CHAPTER 11

Periodic Functors

11.1.1. Let C be a category in which the space of morphisms between any two objects has a given $\bar{\mathbf{Q}}_l$-vector space structure such that composition of morphisms is bilinear and such that finite direct sums exist. We say that C is a *linear* category.

A functor from one linear category to another linear category is said to be linear if it respects the $\bar{\mathbf{Q}}_l$-vector space structures.

11.1.2. Assume that we are given an integer $\mathbf{n} \geq 1$ and a linear functor $a^* : C \to C$ such that $a^{*\mathbf{n}}$ is the identity functor from C to C. We say that a is a *periodic functor*.

We define a new category $\tilde{C}$ as follows. The objects of $\tilde{C}$ are pairs (A, ϕ) where A is an object of C and $\phi : a^*A \to A$ is an isomorphism in C such that the composition

$$a^{*\mathbf{n}}A \xrightarrow{a^{*(\mathbf{n}-1)}\phi} a^{*(\mathbf{n}-1)}A \xrightarrow{a^{*(\mathbf{n}-2)}\phi} a^{*(\mathbf{n}-2)}A \to \cdots \to a^*A \xrightarrow{\phi} A$$

is the identity map of A.

Let (A, ϕ) and (A', ϕ') be two objects of $\tilde{C}$. There is a natural automorphism $u : \mathrm{Hom}(A, A') \to \mathrm{Hom}(A, A')$ given by $u(f) = \phi' a^*(f)\phi^{-1}$. From the definitions it follows that $u^{\mathbf{n}} = 1$. By definition,

$$\mathrm{Hom}_{\tilde{C}}((A, \phi), (A', \phi')) = \{f \in \mathrm{Hom}_C(A, A') | u(f) = f\}.$$

The composition of morphisms in $\tilde{C}$ is induced by that in C. The direct sum of two objects (A, ϕ) and (A', ϕ') of $\tilde{C}$ is $(A \oplus A', \phi \oplus \phi')$. Thus, $\tilde{C}$ is in a natural way a linear category. Clearly, if (A, ϕ) is an object of $\tilde{C}$, then so is $(A, \zeta\phi)$ for any $\zeta \in \bar{\mathbf{Q}}_l$ such that $\zeta^{\mathbf{n}} = 1$.

11.1.3. Assume that we are given three objects $(A, \phi), (A', \phi'), (A'', \phi'')$ of $\tilde{C}$ and morphisms $i' : (A', \phi') \to (A, \phi)$, $p'' : (A, \phi) \to (A'', \phi'')$ in $\tilde{C}$ such that the following holds.

(a) There exist morphisms $i'' : A'' \to A$ and $p' : A \to A'$ in C such that

$$p'i' = 1_{A'},\ p'i'' = 0,\ p''i' = 0,\ p''i'' = 1_{A''},\ i'p' + i''p'' = 1_A.$$

We show that $(A', \phi') \oplus (A'', \phi'') \cong (A, \phi)$ in $\tilde{C}$. Recall that $u^{\mathbf{n}}(i'') = i''$ where $u : \mathrm{Hom}_C(A'', A) \to \mathrm{Hom}_C(A'', A)$ is as in 11.1.2. Let $\tilde{i}'' = \sum_{j=1}^{\mathbf{n}} u^j(i'')/\mathbf{n} : A'' \to A$. Then $\tilde{i}'' \in \mathrm{Hom}_{\tilde{C}}(A'', A')$ and $p''\tilde{i}'' = 1_{A''}$.
We set the $\tilde{p}' = p' - p'\tilde{i}''p'' : A \to A'$. Then

$$\tilde{p}'i' = 1_{A'},\ \tilde{p}'\tilde{i}'' = 0,\ p''i' = 0,\ i'\tilde{p}' + \tilde{i}''p'' = 1_A.$$

It follows that $(i', \tilde{i}'')$ define an isomorphism $(A', \phi') \oplus (A'', \phi'') \to (A, \phi)$ (in $\tilde{C}$). Our assertion is proved.

11.1.4. An object (A, ϕ) of $\tilde{C}$ is said to be *traceless* if there exists an object B of C, an integer $t \geq 2$ dividing $\mathbf{n}$, such that $a^{*t}B \cong B$, and an isomorphism $A \cong B \oplus a^*B \oplus \cdots \oplus a^{*(t-1)}B$ under which ϕ corresponds to an isomorphism $a^*B \oplus a^{*2}B \oplus \cdots \oplus a^{*t}B \cong B \oplus a^*B \oplus \cdots \oplus a^{*(t-1)}B$ carrying the summand $a^{*j}B$ onto the summand $a^{*j}B$ (for $1 \leq j \leq t-1$) and the summand $a^{*t}B$ onto the summand B.

11.1.5. Let $\mathcal{O}$ be the subring of $\bar{\mathbf{Q}}_l$ consisting of all $\mathbf{Z}$-linear combinations of $\mathbf{n}$-th roots of 1. We associate to C and a^* an $\mathcal{O}$-module $\mathcal{K}(C)$. By definition, $\mathcal{K}(C)$ is the $\mathcal{O}$-module generated by symbols $[B, \phi]$, one for each isomorphism class of objects (B, ϕ) of $\tilde{C}$, subject to the following relations:

(a) $[B, \phi] + [B', \phi'] = [B \oplus B', \phi \oplus \phi']$;

(b) $[B, \phi] = 0$ if (B, ϕ) is traceless;

(c) $[B, \zeta\phi] = \zeta[B, \phi]$ if $\zeta \in \bar{\mathbf{Q}}_l$ satisfies $\zeta^{\mathbf{n}} = 1$.

This definition is similar to that of a Grothendieck group.

11.1.6. Now let C' be another linear category with a given functor $a^* : C' \to C'$ such that $a^{*\mathbf{n}}$ is the identity functor from C' to C'. Let $b : C \to C'$ be a linear functor. Assume that we are given an isomorphism of functors $ba^* = a^*b : C \to C'$. Then b induces a linear functor $\tilde{b} : \tilde{C} \to \tilde{C}'$ by $b(A, \phi) = (bA, \phi')$ where $\phi' : a^*bA \to bA$ is the composition $a^*bA = ba^*A \xrightarrow{b(\phi)} bA$. It is clear that $[A, \phi] \mapsto [bA, \phi']$ respects the relations of $\mathcal{K}(C), \mathcal{K}(C')$ and hence defines an $\mathcal{O}$-linear map $\mathcal{K}(C) \to \mathcal{K}(C')$.

11.1.7. Assume now that C is, in addition, an abelian category in which any object is a direct sum of finitely many simple objects. Let B be a simple object of C. Let t_B be the smallest integer ≥ 1 such that $(a^*)^{t_B}B$ is isomorphic to B; let $f_B : (a^*)^{t_B}B \to B$ be an isomorphism. We have $\mathbf{n} = n't_B$ where n' is an integer ≥ 1.

The composition

(a) $$B = (a^*)^{n' t_B} B \longrightarrow \cdots \xrightarrow{a^{*2t_B} f_B} a^{*2t_B} B \xrightarrow{a^{*t_B} f_B} a^{*t_B} B \xrightarrow{f_B} B$$

is a non-zero scalar times identity (since B is simple); hence by changing f_B by a non-zero multiple of f_B, we can assume that the composition (a) is the identity.

Consider the isomorphism

$$\phi_B : a^*(B \oplus a^*B \oplus \cdots (a^*)^{t_B-1}B) = a^*B \oplus a^{*2}B \oplus \cdots (a^*)^{t_B}B$$
$$\rightarrow B \oplus a^*B \oplus \cdots (a^*)^{t_B-1}B$$

which maps the summand $a^{*j}B$ onto the summand $a^{*j}B$ by the identity map (for $1 \leq j \leq t_B - 1$) and maps the summand $(a^*)^{t_B}B$ onto the summand B by f_B. From the definitions we see that

$$(B \oplus a^*B \oplus \cdots (a^*)^{t_B-1}B, \phi_B)$$

is an object of $\tilde{C}$.

Let $\mathcal{S}$ be a set of simple objects of C with the following property: any simple object in C is isomorphic to $a^{*j}B$ for a unique B in $\mathcal{S}$ and some $j \geq 0$. For each B in $\mathcal{S}$ we choose ϕ_B as above. It is easy to see that any object of $\tilde{C}$ is isomorphic to

(b) $$\oplus_{B\in\mathcal{S}}((B \oplus a^*B \oplus \cdots (a^*)^{t_B-1}B) \otimes E_B, \phi_B \otimes \psi_B)$$

where for each B, E_B is a finite dimensional $\bar{\mathbf{Q}}_l$-vector space with a given automorphism $\psi_B : E_B \rightarrow E_B$ such that $\psi_B^{\mathbf{n}} = 1$ and $E_B = 0$ for all but finitely many B. Note that the summands corresponding to B such that $t_B \geq 2$ are traceless.

Let B be a simple object of C such that $a^*B \cong B$. The isomorphisms $\phi : a^*B \cong B$ such that $(B, \phi) \in \tilde{C}$, generate a free $\mathcal{O}$-submodule of rank 1 of $\mathrm{Hom}_C(a^*B, B)$; we denote this $\mathcal{O}$-submodule by $\mathcal{O}_B$. It is easy to see that $\mathcal{O}_B$ depends only on the isomorphism class of B.

11.1.8. From 11.1.7 it follows easily that

(a) $$\mathcal{K}(C) = \oplus_B \mathcal{O}_B$$

as $\mathcal{O}$-modules (the sum is taken over the isomorphism classes of simple objects B such that $a^*B \cong B$); to the element $\phi \in \mathcal{O}_B$ such that $(B, \phi) \in \tilde{C}$ corresponds the element $[B, \phi] \in \mathcal{K}(C)$.