under the operators $\epsilon_i, \phi_i^{(t)} : \mathbf{f} \to \mathbf{f}$, for any $i \in I$.*

(b) *$\mathcal{L}(\mathbf{f})$ is stable under the operators $\tilde{\phi}_i, \tilde{\epsilon}_i : \mathbf{f} \to \mathbf{f}$, for any $i \in I$.*

The stability under $\phi_i^{(t)}$ is clear from definitions. The stability under ϵ_i follows from 13.2.4. This gives (a). Now (b) follows from Lemma 16.2.8(a) applied to $\mathbf{f}, (,)$ and B^i.

Applying Proposition 16.3.5 to our case, we see that the following holds.

Proposition 17.3.5. *Let $b \in B^i(t)$. Let $b_0 \in B^i(0)$ be the unique element such that $\pi_{i,t}b_0 = b$. We have $\tilde{\phi}_i(b) = \pi_{i,t+1}b_0 \mod v^{-1}\mathcal{L}(\mathbf{f})$. We have $\tilde{\epsilon}_i(b) = \pi_{i,t-1}b_0 \mod v^{-1}\mathcal{L}(\mathbf{f})$ if $t \geq 1$ and $\tilde{\epsilon}_i(b) = 0 \mod v^{-1}\mathcal{L}(\mathbf{f})$ if $t = 0$.*

17.3.6. The following result shows that the endomorphisms of the $\mathbf{Z}$-module $\mathcal{L}(\mathbf{f})/v^{-1}\mathcal{L}(\mathbf{f})$ induced by $\tilde{\phi}_i, \tilde{\epsilon}_i$ act, with respect to the signed basis given by the image of $\mathcal{B}$, in a very simple way, described in terms of the bijections $\pi_{i,n}$ in 14.3.2(c).

Corollary 17.3.7. *Let $i \in I$ and let $b \in \mathcal{B}_{i;t}$. Let $b_0 \in \mathcal{B}_{i;0}$ be the unique element such that $\pi_{i,t}b_0 = b$. We have*

(a) *$\tilde{\phi}_i(b) = \pi_{i,t+1}b_0 \mod v^{-1}\mathcal{L}(\mathbf{f})$;*

(b) *$\tilde{\epsilon}_i(b) = \pi_{i,t-1}b_0 \mod v^{-1}\mathcal{L}(\mathbf{f})$ if $t \geq 1$ and $\tilde{\epsilon}_i(b) = 0 \mod v^{-1}\mathcal{L}(\mathbf{f})$ if $t = 0$.*

(c) *If $i \in I$ and $b \in \mathcal{B}$, then we have $\tilde{\phi}_i(b) = b' \mod v^{-1}\mathcal{L}(\mathbf{f})$ for a unique $b' \in \mathcal{B}$. Moreover, $\tilde{\epsilon}_i b' = b \mod v^{-1}\mathcal{L}(\mathbf{f})$.*

(d) *If $i \in I$ and $b \in \mathcal{B}_{i;n}$ for some $n > 0$, then we have $\tilde{\epsilon}_i(b) = b'' \mod v^{-1}\mathcal{L}(\mathbf{f})$ for a unique $b'' \in \mathcal{B}$. Moreover, $\tilde{\phi}_i b'' = b \mod v^{-1}\mathcal{L}(\mathbf{f})$.*

We apply 17.3.5 to b if $b \in B_t^i$ or to $-b$ if $-b \in B_t^i$. This gives (a) and (b).

Let $b' = \pi_{i,n+1}b_0 \in \mathcal{B}_{i;n+1}$. We have $\tilde{\phi}_i(b) = b' \mod v^{-1}\mathcal{L}(\mathbf{f})$ by (a) and $\tilde{\epsilon}_i(b') = b \mod v^{-1}\mathcal{L}(\mathbf{f})$ by (b). This proves (c).

Assume now that $b \in \mathcal{B}_{i;n}$ with $n > 0$. Let $b'' = \pi_{i,n-1}b_0 \in \mathcal{B}_{i;n-1}$. We have $\tilde{\epsilon}_i(b) = b'' \mod v^{-1}\mathcal{L}(\mathbf{f})$ by (b) and $\tilde{\phi}_i(b'') = b \mod v^{-1}\mathcal{L}(\mathbf{f})$ by (a). This proves (d).