Part III

KASHIWARA'S OPERATORS AND APPLICATIONS

In the author's elementary algebraic definition [4] of the canonical basis of $\mathbf{f}$, there were three main ingredients: (a) the basis was assumed to be integral in a suitable sense; (b) the basis was assumed fixed by the involution $^-$; (c) the basis was assumed to be in a specified $\mathbf{Z}[v^{-1}]$-lattice $\mathcal{L}$ and had prescribed image in $\mathcal{L}/v^{-1}\mathcal{L}$.

Of these three ingredients, the last one is the most subtle; in [4], $\mathcal{L}$ and the basis of $\mathcal{L}/v^{-1}\mathcal{L}$ were defined in terms of a braid group action. This definition does not work for Cartan data of infinite type.

Kashiwara's scheme [2] to define a basis of $\mathbf{f}$ involves again the ingredients (a),(b),(c) above, but he proposes a quite different way to construct the lattice $\mathcal{L}$ and the basis of $\mathcal{L}/v^{-1}\mathcal{L}$, which makes sense for any Cartan datum. The main ingredients in his definition were certain operators $\tilde{\epsilon}_i, \tilde{\phi}_i : \mathbf{f} \to \mathbf{f}$ and some analogous operators $\tilde{E}_i, \tilde{F}_i$ on any integrable $\mathbf{U}$-module. (The last operators were already introduced, in a dual form, in an earlier paper [1].)

Part III gives an account of Kashiwara's approach and its applications. (The results in Part III will be needed in Part IV.) Our exposition differs from that of Kashiwara to some extent. In particular, we will make use of the existence of canonical bases (up to sign) established in Part II, while for Kashiwara, that existence was one of the goals.

The algebra $\mathfrak{U}$ in Chapter 15 is defined in a different way than in [2], but eventually, the two definitions agree. The operators $\tilde{\epsilon}_i, \tilde{\phi}_i, \tilde{E}_i, \tilde{F}_i$ are defined in Chapter 16. Chapter 17 contains a proof of a crucial result of Kashiwara on the behaviour of $\tilde{E}_i, \tilde{F}_i$ in a tensor product. Chapters 18 and 19 are concerned with various properties of the canonical basis of Λ_λ, in particular with the fact that this basis is almost orthonormal for the natural inner product. Chapter 20 deals with bases at ∞ (or crystal bases in Kashiwara's terminology). Chapter 21 deals with the special features which hold in the case where the Cartan datum is of finite type. Chapter 22 contains some new positivity results.

In the remainder of this book we assume that, unless otherwise specified, a Cartan datum $(I, \cdot)$ and a root datum $(Y, X, \dots)$ of type $(I, \cdot)$ have been fixed. The notation $\mathbf{f}, \mathbf{U}$, etc. will refer to these fixed data.

CHAPTER 15

The Algebra $\mathfrak{U}$

Lemma 15.1.1. *The algebra homomorphism $\chi : {}'\mathbf{f} \to \mathbf{U}$ given by $\theta_i \mapsto E'_i = (v_i - v_i^{-1})\tilde{K}_{-i}E_i \quad (i \in I)$ factors through an algebra homomorphism $\mathbf{f} \to \mathbf{U}$.*

Let $\chi' : {}'\mathbf{f} \to \mathbf{U}$ be the algebra homomorphism given by $\theta_i \mapsto E_i \quad (i \in I)$. A simple computation shows that, if $f \in {}'\mathbf{f}_\nu$, then

$$\chi(f) = v^N(\prod_i (v_i - v_i^{-1})^{\nu_i})\tilde{K}_{-|f|}\chi'(f)$$

where N depends only on ν and not on f. Hence if f is a homogeneous element in $\mathcal{I}$ (so that $\chi'(f) = 0$), then $\chi(f) = 0$. The lemma is proved.

15.1.2. Let $\tilde{\mathbf{U}}^+$ be the image of χ. Using the previous lemma we see that $E_i \mapsto E'_i$ defines an algebra isomorphism $\mathbf{U}^+ \cong \tilde{\mathbf{U}}^+$.

Let $\tilde{\mathbf{U}}^0$ be the subalgebra of $\mathbf{U}$ generated by the elements $\tilde{K}_{-i}$ for $i \in I$. From the triangular decomposition of $\mathbf{U}$, we can deduce that multiplication defines injective maps $\mathbf{U}^- \otimes \tilde{\mathbf{U}}^0 \otimes \tilde{\mathbf{U}}^+ \to \mathbf{U}$ and $\tilde{\mathbf{U}}^+ \otimes \tilde{\mathbf{U}}^0 \otimes \mathbf{U}^- \to \mathbf{U}$. These maps have the same image, which is a subalgebra $\tilde{\mathbf{U}}$ of $\mathbf{U}$; this follows from the identity $E'_iF_j = v^{i\cdot j}F_jE'_i + \delta_{i,j}(1 - \tilde{K}^2_{-i})$ for all i, j. Note that the elements $\tilde{K}_{-i}$ which are invertible in $\mathbf{U}$ are not invertible in $\tilde{\mathbf{U}}$. The left ideal generated by them in $\tilde{\mathbf{U}}$ coincides with the right ideal generated by them in $\tilde{\mathbf{U}}$. The quotient of $\tilde{\mathbf{U}}$ by this ideal will be denoted by $\mathfrak{U}$. We have obvious algebra homomorphisms $\mathbf{U}^- \to \mathfrak{U}$ and $\tilde{\mathbf{U}}^+ \to \mathfrak{U}$ and it is clear that

(a) multiplication defines isomorphisms of vector spaces $\mathbf{U}^- \otimes \tilde{\mathbf{U}}^+ \cong \mathfrak{U}$ and $\tilde{\mathbf{U}}^+ \otimes \mathbf{U}^- \cong \mathfrak{U}$;

(b) the algebra $\mathfrak{U}$ is defined by the generators $\epsilon_i, \phi_i \quad (i \in I)$ and the relations $\epsilon_i\phi_j = v^{i\cdot j}\phi_j\epsilon_i + \delta_{i,j}$ for all i, j, together with the relations $f(\epsilon_i) = f(\phi_i) = 0$ for any homogeneous $f = f(\theta_i) \in \mathcal{I}$. Here, ϵ_i, ϕ_i are the images of E'_i, F_i in $\mathfrak{U}$.

15.1.3. There is a unique $\mathbf{Q}$-algebra homomorphism $\omega : \mathfrak{U} \to \mathfrak{U}$ such that $\omega(\epsilon_i) = v_i\phi_i, \quad \omega(\phi_i) = -v_i\epsilon_i, \quad \omega(v) = v^{-1}$. We will not use it. Note that $\omega^2 = 1$.

Lemma 15.1.4. *For each $i \in I$ we define $\phi_i : \mathbf{f} \to \mathbf{f}$ to be left multiplication by θ_i and $\epsilon_i : \mathbf{f} \to \mathbf{f}$ to be the linear map ${}_ir$ in 1.2.13.*

(a) *ϕ_i, ϵ_i make $\mathbf{f}$ into a left $\mathfrak{U}$-module.*

(b) *$\epsilon_i : \mathbf{f} \to \mathbf{f}$ is locally nilpotent for any $i \in I$.*

The identity $\epsilon_i\phi_j = v^{i\cdot j}\phi_j\epsilon_i + \delta_{i,j}$ (as maps $\mathbf{f} \to \mathbf{f}$) follows from ${}_ir(\theta_j y) = {}_ir(\theta_j)y + v^{j\cdot i}\theta_j({}_ir(y))$. Let $f = f(\theta_i)$ be a homogeneous element of $\mathcal{I}$. From the definition we have $f(\phi_i) = 0$ as a linear map $\mathbf{f} \to \mathbf{f}$. We must show that $f(\epsilon_i) = 0$. Let $f' = \sigma(f) \in \mathcal{I}$. From the definition we have

$$(x, \epsilon_i(y)) = (1 - v_i^{-2})(\phi_i x, y)$$

for all $x, y \in \mathbf{f}$. It follows that $(x, f(\epsilon_i)(y)) = c(f'(\phi_i)(x), y)$ where $c \in \mathbf{Q}(v)$. From $f'(\phi_i)(x) = 0$ we deduce that $(x, f(\epsilon_i)(y)) = 0$. By the non-degeneracy of $(,)$, this implies that $f(\epsilon_i) = 0$ as a linear map $\mathbf{f} \to \mathbf{f}$. Thus the relations 15.1.2(b) of $\mathfrak{U}$ are verified; (a) is proved.

If $x \in \mathbf{f}_\nu$, then $\epsilon_i(x) \in \mathbf{f}_{\nu-i}$ if $\nu_i \geq 1$ and $\epsilon_i(x) = 0$ if $\nu_i = 0$. It follows that $\epsilon_i : \mathbf{f} \to \mathbf{f}$ is locally nilpotent. The lemma is proved.

Lemma 15.1.5. *There is a unique algebra homomorphism $d : \mathfrak{U} \to \mathfrak{U} \otimes \mathbf{U}$ such that $d(\phi_i) = \phi_i \otimes \tilde{K}_{-i} + 1 \otimes F_i$ and $d(\epsilon_i) = \epsilon_i \otimes \tilde{K}_{-i} + (v_i - v_i^{-1}) 1 \otimes \tilde{K}_{-i}E_i$ for all $i \in I$.*

The homomorphism $\Delta : \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$, satisfies $\Delta(F_i) = F_i \otimes \tilde{K}_{-i} + 1 \otimes F_i$, $\Delta(E_i') = E_i' \otimes \tilde{K}_{-i} + (v_i - v_i^{-1}) 1 \otimes \tilde{K}_{-i}E_i$, and $\Delta(\tilde{K}_{-i}) = \tilde{K}_{-i} \otimes \tilde{K}_{-i}$. Hence Δ restricts to an algebra homomorphism $\tilde{\mathbf{U}} \to \tilde{\mathbf{U}} \otimes \mathbf{U}$ and this induces an algebra homomorphism $d : \mathfrak{U} \to \mathfrak{U} \otimes \mathbf{U}$ which has the required properties.