CHAPTER 16

Kashiwara's Operators in Rank 1

16.1. Definition of the Operators $\tilde{\phi}_i, \tilde{\epsilon}_i$ and $\tilde{F}_i, \tilde{E}_i$

16.1.1. In this chapter we fix $i \in I$. Besides the category $\mathcal{C}'_i$ (see 5.1.1), we shall consider another category $\mathcal{D}_i$ which shares some of the properties of $\mathcal{C}'_i$.

Let $\mathcal{D}_i$ be the category whose objects are $\mathbf{Q}(v)$-vector spaces P provided with two $\mathbf{Q}(v)$-linear maps $\epsilon_i, \phi_i : P \to P$ such that ϵ_i is locally nilpotent and

(a) $\epsilon_i \phi_i = v_i^2 \phi_i \epsilon_i + 1$;

the morphisms in the category are $\mathbf{Q}(v)$-linear maps commuting with ϵ_i, ϕ_i.

For $P \in \mathcal{D}_i$ and $s \in \mathbf{Z}$, let $\phi_i^{(s)} : P \to P$ be defined as $\phi_i^s/[s]_i^!$ if $s \geq 0$ and as 0, if $s < 0$. From (a) we deduce by induction on N:

(b) $\epsilon_i \phi_i^{(N)} = v_i^{2N} \phi_i^{(N)} \epsilon_i + v_i^{N-1} \phi_i^{(N-1)}$ for all N.

For any $t \geq 0$, we consider the operator

$$\Pi_t = \sum_{s \geq 0} (-1)^s v_i^{s(s-1)/2} \phi_i^{(s)} \epsilon_i^{s+t} : P \to P.$$

This is well-defined, since ϵ_i is locally nilpotent. For $N \geq 0$, we define a subspace $P(N)$ of P by $P(0) = \{x \in P | \epsilon_i(x) = 0\}$ and $P(N) = \phi_i^{(N)} P(0)$.

Lemma 16.1.2. (a) *We have* $\epsilon_i \Pi_t = 0$ *for all* $t \geq 0$.

(b) *We have* $\sum_{t \geq 0} v^{-t(t-1)/2} \phi_i^{(t)} \Pi_t = 1$. *The sum is well-defined since, for any* $x \in P$, *we have* $\Pi_t(x) = 0$, *for large* t.

(c) *We have a direct sum decomposition* $P = \oplus_{N \geq 0} P(N)$ *as a vector space. Moreover, for any* $N \geq 0$, *the map* $\phi_i^{(N)}$ *restricts to an isomorphism of vector spaces* $P(0) \cong P(N)$.

(d) $\phi_i : P \to P$ *is injective.*

Using 16.1.1(b), we have

$$\epsilon_i \Pi_t = \sum_{s \geq 0} (-1)^s v_i^{s(s-1)/2} (v_i^{2s} \phi_i^{(s)} \epsilon_i^{s+t+1} + v_i^{s-1} \phi_i^{(s-1)} \epsilon_i^{s+t})$$
$$= \sum_{s \geq 0} (-1)^s \phi_i^{(s)} \epsilon_i^{s+t+1} (v_i^{s(s-1)/2+2s} - v_i^{s(s+1)/2+s}) = 0$$

and (a) is proved. Now (b) follows immediately from 1.3.4.

We prove (c). If $x \in P$, we have by (b): $x = \sum_N \phi_i^{(N)} x_N$ where $x_N = v_i^{-N(N-1)/2}\Pi_N(x)$. By (a), we have $x_N \in P(0)$. It remains to show the uniqueness of the x_N; it is enough to prove the following statement. If $0 = \sum_{N\geq 0} \phi_i^{(N)} x_N$, where $x_N \in P(0)$ are zero for all $N > N_0$ (for some $N_0 \geq 0$), then $x_{N_0} = 0$.

We argue by induction on N_0. For $N_0 = 0$ there is nothing to prove. Assume that $N_0 \geq 1$. Applying ϵ_i and using 16.1.1(b), we obtain $0 = \sum_{N\geq 0} v_i^{(N-1)} \phi_i^{(N-1)} x_N$. The induction hypothesis is applicable to this equation and gives $x_{N_0} = 0$. This proves (c).

(d) follows immediately from (c).

16.1.3. We define linear maps $\tilde{\phi}_i, \tilde{\epsilon}_i : P \to P$ by
$\tilde{\phi}_i(\phi_i^{(N)} y) = \phi_i^{(N+1)} y$ and $\tilde{\epsilon}_i(\phi_i^{(N)} y) = \phi_i^{(N-1)} y$ for all $y \in P(0)$.

Lemma 16.1.4. *Let $M \in \mathcal{C}'_i$ and let $x \in M^t$.*

(a) *We can write uniquely $x = \sum_{s; s\geq 0; s+t\geq 0} F_i^{(s)} x_s$ where $x_s \in \ker(E_i : M^{t+2s} \to M)$ and $x_s = 0$ for large enough s; we can write uniquely $x = \sum_{s; s\geq 0; s-t\geq 0} E_i^{(s)} x'_s$ where $x'_s \in \ker(F_i : M^{t-2s} \to M)$ and $x'_s = 0$ for large enough s.*

(b) *We have $\sum_{s; s\geq 0; s+t\geq 0} F_i^{(s+1)} x_s = \sum_{s; s\geq 0; s-t\geq 0} E_i^{(s-1)} x'_s$. We denote either of these sums by $\tilde{F}_i(x)$.*

(c) *We have $\sum_{s; s\geq 0; s+t\geq 0} F_i^{(s-1)} x_s = \sum_{s; s\geq 0; s-t\geq 0} E_i^{(s+1)} x'_s$. We denote either of these sums by $\tilde{E}_i(x)$.*

This follows from 5.1.5 (we are reduced by 5.1.4 to the case considered there.)

The operators $\tilde{\phi}_i, \tilde{\epsilon}_i$ and $\tilde{F}_i, \tilde{E}_i$ in this and the previous subsection are called *Kashiwara's operators.*

16.1.5. Let $M \in \mathcal{C}'_i$. Consider the $\mathbf{Q}(v)$-linear maps $\tilde{E}_i, \tilde{F}_i : M \to M$ defined in the previous lemma. We have

$$x \in M^n \implies \tilde{E}_i(x) \in M^{n+2}, \tilde{F}_i(x) \in M^{n-2}.$$

16.2. Admissible Forms

16.2.1. We fix $P \in \mathcal{D}_i, M \in \mathcal{C}'_i$. We will study the properties of the operators $\tilde{\phi}_i : P \to P, \tilde{\epsilon}_i : P \to P$ and $\tilde{F}_i : M \to M, \tilde{E}_i : M \to M$ in parallel.

16.2.2. A symmetric bilinear form $(,) : P \times P \to \mathbf{Q}(v)$ is said to be *admissible* if

(a) $(x, \epsilon_i(y)) = (1 - v_i^{-2})(\phi_i x, y)$ for all $x, y \in P$.

A symmetric bilinear form $(,) : M \times M \to \mathbf{Q}(v)$ is said to be *admissible* if

(a′) $(M^n, M^{n'}) = 0$ for $n \neq n'$ and

(b′) $(E_i x, y) = v_i^{n-1}(x, F_i y)$ for all $x \in M^{n-2}, y \in M^n$.

16.2.3. Besides the subrings $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ and $\mathbf{A} = \mathbf{Q}[[v^{-1}]] \cap \mathbf{Q}(v)$ of $\mathbf{Q}(v)$ we shall need the subrings $\mathbf{A}(\mathbf{Z}) = \mathbf{Z}[[v^{-1}]] \cap \mathbf{Q}(v)$ and $\hat{\mathcal{A}} = \mathbf{Z}((v^{-1})) \cap \mathbf{Q}(v)$ of $\mathbf{Q}((v^{-1}))$.

16.2.4. Let B be a basis of the $\mathbf{Q}(v)$-vector space P (resp. M). We say that B is *integral* if

(a) the $\mathcal{A}$-submodule ${}_{\mathcal{A}}P$ of P generated by B is stable under $\epsilon_i, \phi_i^{(t)} : P \to P$ for all $t \geq 0$ (resp. the $\hat{\mathcal{A}}$-submodule ${}_{\hat{\mathcal{A}}}M$ of M generated by B is stable under $E_i^{(t)}, F_i^{(t)} : M \to M$ for all $t \geq 0$); in the case of M, it is further assumed that $B \cap M^n$ is a basis of M^n for all n.

Assume that we are given an admissible form $(,)$ and an integral basis B for P (resp. M) which is almost orthonormal (see 14.2.1). Let

$$\mathcal{L}(P) = \{x \in {}_{\mathcal{A}}P | (x, x) \in \mathbf{A}\}$$

and

$$\mathcal{L}(M) = \{x \in {}_{\hat{\mathcal{A}}}M | (x, x) \in \mathbf{A}\}.$$

Lemma 16.2.5. (a) *$\mathcal{L}(P)$ is a $\mathbf{Z}[v^{-1}]$-submodule of ${}_{\mathcal{A}}P$ and B is a basis of it.*

(b) *Let $x \in {}_{\mathcal{A}}P$ be such that $(x, x) \in 1 + v^{-1}\mathbf{A}$. Then there exists $b \in B$ such that $x = \pm b \mod v^{-1}\mathcal{L}(P)$.*

(c) *Let $x \in {}_{\mathcal{A}}P$ be such that that $(x, x) \in v^{-1}\mathbf{A}$. Then $x \in v^{-1}\mathcal{L}(P)$.*

(d) *$\mathcal{L}(M)$ is an $\mathbf{A}(\mathbf{Z})$-submodule of ${}_{\hat{\mathcal{A}}}M$ and B is a basis of it.*

(e) *Let $x \in {}_{\hat{\mathcal{A}}}M$ be such that $(x, x) \in 1 + v^{-1}\mathbf{A}$. Then there exists $b \in B$ such that $x = \pm b \mod v^{-1}\mathcal{L}(M)$.*

(f) *Let $x \in {}_{\hat{\mathcal{A}}}M$ be such that that $(x, x) \in v^{-1}\mathbf{A}$. Then $x \in v^{-1}\mathcal{L}(M)$.*

This follows from Lemma 14.2.2.

Lemma 16.2.6. *Let $y \in {}_{\mathcal{A}}P$ (resp. $y \in {}_{\hat{\mathcal{A}}}M \cap M^t$ with $t \geq 0$) be such that $\epsilon_i y = 0$ (resp. $E_i y = 0$); let $n \geq 0$ (resp. $0 \leq n \leq t$). We have $(\phi_i^{(n)}y, \phi_i^{(n)}y) = \pi_n(y,y)$ (resp. $(F_i^{(n)}y, F_i^{(n)}y) = \pi'_n(y,y)$) where $\pi_n \in 1 + v^{-1}\mathbf{Z}[[v^{-1}]]$ (resp. $\pi'_n \in 1 + v^{-1}\mathbf{Z}[[v^{-1}]]$).*

It suffices to show that

$$(\phi_i^{(n+1)}y, \phi_i^{(n+1)}y) = \pi(\phi_i^{(n)}y, \phi_i^{(n)}y)$$

(resp. $(F_i^{(n+1)}y, F_i^{(n+1)}y) = \pi'(F_i^{(n)}y, F_i^{(n)}y)$) where $\pi \in 1 + v^{-1}\mathbf{Z}[[v^{-1}]]$ (resp. $\pi' \in 1 + v^{-1}\mathbf{Z}[[v^{-1}]]$) and $n \geq 0$ (resp. $0 \leq n < t$).

We have

$$\begin{aligned}
&(\phi_i^{(n+1)}y, \phi_i^{(n+1)}y) = ([n+1]_i^{-1}\phi_i\phi_i^{(n)}y, \phi_i^{(n+1)}y)\\
&= (1 - v_i^{-2})^{-1}[n+1]_i^{-1}(\phi_i^{(n)}y, \epsilon_i\phi_i^{(n+1)}y)\\
&= (1 - v_i^{-2})^{-1}[n+1]_i^{-1}v_i^n(\phi_i^{(n)}y, \phi_i^{(n)}y).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&(F_i^{(n+1)}y, F_i^{(n+1)}y) = ([n+1]_i^{-1}F_iF_i^{(n)}y, F_i^{(n+1)}y)\\
&= v_i^{-t+2n+1}[n+1]_i^{-1}(F_i^{(n)}y, E_iF_i^{(n+1)}y)\\
&= v_i^{-t+2n+1}[-n+t]_i[n+1]_i^{-1}(F_i^{(n)}y, F_i^{(n)}y).
\end{aligned}$$

It remains to observe that

$$[n+1]_i^{-1}v_i^n \in 1 + v^{-1}\mathbf{Z}[[v^{-1}]]$$

for $0 \leq n$ and

$$v_i^{-t+2n+1}[-n+t]_i[n+1]_i^{-1} \in 1 + v^{-1}\mathbf{Z}[[v^{-1}]]$$

for $0 \leq n < t$. The lemma follows.

Lemma 16.2.7. (a) *Let $x \in {}_{\mathcal{A}}P$; write $x = \sum_{N \geq 0} y_N$ where $y_N = \phi_i^{(N)}x_N$ and $x_N \in P(0)$ are zero for large N (see 16.1.2(c)). Then $x_N \in {}_{\mathcal{A}}P$ for all N.*

(b) *If $x \in \mathcal{L}(P)$, then each x_N and y_N above is in $\mathcal{L}(P)$. If, in addition, $(x,x) \in 1 + v^{-1}\mathbf{A}$, then there exists $N_0 \geq 0$ and $b \in B$ such that $x_{N_0} = \pm b \mod v^{-1}\mathcal{L}(P)$ and $x_N = 0 \mod v^{-1}\mathcal{L}(P)$, $y_N = 0 \mod v^{-1}\mathcal{L}(P)$ for all $N \neq N_0$.*

(c) *Let* $x \in M^t \cap {}_{\hat{\mathcal{A}}}M$. *We write* $x = \sum_{s;s\geq 0;s+t\geq 0} y_s$ *where* $y_s = F_i^{(s)} x_s$ *and* $x_s \in \ker(E_i : M^{t+2s} \to M)$ *are zero for large enough* s*; then* $x_s \in {}_{\hat{\mathcal{A}}}M$ *for all* s.

(d) *If* $x \in M^t \cap \mathcal{L}(M)$, *then each* x_s *and* y_s *above is in* $\mathcal{L}(M)$. *If, in addition,* $(x,x) \in 1 + v^{-1}\mathbf{A}$, *then there exists* $s_0 \geq 0$ *and* $b \in B$ *such that* $x_{s_0} = \pm b \mod v^{-1}\mathcal{L}(M)$ *and* $x_s = 0 \mod v^{-1}\mathcal{L}(M)$, $y_s = 0 \mod v^{-1}\mathcal{L}(M)$ *for all* $s \neq s_0$.

We prove (a). We have $x_N = v_i^{-N(N-1)/2}\Pi_N(x)$. Since ${}_{\mathcal{A}}P$ is stable under ϵ_i and $\phi_i^{(t)}$ for all t, we see that ${}_{\mathcal{A}}P$ is stable under $\Pi_N : P \to P$. Hence $x_N \in {}_{\mathcal{A}}P$. This proves (a).

Next we show that the subspaces $\phi^{(N)}P(0), \phi^{(N')}P(0)$ are orthogonal to each other under $(,)$, if $N \neq N'$. We argue by induction on $N + N'$. If $N \geq 1$, we have for $z, z' \in P(0)$ that $(\phi_i^{(N)}z, \phi_i^{(N')}z')$ is equal to a scalar times $(\phi_i^{(N-1)}z, \epsilon_i\phi_i^{(N')}z')$, hence to a scalar times $(\phi_i^{(N-1)}z, \phi_i^{(N'-1)}z)$ so that it is zero by the induction hypothesis. We treat similarly the case where $N' \geq 1$. If $N \leq 0$ and $N' \leq 0$, the result is trivial; our assertion follows.

Now let $x \in {}_{\mathcal{A}}P$ be non-zero. We have $(x,x) = \sum_N (y_N, y_N)$. We can find $t \in \mathbf{Z}$ such that $v^{-t}y_N \in \mathcal{L}(P)$ for all N and $v^{-t+1}y_N \notin \mathcal{L}(P)$ for some N. Then there exist integers $a_N \geq 0$, not all equal to 0 such that $v^{-2t}(y_N, y_N) - a_N \in v^{-1}\mathbf{A}$ for all N. Hence

(e) $v^{-2t}(x,x) - \sum_N a_N \in v^{-1}\mathbf{A}$ and $\sum_N a_N > 0$.

If $x \in \mathcal{L}(P)$, then (e) shows that $t \leq 0$; hence $y_N \in \mathcal{L}(P)$ for all N and, using the previous lemma, we see that $x_N \in \mathcal{L}(P)$ for all N. If now $x \in \mathcal{L}(P)$ satisfies $(x,x) \in 1 + v^{-1}\mathbf{A}$, then (e) shows that $t = 0$ and $a_{N_0} = 1$ for some N_0 and $a_N = 0$ for all $N \neq N_0$. In other words, we have $(y_{N_0}, y_{N_0}) \in 1 + v^{-1}\mathbf{A}$ and $(y_N, y_N) \in v^{-1}\mathbf{A}$ for all $N \neq N_0$. Using 16.2.6, we deduce that $(x_{N_0}, x_{N_0}) \in 1 + v^{-1}\mathbf{A}$ and $(x_N, x_N) \in v^{-1}\mathbf{A}$ for all $N \neq N_0$ and the second assertion of (b) follows from 16.2.5.

We prove (c). If $x_s = 0$ for all s, then there is nothing to prove. Hence we may assume that $x_s \neq 0$ for some s and we denote by N the largest index such that $x_N \neq 0$. We have $N \geq 0, N + t \geq 0$. We argue by induction on N. If $N = 0$, there is nothing to prove; hence we may assume that $N > 0$. We have $E_i^{(N)}x = \sum_{s;s\geq 0;s+t\geq 0} E_i^{(N)}F_i^{(s)}x_s = \begin{bmatrix} 2N+t \\ N \end{bmatrix}_i x_N$. Since $E_i^{(N)}\,{}_{\hat{\mathcal{A}}}M \subset {}_{\hat{\mathcal{A}}}M$, we have $\begin{bmatrix} 2N+t \\ N \end{bmatrix}_i x_N \in {}_{\hat{\mathcal{A}}}M$. We have $\begin{bmatrix} 2N+t \\ N \end{bmatrix}_i^{-1} \in \hat{\mathcal{A}}$, hence $x_N \in {}_{\hat{\mathcal{A}}}M$. Then $x' = x - F_i^{(N)}x_N \in {}_{\hat{\mathcal{A}}}M$. The induction hypothesis is applicable to x' and (c) follows.

Next we show that $(F_i^{(N)}z, F_i^{(N')}z') = 0$ if $N \neq N'$ and z, z' are homogeneous elements in the kernel of E_i. We argue by induction on $N + N'$. If $N \geq 1$, we have that $(F_i^{(N)}z, F_i^{(N')}z')$ is equal to a scalar times $(F_i^{(N-1)}z, E_i F_i^{(N')}z')$, hence to a scalar times $(F_i^{(N-1)}z, F_i^{(N'-1)}z)$ so that it is zero, by the induction hypothesis. We treat similarly the case where $N' \geq 1$. If $N \leq 0$ and $N' \leq 0$, the result is trivial; our assertion follows. The remainder of the proof is entirely similar to that of (b).

Lemma 16.2.8. (a) *$\tilde{\phi}_i, \tilde{\epsilon}_i : P \to P$ map $\mathcal{L}(P)$ into itself.*

(b) *$\tilde{F}_i, \tilde{E}_i : M \to M$ map $\mathcal{L}(M)$ into itself.*

Let $x \in \mathcal{L}(P)$. We must show that $\tilde{\phi}_i x \in \mathcal{L}(P), \tilde{\epsilon}_i x \in \mathcal{L}(P)$. By 16.2.7, we may assume that $x = \phi_i^{(N)} x_N$ for some x_N as in that lemma. But then $\tilde{\phi}_i x = \phi_i^{(N+1)} x_N \in \mathcal{L}(P)$ and $\tilde{\epsilon}_i x = \phi_i^{(N-1)} x_N \in \mathcal{L}(P)$, by 16.2.6. We argue in the same way for M.

16.2.9. For any $N \geq 0$, we denote by $T_N(P)$ the set of all $x \in {}_{\mathcal{A}}P$ such that $x = \phi_i^{(N)} x'$ for some $x' \in P(0) \cap {}_{\mathcal{A}}P$ with $(x', x') = 1 \mod v^{-1}\mathbf{A}$.

For any s, t such that $s \geq 0, s + t \geq 0$, we denote by $T_{s,t}(M)$ the set of all $x \in {}_{\hat{\mathcal{A}}}M$ such that $x = F_i^{(s)} x'$ for some $x' \in \ker(E_i : M^{t+2s} \to M) \cap {}_{\hat{\mathcal{A}}}M$ with $(x', x') = 1 \mod v^{-1}\mathbf{A}$.

From the definitions we see that

(a) $\tilde{\phi}_i(T_N(P)) \subset T_{N+1}(P)$;

(b) $\tilde{\epsilon}_i(T_N(P)) \subset T_{N-1}(P)$ for $N \geq 1$, $\tilde{\epsilon}_i(T_0(P)) = 0$;

(c) if $N \geq 0$, then $\tilde{\phi}_i : T_N(P) \to T_{N+1}(P)$ and $\tilde{\epsilon}_i : T_{N+1}(P) \to T_N(P)$ are inverse bijections;

(d) $\tilde{F}_i(T_{s,t}(M)) \subset T_{s+1,t-2}(M)$ if $s \geq 0, s + t \geq 1$, and $\tilde{F}_i(T_{s,t}(M)) = 0$ if $s \geq 0, s + t = 0$;

(e) $\tilde{E}_i(T_{s,t}(M)) \subset T_{s-1,t+2}(M)$ if $s \geq 1, s + t \geq 0$, and $\tilde{E}_i(T_{s,t}(M)) = 0$ if $s = 0, t \geq 0$;

(f) if $s \geq 0, s + t \geq 1$, then $\tilde{F}_i : T_{s,t}(M) \to T_{s+1,t-2}(M)$ and $\tilde{E}_i : T_{s+1,t-2}(M)) \to T_{s,t}(M)$ are inverse bijections.

Lemma 16.2.10. (a) *Case of P. We have*

$$\pm B + v^{-1}\mathcal{L}(P) = \cup_{N \geq 0} T_N(P) + v^{-1}\mathcal{L}(P).$$

Moreover, the sets $B_N = B \cap (T_N(P) + v^{-1}\mathcal{L}(P))$ $(N \geq 0)$ form a partition of B.

(b) *Case of M. We have*

$$\pm B + v^{-1}\mathcal{L}(M) = \cup_{s,t;s\geq 0,s+t\geq 0} T_{s,t}(M) + v^{-1}\mathcal{L}(M).$$

Moreover, the sets $B_{s,t} = B \cap (T_{s,t}(M) + v^{-1}\mathcal{L}(M))$ $(s \geq 0, s+t \geq 0)$ form a partition of B.

By 16.2.6 and 16.2.5, we have $T_N(P) \subset \pm B + v^{-1}\mathcal{L}(P)$. Conversely, let $x \in \pm B$. We have $(x,x) \in 1 + v^{-1}\mathbf{A}$. Hence, by 16.2.7, we have $x = y' + y''$ where $y'' \in v^{-1}\mathcal{L}(P)$ and $y' = \phi_i^{(N)} x'$ for some $x' \in P(0) \cap {}_{\mathcal{A}}P$ such that $x' \in \pm B + v^{-1}\mathcal{L}(P)$ and some $N \geq 0$. Thus $x \in T_N(P) + v^{-1}\mathcal{L}(P)$ and the first assertion of (a) follows. To prove the second assertion of (a), it is enough to show that $T_{N_1}(P) \cap (T_{N_2}(P) + v^{-1}\mathcal{L}(P))$ is empty for $N_1 \neq N_2$. Assume that $\phi_i^{(N_1)} x_1 = \phi_i^{(N_2)} x_2 + v^{-1} z$ where $z \in \mathcal{L}(P)$ and $x_1, x_2 \in P(0) \cap ({}_{\mathcal{A}}P)$ satisfy $(x_1, x_1) = 1 \mod v^{-1}\mathbf{A}$ and $(x_2, x_2) = 1 \mod v^{-1}\mathbf{A}$. By 16.2.7, we can write $z = \sum_{N\geq 0} \phi_i^{(N)} z_N$ where $z_N \in \mathcal{L}(P) \cap P(0)$. We have $\phi_i^{(N_1)} x_1 = \phi_i^{(N_2)} x_2 + v^{-1} \sum_{N\geq 0} \phi_i^{(N)} z_N$. This implies, by 16.1.2(c), that $z_N = 0$ for $N \neq N_1, N_2$, $v^{-1} z_{N_1} = x_1$ and $v^{-1} z_{N_2} = -x_2$. From the last equality we deduce that $(x_2, x_2) = v^{-2}(z_{N_2}, z_{N_2}) \in v^{-2}\mathbf{A}$, a contradiction. Thus (a) is proved. The proof of (b) is entirely similar.

16.2.11. Using the previous lemma and the results in 16.2.9, we deduce the following.

In the case of P we have:

(a) $\tilde{\phi}_i(\pm B_N + v^{-1}\mathcal{L}(P)) \subset \pm B_{N+1} + v^{-1}\mathcal{L}(P)$;

(b) $\tilde{\epsilon}_i(\pm B_N + v^{-1}\mathcal{L}(P)) \subset \pm B_{N-1} + v^{-1}\mathcal{L}(P)$ for $N \geq 1$, and $\tilde{\epsilon}_i(\pm B_0 + v^{-1}\mathcal{L}(P)) \subset v^{-1}\mathcal{L}(P)$;

(c) if $N \geq 0$, then $\tilde{\phi}_i : \pm B_N + v^{-1}\mathcal{L}(P) \to \pm B_{N+1} + v^{-1}\mathcal{L}(P)$ and $\tilde{\epsilon}_i : \pm B_{N+1} + v^{-1}\mathcal{L}(P) \to \pm B_N + v^{-1}\mathcal{L}(P)$ are inverse bijections.

In the case of M we have:

(d) $\tilde{F}_i(\pm B_{s,t} + v^{-1}\mathcal{L}(M)) \subset \pm B_{s+1,t-2} + v^{-1}\mathcal{L}(M))$ if $s \geq 0, s+t \geq 1$, and $\tilde{F}_i(\pm B_{s,t} + v^{-1}\mathcal{L}(M)) = 0$ if $s \geq 0, s+t = 0$;

(e) $\tilde{E}_i(\pm B_{s,t} + v^{-1}\mathcal{L}(M)) \subset \pm B_{s-1,t+2} + v^{-1}\mathcal{L}(M))$ if $s \geq 1, s+t \geq 0$, and $\tilde{E}_i(\pm B_{s,t} + v^{-1}\mathcal{L}(M)) = 0$ if $s = 0, t \geq 0$;

(f) if $s \geq 0, s+t \geq 1$, then $\tilde{F}_i : \pm B_{s,t} + v^{-1}\mathcal{L}(M)) \to \pm B_{s+1,t-2} + v^{-1}\mathcal{L}(M))$ and $\tilde{E}_i : \pm B_{s+1,t-2} + v^{-1}\mathcal{L}(M)) \to \pm B_{s,t} + v^{-1}\mathcal{L}(M))$ are inverse bijections.

16.3. Adapted Bases

16.3.1. $P, M, (,)$ are as in the previous section. We say that a basis B of P is *adapted* if there exists a partition $B = \cup_{n\geq 0} B(n)$ and bijections $\pi_n : B(0) \to B(n)$ for all $n \geq 0$ such that

(a) for any $N \geq 0$, $B(N) \cup B(N+1) \cup B(N+2) \cup \dots$ is a basis of $\phi_i^{(N)} P$;

(b) for any $b \in B(0)$ and any $N \geq 0$ we have $\phi_i^{(N)} b - \pi_N(b) \in \phi_i^{(N+1)} P$.

We say that a basis B of M is *adapted* if there exists a partition

$$B = \cup_{s,t;s\geq 0, s+t\geq 0} B(s,t)$$

and bijections

$$\pi_{s,t} : B(0, 2s+t) \to B(s,t)$$

for all s, t as above, such that

(a) $B(s,t) \cup B(s+1,t) \cup B(s+2,t) \cup \dots$ is a basis of $M^t \cap F_i^{(s)} M$;

(b) for any $b \in B(0, 2s+t)$, we have $F_i^{(s)} b - \pi_{s,t}(b) \in F_i^{(s+1)} M$.

In this section it is assumed that B is integral, almost orthonormal (with respect to $(,)$) and adapted.

Lemma 16.3.2. *Let $b \in B$.*

(a) *Case of P. We have $b \in B_0$ if and only if $b \notin \phi_i(P)$.*

(b) *Case of M. We have $b \in \cup_{t\geq 0} B_{0,t}$ if and only if $b \notin F_i M$.*

We prove (a). Assume first that $b \in B_N$ with $N > 0$. Then $b = \phi_i^{(N)} x' + v^{-1} z$ where $z \in \mathcal{L}(P)$ and $x' \in P$. Since B is adapted, we can write $\phi_i^{(N)} x' = \sum c_{b'} b'$ where b' runs over $B \cap \phi_i^{(N)} P$ and $c_{b'} \in \mathbf{Q}(v)$. We can also write $z = \sum_{b''} d_{b''} b''$ where b'' runs over B and $d_{b''} \in \mathbf{Z}[v^{-1}]$. If $b \notin \phi_i^{(N)} P$, then by comparing the coefficients of b, we obtain $1 = v^{-1} d_b$, a contradiction. Thus, we have $b \in \phi_i^{(N)} P$. Since $N > 0$, we have $b \in \phi_i P$. Conversely, assume that $b \in \phi_i P$ and $b \in B_0$. Then $b = x' + v^{-1} z$ where $z \in \mathcal{L}(P), x' \in P(0)$, and $b \in \sum_{N>0} \phi^{(N)} P(0)$; using the equation $(P(0), \phi_i^{(N)} P(0)) = 0$ for $N > 0$, we deduce that $(x', b) = 0$, hence $(b, b) = v^{-1}(z, b) \in v^{-1}\mathbf{A}$, a contradiction. This proves (a). The proof of (b) is entirely similar.

Lemma 16.3.3. (a) *Case of* P. *Let* $b \in B_0, N \geq 0$ *and let* b' *be the unique element of* $\pm B$ *such that* $\tilde{\phi}_i^N(b) = b'$ *mod* $v^{-1}\mathcal{L}(P)$. *Then* $b' = \pi_N b$.

(b) *Case of* M. *Let* $b \in B_{0,s+2t}$ *where* $s \geq 0$. *If* $s + t \geq 0$, *then there is a unique element* $b' \in \pm B$ *such that* $\tilde{F}_i^s(b) = b' \mod v^{-1}\mathcal{L}(M)$ *and* $b' = \pi_{s,t} b$. *If* $s + t < 0$, *then* $\tilde{F}_i^s(b) = 0 \mod v^{-1}\mathcal{L}(M)$.

We prove (a). We write $b = x + v^{-1}z$ where $z \in \mathcal{L}(P)$ and $x \in P(0) \cap_{\mathcal{A}} P$ satisfies $(x, x) = 1 \mod v^{-1}\mathbf{A}$. Using 16.2.7, we write $z = \sum_{N'} z_{N'}$ where $z_{N'} \in \mathcal{L}(P) \cap \phi_i^{(N')} P(0)$ for all N'. Replacing x by $x + v^{-1}z_0$ and z by $z - z_0$, we see that we may assume that z satisfies in addition $z \in \phi_i P$. The equalities $\tilde{\phi}_i^N b = \phi_i^{(N)} x + v^{-1}\tilde{\phi}_i^N z$ and $\phi_i^{(N)} b = \phi_i^{(N)} x + v^{-1}\phi_i^{(N)} z$, together with $\tilde{\phi}_i^N z \in \mathcal{L}(P)$ and $\phi_i^{(N)} z \in \phi_i^{(N+1)} P$, imply

$$\tilde{\phi}_i^N b = \phi_i^{(N)} b \mod v^{-1}\mathcal{L}(P) + \phi_i^{(N+1)} P.$$

By assumption we have $\tilde{\phi}_i^N(b) = b' \mod v^{-1}\mathcal{L}(P)$. Hence

$$b' = \phi_i^{(N)} b \mod v^{-1}\mathcal{L}(P) + \phi_i^{(N+1)} P.$$

Moreover, we have

$$\phi_i^{(N)} b = b_1 \mod \phi_i^{(N+1)} P$$

where $b_1 = \pi_N b \in B$.

We must prove that $b' = b_1$. We have $b_1 + c_1 = b' + c'$ where $c_1 \in \phi_i^{(N+1)} P$ and $c' \in v^{-1}\mathcal{L}(P)$. We have $b_1 \notin \phi_i^{(N+1)} P$. (Otherwise, we would have $\phi_i^{(N)} b \in \phi_i^{(N+1)} P$; hence $b \in \phi_i P$, contradicting the previous lemma.) Hence, if we express $b_1 + c_1$ as a $\mathbf{Q}(v)$-linear combination of elements of B, the element $b_1 \in B$ will appear with coefficient 1. On the other hand, if we express $b' + c'$ as a $\mathbf{Q}(v)$-linear combination of elements of B, then all coefficients are in $v^{-1}\mathbf{Z}[v^{-1}]$ except that of $\pm b'$.

This forces $b_1 = b'$ or $b_1 = -b'$. If $b_1 = -b'$, then we have $2b_1 + c_1 = c'$ and $\pm b_1$ appears in the left hand side with coefficient 2 and in the right hand side with coefficient in $v^{-1}\mathbf{A}$, a contradiction. Hence we have $b_1 = b'$ and (a) is proved.

The proof of (b) is entirely similar.

16.3.4. The following result shows that the action of the operators $\tilde{\phi}_i, \tilde{\epsilon}_i$ (resp. $\tilde{F}_i, \tilde{E}_i$) on the elements of B is described up to elements in $v^{-1}\mathcal{L}(P)$ (resp. $v^{-1}\mathcal{L}(M)$) in terms of the bijections π_n (resp. $\pi_{s,t}$) in 16.3.1.

Proposition 16.3.5.

(a) *Case of P. Let* $b \in B(N)$. *Let* $b_0 \in B(0)$ *be the unique element such that* $\pi_N b_0 = b$. *We have* $\tilde{\phi}_i(b) = \pi_{N+1} b_0 \mod v^{-1}\mathcal{L}(P)$. *We have* $\tilde{\epsilon}_i(b) = \pi_{N-1} b_0 \mod v^{-1}\mathcal{L}(P)$ *if* $N \geq 1$ *and* $\tilde{\epsilon}_i(b) = 0 \mod v^{-1}\mathcal{L}(P)$ *if* $N = 0$. *In particular, we have* $B_N = B(N)$ *for all* N.

(b) *Case of M. Let* $b \in B(s,t)$. *Let* $b_0 \in B(0, 2s+t)$ *be the unique element such that* $\pi_{s,t} b_0 = b$. *We have* $\tilde{F}_i(b) = \pi_{s+1,t-2} b_0 \mod v^{-1}\mathcal{L}(M)$ *if* $s+t \geq 1$ *and* $\tilde{F}_i(b) = 0 \mod v^{-1}\mathcal{L}(M)$ *if* $s+t=0$. *We have* $\tilde{E}_i(b) = \pi_{s-1,t+2} b_0 \mod v^{-1}\mathcal{L}(M)$ *if* $s \geq 1$ *and* $\tilde{E}_i(b) = 0 \mod v^{-1}\mathcal{L}(M)$ *if* $s = 0$. *In particular, we have* $B_{s,t} = B(s,t)$ *for all* s, t.

This follows from 16.3.3.