Lie theory over a semifield

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Let $C = (i : j)$ be a (positive definite) Cartan matrix of simply laced type (i, j run through I). For any field k, Chevalley (1950's) associated to C a (simply connected) group G_k. We often assume $k = \mathbf{C}$ and write $G = G_{\mathbf{C}}$. The definition of G includes a torus $T \subset G$, the Borel subgroups B^+, B^-, their "unipotent radicals" U^+, U^-

and injective (root) homomorphisms $x_i : \mathbf{C} \to U^+$, $y_i : \mathbf{C} \to U^-$ (with $i \in I$). Let W be the Weyl group of G, $\{s_i; i \in I\}$ the simple reflections, $l : W \to \mathbf{N}$ the length function, w_0 the longest element of W. Let $\mathcal{B}$ be the variety of Borel subgroups of G. For B, B' in $\mathcal{B}$ the relative position $pos(B, B') \in W$ is well defined.

A *semifield* is a set with two operations, $+$, $\times$, which is an abelian group with respect to $\times$, an abelian semigroup with respect to $+$ and with $(a+b)c = ac + bc$ for all a, b, c. Thus addition, multiplication, division (but no substraction) are defined.

Examples of semifields:

(i) $K = \mathbf{R}_{>0}$; sum and product are induced from $\mathbf{C}$;

(ii) $K = \mathbf{Z}$; new sum $(a, b) \mapsto \min(a, b)$,

new product $(a, b) \mapsto a + b$;

(iii) $K = \{1\}$ with $1 + 1 = 1, 1 \times 1 = 1$.

The main theme of this talk is that G_k and various related objects can also be defined when the field k is replaced by a semifield K. For evidence of this, assume $G = SL_n$. Then there is a classical submonoid of G, the "totally positive" (TP) part G^{TP} of G introduced by Schoenberg (1930), Gantmacher-Krein (1935). It consists of all matrices in G all of whose $s \times s$ minors are in $\mathbf{R}_{\geq 0}$ for $s = 1, 2, ..., n-1$.

We can view G^{TP} as being obtained from G by replacing $\mathbf{C}$ by the semifield $\mathbf{R}_{>0}$. Return to the general case. Assume $K = \mathbf{R}_{>0}$. In [L1994] I defined

-the TP-part G_K of G as the submonoid of G generated by $\{x_i(a), y_i(a); i \in I, a \in K\}$ and by $\{\chi(a); \chi \in \mathrm{Hom}(\mathbf{C}^*, \mathrm{T}), \mathrm{a} \in \mathrm{K}\}$.

(When $G = SL_n$ this is the same as G^{TP} by results of Whitney, Loewner in the 1950's.)

-the TP-part U_K^+ of U^+ as the submonoid generated by

$\{x_i(a); i \in I, a \in K\}$

-the TP-part U_K^- of U^- as the submonoid generated by

$\{y_i(a); i \in I, a \in K\}$.

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G_K is closed in G. (The proof uses the theory of

canonical bases [L1990].)

The theory in [L1994] was a starting point for

-the theory of cluster algebras: Fomin, Zelevinsky 2002;

-higher Teichmuller theory: Goncharov, Fock 2006;

-the use of the TP grassmannian in physics: Postnikov 2007,

Arkani-Hamed, Trnka 2014.

For any semifield K, we define U_K^+ (or $U^-(K)$) as the monoid

(with 1) with generators i^a with $i \in I$, $a \in K$ and relations

(similar to those of a Coxeter group):

$i^a i^b = i^{a+b}$ for $i \in I$, a, b in K;

$i^a j^b i^c = j^{bc/(a+c)} i^{a+c} j^{ab/(a+c)}$ for $i, j \in I$ with $i : j = -1$, a, b, c in K

$i^a j^b = j^b i^a$ for $i, j \in I$ with $i : j = 0$, a, b in K.

When $K = \mathbf{R}_{>0}$ we recover $U_K^\pm$ defined earlier.

(This definition makes sense even if C is not positive definite.)

In the case where $K = \mathbf{Z}$, relations of the type considered above involve piecewise-linear functions; they first appeared in [L1990] in connection with the parametrization of the canonical basis.

Example: $U^{\pm}_{\{1\}}$ is the monoid with generators i^1 with $i \in I$ and

relations

$i^1 i^1 = i^1$ for $i \in I$;

$i^1 j^1 i^1 = j^1 i^1 j^1$ for $i, j \in I$ with $i : j = -1$;

$i^1 j^1 = j^1 i^1$ for $i, j \in I$ with $i : j = 0$.

We can identify $U^{\pm}_{\{1\}} = W$ as a set (not as a monoid)

by $i^1_1 \dots i^1_m \mapsto s_{i_1} \dots s_{i_m}$ whenever $l(s_{i_1} \dots s_{i_m}) = m$.

We consider besides I, two other copies $-I=\{-i; i\in I\}$,

$\underline{I}=\{\underline{i}; i\in I\}$ of I, in obvious bijection with I. For $\epsilon=\pm 1$,

$i\in I$ we write $\epsilon i=i$ if $\epsilon=1$, $\epsilon i=-i$ if $\epsilon=-1$.

For any semifield K, we define G_K as

the monoid (with 1) with generators $i^a, (-i)^a, \underline{i}^a$

with $i\in I, a\in K$ and the relations below.

$$(\epsilon i)^a(\epsilon i)^b = (\epsilon i)^{a+b} \text{ for } i \in I,\ \epsilon = \pm 1,\ a, b \text{ in } K;$$

$$(\epsilon i)^a(\epsilon j)^b(\epsilon i)^c = (\epsilon j)^{bc/(a+c)}(\epsilon i)^{a+c}(\epsilon j)^{ab/(a+c)}$$

for i, j in I with $i : j = -1$, $\epsilon = \pm 1$, a, b, c in K;

$$(\epsilon i)^a(\epsilon j)^b = (\epsilon j)^b(\epsilon i)^a$$

for i, j in I with $i : j = 0$, $\epsilon = \pm 1$, a, b in K;

$$(\epsilon i)^a(-\epsilon i)^b = (-\epsilon i)^{b/(1+ab)}\underline{i}^{(1+ab)^\epsilon}(\epsilon i)^{a/(1+ab)}$$

for $i \in I$, $\epsilon = \pm 1$, a, b in K;

$\underline{i}^a \underline{i}^b = \underline{i}^{ab}$, $\underline{i}^{(1)} = 1$ for $i \in I$, a, b in K;

$\underline{i}^a \underline{j}^b = \underline{j}^b \underline{i}^a$ for i, j in I, a, b in K;

$\underline{j}^a (\epsilon i)^b = (\epsilon i)^{a^{\epsilon(i:j)} b} \underline{j}^a$ for i, j in I, $\epsilon = \pm 1$, a, b in K;

$(\epsilon i)^a (-\epsilon j)^b = (-\epsilon j)^b (\epsilon i)^a$ for $i \neq j$ in I, $\epsilon = \pm 1$, a, b in K.

When $K = \mathbf{R}_{>0}$ we recover G_K defined earlier.

(This definition makes sense even if C is not positive definite.)

We have $G_{\{1\}} = W \times W$ (as sets, not as monoids).

Tits has said that W ought to be regarded as the Chevalley

group G_k where k is the (non-existent) field with one element.

But $G_{\{1\}}$ is defined for the semifield $\{1\}$. The bijection

$W \times W \to G_{\{1\}}$ almost realizes the wish of Tits.

For any semifield K the obvious map $K \to \{1\}$ is compatible with the semifield structure. It induces homomorphisms of monoids $U_K^{\pm} \to U_{\{1\}}^{\pm} = W$ (with fibre $U_K^{\pm}(w)$ over w), $G_K \to G_{\{1\}} = W \times W$. Assume $K = \mathbf{R}_{>0}$. In each case $X = G, U^+, U^-$, the fibres of $X_K \to X_{\{1\}}$ are cells ($\cong K^m$ for some m); they give a canonical cell decomposition of X_K and $X_{\{1\}}$ can be viewed as the set of cells.

This pattern extends to other basic objects of Lie theory.

Let $\mathcal{U}$ be the set of unipotent elements in G. Assume $K = \mathbf{R}_{>0}$.

The TP-part of $\mathcal{U}$ is by definition $\mathcal{U}_K = \mathcal{U} \cap G_K$. For $w \in W$, let

$\text{supp}(w) = \{i \in I; s_i$ appears in a reduced expression of $w\}$. By

[L1994],

$\mathcal{U}_K = \sqcup_{(w,w') \in W \times W; \text{supp}(w) \cap \text{supp}(w') = \emptyset} \mathcal{U}_K(w, w') \subset G_K$

where $\mathcal{U}_K(w, w') = U_K^+(w) U_K^-(w') = U_K^-(w') U_K^+(w) \subset G_K$ are cells.

The same formula can be used to define $\mathcal{U}_K$ for any semifield K.

For example $\mathcal{U}_{\{1\}} = \{(w, w') \in W \times W; supp(w) \cap supp(w') = \emptyset\}$.

From now on assume $K = \mathbf{R}_{>0}$. In [L1994] I defined the TP-part

$\mathcal{B}_K$ of $\mathcal{B}$ as the closure in $\mathcal{B}$ of the set

$\{uB^+u^{-1}; u \in U_K^-(w_0)\} = \{u'B^-u'^{-1}; u' \in U_K^+(w_0)\}$.

When $G = SL_2$, $\mathcal{B}_K$ is a closed half circle.

Following [L1994] we give a second definition of $\mathcal{B}_K$.

Let V be the irreducible G-module over $\mathbf{C}$ with highest weight ρ (which takes value 1 at any simple coroot). Let $\mathbf{B}$ be the canonical basis [L1990] of V. Let $V_+ = \sum_{b \in B} \mathbf{R}_{\geq 0} b \subset V$.

Let $\mathcal{X}$ be the set of lines L in V such that L contains some vector in the G-orbit of a highest weight vector of V. Let

$\mathcal{X}_K = \{L \in \mathcal{X}; L \cap (V_+ - \{0\}) \neq \emptyset.$

We can identify $\mathcal{X} = \mathcal{B}, \mathcal{X}_K = \mathcal{B}_K$ by $L \mapsto$ stabilizer of L in G.

This second definition of $\mathcal{B}_K$ makes sense even if C is not

positive definite. (The first one doesn't.)

Example: $G = SL_3$. The canonical basis of V can be denoted by

$X_{-12}, X_{-1}, X_{-2}, t_1, t_2, X_1, X_2, X_{12}$.

The set $\mathcal{X}_K$ consists of all $a_{-12}X_{-12} + a_{-1}X_{-1} + a_{-2}X_{-2} + c_1t_1 +$

$+c_2t_2 + a_1X_1 + a_2X_2 + a_{12}X_{12} \in V$

with $a_{-12}, a_{-1}, a_{-2}, c_1, c_2, a_1, a_2, a_{12}$ in $\mathbf{R}_{\geq 0}$ (not all 0) such that

$a_2a_{-12} = c_2a_{-1}, a_1a_{-12} = c_1a_{-2}, a_{-1}a_{12} = c_1a_2,$

$a_{-2}a_{12} = c_2a_1, a_{12}(c_1 + c_2) = a_1a_2, a_{-12}(c_1 + c_2) = a_{-1}a_{-2},$

$c_1c_2 = a_{12}a_{-12}, c_1(c_1 + c_2) = a_1a_{-1}, c_2(c_1 + c_2) = a_2a_{-2},$

modulo the homothety action of $K = \mathbf{R}_{>0}$.

In [L1994] I described a decomposition of $\mathcal{B}_K$ into pieces

$$\mathcal{B}_{K;a\le b} = \{B \in \mathcal{B}_K; pos(B^+, B) = b, pos(B^-, B) = w_0 a\}$$

indexed by pairs $(a, b) \in W \times W$ such that $a \le b$ ($\le$ is the standard partial order on W) and conjectured that

$\mathcal{B}_{K;a\le b} \cong K^{l(b)-l(a)}$. (In the example of SL_3 there are 19 pieces.) The conjecture was proved by Rietsch [1998 MIT Ph.D.thesis]. Hence $\mathcal{B}_{\{1\}} = \{(a, b) \in W \times W; a \le b\}$ is defined.

The natural action of G on $\mathcal{B}$ induces an action of the monoid G_K on $\mathcal{B}_K$. This induces an action of the monoid $G_{\{1\}} = W \times W$ on $\mathcal{B}_{\{1\}}$. It can be described as follows (here $i \in I$):

$(s_i, 1) : (a, b) \mapsto (a, s_i b)$ if $s_i b \geq b$

$(s_i, 1) : (a, b) \mapsto (a, b)$ if $s_i b \leq b$

$(1, s_i) : (a, b) \mapsto (s_i a, b)$ if $s_i a \leq a$

$(1, s_i) : (a, b) \mapsto (a, b)$ if $s_i a \geq a$.

Let $\bar{G}$ be the De Concini-Procesi compactification of G. We can define the TP-part $\bar{G}_K$ of $\bar{G}$ as the closure of G_K in $\bar{G}$.

In the early 2000's I conjectured an explicit cell decomposition for $\bar{G}$ extending the cell decomposition of $\mathcal{B}_K \times \mathcal{B}_K \subset \bar{G}_K$; this was established by Xuhua He [2005 MIT Ph.D.Thesis].

Hence $\bar{G}_{\{1\}}$ is defined (in terms of W) as the indexing set of the set of cells.

Let $u \in G$ be a unipotent element. The Springer fibre $\mathcal{B}_u = \{B \in \mathcal{B}; u \in B\}$ is a much studied variety. (See for example Spaltenstein's 1982 book, which is an extension of his Warwick 1977 Ph.D. thesis). It plays a key role in many questions of representation theory, such as character formulas of complex representations of finite reductive groups.

In 1985/86 (while I was on sabbatical in Rome) I was involved

in a joint work with De Concini and Procesi where we showed

that $\mathcal{B}_u$ has something very close to a cell decomposition

and that its homology is generated by algebraic cycles.

Now assume that $u \in G_K$ is unipotent. We define the

TP-part of the Springer fibre $\mathcal{B}_u$ to be

$\mathcal{B}_{u,K} = \{B \in \mathcal{B}_K; u \in B\} = \mathcal{B}_u \cap \mathcal{B}_K.$

One can show that $\mathcal{B}_{u,K} \neq \emptyset$. Surprisingly, $\mathcal{B}_{u,K}$ has a canonical

cell decomposition. Now u is contained in a unique cell

$\mathcal{U}_K(z, z') = U_K^+(z)U_K^-(z') = U_K^-(z')U_K^+(z)$ of $\mathcal{U}_K$ where

$(z, z') \in W \times W$ and $J = \mathrm{supp}(z)$, $J' = \mathrm{supp}(z')$ are disjoint.

Let $Z_{J,J'} = \{(v,w) \in W \times W;$

$v \leq w; s_i w \leq w, v \not\leq s_i w \quad \forall i \in J; v \leq s_j v, s_j v \not\leq w \quad \forall j \in J'\}.$

Theorem: $\mathcal{B}_{u,K} = \cup_{(v,w)\in Z_{J,J'}} \mathcal{B}_{K;v,w}$.

Thus $\mathcal{B}_{u,K}$ has a canonical cell decomposition with each cell

being a part of the canonical cell decomposition of $\mathcal{B}_K$. Hence

$\mathcal{B}_{u,\{1\}} = Z_{J,J'} \subset \mathcal{B}_{\{1\}}$.

Let $\tilde{\mathcal{B}} = \{(u, B) \in \mathcal{U} \times \mathcal{B}; u \in B\}$. Let $\tilde{\mathcal{B}}_{\{1\}}$ be the set of all $(z, z', v, w) \in W^4$ such that $J = \mathrm{supp}(z)$, $J' = \mathrm{supp}(z')$ are disjoint and $(v, w) \in Z_{J,J'}$. We define the TP-part of $\tilde{\mathcal{B}}$ to be

$\tilde{\mathcal{B}}_K = \{(u, B) \in \mathcal{U}_K \times \mathcal{B}_K; u \in B\}$.

We have a canonical cell decomposition $\tilde{\mathcal{B}}_K = \sqcup_{z,z',v,w} \tilde{\mathcal{B}}_{K,z,z',v,w}$ where $\tilde{\mathcal{B}}_{K,z,z',v,w} = \{(u, B) \in \mathcal{U}_K(z, z') \times \mathcal{B}_{K;v,w}\}$ is a cell of dimension $l(z) + l(z') + l(w) - l(v)$.

Another example of a semifield is $K' = \mathbf{R}(t)_{>0}$, the set of $f \in \mathbf{R}(t)$ of form $f = t^e f_0/f_1$ for some f_0, f_1 in $\mathbf{R}[t]$ with constant term in $\mathbf{R}_{>0}, e \in \mathbf{Z}$ (t is an indeterminate); sum and product are induced from $\mathbf{R}(t)$.

Remark: The map $\alpha : K' \to \mathbf{Z}$, $t^e f_0/f_1 \to e$ is a semifield homomorphism.

Let $\overset{\circ}{\mathcal{B}} = \{B \in \mathcal{B}; pos(B^+, B) = pos(B^-, B) = w_0\}$ an open subset

of $\mathcal{B}$. Define its TP-part as

$$\overset{\circ}{\mathcal{B}}_K = \{uB^+u^{-1}; u \in U_K^-(w_0)\} = \{u'B^-u'^{-1}; u' \in U_K^+(w_0)\}.$$

Now $\overset{\circ}{\mathcal{B}}$ makes sense over any field, in particular over $\mathbf{C}(t)$ and

then it contains

$$\overset{\circ}{\mathcal{B}}_{K'} := \{uB^+u^{-1}; u \in U_{K'}^-(w_0)\} = \{u'B^-u'^{-1}; u' \in U_{K'}^+(w_0)\}$$

as a subset.

We have bijections $U^-_{K'}(w_0) \to \mathring{\mathcal{B}}_{K'}$, $u \mapsto uB^+u^{-1}$ and $U^+_{K'}(w_0) \to \mathring{\mathcal{B}}_{K'}$, $u' \mapsto u'B^-u'^{-1}$. The composition of the first bijection with the inverse of the second bijection is a bijection $U^-_{K'}(w_0) \to U^+_{K'}(w_0)$.

One can show that there is a unique bijection $U^-_{\mathbf{Z}}(w_0) \to U^-_{\mathbf{Z}}(w_0)$ such that we have a commutative diagram

$$\begin{tikzcd} U^-_{K'}(w_0) \arrow[r] \arrow[d] & U^+_{K'}(w_0) \arrow[d] \\ U^-_{\mathbf{Z}}(w_0) \arrow[r] & U^+_{\mathbf{Z}}(w_0) \end{tikzcd}$$

with vertical maps induced by $\alpha : K' \to \mathbf{Z}$.

We define $\overset{\circ}{\mathcal{B}}_{\mathbf{Z}}$ to be the set of pairs $(\xi^+, \xi^-) \in U^-_{\mathbf{Z}}(w_0) \times U^+_{\mathbf{Z}}(w_0)$ such that ξ^+, ξ^- correspond to each other under the bijection $U^-_{\mathbf{Z}}(w_0) \to U^+_{\mathbf{Z}}(w_0)$ above. Thus

(a) $\overset{\circ}{\mathcal{B}}_K, \overset{\circ}{\mathcal{B}}_{K'}, \overset{\circ}{\mathcal{B}}_{\mathbf{Z}}$

are defined. Note that $\overset{\circ}{\mathcal{B}}_{\mathbf{Z}}$ is some kind of flag manifold over the semifield $\mathbf{Z}$. One can show that $G_K, G_{K'}, G_{\mathbf{Z}}$ acts naturally on (a).