CHAPTER 10

Fourier-Deligne Transform

10.1. Fourier-Deligne Transform and Restriction

10.1.1. In addition to the orientation $h \to h'$, $h \to h''$ in 9.1.1, we shall consider a new orientation of our graph. Thus, we assume we are given two new maps $H \to I$ denoted $h \mapsto {}'h$ and $h \mapsto {}''h$, such that for any $h \in H$, the subset $[h]$ of $\mathbf{I}$ consists precisely of ${}'h, {}''h$. Let

$$H_1 = \{h \in H | {}'h = h' \text{ and } {}''h = h''\}; \quad H_2 = \{h \in H | {}'h = h'' \text{ and } {}''h = h'\}.$$

Then H_1, H_2 form a partition of H.

For $\mathbf{V} \in \mathcal{V}$, we define ${}'\mathbf{E_V}$ like $\mathbf{E_V}$ in 9.1.2, but using the new orientation: ${}'\mathbf{E_V} = \oplus_{h\in H}\mathrm{Hom}(\mathbf{V}_{'h}, \mathbf{V}_{''h})$. This has a natural $G_\mathbf{V}$-action just like $\mathbf{E_V}$.

We have

$$\mathbf{E_V} = \oplus_{h\in H_1}\mathrm{Hom}(\mathbf{V}_{h'}, \mathbf{V}_{h''}) \oplus (\oplus_{h\in H_2}\mathrm{Hom}(\mathbf{V}_{h'}, \mathbf{V}_{h''})),$$

$${}'\mathbf{E_V} = \oplus_{h\in H_1}\mathrm{Hom}(\mathbf{V}_{h'}, \mathbf{V}_{h''}) \oplus (\oplus_{h\in H_2}\mathrm{Hom}(\mathbf{V}_{h''}, \mathbf{V}_{h'})).$$

Let $\dot{\mathbf{E}}_\mathbf{V}$ be the vector space

$$\oplus_{h\in H_1}\mathrm{Hom}(\mathbf{V}_{h'}, \mathbf{V}_{h''}) \oplus (\oplus_{h\in H_2}\mathrm{Hom}(\mathbf{V}_{h'}, \mathbf{V}_{h''})) \oplus (\oplus_{h\in H_2}\mathrm{Hom}(\mathbf{V}_{h''}, \mathbf{V}_{h'})).$$

We have the diagram

(a) $$\mathbf{E_V} \xleftarrow{s} \dot{\mathbf{E}}_\mathbf{V} \xrightarrow{t} {}'\mathbf{E_V}$$

where s, t are the obvious projections.

Let $T : \dot{\mathbf{E}}_\mathbf{V} \to k$ be the map given by $T(e) = \sum_{h\in H_2} \mathrm{tr}\ (\mathbf{V}_{h'} \to \mathbf{V}_{h''} \to \mathbf{V}_{h'})$ where the two unnamed maps are components of e. Let us consider the Fourier-Deligne transform $\Phi : \mathcal{D}(\mathbf{E_V}) \to \mathcal{D}({}'\mathbf{E_V})$ defined by $\Phi(K) = t_!(s^*(K) \otimes \mathcal{L}_T)[d_\mathbf{V}]$ where $d_\mathbf{V} = \sum_{h\in H_2} \dim \mathbf{V}_{h'} \dim \mathbf{V}_{h''}$. (See 8.1.11.) Now let $\mathbf{T}, \mathbf{W}$ be as in 9.2.1. We may consider a diagram like (a) for $\mathbf{T}$ and for $\mathbf{W}$ instead of $\mathbf{V}$; taking direct products, we obtain the diagram

$$\mathbf{E_T} \times \mathbf{E_W} \xleftarrow{\bar{s}} \dot{\mathbf{E}}_\mathbf{T} \times \dot{\mathbf{E}}_\mathbf{W} \xrightarrow{\bar{t}} {}'\mathbf{E_T} \times {}'\mathbf{E_W}.$$

On each of $\dot{\mathbf{E}}_{\mathbf{T}}$ and $\dot{\mathbf{E}}_{\mathbf{W}}$ we have a linear form like T above; the sum of these gives a linear form $\bar{T} : \dot{\mathbf{E}}_{\mathbf{T}} \times \dot{\mathbf{E}}_{\mathbf{W}} \to k$. The Fourier-Deligne transform $\Phi : \mathcal{D}(\mathbf{E}_{\mathbf{T}} \times \mathbf{E}_{\mathbf{W}}) \to \mathcal{D}('\mathbf{E}_{\mathbf{T}} \times '\mathbf{E}_{\mathbf{W}})$ is given by

$$\Phi(K) = \bar{t}_!(\bar{s}^*(K) \otimes \mathcal{L}_{\bar{T}})[d_{\mathbf{T}} + d_{\mathbf{W}}].$$

The following result shows the relation between the Fourier-Deligne transform and the restriction functor.

Proposition 10.1.2. *For any $K \in \mathcal{Q}_{\mathbf{V}}$ we have*

$$\Phi(\tilde{Res}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}} K) = \tilde{Res}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}(\Phi(K))[\pi]$$

where

$$\pi = \sum_{h \in H_2} (\dim \mathbf{T}_{h''} \dim \mathbf{W}_{h'} - \dim \mathbf{T}_{h'} \dim \mathbf{W}_{h''}).$$

We consider the commutative diagram of vector spaces and linear maps

$$\begin{array}{ccccccccc}
\mathbf{E}_{\mathbf{T}} \times \mathbf{E}_{\mathbf{W}} & \xleftarrow{p} & F & & & \xrightarrow{\iota} & & & \mathbf{E}_{\mathbf{V}} \\
\uparrow{\scriptstyle \bar{s}} & & \uparrow{\scriptstyle \dot{s}} & & & & & & \uparrow{\scriptstyle s} \\
\dot{\mathbf{E}}_{\mathbf{T}} \times \dot{\mathbf{E}}_{\mathbf{W}} & \xleftarrow{\dot{p}} & \Psi & \xleftarrow{\dot{q}} & \dot{F} & \xrightarrow{\dot{\zeta}} & \Xi & \xrightarrow{\dot{\iota}} & \dot{\mathbf{E}}_{\mathbf{V}} \\
\downarrow{\scriptstyle \bar{t}} & & & & & & \downarrow{\scriptstyle \dot{t}} & & \downarrow{\scriptstyle t} \\
'\mathbf{E}_{\mathbf{T}} \times '\mathbf{E}_{\mathbf{W}} & & & \xleftarrow{'p} & & & 'F & \xrightarrow{'\iota} & '\mathbf{E}_{\mathbf{V}}
\end{array}$$

where the following notation is used.

F is the set of all $x \in \mathbf{E}_{\mathbf{V}}$ such that $x_h(\mathbf{W}_{h'}) \subset \mathbf{W}_{h''}$ for all $h \in H$; p is the obvious surjective map and ι is the obvious imbedding.

$'F$ is the set of all $x \in '\mathbf{E}_{\mathbf{V}}$ such that $x_h(\mathbf{W}_{'h}) \subset \mathbf{W}_{''h}$ for all $h \in H$; $'p$ is the obvious surjective map and $'\iota$ is the obvious imbedding.

$\dot{F}$ is the set of all $x \in \dot{\mathbf{E}}_{\mathbf{V}}$ such that $sx \in F$ and $tx \in 'F$.

Ξ is defined by the condition that $(\dot{\iota}, t, \dot{t}, '\iota)$ is a cartesian diagram.

Ψ is defined by the condition that $(\dot{s}, p, \dot{p}, \bar{s})$ is a cartesian diagram.

$\dot{q}$ is such that $\dot{s}\dot{q}$ and $\dot{p}\dot{q}$ are the obvious surjective maps.

$\dot{\zeta}$ is such that $\dot{\iota}\dot{\zeta}$ and $\dot{t}\dot{\zeta}$ are the obvious imbeddings.

We have $\Xi = 'F \oplus (\oplus_{h \in H_2} \mathrm{Hom}(\mathbf{V}_{h'}, \mathbf{V}_{h''}))$. Let Z be the subspace of Ξ consisting of the elements such that each component $\mathbf{V}_{h''} \to \mathbf{V}_{h'}$ $(h \in H_2)$ carries $\mathbf{W}_{h''}$ to 0 and all other components are zero. Let $c : \Xi \to \Xi/Z$ be the canonical map. Let $\tilde{T} : \Xi \to k$ be given by $\tilde{T}(x) = T(\iota(x))$. From definitions, it follows immediately that the restriction of $\tilde{T}$ to a fibre ($\cong Z$) of $c : \Xi \to \Xi/Z$ is an affine-linear function which is constant if and only if that fibre is contained in the subspace $\dot{\zeta}(\dot{F})$.

Let $\Xi' = \Xi - \dot{\zeta}(\dot{F})$, and let $(\Xi/Z)' = c(\Xi')$. We have $Z \subset \dot{\zeta}(\dot{F})$; hence all fibres of $c' : \Xi' \to (\Xi/Z)'$ (restriction of c) are isomorphic to Z.

Let $T' : \Xi' \to k$ be the restriction of $\tilde{T}$. As we have seen above, the restriction of T' to any fibre of $c' : \Xi' \to (\Xi/Z)'$ is a non-constant affine-linear function. Hence the local system $\mathcal{L}_{T'}$ on Ξ' satisfies $c'_!(\mathcal{L}_{T'}) = 0$ (see 8.1.13). Using the distinguished triangle associated to the partition $\Xi = \Xi' \cup \dot{\zeta}(\dot{F})$, we deduce that $c_! \dot{\zeta}_! (\dot{\zeta}^* \mathcal{L}_{\tilde{T}}) = c_! \mathcal{L}_{\tilde{T}}$. It is clear that the composition $si : \Xi \to \mathbf{E_V}$ factors through Ξ/Z; hence $i^* s^* K$ is in the image of c^* so that the previous equality implies

$$c_!(\dot{\zeta}_!(\dot{\zeta}^* \mathcal{L}_{\tilde{T}}) \otimes i^* s^* K) = c_!(\mathcal{L}_{\tilde{T}} \otimes i^* s^* K).$$

It is also clear that the composition $'p\dot{t} : \Xi \to {}'\mathbf{E_T} \times {}'\mathbf{E_W}$ factors through Ξ/Z. Hence the previous equality implies

$$'p_! \dot{t}_! (\dot{\zeta}_!(\dot{\zeta}^* \mathcal{L}_{\tilde{T}}) \otimes i^* s^* K) = {}'p_! \dot{t}_! (\mathcal{L}_{\tilde{T}} \otimes i^* s^* K).$$

We have $Ti\dot{\zeta} = \tilde{T}\dot{p}\dot{q}$; hence $\dot{p}^* \dot{q}^* \mathcal{L}_{\tilde{T}} = \dot{\zeta}^* i^* \mathcal{L}_T = \dot{\zeta}^* \mathcal{L}_{\tilde{T}}$. Since $\dot{q}$ is a surjective linear map with kernel of dimension

$$m = \sum_{h \in H_2} \dim \mathbf{T}_{h''} \dim \mathbf{W}_{h'},$$

we obtain $\dot{q}_! \dot{q}^* L = L[-2m]$ for all $L \in \mathcal{D}(\Psi)$. We have

$$\begin{aligned}
\Phi(\mathrm{Res}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}} K) &= \bar{t}_!(\mathcal{L}_T \otimes \bar{s}^* p_! \iota^* K)[d_\mathbf{T} + d_\mathbf{W}] \\
&= \bar{t}_!(\mathcal{L}_T \otimes \dot{p}_! \dot{s}^* \iota^* K)[d_\mathbf{T} + d_\mathbf{W}] \\
&= \bar{t}_!(\mathcal{L}_{\tilde{T}} \otimes \dot{p}_! \dot{q}_! \dot{q}^* \dot{s}^* \iota^* K[2m])[d_\mathbf{T} + d_\mathbf{W}] \\
&= \bar{t}_! \dot{p}_! \dot{q}_! (\dot{p}^* \dot{q}^* (\mathcal{L}_{\tilde{T}}) \otimes \dot{q}^* \dot{s}^* \iota^* K)[2m + d_\mathbf{T} + d_\mathbf{W}] \\
&= {}'p_! \dot{t}_! \dot{\zeta}_! (\dot{p}^* \dot{q}^* (\mathcal{L}_{\tilde{T}}) \otimes \dot{\zeta}^* i^* s^* K)[2m + d_\mathbf{T} + d_\mathbf{W}] \\
&= {}'p_! \dot{t}_! (\dot{\zeta}_! (\dot{p}^* \dot{q}^* \mathcal{L}_{\tilde{T}}) \otimes i^* s^* K)[2m + d_\mathbf{T} + d_\mathbf{W}] \\
&= {}'p_! \dot{t}_! (\dot{\zeta}_! (\dot{\zeta}^* \mathcal{L}_{\tilde{T}}) \otimes i^* s^* K)[2m + d_\mathbf{T} + d_\mathbf{W}] \\
&= {}'p_! \dot{t}_! (\mathcal{L}_{\tilde{T}} \otimes i^* s^* K)[2m + d_\mathbf{T} + d_\mathbf{W}]
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Res}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}(\Phi(K))[\pi] &= {}'p_! {}'\iota^* t_!(\mathcal{L}_T \otimes s^* K)[\pi + d_\mathbf{V}] \\
&= {}'p_! \dot{t}_! i^* (\mathcal{L}_T \otimes s^* K)[\pi + d_\mathbf{V}] \\
&= {}'p_! \dot{t}_! (\mathcal{L}_{\tilde{T}} \otimes i^* s^* K)[\pi + d_\mathbf{V}].
\end{aligned}$$

It remains for us to observe that $\pi + d_\mathbf{V} = 2m + d_\mathbf{T} + d_\mathbf{W}$. The proposition is proved.

10.1.3. We can reformulate the previous proposition using $\operatorname{Res}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}$ instead of $\tilde{\operatorname{Res}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}$; the shift by π will then disappear:

$$\Phi(\operatorname{Res}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}} K) = \operatorname{Res}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}(\Phi(K)).$$

10.2. Fourier-Deligne Transform and Induction

10.2.1. Let $\boldsymbol{\nu} = (\nu^1, \nu^2, \dots, \nu^m) \in \mathcal{X}$ be such that $\dim \mathbf{V_i} = \sum_l \nu^l_{\mathbf{i}}$ for all $\mathbf{i}$. Recall that we have a natural proper morphism $\pi_{\boldsymbol{\nu}} : \tilde{\mathcal{F}}_{\boldsymbol{\nu}} \to \mathbf{E_V}$. The same definition with the new orientation for our graph gives a proper morphism $'\pi_{\boldsymbol{\nu}} : {}'\tilde{\mathcal{F}}_{\boldsymbol{\nu}} \to {}'\mathbf{E_V}$, where $'\tilde{\mathcal{F}}_{\boldsymbol{\nu}}$ is the variety of all pairs (x, f) such that $x \in {}'\mathbf{E_V}$ and $f \in \mathcal{F}_{\boldsymbol{\nu}}$ is x-stable; $'\pi_{\boldsymbol{\nu}}$ is the first projection.

Recall the definition $\tilde{L}_{\boldsymbol{\nu}} = (\pi_{\boldsymbol{\nu}})_! \mathbf{1} \in \mathcal{D}(\mathbf{E_V})$. Similarly, we set $'\tilde{L}_{\boldsymbol{\nu}} = ({}'\pi_{\boldsymbol{\nu}})_! \mathbf{1} \in \mathcal{D}({}'\mathbf{E_V})$.

Proposition 10.2.2. $\Phi(\tilde{L}_{\boldsymbol{\nu}}) = {}'\tilde{L}_{\boldsymbol{\nu}}[M]$ *where*

$$M = \sum_{h \in H_2; l > l'} (\nu^l_{h'} \nu^{l'}_{h''} - \nu^{l'}_{h'} \nu^{l'}_{h''}).$$

Consider the commutative diagram

$$\begin{array}{ccccc}
\tilde{\mathcal{F}}_{\boldsymbol{\nu}} & \longrightarrow & \Xi & \xrightarrow{c} & \bar{\Xi} \\
{\scriptstyle \pi_{\boldsymbol{\nu}}}\downarrow & & {\scriptstyle \rho}\downarrow & & \downarrow \\
\mathbf{E_V} & \xleftarrow{s} & \dot{\mathbf{E}}_\mathbf{V} & \xrightarrow{t} & {}'\mathbf{E_V}
\end{array}$$

where the following notation is used.

Ξ is the set of all (x, y, f) where $x \in \mathbf{E_V}$, f is an x-stable flag in $\mathcal{F}_{\boldsymbol{\nu}}$ and $y \in {}'\mathbf{E_V}$ is such that $y_h = x_h : \mathbf{V}_{h'} \to \mathbf{V}_{h''}$ for any $h \in H_1$.

$\bar{\Xi}$ is the set of all (y, f) where $y \in {}'\mathbf{E_V}$ and $f = (\mathbf{V} = \mathbf{V}^0 \supset \mathbf{V}^1 \supset \cdots \supset \mathbf{V}^m = 0)$ is a flag in $\mathcal{F}_{\boldsymbol{\nu}}$ such that $y_h(\mathbf{V}^l_{h'}) \subset \mathbf{V}^l_{h''}$ for all l and all $h \in H_1$.

The lower horizontal maps are as in 10.1.1(a); the other maps are the obvious ones. The left square is cartesian. We have $s^*(\pi_{\boldsymbol{\nu}})_! \mathbf{1} = \rho_! \mathbf{1}$. Hence

$$\Phi(\tilde{L}_{\boldsymbol{\nu}}) = t_!(\mathcal{L}_T \otimes \rho_! \mathbf{1})[d_\mathbf{V}] = \tilde{t}_!(\mathcal{L}_{\tilde{T}})[d_\mathbf{V}]$$

where $T : \dot{\mathbf{E}}_\mathbf{V} \to k$ is as in 10.1.1, $\tilde{T} : \Xi \to k$ is given by $\tilde{T} = T\rho$ and $\tilde{t} = t\rho : \Xi \to {}'\mathbf{E_V}$.

The fibres of c are affine spaces of dimension $N = \sum_{h\in H_2; l<l'} \nu^l_{h'}\nu^{l'}_{h''}$. (In the formula for N we have $\nu^l_{h'}\nu^{l'}_{h''} = 0$ for $l = l'$, since ν^l is discrete.)

We have a partition $\Xi = \Xi_0 \cup \Xi_1$ where Ξ_0 is the closed subset of Ξ consisting of those (x, y, f) such that f is y-stable. It can be verified that the restriction of $\tilde{T}$ to the fibre of c at $c(x, y, f)$ is an affine-linear function and that this function is constant if and only if $(x, y, f) \in \Xi_0$. Note that Ξ_0 is a union of fibres of c.

Using 8.1.13, it follows that $(c_1)_!(\mathcal{L}_{\tilde{T}}|_{\Xi_1}) = 0$, where $c' : \Xi_1 \to \bar{\Xi}$ is the restriction of c. Hence, if $j : \Xi_0 \to \Xi$ is the inclusion, we have $c_! j_!(j^*\mathcal{L}_{\tilde{T}}) = c_!\mathcal{L}_{\tilde{T}}$. From the commutative diagram above, it then follows that

$$(t\rho)_!(\mathcal{L}_{\tilde{T}} = (t_0)_!(\mathcal{L}_{\tilde{T}}|_{\Xi_0})$$

where $t_0 : \Xi_0 \to {}'\mathbf{E}_\mathbf{V}$ is the restriction of $t\rho$.

Let $(x, y, f) \in \Xi_0$ with f as above. We have

$$\tilde{T}(x, y, f) = T(x, y) = \sum_{h\in H_2} \mathrm{tr}\ (y_h x_h : \mathbf{V}_{h'} \to \mathbf{V}_{h'}).$$

Since f is stable under both x and y, we have

$$\mathrm{tr}\ (y_h x_h : \mathbf{V}_{h'} \to \mathbf{V}_{h'}) = \sum_l \mathrm{tr}\ (y_h x_h : \mathbf{V}^{l-1}_{h'}/\mathbf{V}^l_{h'} \to \mathbf{V}^{l-1}_{h''}/\mathbf{V}^l_{h''}).$$

For any l, at least one of the vector spaces $\mathbf{V}^{l-1}_{h'}/\mathbf{V}^l_{h'}, \mathbf{V}^{l-1}_{h''}/\mathbf{V}^l_{h''}$ is zero, since ν^l is discrete. Thus, $\mathrm{tr}\ (y_h x_h : \mathbf{V}_{h'} \to \mathbf{V}_{h'}) = 0$ for each $h \in H_2$, so that $\tilde{T}(x, y, f) = 0$. Since $\tilde{T}$ is identically zero on Ξ_0, we have $\mathcal{L}_{\tilde{T}}|_{\Xi_0} = \mathbf{1}$ and we see that

$$(t\rho)_!(\mathcal{L}_{\tilde{T}} = (t_0)_!\mathbf{1}.$$

Now t_0 can be factored as a composition $\Xi_0 \to {}'\tilde{\mathcal{F}}_\nu \xrightarrow{{}'\pi_\nu} {}'\mathbf{E}_\mathbf{V}$, where the first map (restriction of c) is a vector bundle of dimension N. Hence

$$(t_0)_!\mathbf{1} = ({}'\pi_\nu)_!\mathbf{1}[-2N] = {}'L_\nu[-2N].$$

It follows that $(t\rho)_!(\mathcal{L}_{\tilde{T}} = (t_0)_!\mathbf{1} = {}'L_\nu[-2N]$. It remains for us to observe that $d_\mathbf{V} - 2N = M$. The proposition is proved.

10.2.3. Using the proposition and the general properties of the Fourier-Deligne transform (see 8.1.11) we see that $\Phi : \mathcal{D}(\mathbf{E}_\mathbf{V}) \to \mathcal{D}({}'\mathbf{E}_\mathbf{V})$ defines an equivalence of categories $\mathcal{Q}_\mathbf{V} \to {}'\mathcal{Q}_\mathbf{V}$ and $\mathcal{P}_\mathbf{V} \to {}'\mathcal{P}_\mathbf{V}$, where ${}'\mathcal{Q}_\mathbf{V}, {}'\mathcal{P}_\mathbf{V}$ are defined as $\mathcal{Q}_\mathbf{V}, \mathcal{P}_\mathbf{V}$ but using the new orientation of our graph. Hence Φ induces a bijection between the set of simple objects in $\mathcal{P}_\mathbf{V}$ and that in ${}'\mathcal{P}_\mathbf{V}$.

10.2.4. We have a natural action of $(k^*)^H$ on $\mathbf{E_V}$ (resp. on $\tilde{\mathcal{F}}_\nu$) given by $(\zeta_h) : (x_h) \mapsto (\zeta_h x_h)$ (resp. $(\zeta_h) : ((x_h), f) \mapsto ((\zeta_h x_h), f)$. The map π_ν is compatible with these actions. It follows that $H^n L_\nu$ is $(k^*)^H$-equivariant for any n. Hence any $K \in \mathcal{P}_\mathbf{V}$ is $(k^*)^H$-equivariant. In particular, we have $j^*K = K$, where $j : \mathbf{E_V} \to \mathbf{E_V}$ is the involution which acts as -1 on the summands $\mathrm{Hom}(\mathbf{V}_{h'}, \mathbf{V}_{h''})$ for $h \in H_2$ and as 1 on the other summands. Hence for $K \in \mathcal{P}_\mathbf{V}$, the Fourier inversion formula (see 8.1.11) simplifies to $\Phi(\Phi(K)) = K$.

10.2.5. Let $A \in \mathcal{Q}_\mathbf{V}$ and let $A' \in {}'\mathcal{Q}_\mathbf{V}$. For any $j \in \mathbf{Z}$, we have a canonical isomorphism

$$\mathbf{D}_j(\mathbf{E_V}, G_\mathbf{V}; A, \Phi(A')) = \mathbf{D}_j({}'\mathbf{E_V}, G_\mathbf{V}; \Phi(A), A').$$

This follows by applying 8.1.12 to the diagram

$$G_\mathbf{V}\backslash(\Gamma \times \mathbf{E_V}) \leftarrow G_\mathbf{V}\backslash(\Gamma \times \dot{\mathbf{E}}_\mathbf{V}) \to G_\mathbf{V}\backslash(\Gamma \times {}'\mathbf{E_V})$$

obtained from 10.1.1(a), where Γ is a suitable smooth variety with a free $G_\mathbf{V}$-action.

Proposition 10.2.6. *With the notations of Proposition 10.1.2, let $L \in \mathcal{Q}_{\mathbf{T},\mathbf{W}}$. There exists an isomorphism in ${}'\mathcal{Q}_\mathbf{V}$:*

$$\Phi(Ind_{\mathbf{T},\mathbf{W}}^\mathbf{V} L) \cong Ind_{\mathbf{T},\mathbf{W}}^\mathbf{V}(\Phi L).$$

Since ${}'\mathcal{P}_\mathbf{V}$ is stable under Verdier duality, we see from 9.1.6 that it suffices to check that

(a)

$$\dim \mathbf{D}_j({}'\mathbf{E_V}, G_\mathbf{V}; \Phi(\mathrm{Ind}_{\mathbf{T},\mathbf{W}}^\mathbf{V} L), \Phi K) = \dim \mathbf{D}_j({}'\mathbf{E_V}, G_\mathbf{V}; \mathrm{Ind}_{\mathbf{T},\mathbf{W}}^\mathbf{V}(\Phi L), \Phi K)$$

for any $K \in \mathcal{P}_\mathbf{V}$ and any $j \in \mathbf{Z}$.

By 10.2.5, the left hand side of (a) is equal to

$$\dim \mathbf{D}_j(\mathbf{E_V}, G_\mathbf{V}; \mathrm{Ind}_{\mathbf{T},\mathbf{W}}^\mathbf{V} L, K)$$

and by 9.2.9, this is equal to

$$\dim \mathbf{D}_j(\mathbf{E_T} \times \mathbf{E_W}, G_\mathbf{T} \times G_\mathbf{W}; L, \mathrm{Res}_{\mathbf{T},\mathbf{W}}^\mathbf{V} K).$$

By 9.2.9, the right hand side of (a) is equal to

$$\dim \mathbf{D}_j({}'\mathbf{E_T} \times {}'\mathbf{E_W}, G_\mathbf{T} \times G_\mathbf{W}; \Phi L, \mathrm{Res}_{\mathbf{T},\mathbf{W}}^\mathbf{V}(\Phi K))$$

and by 10.2.5, this is equal to

$$\dim \mathbf{D}_j(\mathbf{E_T} \times \mathbf{E_W}, G_\mathbf{T} \times G_\mathbf{W}; L, \Phi(\mathrm{Res}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}(\Phi K))).$$

Hence (a) is equivalent to

$$\begin{aligned}&\dim \mathbf{D}_j(\mathbf{E_T} \times \mathbf{E_W}, G_\mathbf{T} \times G_\mathbf{W}; L, \mathrm{Res}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}} K)\\ &= \dim \mathbf{D}_j(\mathbf{E_T} \times \mathbf{E_W}, G_\mathbf{T} \times G_\mathbf{W}; L, \Phi(\mathrm{Res}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}(\Phi K))).\end{aligned}$$

But this follows from $\mathrm{Res}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}} K = \Phi(\mathrm{Res}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}(\Phi K))$ (see 10.1.3). The proposition is proved.

10.3. A Key Inductive Step

Lemma 10.3.1. *Let $\mathbf{I}', \gamma$ be as in 9.3.1. The Fourier-Deligne transform $\Phi : \mathcal{P}_\mathbf{V} \to {}'\mathcal{P}_\mathbf{V}$ defines an equivalence of categories between $\mathcal{P}_{\mathbf{V};\mathbf{I}';\gamma}$ and the analogous category ${}'\mathcal{P}_{\mathbf{V};\mathbf{I}';\gamma}$ defined as $\mathcal{P}_{\mathbf{V};\mathbf{I}';\gamma}$ with respect to the new orientation.*

This follows immediately from the definitions since the Fourier-Deligne transform commutes with Ind.

Proposition 10.3.2. *Let $\mathbf{I}', \gamma$ be as in 9.3.1. Let $\mathbf{W}$ be a graded subspace of $\mathbf{V}$ such that $\mathbf{T} = \mathbf{V}/\mathbf{W}$ satisfies $\dim \mathbf{T_i} = \gamma_\mathbf{i}$ for all $\mathbf{i} \in \mathbf{I}'$ and $\mathbf{T}_{\mathbf{i}'} = 0$ for all $\mathbf{i}' \in \mathbf{I} - \mathbf{I}'$.*

(a) *Let B be a simple object of $\mathcal{P}_{\mathbf{V};\mathbf{I}';\gamma}$. We have*

$$Res^{\mathbf{V}}_{\mathbf{T},\mathbf{W}} B \cong A \oplus (\oplus_j L_j[j])$$

where A is a simple object of $\mathcal{P}_{\mathbf{W};\mathbf{I}';0}$ and $L_j \in \mathcal{P}_{\mathbf{W};\mathbf{I}';>0}$ for all j.

(b) *Let A be a simple object of $\mathcal{P}_{\mathbf{W};\mathbf{I}';0}$. We have*

$$Ind^{\mathbf{V}}_{\mathbf{T},\mathbf{W}} A \cong B \oplus (\oplus_j C_j[j])$$

where B is a simple object of $\mathcal{P}_{\mathbf{V};\mathbf{I}';\gamma}$ and

$$C_j \in \mathcal{P}_{\mathbf{V};\mathbf{I}';>\gamma}$$

for all j.

(c) *The maps $B \mapsto A$ in (a) and $A \mapsto B$ in (b) are inverse bijections between the set of isomorphism classes of simple objects in $\mathcal{P}_{\mathbf{V};\mathbf{I}';\gamma}$ and the analogous set for $\mathcal{P}_{\mathbf{W};\mathbf{I}';0}$.*

This statement is independent of the orientation of our graph: we use the previous lemma and the fact that the Fourier-Deligne transform commutes with Ind and Res. Hence it is enough to prove the proposition under the additional assumption that $h' \notin \mathbf{I}'$ for any $h \in H$. We can achieve this by a change of orientation.

Let A be as in (b). By Lemma 9.3.5 , the support of A meets $\mathbf{E}_{\mathbf{W};0}$. Hence Proposition 9.3.3 is applicable to $A, \mathbf{I}'$; it shows that $\mathrm{Ind}_{\mathbf{T},\mathbf{W}}^{\mathbf{V}} A \cong B \oplus (\oplus_j C_j[j])$ where B is a simple object of $\mathcal{P}_{\mathbf{V}}$ such that the support of B is contained in $\mathbf{E}_{\mathbf{V};\geq\gamma}$ and meets $\mathbf{E}_{\mathbf{V};\gamma}$; $C_j \in \mathcal{P}_{\mathbf{V}}$ has support contained in $\mathbf{E}_{\mathbf{V};\geq\gamma}$ and is disjoint from $\mathbf{E}_{\mathbf{V};\gamma}$ for any j. By Lemma 9.3.5, we then have $B \in \mathcal{P}_{\mathbf{V};\mathbf{I}';\gamma}$ and $C_j \in \mathcal{P}_{\mathbf{V};\mathbf{I}';>\gamma}$.

Conversely, let B be as in (a). By Lemma 9.3.5, we have that the support of B is contained in $\mathbf{E}_{\mathbf{V};\geq\gamma}$ and meets $\mathbf{E}_{\mathbf{V};\gamma}$. Hence Proposition 9.3.3 is applicable to B and $\mathbf{I}'$. It shows that $\mathrm{Res}_{\mathbf{T},\mathbf{W}}^{\mathbf{V}} B \cong A \oplus (\oplus_j L_j[j])$ where A is a simple object of $\mathcal{P}_{\mathbf{W}}$ such that the support of A meets $\mathbf{E}_{\mathbf{W};0}$ and $L_j \in \mathcal{P}_{\mathbf{W}}$ has support disjoint from $\mathbf{E}_{\mathbf{W};0}$ for any j. By Lemma 9.3.5 we then have $A \in \mathcal{P}_{\mathbf{W};\mathbf{I}';0}$ and $L_j \in \mathcal{P}_{\mathbf{W};\mathbf{I}';>0}$. This proves (a), (b). Statement (c) follows from the last assertion of Proposition 9.3.3.

10.3.3. Remark. The previous proof shows that, given $\mathbf{I}'$ as above and a simple object B in $\mathcal{P}_{\mathbf{V}}$, there is a unique $\gamma \in \mathbf{N}[\mathbf{I}]$ with support contained in $\mathbf{I}'$ such that $B \in \mathcal{P}_{\mathbf{V};\mathbf{I}';\gamma}$.

The existence of γ is obvious. To prove uniqueness, we may assume that the orientation has been chosen as in the previous proof; but then γ is such that the support of B is contained in $\mathbf{E}_{\mathbf{V};\geq\gamma}$ and meets $\mathbf{E}_{\mathbf{V};\gamma}$ and these conditions determine γ uniquely since the support of B is irreducible.

10.3.4. Passage to the opposite orientation. Let $\mathbf{V} \in \mathcal{V}$. For each $\mathbf{i} \in \mathbf{I}$, let $\mathbf{V}_{\mathbf{i}}^*$ be the dual space of $\mathbf{V}_{\mathbf{i}}$ and let $\mathbf{V}^* = \oplus_{\mathbf{i}} \mathbf{V}_{\mathbf{i}}^* \in \mathcal{V}$. Assume now that the new orientation (see 10.1.1) of our graph is the opposite of the old one, that is, $'h = h''$ and $''h = h'$ for all $h \in H$. We have an isomorphism $\rho : \mathbf{E}_{\mathbf{V}} \cong {}'\mathbf{E}_{\mathbf{V}^*}$ given by $\rho(x) = x'$ where $x'_h : \mathbf{V}_{h''}^* \to \mathbf{V}_{h'}^*$ is the transpose of $x_h : \mathbf{V}_{h'} \to \mathbf{V}_{h''}$. This induces an equivalence of categories $\rho_! : \mathcal{D}(\mathbf{E}_{\mathbf{V}}) \cong \mathcal{D}(\mathbf{E}_{\mathbf{V}^*})$ with inverse ρ^*.

Let $\boldsymbol{\nu} = (\nu^1, \nu^2, \dots, \nu^m) \in \mathcal{X}$ be such that $\dim \mathbf{V}_{\mathbf{i}} = \sum_l \nu_{\mathbf{i}}^l$ for all $\mathbf{i} \in \mathbf{I}$. Let $\boldsymbol{\nu}' = (\nu^m, \nu^{m-1}, \dots, \nu^1) \in \mathcal{X}$. It follows immediately from definitions that $\rho_! L_{\boldsymbol{\nu}} = L_{\boldsymbol{\nu}'} \in \mathcal{D}(\mathbf{E}_{\mathbf{V}^*})$. From this we deduce that $\rho_!$ defines equivalences of categories $\mathcal{P}_{\mathbf{V}} \to {}'\mathcal{P}_{\mathbf{V}^*}$ and $\mathcal{Q}_{\mathbf{V}} \to {}'\mathcal{Q}_{\mathbf{V}^*}$.