CHAPTER 14

The Signed Basis of f

14.1. Cartan Data and Graphs with Automorphisms

14.1.1. There is a very close connection between Cartan data and graphs with automorphisms. Given an admissible automorphism a of a finite graph $(\mathbf{I}, H, h \mapsto [h])$ (see 12.1.1), we define I to be the set of a-orbits on $\mathbf{I}$. For $i, j \in I$, we define $i \cdot j \in \mathbf{Z}$ as follows: if $i \neq j$ in I, then $i \cdot j$ is -1 times the number of edges which join some vertex in the a-orbit i to some vertex in the a-orbit j; $i \cdot i$ is 2 times the number of vertices in the a-orbit i. As shown in 13.2.9, $(I, \cdot)$ is a Cartan datum. Conversely, we have the following result.

Proposition 14.1.2. *Let $(I, \cdot)$ be a Cartan datum. There exists a finite graph $(\mathbf{I}, H, h \mapsto [h])$ and an admissible automorphism a of this graph such that $(I, \cdot)$ is obtained from them by the construction in 14.1.1.*

For each $i \in I$, we consider a set D_i of cardinal $d_i = i \cdot i/2$ and a cyclic permutation $a : D_i \to D_i$. Let $\mathbf{I} = \sqcup_{i \in I} D_i$ and let $a : \mathbf{I} \to \mathbf{I}$ be the permutation whose restriction to each D_i is the permutation $a : D_i \to D_i$ considered above.

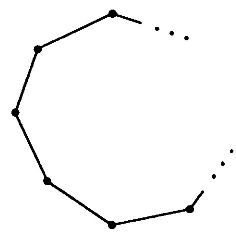
For each unordered pair i, j of distinct elements of I, we choose an a-orbit $\wp$ of the permutation $a \times a : D_i \times D_j \to D_i \times D_j$. Then $\wp$ has cardinality equal to the lowest common multiple $l(d_i, d_j)$ of d_i and d_j, which by the definition of a Cartan datum, divides $-i \cdot j$. Hence we may consider a set $H_{i,j}$ which is a disjoint union of $-i \cdot j / l(d_i, d_j)$ copies of $\wp$ with a permutation $a : H_{i,j} \to H_{i,j}$ whose restriction to each copy of $\wp$ is the permutation defined by $a \times a$. We have a natural map $H_{i,j} \to D_i \times D_j$ whose restriction to each copy of $\wp$ is the imbedding $\wp \to D_i \times D_j$.

Let $H = \sqcup H_{i,j}$ (union over the unordered pairs i, j of distinct elements in I). This has a permutation $a : H \to H$ (defined by the permutations $a : H_{i,j} \to H_{i,j}$) and a map $H \to \sqcup D_i \times D_j$ (union over the unordered pairs i, j of distinct elements in I) induced by $H_{i,j} \to D_i \times D_j$. This defines a structure of a graph on $\mathbf{I}, H$. This clearly has the required properties.

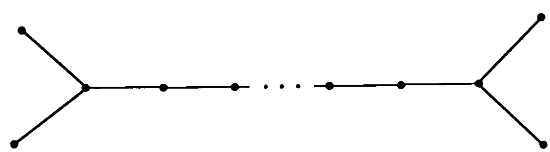
14.1.3. Remark. In general, the graph with automorphism whose existence is asserted in the previous proposition is not uniquely determined by $(I,\cdot)$. However, if the Cartan datum $(I,\cdot)$ is symmetric, the construction in the previous proposition attaches to $(I,\cdot)$ a graph $(\mathbf{I}, H, h \mapsto [h])$, called *the graph of* $(I,\cdot)$, which is unique up to isomorphism; in this case, $\mathbf{I} = I$, $H_{i,j}$ is a set with $-i\cdot j$ elements and a is the identity automorphism.

14.1.4. Classification of symmetric Cartan data of affine or finite type. The symmetric Cartan data of affine type are completely described by their graphs. We enumerate below the graphs that appear in this way.

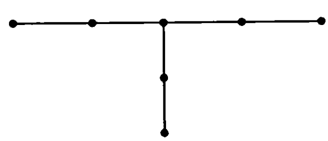
$\tilde{A}_n (n \geq 1)$; a polygon with $n+1$ vertices; for $n = 1$, this is the graph with two vertices which are joined with two edges.



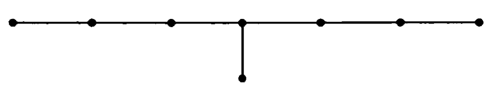
$\tilde{D}_n (n \geq 4)$ (a graph with $n+1$ vertices):



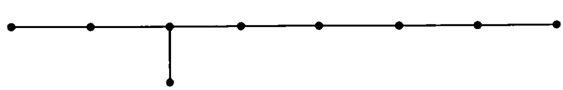
$\tilde{E}_6$:



$\tilde{E}_7$:



$\tilde{E}_8$:



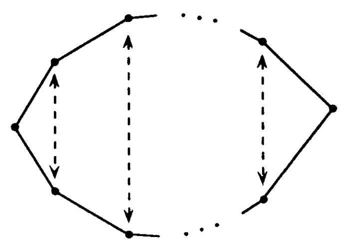
According to McKay, these graphs are in 1-1 correspondence with the various finite subgroups of $SL_2(\mathbf{C})$, up to isomorphism.

Certain vertices of these graphs are said to be *special* : namely, all vertices for $\tilde{A}_n$, the four end points for $\tilde{D}_n$, the three end points for $\tilde{E}_6$, the two end points furthest from the branch point for $\tilde{E}_7$, the end point furthest from the branch point for $\tilde{E}_8$.

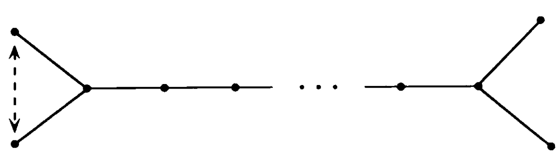
The group of automorphisms of any of the graphs above acts transitively on the set of special vertices. Therefore, by removing a special vertex from one of the graphs above, we obtain a graph which is independent of the special vertex chosen. The resulting graphs are denoted A_n, D_n, E_6, E_7, E_8. We get in this way the various graphs corresponding to irreducible, simply laced Cartan data of finite type.

14.1.5. Classification of non-symmetric Cartan data of affine type. Let us consider one of the graphs $\tilde{A}_n, \ldots, \tilde{E}_8$, together with an admissible automorphism a of order $\mathbf{n} > 1$, which has at least one fixed vertex. We enumerate the various possibilities (up to isomorphism).

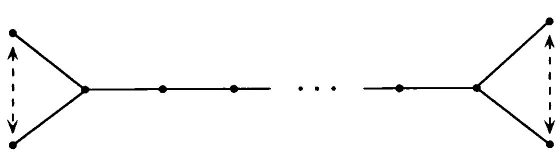
(a) $\tilde{A}_n$ ($n \geq 3$, odd), $\mathbf{n} = 2$ and a has 2 fixed vertices:



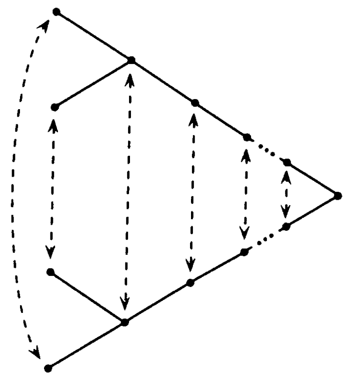
(b) $\tilde{D}_n$, $\mathbf{n} = 2$ and a has $n - 1$ fixed vertices:



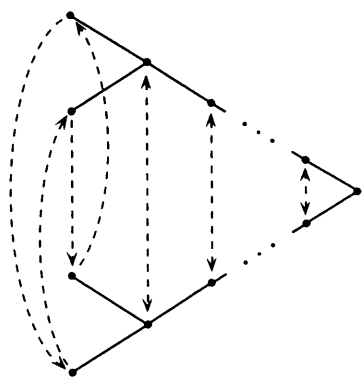
(c) $\tilde{D}_n$, ($n \geq 5$), $\mathbf{n} = 2$ and a has $n - 3$ fixed vertices.



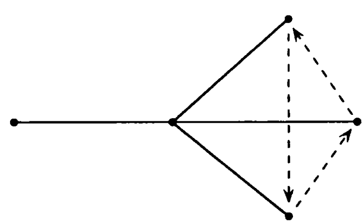
(d) $\tilde{D}_n$ (n even), $\mathbf{n} = 2$ and a has 1 fixed vertex:



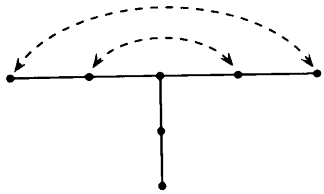
(e) $\tilde{D}_n$ (n even), $\mathbf{n} = 4$ and a has 1 fixed vertex:



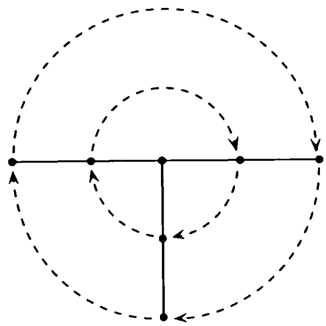
(f) $\tilde{D}_4$, $\mathbf{n} = 3$ and a has 2 fixed vertices:



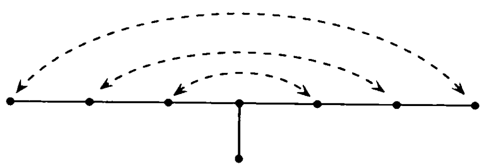
(g) $\tilde{E}_6$, $\mathbf{n} = 2$ and a has 3 fixed vertices:



(h) $\tilde{E}_6$, $\mathbf{n} = 3$ and a has 1 fixed vertex:



(i) $\tilde{E}_7$, $\mathbf{n} = 2$ and a has 2 fixed vertices:



In each case (a)–(i), we may define a Cartan datum as in 14.1.1. We then obtain exactly the various affine non-symmetric Cartan data, up to proportionality (see 1.1.1) which were classified by Kac, Macdonald, Moody and Bruhat-Tits.

14.1.6. Classification of irreducible, non-symmetric Cartan data of finite type. We consider one of the graphs $A_n, \ldots, E_8$, together with an admissible automorphism a of order $\mathbf{n} > 1$.

We enumerate the various possibilities (up to isomorphism).

(a) A_n ($n \geq 3$, odd), $\mathbf{n} = 2$.

(b) D_n, $\mathbf{n} = 2$.

(c) D_4, $\mathbf{n} = 3$.

(d) E_6, $\mathbf{n} = 2$.

In each case (a)–(d), we may define a Cartan datum as in 14.1.1. We then obtain exactly the irreducible non-symmetric Cartan data of finite type, up to proportionality.

14.2. The Signed Basis $\mathcal{B}$

14.2.1. Let V be a $\mathbf{Q}(v)$-vector space with a given basis B and a given symmetric bilinear form $(,) : V \times V \to \mathbf{Q}(v)$. We say that B is *almost orthonormal* for $(,)$ if

(a) $(b, b') \in \delta_{b,b'} + v^{-1}\mathbf{Z}[[v^{-1}]] \cap \mathbf{Q}(v)$ for all $b, b' \in B$.

Let $\mathbf{A} = \mathbf{Q}[[v^{-1}]] \cap \mathbf{Q}(v)$. Let ${}_{\mathcal{A}}V$ be the $\mathcal{A}$-submodule of V generated by B and let $L(V) = \{x \in V | (x, x) \in \mathbf{A}\}$.

Lemma 14.2.2. *In the setup above, the following hold.*

(a) *$L(V)$ is an $\mathbf{A}$-submodule of V and B is a basis of it.*

(b) *Let $x \in {}_{\mathcal{A}}V$ be such that $(x, x) \in 1 + v^{-1}\mathbf{A}$. Then there exists $b \in B$ such that $x = \pm b \mod v^{-1}L(V)$.*

(c) *Let $x \in V$ be such that $(x, x) \in v^{-1}\mathbf{A}$. Then $x \in v^{-1}L(V)$.*

Let $x \in V$. Assume that $x \neq 0$. We can write uniquely $x = \sum_{b \in B} c_b b$ with $c_b \in \mathbf{Q}(v)$. Since only finitely many c_b are non-zero, we can find uniquely $t \in \mathbf{Z}$ and $p_b \in \mathbf{Z}$ (zero for all but finitely many b, but non-zero for some b) such that, for all b, we have $v^{-t}c_b - p_b \in v^{-1}\mathbf{A}$.

We have $(x, x) = (\sum_b p_b^2)v^{2t} \mod v^{2t-1}\mathbf{A}$. Note that $\sum_b p_b^2$ is a rational number > 0. Hence $(x, x) \in \mathbf{A}$ if and only if $t \leq 0$; this is equivalent to the condition that $c_b \in \mathbf{A}$ for all b and (a) follows.

If $(x, x) \in v^{-1}\mathbf{A}$, then we must have $t < 0$; hence $c_b \in v^{-1}\mathbf{A}$ for all b; (c) follows. If $x \in {}_{\mathcal{A}}V$ and $(x, x) \in 1 + v^{-1}\mathbf{A}$, then we must have $t = 0$ and $\sum_b p_b^2 = 1$ with $p_b \in \mathbf{Z}$; hence $p_b = \pm 1$ for some b and $p_b = 0$ for all other b; thus, (b) follows. The lemma is proved.

In the remainder of this chapter we fix a Cartan datum $(I, \cdot)$. Let $\mathbf{f}, {}_{\mathcal{A}}\mathbf{f}$ be defined in terms of $(I, \cdot)$ as in 1.2.5, 1.4.7.

Theorem 14.2.3. *Let $\mathcal{B}$ be the set of all $x \in \mathbf{f}$ such that $x \in {}_{\mathcal{A}}\mathbf{f}$, $\bar{x} = x$ and $(x, x) \in 1 + v^{-1}\mathbf{Z}[[v^{-1}]]$. (The last condition is equivalent to $\{x, x\} \in 1 + v\mathbf{Z}[[v]]$ since $\bar{x} = x$.)*

(a) *$\mathcal{B}$ is a signed basis of the $\mathcal{A}$-module ${}_{\mathcal{A}}\mathbf{f}$ and of the $\mathbf{Q}(v)$-vector space* $\mathbf{f}$.

(b) *If $b, b' \in \mathcal{B}$ and $b' \neq \pm b$, then $(b,b') \in v^{-1}\mathbf{Z}[[v^{-1}]]$ and $\{b,b'\} \in v\mathbf{Z}[[v]]$.*

By 14.1.2, we can find a finite graph $(\mathbf{I}, H, h \mapsto [h])$ and an admissible automorphism a of this graph such that $(I,\cdot)$ is obtained from these data by the construction in 14.1.1. Let $\mathbf{n} \geq 1$ be such that $a^{\mathbf{n}} = 1$. Then, by 13.2.11, $\mathbf{f}$ has a natural isomorphism, say χ, onto the corresponding algebra $\mathbf{k}$ (see 13.2.6). Under χ, the pairings $(,)$ and $(,)$ on $\mathbf{k}$ and $\mathbf{f}$ correspond to each other. This has been seen in the proof of 13.2.11. Moreover, the involutions $D : \mathbf{k} \to \mathbf{k}$ and $^- : \mathbf{f} \to \mathbf{f}$ correspond to each other (they both map the generators $\mathbf{1}_i$ and θ_i to themselves).

Note that χ carries $\mathbf{1}_{ni} \in \mathbf{k}$ to $\theta_i^{(n)}$ for any $i \in I$ and any $n \geq 0$ (this follows from 13.1.12(c)); hence it carries the $\mathcal{A}$-subalgebra ${}_{\mathcal{A}}\mathbf{k}$ (which is generated by the $\mathbf{1}_{ni}$) onto ${}_{\mathcal{A}}\mathbf{f}$ (see 1.4.7). Moreover, χ carries the signed basis $\mathcal{B}$ of $\mathbf{k}$ (see 13.1.2) onto a signed basis of $\mathbf{f}$, which we denote by the same letter. By the already known properties of the signed basis of $\mathbf{k}$, it remains to prove the following statement: let $x \in {}_{\mathcal{A}}\mathbf{f}$ be such that $\bar{x} = x$ and $(x,x) \in 1 + v^{-1}\mathbf{Z}[[v^{-1}]]$; then $x \in \mathcal{B}$. Let B be a basis of $\mathbf{f}$ such that $\mathcal{B} = B \cup (-B)$. We can write uniquely $x = \sum_b c_b b$ where b runs over B and $c_b \in \mathcal{A}$ are zero except for finitely many b. Using 14.2.2(b), we see that there is a unique $b \in B$ such that $c_b \in \pm 1 + v^{-1}\mathbf{Z}[v^{-1}]$ and $c_{b'} \in v^{-1}\mathbf{Z}[v^{-1}]$ for $b' \neq b$. From $\bar{x} = x$ and $\bar{b}' = b'$ for all $b' \in B$, it follows that $\bar{c}_{b'} = c_{b'}$ for all $b' \in B$. It follows that $c_b = \pm 1$ and $c_{b'} = 0$ for all $b' \neq b$. Thus, $x \in B \cup (-B)$. The theorem is proved.

14.2.4. Definition. $\mathcal{B}$ is called the *canonical signed basis* of $\mathbf{f}$.

Although the proof of the existence of $\mathcal{B}$ requires a choice of a graph with automorphism, which is not unique in general, the resulting signed basis is independent of any choice, hence the word *canonical.*

14.2.5. The following properties of $\mathcal{B}$ follow immediately from the definitions.

(a) We have $\mathcal{B} = \sqcup_\nu \mathcal{B}_\nu$ where $\mathcal{B}_\nu = \mathcal{B} \cap \mathbf{f}_\nu$.

(b) We have $\theta_i^{(s)} \in \mathcal{B}$ for any $i \in I$ and $s \geq 0$; in particular, $1 \in \mathcal{B}$.

Using 1.2.8(b), we see that

(c) $$\sigma(\mathcal{B}) = \mathcal{B}.$$

Proposition 14.2.6. (a) *r and $\bar{r}$ map ${}_{\mathcal{A}}\mathbf{f}$ into the $\mathcal{A}$-submodule ${}_{\mathcal{A}}\mathbf{f}\otimes_{\mathcal{A}}({}_{\mathcal{A}}\mathbf{f})$ of $\mathbf{f}\otimes\mathbf{f}$.*

(b) *For any $x,y\in{}_{\mathcal{A}}\mathbf{f}$ we have $(x,y)\in\mathbf{Z}[[v^{-1}]]\cap\mathbf{Q}(v)$ and $\{x,y\}\in\mathbf{Z}[[v]]\cap\mathbf{Q}(v)$.*

This follows immediately from the analogous properties of ${}_{\mathcal{A}}\mathbf{k}$ (see 13.2.2, 13.2.4, 13.2.5) which are already known.

14.3. The Subsets $\mathcal{B}_{i;n}$ of $\mathcal{B}$

14.3.1. Given $i\in I$ and $n\geq 0$, we set $\mathcal{B}_{i;\geq n}=\mathcal{B}\cap\theta_i^n\mathbf{f}$ and ${}^{\sigma}\mathcal{B}_{i;\geq n}=\mathcal{B}\cap\mathbf{f}\theta_i^n$. Let $\mathcal{B}_{i;n}=\mathcal{B}_{i;\geq n}-\mathcal{B}_{i;\geq n+1}$ and ${}^{\sigma}\mathcal{B}_{i;n}={}^{\sigma}\mathcal{B}_{i;\geq n}-{}^{\sigma}\mathcal{B}_{i;\geq n+1}$. Thus, we have partitions $\mathcal{B}_{i;\geq n}=\sqcup_{n'\geq n}\mathcal{B}_{i;n'}$ and ${}^{\sigma}\mathcal{B}_{i;\geq n}=\sqcup_{n'\geq n}{}^{\sigma}\mathcal{B}_{i;n'}$. Since $\sigma(\mathcal{B})=\mathcal{B}$ and $\sigma(\theta_i^n\mathbf{f})=\mathbf{f}\theta_i^n$, we have ${}^{\sigma}\mathcal{B}_{i;\geq n}=\sigma(\mathcal{B}_{i;\geq n})$ and ${}^{\sigma}\mathcal{B}_{i;n}=\sigma(\mathcal{B}_{i;n})$.

Theorem 14.3.2. (a) *$\mathcal{B}_{i;\geq n}$ is a signed basis of the $\mathbf{Q}(v)$-vector space $\theta_i^n\mathbf{f}$ and of the $\mathcal{A}$-module $\sum_{n':n'\geq n}\theta_i^{(n')}{}_{\mathcal{A}}\mathbf{f}$.*

(b) *${}^{\sigma}\mathcal{B}_{i;\geq n}$ is a signed basis of the $\mathbf{Q}(v)$-vector space $\mathbf{f}\theta_i^n$ and of the $\mathcal{A}$-module $\sum_{n':n'\geq n}({}_{\mathcal{A}}\mathbf{f}\theta_i^{(n')})$.*

(c) *If $b\in\mathcal{B}_{i;0}$, then there is a unique element $b'\in\mathcal{B}_{i;n}$ such that $\theta_i^{(n)}b=b'$ plus an $\mathcal{A}$-linear combination of elements in $\mathcal{B}_{i;\geq n+1}$. Moreover, $b\mapsto b'$ is a bijection $\pi_{i,n}:\mathcal{B}_{i;0}\to\mathcal{B}_{i;n}$.*

(d) *If $b\in{}^{\sigma}\mathcal{B}_{i;0}$, then there is a unique element $b''\in{}^{\sigma}\mathcal{B}_{i;n}$ such that $b\theta_i^{(n)}=b''$ plus an $\mathcal{A}$-linear combination of elements in ${}^{\sigma}\mathcal{B}_{i;\geq n+1}$. Moreover, $b\mapsto b''$ is a bijection ${}^{\sigma}\pi_{i,n}:{}^{\sigma}\mathcal{B}_{i;0}\to{}^{\sigma}\mathcal{B}_{i;n}$.*

For the proof we place ourselves in the setup considered in the proof of Theorem 14.2.3. Thus i is now regarded as an a-orbit in $\mathbf{I}$. Let $\mathbf{V}\in\mathcal{V}_\nu^a$. For any $n\geq 0$, let $\mathcal{B}_{\mathbf{V};i;n}$ be the set of all $\pm[B,\phi]\in\mathcal{B}_{\mathbf{V}}$ (see 12.6.4) such that $B\in\mathcal{P}_{\mathbf{V};i;n\gamma}$ where $\gamma=\sum_{\mathbf{i}\in i}\mathbf{i}$. By 10.3.3 and 12.5.1, we have a partition $\mathcal{B}_{\mathbf{V}}=\sqcup_{n\geq 0}\mathcal{B}_{\mathbf{V};i;n}$. By our identification $\mathcal{B}_{\mathbf{V}}=\mathcal{B}_\nu$, this becomes a partition $\mathcal{B}_\nu=\sqcup_{n\geq 0}\mathcal{B}_{\nu;i;n}$.

Let $\mathcal{B}'_{i;n}=\cup_\nu\mathcal{B}_{\nu;i;n}$. We will show below that $\mathcal{B}'_{i,n}$ just defined is the same as $\mathcal{B}_{i;n}$ in 14.3.1; see (h) below. We then have a partition $\mathcal{B}=\sqcup_{n\geq 0}\mathcal{B}'_{i;n}$.

Translating the geometric properties of $\mathcal{B}_{\mathbf{V};i;n}$ expressed by 10.3.2(c) we obtain the following property of $\mathcal{B}'_{i;n}$.

(e) For any $n\geq 0$, there is a unique 1-1 correspondence $b\leftrightarrow b'$ between $\mathcal{B}'_{i;0}$ and $\mathcal{B}'_{i;n}$ such that $\theta_i^{(n)}b=b'$ plus an $\mathcal{A}$-linear combination of elements in $\cup_{n'>n}\mathcal{B}'_{i;n'}$.

Let $M_n = \sum_{n':n'\geq n} \theta_i^{(n')}{}_{\mathcal{A}}\mathbf{f}$ and let M'_n be the $\mathcal{A}$-submodule of $\mathbf{f}$ generated by $\cup_{n':n'\geq n}\mathcal{B}'_{i;n'}$. We show that for fixed ν,

(f) any $b \in \mathcal{B}_{\nu;i;n}$ is contained in M_n.

We argue by descending induction on n; note that $\mathcal{B}_{\nu;i;n}$ is empty unless $n \leq \nu_{\mathbf{i}}$ for any $\mathbf{i} \in i$. By (e), we have

$$b \in \theta_i^{(n)}{}_{\mathcal{A}}\mathbf{f} + \sum_{n':n'>s} \mathcal{A}\mathcal{B}_{\nu;i,n'};$$

by the induction hypothesis, we have $\mathcal{B}_{\nu;i,n'} \subset M_{n'}$ and it follows that $b \in M_n$. Thus (f) is proved. Thus we have $\mathcal{B}'_{i;n} \subset M_n$. If $n' \geq n$, we have $\mathcal{B}'_{i;n'} \subset M_{n'} \subset M_n$. It follows that, for any $n \geq 0$, we have $M'_n \subset M_n$.

Next we show that

(g) for any $b \in \mathcal{B}_\nu$, we have $\theta_i^{(n)} b \in M'_n$.

We argue by induction on $c(b) = \sum_{\mathbf{i}} \nu_{\mathbf{i}}$. If $b \in \mathcal{B}_{\nu;i;t}$ where $t > 0$, then as we have seen, we have $b \in M_t$; hence b is an $\mathcal{A}$-linear combination of elements $\theta_i^{(m)} b_1$ with $m \geq t$ and with $b_1 \in \mathcal{B}$ such that the induction hypothesis applies to b_1. Then $\theta_i^{(n)} b$ is an $\mathcal{A}$-linear combination of elements $\theta_i^{(m)}\theta_i^{(n)} b_1$ with b_1 as above. By the induction hypothesis, we have $\theta_i^{(n)} b_1 \in M'_n$. Hence $\theta_i^{(n)} b \in \sum_m \theta_i^{(m)} M'_n \subset M'_n$, as required.

Next we assume that $b \in \mathcal{B}_{\nu;i;0}$. Then $\theta_i^{(n)} b \in M'_n$ by (e). Thus, (g) is proved. It follows that, for any $n \geq 0$, we have

$$M_n = \sum_{n':n'\geq n} \theta_i^{(n')}{}_{\mathcal{A}}\mathbf{k} \subset \sum_{n':n'\geq n} M'_{n'} \subset M'_n.$$

We have proved that $M_n \subset M_{n'} \subset M_n$. Thus, $M'_n = M_n$. It follows that (a) holds and $\mathcal{B}_{i;\geq n} = \cup_{n':n'\geq n}\mathcal{B}'_{i;n'}$. In particular, we have

(h) $\mathcal{B}_{i;n} = \mathcal{B}'_{i;n}$.

We now prove (c). The existence of b' asserted in (c) follows immediately from (e). We now prove uniqueness. Assume that $b'_1 \in \mathcal{B}_{i;n}$ has the same property as that asserted for b' in (c). Then $b' - b'_1$ is on the one hand a linear combination of elements in $\cup_{n'>n}\mathcal{B}_{i;n'}$ and on the other hand it is a linear combination of elements in $\mathcal{B}_{i;n}$. It follows that $b' = b'_1$ and (c) is proved.

Now (b) and (d) follow from (a),(c) by applying σ. The theorem is proved.

14.3.3. By 12.5.1(c), we have

$$\mathcal{B} - \{\pm 1\} = \cup_{i \in I; n > 0} \mathcal{B}_{i;n}.$$

14.4. The Canonical Basis B of f

14.4.1. We would like to find in a natural way a basis of $\mathbf{f}$ contained in its canonical signed basis. If $(I, \cdot)$ is symmetric, such a basis is given by geometry. In this case, a is the identity automorphism of our graph and we can take $\mathbf{n} = 1$. Hence we have $\mathcal{O}' = \mathcal{A}$ and $\mathcal{K}(\mathcal{Q}_{\mathbf{V}}) = {}_{\mathcal{A}}\mathcal{K}(\mathcal{Q}_{\mathbf{V}})$ (for $\mathbf{V} \in \mathcal{V}$) has not only a natural signed basis, but a natural basis consisting of the elements $[B, 1]$ where B is a simple object of $\mathcal{P}_{\mathbf{V}}$ and 1 is the identity isomorphism $1 : B \cong B$.

14.4.2. In the general case, the descent from a signed basis to a basis will be non-geometric. We lay the groundwork with some definitions.

For any $\nu \in \mathbf{N}[I]$ we define a subset $\mathbf{B}_\nu$ of $\mathcal{B}_\nu$ by induction on $\operatorname{tr} \nu$ as follows. If $\nu = 0$, we set $\mathbf{B}_\nu = \{1\}$. If $\operatorname{tr} \nu > 0$, we set $\mathbf{B}_\nu = \cup_{i \in I, n > 0; \nu_i \geq n} \pi_{i,n}(\mathbf{B}_{\nu - ni} \cap \mathcal{B}_{i;0})$.

Let $\mathbf{B} = \sqcup_\nu \mathbf{B}_\nu \subset \mathcal{B}$. By 14.3.3, we have that $\mathcal{B} = \mathbf{B} \cup (-\mathbf{B})$. We can now state the following result.

Theorem 14.4.3. *Let* $\nu \in \mathbf{N}[I]$. *Then*

(a) $\mathbf{B}_\nu \cap (-\mathbf{B}_\nu) = \emptyset$;

(b) $\mathbf{B}_\nu \cap (-\sigma(\mathbf{B}_\nu)) = \emptyset$;

(c) $\sigma(\mathbf{B}_\nu) = \mathbf{B}_\nu$.

(d) $\mathbf{B}$ *is a basis of the* $\mathcal{A}$*-module* ${}_{\mathcal{A}}\mathbf{f}$ *and a basis of the* $\mathbf{Q}(v)$*-vector space* $\mathbf{f}$.

(e) *For any* ν, $\mathbf{B}_\nu$ *is a basis of the* $\mathcal{A}$*-module* ${}_{\mathcal{A}}\mathbf{f}_\nu$ *and a basis of the* $\mathbf{Q}(v)$*-vector space* $\mathbf{f}_\nu$.

14.4.4. Proof of the theorem, assuming that $(I, \cdot)$ is symmetric. In this case, as in the proof of Theorem 14.2.3, we have a natural choice for the graph (with identity automorphism a), see Remark 14.1.3. Moreover since a=1, the corresponding algebra $\mathbf{k}$ has a natural basis inside its natural signed basis, defined as in 14.4.1. From the definitions, it is clear that this basis (transferred to $\mathbf{f}$) coincides with $\mathbf{B}$ and has all the required properties. This completes the proof (in the symmetric case).

14.4.5. The proof of Theorem 14.4.3 in the general case will be given in 19.2.3; in the remainder of this section we shall assume that the theorem is known in general.

14.4.6. Definition. $\mathbf{B}$ is called the *canonical basis* of $\mathbf{f}$.

We shall use the following notation: $\mathbf{B}_{i;n} = \mathcal{B}_{i;n} \cap \mathbf{B}$ for any $i \in I$ and $n \in \mathbf{N}$; note that $\pi_{i,n}$ defines a bijection $\mathbf{B}_{i;0} \cong \mathbf{B}_{i;n}$. Set ${}^\sigma\mathbf{B}_{i;n} = \sigma(\mathbf{B}_{i;n})$. Then ${}^\sigma\pi_{i,n}$ defines a bijection ${}^\sigma\mathbf{B}_{i;0} \cong {}^\sigma\mathbf{B}_{i;n}$.

14.4.7. We can regard $\mathbf{B}$ as the set of vertices of a graph colored by $I \times \{1, 2, \dots\}$ in which b, b' are joined by an edge of color (i, n) if $b \in \mathbf{B}_{i;0}, b' \in \mathbf{B}_{i;n}$ and $b' = \pi_{i,n}(b)$. This is called the *left graph* on $\mathbf{B}$.

Similarly, we can regard $\mathbf{B}$ as the set of vertices of a graph colored by $I \times \{1, 2, \dots\}$ in which b, b'' are joined by an edge of color (i, n) if $b \in {}^\sigma\mathbf{B}_{i;0}, b'' \in {}^\sigma\mathbf{B}_{i;n}$ and $b'' = {}^\sigma\pi_{i,n}(b)$. This is called the *right graph* on $\mathbf{B}$.

14.4.8. Let us choose a finite graph $(\mathbf{I}, H, h \mapsto [h])$ and an admissible automorphism a of this graph such that $(I, \cdot)$ is obtained from them by the construction in 14.1.1. We define a new (symmetric) Cartan datum $(\tilde{I}, \cdot)$ associated to the same graph and to its identity automorphism, as in 14.1.3. More precisely, we have $\tilde{I} = \mathbf{I}$, $\mathbf{i} \cdot \mathbf{i} = 2$ and, for $\mathbf{i} \neq \mathbf{j} \in \mathbf{I}$, we have that $\mathbf{i} \cdot \mathbf{j}$ is -1 times the number of edges joining $\mathbf{i}$ to $\mathbf{j}$.

Let $\tilde{\mathbf{f}}$ be the algebra defined like $\mathbf{f}$, in terms of the Cartan datum $(\tilde{I}, \cdot)$ and let $\tilde{\mathbf{B}} \subset \tilde{\mathbf{f}}$ be its canonical basis. Similarly, let $\pi_{\mathbf{i},n} : \tilde{\mathbf{B}}_{\mathbf{i};0} \to \tilde{\mathbf{B}}_{\mathbf{i};n}$ and ${}^\sigma\pi_{\mathbf{i},n} : {}^\sigma\tilde{\mathbf{B}}_{\mathbf{i};0} \to {}^\sigma\tilde{\mathbf{B}}_{\mathbf{i};n}$ be the bijections analogous to $\pi_{i,n} : \mathbf{B}_{i;0} \to \mathbf{B}_{i;n}$ and ${}^\sigma\pi_{i,n} : {}^\sigma\mathbf{B}_{i;0} \to {}^\sigma\mathbf{B}_{i;n}$ in 14.4.6. Now $a : \mathbf{I} \to \mathbf{I}$ induces an algebra automorphism $a : \tilde{\mathbf{f}} \to \tilde{\mathbf{f}}$ which restricts to a bijection $a : \tilde{\mathbf{B}} \to \tilde{\mathbf{B}}$ whose fixed point set is denoted by $\tilde{\mathbf{B}}^a$.

The $\mathbf{I} \times \{1, 2, \dots\}$-colored left graph structure on $\tilde{\mathbf{B}}$ (as in 14.4.7) defines a $I \times \{1, 2, \dots\}$-colored graph structure on the subset $\tilde{\mathbf{B}}^a$ as follows. We say that $b, b' \in \tilde{\mathbf{B}}^a$ are joined by an edge of color (i, n) if they can be joined in the left graph on $\tilde{\mathbf{B}}$ by a sequence of edges of colors $(\mathbf{i}_1, n), (\mathbf{i}_2, n), \dots, (\mathbf{i}_s, n)$ where $\mathbf{i}_1, \mathbf{i}_2, \dots, \mathbf{i}_s$ is an enumeration of the elements of i in some order. This is called the left graph on $\tilde{\mathbf{B}}^a$. By replacing "left" by "right" we obtain a $I \times \{1, 2, \dots\}$-colored graph structure on $\tilde{\mathbf{B}}^a$, called the right graph on $\tilde{\mathbf{B}}^a$.

Theorem 14.4.9. *There is a unique bijection $\chi : \mathbf{B} \to \tilde{\mathbf{B}}^a$ compatible with the structures of $I \times \{1, 2, \dots\}$-colored left (resp. right) graphs and such that $\chi(1) = 1$. The two bijection corresponding to "left" and "right" coincide.*

The inverse bijection is obtained geometrically by attaching to a pair (B, ϕ) where B is a simple object of a suitable $\mathcal{P}_{\mathbf{V}}$ and ϕ is an isomorphism $a^*B \cong B$, the simple object B without specifying ϕ. The fact that this bijection is compatible with the colored graph structures is also clear geometrically (using, in particular, 12.5.1(a)).

14.4.10. Remark. This theorem shows that to describe the left or right graph structure on $\mathbf{B}$ it is enough to do the same in the case where the Cartan datum is symmetric.

Theorem 14.4.11. *Assume that the root datum is Y-regular. Let $\lambda \in X^+$ and let $\Lambda_\lambda = \mathbf{f}/\sum_i \mathbf{f}\theta^{\langle i,\lambda\rangle+1}$ be the $\mathbf{U}$-module defined in 3.5.6. As in 3.5.7, let $\eta \in \Lambda_\lambda$ be the image of 1. Let $\mathbf{B}(\lambda) = \cap_{i\in I}(\cup_{n;0\le n\le\langle i,\lambda\rangle}{}^\sigma\mathbf{B}_{i,n})$.*

(a) *The map $b \to b^-\eta$ define a bijection of $\mathbf{B}(\lambda)$ onto a basis $\mathbf{B}(\Lambda_\lambda)$ of Λ_λ.*

(b) *If $b \in \mathbf{B} - \mathbf{B}(\lambda)$, then $b^-\eta = 0$.*

An equivalent statement is that

$$\cup_{i,n;n\ge\langle i,\lambda\rangle+1}{}^\sigma\mathbf{B}_{i,n}$$

is a basis of $\sum_i \mathbf{f}\theta^{\langle i,\lambda\rangle+1}$. This follows immediately from Theorem 14.3.2.

14.4.12. Definition. $\mathbf{B}(\Lambda_\lambda)$ is called the *canonical basis* of Λ_λ.

Theorem 14.4.13 (Positivity). *Assume that $(I,\cdot)$ is symmetric.*

(a) *For any $b, b' \in \mathbf{B}$, we have*

$$bb' = \sum_{b''\in\mathbf{B};n\in\mathbf{Z}} c_{b,b',b'',n}v^n b''$$

where $c_{b,b',b'',n} \in \mathbf{N}$ are zero except for finitely many b'', n.

(b) *For any $b \in \mathbf{B}$ we have*

$$r(b) = \sum_{b',b''\in\mathbf{B};n\in\mathbf{Z}} d_{b,b',b'',n}v^n b' \otimes b''$$

where $d_{b,b',b'',n} \in \mathbf{N}$ are zero except for finitely many b', b'', n.

(c) *For any $b, b' \in \mathbf{B}$ we have*

$$(b, b') = \sum_{n\in\mathbf{N}} f_{b,b',n}v^{-n}$$

where $f_{b,b',n} \in \mathbf{N}$.

The theorem asserts the positivity of certain integers; in our definition in the framework of perverse sheaves, these integers are the dimensions of certain $\bar{\mathbf{Q}}_l$-vector spaces. The theorem follows.

14.4.14. Remark. For non-symmetric $(I,\cdot)$, the integers in question are not dimensions, but traces of automorphisms of finite order of certain $\bar{\mathbf{Q}}_l$-vector spaces and it is not clear whether they are positive or not.

14.4.15. In the case where $(I,\cdot)$ is symmetric, the set $\mathbf{B}_\nu$ is (conjecturally) in natural $1-1$ correspondence with the set X_ν of irreducible components of a certain Lagrangian variety naturally attached to $(I,\cdot)$ and to ν (see [9, 13.7]). The union $\sqcup_\nu X_\nu$ has a natural colored graph structure (defined as in [8]) and one can hope that the previous bijection respects the colored graph structures.

14.5. Examples

14.5.1. Assume that $(I,\cdot)$ is a simply laced Cartan datum of finite type. Let $(\mathbf{I}, H, \dots)$ be the graph of $(I,\cdot)$ (see 14.1.3); note that $\mathbf{I} = I$. We choose an orientation of this graph. Let $\mathbf{V} \in \mathcal{V}$ and let $G_\mathbf{V}, \mathbf{E_V}$ be as in 9.1.2.

From the results in [9], it follows that there is a 1-1 correspondence between the set of orbits of $G_\mathbf{V}$ on $\mathbf{E_V}$ (a finite set, by Gabriel's theorem) and the set of isomorphism classes of objects of $\mathcal{P}_\mathbf{V}$ (see 9.1.3): to an orbit of $G_\mathbf{V}$ corresponds the $G_\mathbf{V}$-equivariant simple perverse sheaf whose support is the closure of that orbit. This is well-defined since the action of $G_\mathbf{V}$ has connected isotropy groups.

14.5.2. Assume that $(I,\cdot)$ is such that $I = \{i,j\}$ and $i\cdot i = j\cdot j = 2, i\cdot j = j\cdot i = -2$. Then $(I,\cdot)$ is a symmetric Cartan datum of affine type.

Let $(\mathbf{I}, H, \dots)$ be the graph of $(I,\cdot)$ (see 14.1.3); note that $\mathbf{I} = I$ and H has two elements. We orient this graph so that $h' = i$ for both $h \in H$. Let $\mathbf{V} \in \mathcal{V}$ and let $G_\mathbf{V}, \mathbf{E_V}$ be as in 9.1.2. Note that $\mathbf{E_V}$ consists of all pairs T, T' of linear maps $\mathbf{V}_i \to \mathbf{V}_j$. Assume that both $\mathbf{V}_i$ and $\mathbf{V}_j$ are n-dimensional and $n \geq 2$. Then $G_\mathbf{V}$ acts on $\mathbf{E_V}$ with infinitely many orbits.

Let $\boldsymbol{\nu} = (i,j,i,j,\dots)$ ($2n$ terms). Then $\pi_{\boldsymbol{\nu}} : \tilde{\mathcal{F}}_{\boldsymbol{\nu}} \to \mathbf{E_V}$ (see 9.1.3) is a principal covering with group S_n (the symmetric group) over an open dense subset of $\mathbf{E_V}$. This gives rise to irreducible local systems over an open dense subset of $\mathbf{E_V}$, and hence to simple perverse sheaves on $\mathbf{E_V}$, indexed by the irreducible representations of S_n. These simple perverse sheaves belong to $\mathcal{P}_\mathbf{V}$.

14.5.3. Assume that $(I,\cdot)$ is such that $I = \{i\}$ and $i\cdot i = 2$. The canonical basis $\mathbf{B}$ of $\mathbf{f}$ consists of the elements $\theta_i^{(a)}$ $(a \in \mathbf{N})$.

14.5.4. Assume that $(I, \cdot)$ is such that $I = \{i, j\}$ and $i \cdot i = j \cdot j = 2$ and $i \cdot j = j \cdot i = -1$. The canonical basis $\mathbf{B}$ of $\mathbf{f}$ consists of the elements $\theta_i^{(a)}\theta_j^{(b)}\theta_i^{(c)}$ $(a, b, c \in \mathbf{N}, b \geq a + c)$ and of the elements $\theta_j^{(c)}\theta_i^{(b)}\theta_j^{(a)}$ $(a, b, c \in \mathbf{N}, b \geq a + c)$ with the identification $\theta_i^{(a)}\theta_j^{(b)}\theta_i^{(c)} = \theta_j^{(c)}\theta_i^{(b)}\theta_j^{(a)}$ for $b = a + c$.

14.5.5. Assume that $(I, \cdot)$ is as in 14.5.2. The elements of $\mathbf{B}_{i+j}$ are $\theta_i\theta_j, \theta_j\theta_i$.

The elements of $\mathbf{B}_{2i+2j}$ are:

$$\theta_i^{(2)}\theta_j^{(2)}, \theta_j^{(2)}\theta_i^{(2)}, \theta_i\theta_j^{(2)}\theta_i, \theta_j\theta_i^{(2)}\theta_j, \theta_i\theta_j\theta_i\theta_j - \theta_i^{(2)}\theta_j^{(2)}, \theta_j\theta_i\theta_j\theta_i - \theta_j^{(2)}\theta_i^{(2)}.$$

For further examples, see [11].

Notes on Part II

1. The canonical basis of **f** has been introduced by the author in [7], assuming that the Cartan datum is symmetric, of finite type. In fact, in [7] two definitions for the canonical basis were given: an elementary algebraic one, involving braid group actions, and a topological one, based on quivers and perverse sheaves. (The elementary definition applies essentially without change to not necessarily symmetric Cartan data of finite type.) The topological definition in [7] was in terms of intersection cohomology of certain singular varieties arising from quivers by a construction reminiscent of that in [4] of the new basis of a Hecke algebra (which used the intersection cohomology of Schubert varieties). One of the main observations of [7] was that the canonical basis of **f** gives rise simultaneoulsy to a canonical basis in each **U**-module Λ_λ, which had rather favourable properties.
2. After [7] became available, Kashiwara announced an elementary algebraic definition of the canonical basis which applied to an arbitrary Cartan datum. Kashiwara's paper [3] contains an inductive construction of the canonical basis, both of **f** and of Λ_λ, which advances like a huge spiral. His construction agrees with that in [7], as shown in [8].
   On the other hand, the author [9] extended the topological definition [7] of the canonical basis to arbitrary (symmetric) Cartan data. (The case of not necessarily symmetric Cartan data was only sketched in [9].) The definition of [9] resembles that of character sheaves [5]. While the method of [9] is not elementary, it has the advantage of being more global and to yield positivity results which cannot be obtained by the elementary approach. The agreement of the definitions in [3], [9], was proved in [2].
3. The exposition in Part II essentially follows the treatment in [9], with two main differences. First, in order to include not necessarily symmetric Cartan data in our treatment, we have to take into account the action of a cyclic group, which is a complicating factor, not present in [9], where only symmetric Cartan data were treated. In addition, we make use of the geometric interpretation of the inner product on **f** given in [2]; this simplifies somewhat the original proof in [9] and provides the link with [3].
4. The basic reference for the theory of perverse sheaves on algebraic varieties is the work of Beilinson, Bernstein, Deligne and Gabber [1].
5. The representation theory of quivers (which is implicit in the constructions in Chapter 9) has a long history going back to Kronecker. In Ringel's work [12], the connection between the representations of a quiver of finite type over a finite field F_q and the plus part of the corresponding Drinfeld-Jimbo algebra at parameter $\sqrt{q}$ was observed for the first time. This work of Ringel was an important source of inspiration for the author's work on the canonical basis. In particular, the definition of the induction functor in 9.2 was inspired by Ringel's definition of the Hall algebra arising from quivers over F_q. On the other hand, the definition of the restriction functor in 9.2 was inspired by the analogous concept for character sheaves [5].
6. The geometric definition of the inner product in 12.2 is taken from [2] where,

however, the cyclic group action was not present.

7. The idea that the canonical basis can be characterized by an almost orthonormality property for the natural inner product, has originally appeared in Kashiwara's paper [3] and has been later used in [2]. This is analogous to the orthogonality properties of character sheaves [5]; it is a hallmark of intersection cohomology.
8. The description of non-symmetric affine Cartan data given in 14.1.5 is different, as far as I know, from the ones in the literature.
9. The ingredients for the definition of the colored graph in 14.4.7 were introduced in [9]; it turns out that that graph contains the same information as the colored graph defined by Kashiwara (but the two graphs are different).
10. The statement 14.4.13 appeared in [2].
11. The example in 14.5.2 is a special case of the results in [10] where the perverse sheaves which constitute the canonical basis in the affine case are described explicitly. (The results in [10] dealt with symmetric affine Cartan data; but in view of Theorem 14.4.9, the same results can be applied in the case of non-symmetric affine Cartan data.)
12. The geometric method used here to construct canonical bases can be applied, more or less word by word, to quivers in which edges joining a vertex with itself are allowed. (This includes, for example, the classical Hall algebra with its canonical basis.) We have not included this more general case in our discussion (but see [11]).

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