CHAPTER 13

The Algebras ${}_{\mathcal{O}'}\mathbf{k}$ and $\mathbf{k}$

13.1. THE ALGEBRA ${}_{\mathcal{O}'}\mathbf{k}$

13.1.1. We preserve the setup of the previous chapter. Given $\nu \in \mathbf{N}[\mathbf{I}]^a$, we may regard $\mathbf{V} \mapsto \mathcal{K}(\mathcal{Q}_{\mathbf{V}})$ as a functor on the category $\mathcal{V}_\nu^a$ with values in the category of $\mathcal{O}'$-modules. An isomorphism $\mathbf{V} \cong \mathbf{V}'$ in $\mathcal{V}_\nu^a$ induces an isomorphism $\mathbf{E}_{\mathbf{V}} \cong \mathbf{E}_{\mathbf{V}'}$ compatible with the a-actions; this induces an isomorphism $\mathcal{Q}_{\mathbf{V}} \cong \mathcal{Q}_{\mathbf{V}'}$ which induces an isomorphism $\mathcal{K}(\mathcal{Q}_{\mathbf{V}}) \cong \mathcal{K}(\mathcal{Q}_{\mathbf{V}'})$ that is actually independent of the choice of the isomorphism $\mathbf{V} \cong \mathbf{V}'$ by the equivariance properties of the complexes considered. Hence we may take the direct limit $\varinjlim_{\mathbf{V}} \mathcal{K}(\mathcal{Q}_{\mathbf{V}})$ over the category $\mathcal{V}_\nu^a$. This direct limit is denoted by ${}_{\mathcal{O}'}\mathbf{k}_\nu$. By the previous discussion, the natural homomorphism $\mathcal{K}(\mathcal{Q}_{\mathbf{V}}) \to {}_{\mathcal{O}'}\mathbf{k}_\nu$ is an isomorphism for any $\mathbf{V} \in \mathcal{V}_\nu^a$.

The signed basis $\mathcal{B}_{\mathbf{V}}$ of $\mathcal{K}(\mathcal{Q}_{\mathbf{V}})$ (see 12.6.4) (where $\mathbf{V} \in \mathcal{V}_\nu^a$) can be regarded as a signed basis of the $\mathcal{O}'$-module ${}_{\mathcal{O}'}\mathbf{k}_\nu$, independent of $\mathbf{V}$; we denote it by $\mathcal{B}_\nu$. It is a finite set.

13.1.2. Let ${}_{\mathcal{O}'}\mathbf{k} = \oplus_\nu ({}_{\mathcal{O}'}\mathbf{k}_\nu)$ (ν runs over $\mathbf{N}[\mathbf{I}]^a$). Let $\mathcal{B} = \sqcup_\nu \mathcal{B}_\nu$, a signed basis of the $\mathcal{O}'$-module ${}_{\mathcal{O}'}\mathbf{k}$. An element $x \in {}_{\mathcal{O}'}\mathbf{k}$ is said to be *homogeneous* if it belongs to ${}_{\mathcal{O}'}\mathbf{k}_\nu$ for some ν; we then write $|x| = \nu$.

The homomorphisms $\mathrm{ind}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}$ can be regarded as $\mathcal{O}'$-linear maps ${}_{\mathcal{O}'}\mathbf{k}_\tau \otimes_{\mathcal{O}'} ({}_{\mathcal{O}'}\mathbf{k}_\omega) \to {}_{\mathcal{O}'}\mathbf{k}_\nu$, defined whenever $\tau, \omega, \nu \in \mathbf{N}[\mathbf{I}]^a$ satisfy $\tau + \omega = \nu$. They define a multiplication operation, hence they define a structure of $\mathcal{O}'$-algebra on ${}_{\mathcal{O}'}\mathbf{k}$. For any $\boldsymbol{\nu} = (\nu^1, \dots, \nu^m) \in \mathcal{X}^a$, we may regard $L_{\boldsymbol{\nu}}$ as an element in ${}_{\mathcal{O}'}\mathbf{k}_\nu$ where $\nu_{\mathbf{i}} = \sum_l \nu_{\mathbf{i}}^l$ for all $\mathbf{i}$.

Lemma 12.3.3 can be now restated as follows:

(a) $$L_{\boldsymbol{\nu}'} L_{\boldsymbol{\nu}''} = L_{\boldsymbol{\nu}'\boldsymbol{\nu}''}.$$

Since the elements $L_{\boldsymbol{\nu}}$ generate ${}_{\mathcal{O}'}\mathbf{k}$ as a $\mathcal{O}'$-module (see 12.6.3), it follows that the algebra structure on ${}_{\mathcal{O}'}\mathbf{k}$ is associative. One can also see this more directly.

13.1.3. The homomorphisms $\mathrm{res}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}$ can be regarded as $\mathcal{O}'$-linear maps ${}_{\mathcal{O}'}\mathbf{k}_\nu \to {}_{\mathcal{O}'}\mathbf{k}_\tau \otimes_{\mathcal{O}'} ({}_{\mathcal{O}'}\mathbf{k}_\omega)$, defined whenever $\tau, \omega, \nu \in \mathbf{N}[\mathbf{I}]^a$ satisfy $\tau + \omega = \nu$. By taking direct sums, we obtain an $\mathcal{O}'$-linear map $\bar{r} : {}_{\mathcal{O}'}\mathbf{k} \to {}_{\mathcal{O}'}\mathbf{k} \otimes_{\mathcal{O}'} ({}_{\mathcal{O}'}\mathbf{k})$.

13.1.4. We have a symmetric bilinear pairing $\mathbf{Z}[\mathbf{I}] \times \mathbf{Z}[\mathbf{I}] \to \mathbf{Z}$ given by

$$\nu \cdot \nu' = 2\sum_{\mathbf{i}} \nu_{\mathbf{i}}\nu'_{\mathbf{i}} - \sum_{h}(\nu_{h'}\nu'_{h''} + \nu_{h''}\nu'_{h'}).$$

This bilinear form is independent of orientation. Let ${}_{\mathcal{O}'}\mathbf{k}\bar{\otimes}_{\mathcal{O}'}({}_{\mathcal{O}'}\mathbf{k})$ be the $\mathcal{O}'$-module ${}_{\mathcal{O}'}\mathbf{k} \otimes_{\mathcal{O}'} ({}_{\mathcal{O}'}\mathbf{k})$ with the $\mathcal{O}'$-algebra structure given by

$$(x \otimes y)(x' \otimes y') = v^{-|x'|\cdot|y|}xx' \otimes yy'$$

for x, x', y, y' homogeneous.

Lemma 13.1.5. *$\bar{r} : {}_{\mathcal{O}'}\mathbf{k} \to {}_{\mathcal{O}'}\mathbf{k}\bar{\otimes}_{\mathcal{O}'}({}_{\mathcal{O}'}\mathbf{k})$ is a homomorphism of $\mathcal{O}'$-algebras.*

We must check that $\bar{r}(xy) = \bar{r}(x)\bar{r}(y)$ for any $x, y \in {}_{\mathcal{O}'}\mathbf{k}$. Since the elements $L_{\boldsymbol{\nu}}$ generate the $\mathcal{O}'$-module ${}_{\mathcal{O}'}\mathbf{k}$ (see 12.6.3), we may assume that $x = L_{\boldsymbol{\nu}'}, y = L_{\boldsymbol{\nu}''}$, where $\boldsymbol{\nu}' = (\nu'^1, \dots, \nu'^m)$ and $\boldsymbol{\nu}'' = (\nu''^1, \dots, \nu''^n)$ are elements of $\mathcal{X}^a$. We have

(a) $$\bar{r}(L_{\boldsymbol{\nu}'}) = \sum v^{M'(\boldsymbol{\tau}',\boldsymbol{\omega}')} L_{\boldsymbol{\tau}'} \otimes L_{\boldsymbol{\omega}'}$$

where the sum is taken over all $\boldsymbol{\tau}' = (\tau'^1, \dots, \tau'^m)$ and $\boldsymbol{\omega}' = (\omega'^1, \dots, \omega'^m)$ in $\mathcal{X}^a$ such that $\tau'^l + \omega'^l = \nu'^l$ for $1 \le l \le m$; $M'(\boldsymbol{\tau}', \boldsymbol{\omega}')$ is as in 9.2.11.

Similarly, we have

$$\bar{r}(L_{\boldsymbol{\nu}''}) = \sum v^{M'(\boldsymbol{\tau}'',\boldsymbol{\omega}'')} L_{\boldsymbol{\tau}''} \otimes L_{\boldsymbol{\omega}''}$$

where the sum is taken over all $\boldsymbol{\tau}'' = (\tau''^{(m+1)}, \dots, \tau''^{(m+n)})$ and $\boldsymbol{\omega}'' = (\omega''^{(m+1)}, \dots, \omega''^{(m+n)})$ in $\mathcal{X}^a$ such that $\tau''^l + \omega''^l = \nu''^{l-m}$ for $m+1 \le l \le m+n$. Hence

$$\bar{r}(L_{\boldsymbol{\nu}'})\bar{r}(L_{\boldsymbol{\nu}''}) = \sum v^{M'(\boldsymbol{\tau}',\boldsymbol{\omega}')+M'(\boldsymbol{\tau}'',\boldsymbol{\omega}'')+|L_{\boldsymbol{\omega}'}|\cdot|L_{\boldsymbol{\tau}''}|} L_{\boldsymbol{\tau}'\boldsymbol{\tau}''} \otimes L_{\boldsymbol{\omega}'\boldsymbol{\omega}''}$$

where the sum is taken over all $\boldsymbol{\tau}' = (\tau'^1, \dots, \tau'^m)$, $\boldsymbol{\omega}' = (\omega'^1, \dots, \omega'^m)$, $\boldsymbol{\tau}'' = (\tau''^{(m+1)}, \dots, \tau''^{(m+n)})$, $\boldsymbol{\omega}'' = (\omega''^{(m+1)}, \dots, \omega''^{(m+n)})$ in $\mathcal{X}^a$ such that $\tau'^l + \omega'^l = \nu'^l$ for $1 \le l \le m$ and $\tau''^l + \omega''^l = \nu''^{l-m}$ for $m+1 \le l \le m+n$.

We have

$$\bar{r}(L_{\boldsymbol{\nu}'}L_{\boldsymbol{\nu}''}) = \bar{r}(L_{\boldsymbol{\nu}'\boldsymbol{\nu}''}) = \sum v^{M'(\boldsymbol{\tau},\boldsymbol{\omega})} L_{\boldsymbol{\tau}} \otimes L_{\boldsymbol{\omega}}$$

where the sum is taken over all $\boldsymbol{\tau} = (\tau^1, \dots, \tau^{m+n})$ and $\boldsymbol{\omega} = (\omega^1, \dots, \omega^{m+n})$ in $\mathcal{X}^a$ such that $\tau^l + \omega^l = \nu'^l$ for $1 \le l \le m$ and $\tau^l + \omega^l = \nu''^{l-m}$ for $m+1 \le l \le m+n$.

It remains to show that

$$|L_{\boldsymbol{\omega}'}| \cdot |L_{\boldsymbol{\tau}''}| = M'(\boldsymbol{\tau}'\boldsymbol{\tau}'', \boldsymbol{\omega}'\boldsymbol{\omega}'') - M'(\boldsymbol{\tau}', \boldsymbol{\omega}') - M'(\boldsymbol{\tau}'', \boldsymbol{\omega}'').$$

This follows by a straightforward computation. The lemma is proved.

13.1.6. The pairing $\{,\}$ on $\mathcal{K}(\mathcal{Q}_{\mathbf{V}})$ (see 12.1.2) (where $\mathbf{V} \in \mathcal{V}_\nu^a$) can be regarded as an $\mathcal{O}'$-bilinear pairing $\{,\} : {}_{\mathcal{O}'}\mathbf{k}_\nu \times {}_{\mathcal{O}'}\mathbf{k}_\nu \to \mathcal{O}((v))$, which is independent of $\mathbf{V}$. This extends to an $\mathcal{O}'$-bilinear pairing $\{,\} : {}_{\mathcal{O}'}\mathbf{k} \times {}_{\mathcal{O}'}\mathbf{k} \to \mathcal{O}((v))$ such that for homogeneous x, y, $\{x, y\}$ is given by the previous pairing if $|x| = |y|$, and is zero if $|x| \neq |y|$.

13.1.7. We define a $\mathcal{O}'$-bilinear pairing $\{,\}$ on ${}_{\mathcal{O}'}\mathbf{k} \otimes_{\mathcal{O}'} ({}_{\mathcal{O}'}\mathbf{k})$ by

$$\{x' \otimes x'', y' \otimes y''\} = \{x', y'\}\{x'', y''\}.$$

The identity

$$\{x, y'y''\} = \{\bar{r}(x), y' \otimes y''\}$$

for all $x, y', y'' \in {}_{\mathcal{O}'}\mathbf{k}$, follows immediately from 12.2.2.

13.1.8. The homomorphism $D : \mathcal{K}(\mathcal{Q}_{\mathbf{V}}) \to \mathcal{K}(\mathcal{Q}_{\mathbf{V}})$ (where $\mathbf{V} \in \mathcal{V}_\nu^a$) can be regarded as a group homomorphism $D : {}_{\mathcal{O}'}\mathbf{k}_\nu \to {}_{\mathcal{O}'}\mathbf{k}_\nu$ that has square 1 and is semi-linear with respect to the ring involution $^- : \mathcal{O}' \to \mathcal{O}'$ given by $v^n \mapsto v^{-n}$ and $\zeta \mapsto \zeta^{-1}$ for $\zeta \in \mathcal{O}, \zeta^{\mathbf{n}} = 1$. By taking direct sums we obtain $D : {}_{\mathcal{O}'}\mathbf{k} \to {}_{\mathcal{O}'}\mathbf{k}$ which, by 12.4.3, is a ring homomorphism.

13.1.9. We shall regard ${}_{\mathcal{O}'}\mathbf{k} \otimes_{\mathcal{O}'} ({}_{\mathcal{O}'}\mathbf{k})$ as an $\mathcal{O}'$-algebra with

$$(x \otimes y)(x' \otimes y') = v^{|x'| \cdot |y|} x x' \otimes y y'$$

for x, x', y, y' homogeneous. This should be distinguished from the algebra ${}_{\mathcal{O}'}\mathbf{k} \bar{\otimes}_{\mathcal{O}'} ({}_{\mathcal{O}'}\mathbf{k})$. Let $D : {}_{\mathcal{O}'}\mathbf{k} \bar{\otimes}_{\mathcal{O}'} ({}_{\mathcal{O}'}\mathbf{k}) \to {}_{\mathcal{O}'}\mathbf{k} \otimes_{\mathcal{O}'} ({}_{\mathcal{O}'}\mathbf{k})$ be the ring isomorphism given by $D(x \otimes y) = D(x) \otimes D(y)$ for all x, y.

Let $r : {}_{\mathcal{O}'}\mathbf{k} \to {}_{\mathcal{O}'}\mathbf{k} \otimes_{\mathcal{O}'} ({}_{\mathcal{O}'}\mathbf{k})$ be the $\mathcal{O}'$-algebra homomorphism defined as the composition

$${}_{\mathcal{O}'}\mathbf{k} \xrightarrow{D} {}_{\mathcal{O}'}\mathbf{k} \xrightarrow{\bar{r}} {}_{\mathcal{O}'}\mathbf{k} \bar{\otimes}_{\mathcal{O}'} ({}_{\mathcal{O}'}\mathbf{k}) \xrightarrow{D} {}_{\mathcal{O}'}\mathbf{k} \otimes_{\mathcal{O}'} ({}_{\mathcal{O}'}\mathbf{k}).$$

13.1.10. Let $(,) : {}_{\mathcal{O}'}\mathbf{k} \times {}_{\mathcal{O}'}\mathbf{k} \to \mathcal{O}((v^{-1}))$ be the $\mathcal{O}'$-bilinear pairing given by $(x, y) = \overline{\{D(x), D(y)\}}$. Here, $^- : \mathcal{O}((v)) \to \mathcal{O}((v^{-1}))$ is given by $\sum_n a_n v^n \mapsto \sum_n \bar{a}_n v^{-n}$ $(a_n \in \mathcal{O})$.

From 13.1.7, we deduce the identity

(a) $$(x, y'y'') = (r(x), y' \otimes y'')$$

for all $x, y', y'' \in {}_{\mathcal{O}'}\mathbf{k}$.

13.1.11. From the definition we have

(a) $D(b) = b$ for all $b \in \mathcal{B}$.

We have

(b) $\{b, b'\} \in v\mathbf{Z}[[v]] \cap \mathbf{Q}(v)$ for any $b, b' \in \mathcal{B}$ such that $b' \neq \pm b$.

Indeed, $\{b, b'\}$ is in $v\mathcal{O}[[v]]$ by 12.5.3 and in $\mathbf{Z}((v))$ by 12.6.2 and 12.6.3, and hence in $v\mathbf{Z}[[v]]$.

We have

(c) $\{b, b\} \in 1 + v\mathbf{Z}[[v]] \cap \mathbf{Q}(v)$ for all $b \in \mathcal{B}$.

Indeed, $\{b, b\}$ is in $1 + v\mathcal{O}[[v]]$ by 12.5.3 and in $\mathbf{Z}((v))$ by 12.6.2 and 12.6.3, and hence in $1 + v\mathbf{Z}[[v]]$.

From (a),(b),(c) we deduce:

(b′) $(b, b') \in v^{-1}\mathbf{Z}[[v^{-1}]] \cap \mathbf{Q}(v)$ for any $b, b' \in \mathcal{B}$ such that $b' \neq \pm b$.

(c′) $(b, b) \in 1 + v^{-1}\mathbf{Z}[[v^{-1}]] \cap \mathbf{Q}(v)$ for all $b \in \mathcal{B}$.

13.1.12. Let i be an a-orbit on $\mathbf{I}$ and let $\gamma = \sum_{\mathbf{i} \in i} \mathbf{i}$. For any $n \geq 0$, ${}_{\mathcal{O}'}\mathbf{k}_{n\gamma}$ has a distinguished element denoted $\mathbf{1}_{ni}$; it corresponds to $[\mathbf{1}, 1] \in \mathcal{K}(\mathcal{Q}_{\mathbf{V}})$ where $\mathbf{V} \in \mathcal{V}^a_{n\gamma}$. This element forms a basis of the $\mathcal{O}'$-module ${}_{\mathcal{O}'}\mathbf{k}_{n\gamma}$. When $n = 0$, this is independent of i and is denoted simply by $\mathbf{1} \in {}_{\mathcal{O}'}\mathbf{k}_0$. Note that

(a) $\mathbf{1}$ is the unit element of the algebra ${}_{\mathcal{O}'}\mathbf{k}$.

(b) The elements $L_{\nu} \in {}_{\mathcal{O}'}\mathbf{k}$ are precisely the elements of ${}_{\mathcal{O}'}\mathbf{k}$ which are products of elements of form $\mathbf{1}_{ni}$ for various i, n. Hence the elements $\mathbf{1}_{ni}$ generate ${}_{\mathcal{O}'}\mathbf{k}$ as an $\mathcal{O}'$-algebra.

(c) We have $\mathbf{1}_i\mathbf{1}_{(n-1)i} = v^{d(n-1)}(\sum_{s=0}^{n-1} v^{-ds})\mathbf{1}_{ni}$ (for $n \geq 1$), where d is the number of elements in the orbit i. (See 12.3.4.)

From 12.3.6, we have

(d) $\{\mathbf{1}_i, \mathbf{1}_i\} = (1 - v^{2d})^{-1}$ (where d is as above)

or equivalently, since $D(\mathbf{1}_i) = \mathbf{1}_i$:

(d′) $(\mathbf{1}_i, \mathbf{1}_i) = (1 - v^{-2d})^{-1}$.

From (b) and (c) we see that

(e) $\bar{r}(\mathbf{1}_i) = r(\mathbf{1}_i) = \mathbf{1}_i \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1}_i$.

This is obvious from the definitions. It is clear that

(f) $(\mathbf{1}, \mathbf{1}) = \{\mathbf{1}, \mathbf{1}\} = 1$.

13.1.13. From 10.3.4, it follows easily that there is a unique $\mathcal{O}'$-linear map $\sigma : {}_{\mathcal{O}'}\mathbf{k} \to {}_{\mathcal{O}'}\mathbf{k}$ such that $\sigma(L_{\boldsymbol{\nu}}) = L_{\boldsymbol{\nu}'}$ for any $\boldsymbol{\nu} = (\nu^1, \nu^2, \dots, \nu^m) \in \mathcal{X}^a$, where $\boldsymbol{\nu}' = (\nu^m, \nu^{m-1}, \dots, \nu^1) \in \mathcal{X}^a$; moreover, we have $\sigma(\mathcal{B}) = \mathcal{B}$. It follows that σ is the unique isomorphism of ${}_{\mathcal{O}'}\mathbf{k}$ onto the algebra opposed to ${}_{\mathcal{O}'}\mathbf{k}$ such that $\sigma(\mathbf{1}_{ni}) = \mathbf{1}_{ni}$ for all $i \in I$ and $n \geq 0$.

13.2. The Algebra $\mathbf{k}$

13.2.1. Let ${}_{\mathcal{A}}\mathbf{k}$ be the $\mathcal{A}$-submodule of ${}_{\mathcal{O}'}\mathbf{k}$ spanned by $\mathcal{B}$, or equivalently (see 12.6.3), by the elements $L_{\boldsymbol{\nu}}$ for various $\boldsymbol{\nu} \in \mathcal{X}^a$. Thus, on the one hand, ${}_{\mathcal{A}}\mathbf{k}$ is the $\mathcal{A}$-subalgebra of ${}_{\mathcal{O}'}\mathbf{k}$ generated by the elements $\mathbf{1}_{ni}$ as in 13.1.12(b), and on the other hand, $\mathcal{B}$ is a signed basis for the $\mathcal{A}$-module ${}_{\mathcal{A}}\mathbf{k}$. We have ${}_{\mathcal{A}}\mathbf{k} = \oplus_\nu ({}_{\mathcal{A}}\mathbf{k}_\nu)$ where ${}_{\mathcal{A}}\mathbf{k}_\nu$ is the $\mathcal{A}$-submodule generated by $\mathcal{B}_\nu$.

13.2.2. From 13.1.5(a), we see that $\bar{r}$ restricts to an $\mathcal{A}$-linear map ${}_{\mathcal{A}}\mathbf{k} \to {}_{\mathcal{A}}\mathbf{k} \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{k})$, denoted again by $\bar{r}$; this is an $\mathcal{A}$-algebra homomorphism if ${}_{\mathcal{A}}\mathbf{k} \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{k})$ (which is naturally imbedded in ${}_{\mathcal{O}'}\mathbf{k} \bar{\otimes}_{\mathcal{O}'} ({}_{\mathcal{O}'}\mathbf{k})$) is given the induced $\mathcal{A}$-algebra structure (see 13.1.5).

13.2.3. By 13.1.11(a), the ring homomorphism $D : {}_{\mathcal{O}'}\mathbf{k} \to {}_{\mathcal{O}'}\mathbf{k}$ restricts to a ring homomorphism $D : {}_{\mathcal{A}}\mathbf{k} \to {}_{\mathcal{A}}\mathbf{k}$ which has square 1 and is semi-linear with respect to the ring involution $^- : \mathcal{A} \to \mathcal{A}$.

13.2.4. From 13.2.3 and 13.2.2, it follows that the $\mathcal{O}'$-algebra homomorphism $r : {}_{\mathcal{O}'}\mathbf{k} \to {}_{\mathcal{O}'}\mathbf{k} \otimes_{\mathcal{O}'} ({}_{\mathcal{O}'}\mathbf{k})$ (see 13.1.9) restricts to an $\mathcal{A}$-algebra homomorphism ${}_{\mathcal{A}}\mathbf{k} \to {}_{\mathcal{A}}\mathbf{k} \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{k})$, denoted again by r. This is an $\mathcal{A}$-algebra homomorphism if ${}_{\mathcal{A}}\mathbf{k} \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{k})$ (which is naturally imbedded in ${}_{\mathcal{O}'}\mathbf{k} \otimes_{\mathcal{O}'} ({}_{\mathcal{O}'}\mathbf{k})$) is given the induced $\mathcal{A}$-algebra structure (see 13.1.9).

13.2.5. The pairing $(,) : {}_{\mathcal{O}'}\mathbf{k} \times {}_{\mathcal{O}'}\mathbf{k} \to \mathcal{O}((v^{-1}))$ (see 13.1.10) restricts to an $\mathcal{A}$-bilinear pairing $(,) : {}_{\mathcal{A}}\mathbf{k} \times {}_{\mathcal{A}}\mathbf{k} \to \mathbf{Z}((v^{-1})) \cap \mathbf{Q}(v)$ (see 13.1.11(b′),(c′)). The equation analogous to 13.1.10(a) continues of course to hold over $\mathcal{A}$.

13.2.6. Let $\mathbf{k}$ be the $\mathbf{Q}(v)$-algebra $\mathbf{Q}(v) \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{k})$. Note that $\mathcal{B}$ is a signed basis of the $\mathbf{Q}(v)$-vector space $\mathbf{k}$. We have a direct sum decomposition $\mathbf{k} = \oplus_\nu \mathbf{k}_\nu$ where $\mathbf{k}_\nu$ is the subspace spanned by $\mathcal{B}_\nu$.

From 13.2.1 and 13.1.12(c), we see by induction on n that $\mathbf{k}$ is generated as a $\mathbf{Q}(v)$-algebra by the elements $\mathbf{1}_i$ for the various a-orbits on $\mathbf{I}$.

13.2.7. The homomorphism r in 13.2.4 extends to a $\mathbf{Q}(v)$-algebra homomorphism $\mathbf{k} \to \mathbf{k} \otimes_{\mathbf{Q}(v)} \mathbf{k}$ (denoted again by r) where $\mathbf{k} \otimes_{\mathbf{Q}(v)} \mathbf{k}$ is regarded as a $\mathbf{Q}(v)$-algebra by the same rule as in 13.1.9.

13.2.8. The pairing $(,)$ on ${}_{\mathcal{A}}\mathbf{k}$ extends to a $\mathbf{Q}(v)$-bilinear pairing $(,) : \mathbf{k} \times \mathbf{k} \to \mathbf{Q}(v)$. From 13.1.11(b′),(c′), we see that the restriction of this pairing to $\mathbf{k}_\nu$ is non-degenerate, for any ν. (Its determinant with respect to a basis contained in $\mathcal{B}_\nu$ belongs to $1 + v^{-1}\mathbf{Z}[[v^{-1}]] \cap \mathbf{Q}(v)$ and hence is non-zero.)

13.2.9. Let I be the set of a-orbits on $\mathbf{I}$. We identify $\mathbf{Z}[I]$ with the subgroup

$$\mathbf{Z}[\mathbf{I}]^a = \{\nu \in \mathbf{Z}[\mathbf{I}] | \nu_{\mathbf{i}} = \nu_{a(\mathbf{i})} \quad \forall \mathbf{i} \in \mathbf{I}\}$$

of $\mathbf{Z}[\mathbf{I}]$ by associating to each $\nu \in \mathbf{Z}[I]$ the element of $\mathbf{Z}[\mathbf{I}]$ (denoted again ν), in which the coefficient of $\mathbf{i}$ is ν_i where i is the a-orbit of $\mathbf{i}$.

For $\nu, \nu' \in \mathbf{Z}[I]$ we define $\nu \cdot \nu' \in \mathbf{Z}$ by regarding ν, ν' as elements of $\mathbf{Z}[\mathbf{I}]$ as above and then computing $\nu \cdot \nu'$ according to 13.1.4. (This is a symmetric bilinear form). According to this rule, we have, for $i, j \in I$: $i \cdot j =$ minus the number of $h \in H$ such that $[h]$ consists of a point in i and a point in j, if $i \neq j$ and $i \cdot i =$ twice the number of elements in the orbit i. Note that, if $i \neq j$ then $-2\frac{i \cdot j}{i \cdot i} \in \mathbf{N}$; indeed, this is the number of $h \in H$ such that $[h]$ consists of a given point in i and some point in j. Hence we have obtained a Cartan datum $(I, \cdot)$.

13.2.10. Let $\mathbf{f}$ be the $\mathbf{Q}(v)$-algebra, defined as in 1.2.5, in terms of the Cartan datum $(I, \cdot)$ just described. Recall that $\mathbf{f} = {}'\mathbf{f}/\mathcal{I}$ where ${}'\mathbf{f}$ is the free associative $\mathbf{Q}(v)$-algebra on the generators $\theta_i (i \in I)$ and $\mathcal{I}$ is a two-sided ideal defined as the radical of a certain symmetric bilinear form $(,)$ on ${}'\mathbf{f}$. Let $\chi : {}'\mathbf{f} \to \mathbf{k}$ be the unique homomorphism of $\mathbf{Q}(v)$-algebras with 1 such that $\chi(\theta_i) = \mathbf{1}_i$ for each $i \in I$.

Theorem 13.2.11. *χ induces an algebra isomorphism* $\mathbf{f} = {}'\mathbf{f}/\mathcal{I} \cong \mathbf{k}$.

The homomorphism χ is surjective, since $\mathbf{k}$ is generated by the $\mathbf{1}_i$ as a $\mathbf{Q}(v)$-algebra (see 13.2.6.)

The homomorphism $r : {}'\mathbf{f} \to {}'\mathbf{f} \otimes {}'\mathbf{f}$ (see 1.2.2) and the homomorphism $r : \mathbf{k} \to \mathbf{k} \otimes \mathbf{k}$ (see 13.2.7) make the following diagram commutative:

$$\begin{array}{ccc} {}'\mathbf{f} & \xrightarrow{r} & {}'\mathbf{f} \otimes {}'\mathbf{f} \\ \chi \downarrow & & \chi\otimes\chi \downarrow \\ \mathbf{k} & \xrightarrow{r} & \mathbf{k} \otimes \mathbf{k} \end{array}$$

Indeed, first we note that $\chi \otimes \chi$ is an algebra homomorphism, since $\nu \cdot \nu'$ on $\mathbf{Z}[I]$ has been defined in terms of the pairing on $\mathbf{Z}[\mathbf{I}]$. Hence the two possible compositions in the diagram are algebra homomorphisms; to check that they are equal, it suffices to do this on the generators θ_i. But they both take θ_i to $\mathbf{1}_i \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1}_i$ (see 13.1.12(e)).

For $x, y \in {}'\mathbf{f}$, we set $((x, y)) = (\chi(x), \chi(y))$ (the right hand side is as in 13.2.8). We have

(a) $((\theta_i, \theta_i)) = (\mathbf{1}_i, \mathbf{1}_i) = (1 - v^{-i \cdot i/2})^{-1}$ (see 13.1.12(d')); $((\theta_i, \theta_j)) = (\mathbf{1}_i, \mathbf{1}_j) = 0$, if $i \neq j$ (trivially);

$$((x, y'y'')) = (\chi(x), \chi(y')\chi(y'')) = (r(\chi(x)), \chi(y') \otimes \chi(y''))$$
(b) $$= (\chi \otimes \chi)(r(x), \chi(y') \otimes \chi(y'')) = ((r(x), y' \otimes y''));$$

we have used 13.1.10(a) and the commutativity of the diagram above. By (b) and the symmetry of $((,))$, we obtain

(c) $$((xx', y)) = ((x \otimes x', r(y))).$$

We have $((1, 1)) = 1$ (see 13.1.12(f)). Thus, $((x, y))$ satisfies the defining properties of $(,)$ in 1.2.3; hence it coincides with $(,)$. Since $\mathcal{I}$ is defined as the radical of $(,)$ on $'\mathbf{f}$, we also get

(d) $$\mathcal{I} = \{x \in {}'\mathbf{f} | (\chi(x), \chi(y)) = 0 \quad \forall y \in {}'\mathbf{f}\}.$$

Hence, if $x \in {}'\mathbf{f}$ satisfies $\chi(x) = 0$, then $x \in \mathcal{I}$, so that $\operatorname{Ker} \chi \subset \mathcal{I}$. Conversely, assume that $x \in \mathcal{I}$. Let $z \in \mathbf{k}$. We have $z = \chi(y)$ for some $y \in {}'\mathbf{f}$ (recall that χ is surjective). We have $(\chi(x), z) = (\chi(x), \chi(y)) = 0$ by (d). Thus $\chi(x)$ is in the radical of the form $(,)$ on $\mathbf{k}$. But this radical is zero (see 13.2.8). Hence $\chi(x) = 0$. Thus we have proved that $\operatorname{Ker} \chi = \mathcal{I}$. The theorem follows.