CHARACTERS OF SIMPLYLACED NONCONNECTED GROUPS VERSUS CHARACTERS OF NONSIMPLYLACED CONNECTED GROUPS

Shrawan Kumar, George Lusztig and Dipendra Prasad

Abstract. Let G be a connected, simply-connected, almost simple semisimple group over $\mathbf{C}$ of simplylaced type and let σ be a nontrivial diagram automorphism of G. Let $G\langle\sigma\rangle$ be the (disconnected) group generated by G and σ. As a consequence of a theorem of Jantzen the character of an irreducible representation of $G\langle\sigma\rangle$ (also irreducible on G) on $G\sigma$ can be expressed in terms of a character of an irreducible representation of a certain connected simply connected semisimple group G_σ of nonsimplylaced type. We show how Jantzen's theorem can be deduced from properties of the canonical bases.

Let G be a connected, simply-connected, almost simple algebraic group of simplylaced type over $\mathbf{C}$. Let T be a maximal torus of G. Let $x_i : \mathbf{C} \to G$, $y_i : \mathbf{C} \to G$ $(i \in I)$ be homomorphisms which together with T form a pinning (épinglage) of G. We fix a nontrivial automorphism σ of G such that $\sigma(T) = T$, and such that for some permutation $i \mapsto \tilde{i}$ of I we have $\sigma(x_i(a)) = x_{\tilde{i}}(a)$, $\sigma(y_i(a)) = y_{\tilde{i}}(a)$ for all $a \in \mathbf{C}$. For $i \in I$ we write $\sigma(i) = \tilde{i}$. Let $\langle\sigma\rangle$ be the finite subgroup of the automorphism group of G generated by σ and let $G\langle\sigma\rangle$ be the semidirect product of G with $\langle\sigma\rangle$.

Let X be the group of characters $T \to \mathbf{C}^*$; let Y be the group of one parameter subgroups $\mathbf{C}^* \to T$ and let $\langle,\rangle : Y \times X \to \mathbf{Z}$ be the standard pairing. For $i \in I$ we define $\alpha_i \in X$ by $x_i(\alpha_i(t)) = tx_i(1)t^{-1}$, $y_i(\alpha_i(t)^{-1}) = ty_i(1)t^{-1}$ for all $t \in T$. This is a root of G. Let $\check{\alpha}_i \in Y$ be the corresponding coroot. Note that

(a) $(Y, X, \langle,\rangle, \check{\alpha}_i, \alpha_i (i \in I))$

is the root datum of G. Now σ induces automorphisms of X, Y denoted again by σ; these are compatible with $\langle,\rangle$ and we have $\sigma(\alpha_i) = \alpha_{\sigma(i)}, \sigma(\check{\alpha}_i) = \check{\alpha}_{\sigma(i)}$ for $i \in I$. Let $X^+ = \{\lambda \in X; \langle\check{\alpha}_i, \lambda\rangle \in \mathbf{N} \forall i \in I\}$.

We set $Y_\sigma = Y/(\sigma - 1)Y$, ${}^\sigma X = \{\lambda \in X; \sigma(\lambda) = \lambda\}$. Note that $\langle,\rangle : Y \times X \to \mathbf{Z}$ induces a perfect pairing $Y_\sigma \times {}^\sigma X \to \mathbf{Z}$ denoted again by $\langle,\rangle$. Let I_σ be the set of σ-orbits on I. For any $\mathcal{O} \in I_\sigma$ let $\check{\alpha}_\mathcal{O} \in Y_\sigma$ be the image of $\check{\alpha}_i$ under $Y \to Y_\sigma$ where i is any element of $\mathcal{O}$. Since $\{\check{\alpha}_i; i \in I\}$ is a $\mathbf{Z}$-basis of Y we see that $\{\check{\alpha}_\mathcal{O}; \mathcal{O} \in I_\sigma\}$ is a $\mathbf{Z}$-basis of Y_σ. For any $\mathcal{O} \in I_\sigma$ let $\alpha_\mathcal{O} = 2^h \sum_{i \in \mathcal{O}} \alpha_i \in {}^\sigma X$ where h is the

Typeset by $\mathcal{AMS}$-TEX

number of unordered pairs (i,j) such that $i,j \in \mathcal{O}$, and $\alpha_i + \alpha_j$ is a root. Note that $h = 0$ except when G is of type A_{2n} when $h = 0$ for all $\mathcal{O}$ but one and $h = 1$ for one $\mathcal{O}$. Note that

(b) $(Y_\sigma, {}^\sigma X, \langle,\rangle, \check{\alpha}_{\mathcal{O}}, \alpha_{\mathcal{O}}(\mathcal{O} \in I_\sigma))$

is a root datum, see [Ja, p.29]. Let ${}^\sigma X^+ = \{\lambda \in {}^\sigma X; \langle \check{\alpha}_{\mathcal{O}}, \lambda\rangle \in \mathbf{N} \forall \mathcal{O} \in I_\sigma\} = {}^\sigma X \cap X^+$. Let G_σ be the connected semisimple group over $\mathbf{C}$ with root datum (b). By definition, G_σ is provided with an épinglage $(T_\sigma, x_{\mathcal{O}}, y_{\mathcal{O}} \quad (\mathcal{O} \in I_\sigma)$ where $T_\sigma := \mathbf{C}^* \otimes Y_\sigma = T/\{\sigma(t)t^{-1}; t \in T\}$ is a maximal torus of G_σ and $x_{\mathcal{O}} : \mathbf{C} \to G_\sigma$, $y_{\mathcal{O}} : \mathbf{C} \to G_\sigma$ satisfy $x_{\mathcal{O}}(\alpha_{\mathcal{O}}(t_1)) = t_1 x_{\mathcal{O}}(1) t_1^{-1}$, $y_{\mathcal{O}}(\alpha_{\mathcal{O}}(t_1)^{-1}) = t_1 y_{\mathcal{O}}(1) t_1^{-1}$ for all $t_1 \in T_\sigma$. (We have ${}^\sigma X = \mathrm{Hom}(T_\sigma, \mathbf{C}^*)$ canonically.)

Note that G_σ is simply connected and that $G_\sigma \cong {}^L((({}^L G)^\sigma)^0)$ where ${}^L()$ denotes the Langlands dual group and $({}^L G)^\sigma$ denotes the fixed point set of the automorphism of ${}^L G$ induced by σ. Now G_σ is only isogenous to ${}^L(G^\sigma)$ where G^σ is the fixed point set of $\sigma : G \to G$.

Let $\lambda \in {}^\sigma X^+$. We can view λ both as a character of T and as a character of T_σ. Let V (resp. V') be a finite dimensional complex irreducible representation of G (resp. G_σ) with a non-zero vector η (resp. η') such that $x_i(a)\eta = 0$ for all $i \in I, a \in \mathbf{C}$ (resp. $x_{\mathcal{O}}(a)\eta' = 0$ for all $\mathcal{O} \in I_\sigma, a \in \mathbf{C}$) and $t\eta = \lambda(t)\eta$ for all $t \in T$ (resp. $t'\eta' = \lambda(t')\eta'$ for all $t' \in T_\sigma$). Now V can be regarded as a representation of $G\langle\sigma\rangle$ whose restriction to G is as above and on which the action of σ satisfies $\sigma(\eta) = \eta$.

Let $\mu \in X$. Let $V_\mu = \{x \in V; tx = \mu(t)x \quad \forall t \in T\}$. Note that $\sigma : V \to V$ permutes the weight spaces V_μ among themselves. A weight space V_μ is σ-stable if and only if $\mu \in {}^\sigma X$; in this case μ can be viewed as a character of T_σ and we set $V'_\mu = \{x' \in V'; t'x' = \mu(t')x' \forall t' \in T_\sigma\}$.

Theorem (Jantzen [Ja, Satz 9]**).** *For $\mu \in {}^\sigma X$ we have* $\mathrm{tr}(\sigma : V_\mu \to V_\mu) = \dim V'_\mu$.

Corollary. *Let $\varpi : T \to T_\sigma$ be the canonical homomorphism. For any $t \in T$ we have* $\mathrm{tr}(t\sigma : V \to V) = \mathrm{tr}(\varpi(t), V')$.

The corollary describes completely the character of V on $G\sigma$ in terms of the character of V' since any semisimple element in $G\sigma$ is G-conjugate to an element of the form $t\sigma$ with $t \in T$. Note also that there is a well defined bijection between the set of semisimple G-conjugacy classes in $G\sigma$ and the set of semisimple G_σ-conjugacy classes in G_σ which for any $t \in T$ maps the G-conjugacy class of $t\sigma$ to the G_σ-conjugacy class of $\varpi(t)$; see [L2, 6.26], [Mo].

We now show (assuming that G is not of type A_{2n}) how Jantzen's theorem can be deduced from properties of canonical bases in [L1]. According to [L1], V has a canonical basis B_λ and V' has a canonical basis B'_λ. Also, B_λ (resp. B'_λ) can be naturally viewed as a subset of $\mathbf{B}$ (resp. $\mathbf{B}'$), the canonical basis of the + part of the universal enveloping algebra attached to the root datum (a) (resp. (b)). Now σ acts naturally on $\mathbf{B}$ (preserving the subset B_λ) and [L1, Theorem 14.4.9] provides a canonical bijection between $\mathbf{B}'$ and the fixed point set of σ on

B. (This theorem is applicable since the Cartan datum of (b) is obtained from the Cartan datum of (a) by the general "folding" procedure [L1, 14.1] which appplies to any simplylaced Cartan datum of not necessarily finite type together with an admissible automorphism; here we use that G is not of type A_{2n}.) This restricts to a bijection between B'_λ and the fixed point set ${}^\sigma B_\lambda$ of σ on B_λ. Next we note that B_λ (resp. B'_λ) is compatible with the decomposition of V (resp. V') into weight spaces and from the definitions we see that the bijection above carries $B'_\lambda \cap V'_\mu$ bijectively onto ${}^\sigma B_\lambda \cap V_\mu$. Since $B_\lambda \cap V_\mu$ is a basis of V_μ which is σ-stable we have $\mathrm{tr}(\sigma : V_\mu \longrightarrow V_\mu) = \sharp({}^\sigma B_\lambda \cap V_\mu)$. Using the bijection above this equals $\sharp(B'_\lambda \cap V'_\mu)$ and this is equal to $\dim V'_\mu$ since $B'_\lambda \cap V'_\mu$ is a basis of V'_μ. This gives the desired result.

We refer the reader to [FSS, FRS, NS, N1, N2] for other approaches to Jantzen's theorem. We thank S. Naito for pointing out these references to us.

The first two authors were supported in part by the National Science Foundation. The third author thanks the Institute for Advanced Study where this work was done, and gratefully acknowledges receiving support through grants to the Institute by the Friends of the Institute, and the von Neumann Fund.

References

[FSS] J. Fuchs, B. Schellekens and G. Schweigert, *From Dynkin diagrams symmetries to fixed point structures*, Comm. Math. Phys. **180** (1996), 39-97.

[FRS] J. Fuchs, U. Ray and G. Schweigert, *Some automorphisms of generalized Kac-Moody algebras*, J. Algebra **191** (1997), 518-590.

[Ja] J. C. Jantzen, *Darstellungen Halbeinfacher Algebraischer Groupen,*, Bonner Math. Schriften **67** (1973).

[L1] G. Lusztig, *Introduction to quantum groups*, Progress in Math., vol. 110, Birkhäuser, 1993.

[L2] G. Lusztig, *Classification of unipotent representations in simple p-adic groups, II*, Represent. Theory (2002), 243-289.

[Mo] S. Mohrdieck, *Conjugacy classes of non-connected semisimple algebraic groups*, Transfor. Groups **8** (2003), 377-395.

[N1] S.Naito, *Twining character formulas of Borel-Weil-Bott type*, J. Math. Sci. Univ. Tokyo **9** (2002), 637-658.

[N2] S.Naito, *Twining characters, Kostant's homology formula and the Bernstein-Gelfand-Gelfand resolution*, J. Math. Kyoto Univ. **42** (2002), 83-103.

[NS] S. Naito and D. Sagaki, *Lakshmibai-Seshadri paths fixed by a diagram automorphism*, J. Algebra **245** (2001), 395-412.

Addresses:

S.K.: Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA

G.L.: Department of Mathematics, M.I.T., Cambridge, MA 02139, USA

D.P.: School of Mathematics, Tata Institute of Fundamental Research, Colaba, Mumbai 400005, India, and The Institute for Advanced Study, Princeton, NJ 08540, USA