A CLASS OF PERVERSE SHEAVES ON A PARTIAL FLAG MANIFOLD

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Introduction

Let $f : X \to Y$ be a morphism of algebraic varieties over an algebraically closed field $\mathbf{k}$. The theory of perverse sheaves [BBD] associates to f a collection of invariants, namely the collection of simple perverse sheaves on Y which appear as subquotients of the l-adic perverse sheaf $\oplus_{j\in\mathbf{Z}} {}^pH^j(f_!\bar{\mathbf{Q}}_l)$ on Y. More precisely, if f is equivariant for given actions of a finite group Γ on X and Y then we denote by $C_\Gamma(f)$ the (finite) collection of simple Γ-equivariant perverse sheaves on Y which appear as subquotients of the l-adic Γ-equivariant perverse sheaf $\oplus_{j\in\mathbf{Z}} {}^pH^j(f_!\bar{\mathbf{Q}}_l)$ on Y. (The perverse sheaves in $C_\Gamma(f)$ are not necessarily simple if the Γ-equivariant structure is disregarded.)

In this paper we try to understand the collection $C_\Gamma(f)$ in the case where:

$\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$,

G is a connected reductive algebraic group defined over $\mathbf{F}_q$ with Frobenius map $F : G \to G$,

Γ is the group of $\mathbf{F}_q$-rational points of G,

X is the variety of all pairs (B', g) where B' is a Borel subgroup of G and $g \in G$ is such that $g^{-1}F(g)$ is in the unipotent radical of B',

Y is the variety of parabolic subgroups of G of a fixed type,

f associates to $(B', g) \in X$ the unique parabolic subgroup in Y that contains B'.

(The action of Γ on X is $g_0 : (B', g) \mapsto (g_0B'g_0^{-1}, gg_0^{-1})$; the action of Γ on Y is by conjugation.)

Note that the variety X can be viewed as a family of varieties of the type considered in [DL] indexed by the full flag manifold.

In this paper we define a finite collection $\mathbb{S}(Y)$ of simple Γ-equivariant perverse sheaves on Y by two methods (see Sections 3 and 4). These methods and the proof of their equivalence are similar to those used in the theory of parabolic character sheaves [L5]. The second method (see Section 4) gives a description of these perverse sheaves in terms of some explicit local systems on some pieces

Supported in part by the National Science Foundation.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

of a finite partition of Y. This partition, introduced by the author in 1977, and further studied in [Be], reduces in the case where Y is the full flag manifold to the partition introduced in [DL].

In 7.6 we show that $C_f(Y) \subset \mathbb{S}(Y)$.

In Section 6 we construct an explicit basis for the space of intertwining operators between certain cohomologically induced representations of Γ, extending an idea of [L4]. As a bi-product we obtain a disjointness theorem for the objects of $\mathbb{S}(Y)$ which in the special case where $Y = \{G\}$ reduces to the disjointness theorem [DL, 6.2, 6.3] (but the present proof is quite different from that of [DL]).

In Section 7 we study the variety X (see above). In particular we show (using results in Section 6) that X is connected if G is simply connected.

In Section 8 we give a conjecture (based on results in Section 6 and some combinatorial results in Section 5) which should explain in an intrinsic way the "Jordan decomposition" [L1] for irreducible representations of Γ.

Contents

1. Preliminaries

1.1. Let $\mathbf{k}$ be an algebraically closed field. In this paper all algebraic varieties are over $\mathbf{k}$. Let G be a connected reductive algebraic group. Let $\mathcal{B}$ be the variety of Borel subgroups of G. We fix $B \in \mathcal{B}$ and a maximal torus T of B. Let $N(T)$ be the normalizer of T in G. Let $W = N(T)/T$. Note that W acts on T by conjugation; we use this action to identify W with a subgroup of $\mathrm{Aut}(T)$. For any $(B', B'') \in \mathcal{B} \times \mathcal{B}$ there is a unique $w \in W$ such that $gB'g^{-1} = B, gB''g^{-1} = \tilde{w}B\tilde{w}^{-1}$ for some $g \in G$ and some representative $\tilde{w}$ of w in $N(T)$. We then write $\mathrm{pos}(B', B'') = w$. Let $l : W \longrightarrow \mathbf{N}$ be the length function: $l(w) = \dim(B'/(B' \cap B''))$ where $(B', B'') \in \mathcal{B} \times \mathcal{B}$, $\mathrm{pos}(B', B'') = w$. Let $\mathbf{I} = \{w \in W; l(w) = 1\}$.

If H is a group acting on a set X we denote by X^H the fixed point set of H on X.

1.2. Let $\le$ be the standard partial order on W regarded as a Coxeter group with generators $\mathbf{I}$. If X is a subset of W and $w \in X$ we write $w = \min X$ if $l(w) < l(w')$ for all $w' \in W - \{w\}$. For $J \subset \mathbf{I}$ let W_J be the subgroup of W generated by J. For $J, J' \subset W$ let ${}^JW = \{w \in W; w = \min(W_Jw)\}$, $W^{J'} = \{w \in W; w = \min(wW_{J'})\}$, ${}^JW^{J'} = {}^JW \cap W^{J'}$.

Let $w_{\mathbf{I}}$ be the unique element of maximal length in W.

1.3. Let $\mathcal{P}$ be the variety of parabolic subgroups of G. For any $P \in \mathcal{P}$ let U_P be the unipotent radical of P. We set $U = U_B$. For $J \subset \mathbf{I}$ let $P_J \in \mathcal{P}$ be the subgroup of G generated by B and by representatives in $N(T)$ of the various elements of J. Let L_J be the unique Levi subgroup of P that contains T. For $s \in \mathbf{I}$ we write P_s instead of $P_{\{s\}}$. For $J \subset \mathbf{I}$ let $\mathcal{P}_J$ be the G-conjugacy class of parabolic subgroups of G that contains P_J. For $B' \in \mathcal{B}$ let $P_{B',J}$ be the unique subgroup in $\mathcal{P}_J$ that contains B'. For $P \in \mathcal{P}_J$, $Q \in \mathcal{P}_{J'}$ the element $\mathrm{pos}(P,Q) := \min\{w \in W; w = \mathrm{pos}(B',B'') \text{ for some } B' \subset P, B'' \subset Q\}$ is well defined and $\mathrm{pos}(P,Q) \in {}^{J}W^{J'}$. We set

(a) $P^Q = (P \cap Q)U_P \in \mathcal{P}_{J \cap uJ'u^{-1}}$ where $u = \mathrm{pos}(P,Q)$.

For any $g \in G$ we define $k(g) \in N(T)$ by $g \in Uk(g)U$.

1.4. Let $R \subset \mathrm{Hom}(T, \mathbf{k}^*)$ be the set of roots. Let $\check{R} \subset \mathrm{Hom}(\mathbf{k}^*, T)$ be the set of coroots. Let $\check{\alpha} \leftrightarrow \alpha$ be the standard bijection $\check{R} \leftrightarrow R$. For $\alpha \in R$ let U_α be the one dimensional root subgroup (normalized by T) corresponding to α. Let $R^+ = \{a \in R; U_\alpha \subset B\}$, $R^- = R - R^+$. For $s \in \mathbf{I}$ let α_s be the unique root such that $U_{\alpha_s} \subset P_s$. We write $\check{\alpha}_s$ instead of $\check{\alpha_s}$. The natural action of W on T induces an action of W on R.

1.5. For any $s \in \mathbf{I}$ we fix a homomorphism $h_s : SL_2(\mathbf{k}) \longrightarrow G$ such that

$h_s\begin{pmatrix} b & 0 \\ 0 & b^{-1}\end{pmatrix} = \check{\alpha}_s(b)$ for all $b \in \mathbf{k}^*$,

$a \mapsto h_s\begin{pmatrix} 1 & a \\ 0 & 1\end{pmatrix}$ is an isomorphism $x_s : \mathbf{k} \xrightarrow{\sim} U_{\alpha_s}$,

$a \mapsto h_s\begin{pmatrix} 1 & 0 \\ a & 1\end{pmatrix}$ is an isomorphism $y_s : \mathbf{k} \xrightarrow{\sim} U_{\alpha_s^{-1}}$.

We say that $\{B, T, h_s(s \in \mathbf{I})\}$ is an *épinglage* of G. Let $\dot{s} = h_s\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \in N(T)$. For any $w \in W$ we set $\dot{w} = \dot{s}_1\dot{s}_2\dots\dot{s}_n \in N(T)$ where $s_1, s_2, \dots, s_n \in \mathbf{I}$ are chosen so that $w = s_1 s_2 \dots s_n$, $l(w) = n$. (This is independent of the choice.) In particular, $\dot{1} = 1$.

For any sequence $\mathbf{w} = (w_1, w_2, \dots, w_r)$ in W we set $[\mathbf{w}] = w_1 w_2 \dots w_r \in W$ and $[\mathbf{w}]^\bullet = \dot{w}_1\dot{w}_2\dots\dot{w}_r \in N(T)$.

1.6. Equivariant structures. If X is an algebraic variety we write $\mathcal{D}(X)$ for the derived category of bounded constructible $\bar{\mathbf{Q}}_l$-sheaves on X. If $K \in \mathcal{D}(X)$ let $\mathfrak{D}(K) \in \mathcal{D}(X)$ be the Verdier dual of X. Let $\mathcal{M}(X)$ be the subcategory of $\mathcal{D}(X)$ whose objects are perverse sheaves. If $\mathcal{E}$ is a local system on X we denote by $\check{\mathcal{E}}$ the dual local system. For $K \in \mathcal{D}(X)$ we write ${}^pH^\cdot(K)$ instead of $\oplus_j {}^pH^j(K)$. If $f : X \longrightarrow Y$ is a smooth morphism between algebraic varieties with connected fibres of dimension δ, we set $f^\bigstar A = f^*A[\delta]$ for any $A \in \mathcal{D}(Y)$.

Let $m : H \times Y \longrightarrow Y$ be an action of an algebraic group H on an algebraic variety Y. For any connected component C of H define $m_C : C \times Y \longrightarrow Y$ by $(g,y) \mapsto gy$ and $\pi_C : C \times Y \longrightarrow Y$ by $(g,y) \mapsto y$. Let $K \in \mathcal{M}(Y)$. An H-equivariant structure on K is a collection of isomorphisms $\phi_C : \pi_C^\bigstar K \xrightarrow{\sim} m_C^\bigstar K$ (one for each C) such that for any two connected components C, C' of H,

$$(\tilde{m}\boxtimes 1)^\bigstar(\phi_{CC'}) : (\tilde{m}\boxtimes 1)^\bigstar(\pi^\bigstar_{CC'}K) \to (\tilde{m}\boxtimes 1)^\bigstar(m^\bigstar_{CC'}K)$$

is equal to the composition

$$(\tilde{m}\boxtimes 1)^\bigstar(\pi^\bigstar_{CC'}K) \xrightarrow{\overset{(1)}{=}} \tilde{\pi}^\bigstar(\pi^\bigstar_{C'}K) \xrightarrow{\beta} \tilde{\pi}^\bigstar(m^\bigstar_{C'}K)$$
$$\xrightarrow{\overset{(3)}{=}} (1\times m_{C'})^\bigstar(\pi^\bigstar_C K) \xrightarrow{\gamma} (1\times m_{C'})^\bigstar(m^\bigstar_C K) \xrightarrow{\overset{(2)}{=}} (\tilde{m}\boxtimes 1)^\bigstar(m^\bigstar_{CC'}K)$$

where

$\tilde{m} : C\times C' \to CC'$ is $(g,g')\mapsto gg'$, $\tilde{\pi} : C\times C'\times Y \to C'\times Y$ is $(g,g',y)\mapsto (g',y)$;
the equality (1) comes from $\pi_{C'}\tilde{\pi} = \pi_{CC'}(\tilde{m}\times 1) : C\times C'\times Y \to Y$;
the equality (2) comes from $m_{CC'}(1\times m_{C'}) = m_{CC'}(\tilde{m}\times 1) : C\times C'\times Y \to Y$;
the equality (3) comes from $m_{C'}\tilde{\pi} = \pi_C(1\times m_{C'}) : C\times C'\times Y \to Y$;
$\beta = \tilde{\pi}^\bigstar\phi_{C'}$, $\gamma = (1\times m_{C'})^\bigstar\phi_C$.

The notion of H-equivariant structure on a local system on Y is defined in the same way (but replace $()^\bigstar$ by $()^*$).

We denote by $\mathcal{M}_H(Y)$ the (abelian) category of perverse sheaves on Y endowed with an H-equivariant structure; the morphisms are morphisms of perverse sheaves which are compatible with the equivariant structures. Note that any object of $\mathcal{M}_H(Y)$ has finite length. If $K\in\mathcal{M}_H(Y)$ is semisimple as an object of $\mathcal{M}(Y)$ and H is finite then K is semisimple as an object of $\mathcal{M}_H(Y)$.

Let $f : X \to Y$ be a morphism compatible with H-actions on X and Y. If $K\in\mathcal{M}_H(Y)$ then ${}^pH^i(f^*K)$ is naturally an object of $\mathcal{M}_H(X)$. If $K'\in\mathcal{M}_H(X)$ then ${}^pH^i(f_!K')$ is naturally an object of $\mathcal{M}_H(Y)$. Similarly, if $\mathcal{E}$ is a local system on Y with a given H-equivariant structure then the local system $f^*\mathcal{E}$ on X has an induced H-equivariant structure.

If Y_0 is a locally closed smooth H-stable subvariety of pure dimension of Y and $\mathcal{F}$ is an H-equivariant local system on Y_0 we denote by $\mathcal{F}^\sharp$ the complex $IC(\bar{Y}_0,\mathcal{F})$ (where $\bar{Y}_0$ is the closure of Y_0 in Y) extended by 0 on $Y-\bar{Y}_0$. Then $\mathcal{F}^\sharp[\dim Y_0]$ is an object of $\mathcal{M}_H(Y)$.

If K is a simple object of $\mathcal{M}_H(Y)$ then there exists $Y_0,\mathcal{F}$ as above so that the connected components of Y_0 are permuted transitively by the H-action and $K=\mathcal{F}^\sharp[\dim Y_0]$. Note that K is not necessarily simple as an object of $\mathcal{M}(Y)$. If K, K' are objects of $\mathcal{M}_H(Y)$ we write $K \dashv_\Gamma K'$ if K is isomorphic to a subquotient of K'.

If $K\in\mathcal{M}_H(Y)$ then $\mathfrak{D}(K)$ is naturally an object of $\mathcal{M}_H(Y)$.

If H is connected then K has at most one H-equivariant structure.

Now assume that H is finite. Let $m_g : Y\to Y, y\mapsto gy$. An H-equivariant structure on K is a collection of isomorphisms $\phi_g : K \to m_g^*K$ (one for each $g\in H$) such that for any g,g' in H, $K\xrightarrow{\phi_{gg'}} m^*_{gg'}K$ is equal to the composition $K\xrightarrow{\phi_{g'}} m^*_{g'}K \xrightarrow{m^*_{g'}\phi_g} m^*_{gg'}K$. The same definition applies to H-equivariant structures on local systems on Y.

Assume now that A, A' are two simple objects of $\mathcal{M}_H(Y)$. Let $S = \mathrm{supp}(A)$, $S' = \mathrm{supp}(A')$. Note that the irreducible components of S (resp. S') are permuted transitively by H hence S (resp. S') has pure dimmension d (resp. d'). We show:

(a) *$H^0_c(Y, A \otimes A')^H = \bar{\mathbf{Q}}_l(-d)$ if $A' = \mathfrak{D}(A)$ in $\mathcal{M}_H(Y)$ and $H^0_c(Y, A \otimes A')^H = 0$ if $A' \not\cong \mathfrak{D}(A)$ in $\mathcal{M}_(Y)$.*

Note that there exists an open dense smooth H-stable subset S_0 (resp. S'_0) of S (resp. S') and an H-equivariant local system $\mathcal{E}$ (resp. $\mathcal{E}'$) on S_0 (resp. S'_0) such that $A = \mathcal{E}^\sharp[d]$, $\mathfrak{D}(A) = \check{\mathcal{E}}^\sharp[d]$, $A' = \mathcal{E}'^\sharp[d']$. By an argument as in [L2, II, 7.4] we see that $H^0_c(Y, A \otimes A') = 0$ if $S \neq S'$. In the rest of the proof we assume that $S = S'$. Then we can also assume that $S_0 = S'_0$. Again by the argument in [L2, II, 7.4] we see that $H^0_c(Y, A \otimes A') = H^{2d}_c(S_0, \mathcal{E} \otimes \mathcal{E}')$. By Poincaré duality this can be identified with $H^0(S_0, \check{\mathcal{E}} \otimes \check{\mathcal{E}}')^*(-d)$. It remains to describe $H^0(S_0, \check{\mathcal{E}} \otimes \check{\mathcal{E}}')^H$. This is the vector space of homomorphisms of local systems $\mathcal{E}' \longrightarrow \check{\mathcal{E}}$ which are compatible with the H-equivariant structures. This is the same as the space of morphisms from A' to $\mathfrak{D}(A)$ in $\mathcal{M}_H(Y)$ which is $\bar{\mathbf{Q}}_l$ if $A' = \mathfrak{D}(A)$ and is 0 if $A' \not\cong \mathfrak{D}(A)$.

Let E be a finite dimensional $\bar{\mathbf{Q}}_l$-vector space and let $r : H \longrightarrow GL(E)$ be a homomorphism. We regard E as an H-equivariant local system over a point in an obvious way. If X is an algebraic variety with H-action, we denote by $\epsilon : X \longrightarrow$ point the obvious map and we set $E_X = \epsilon^* E$; this is naturally an H-equivariant local system on X (since ϵ is compatible with the H-action on X and the trivial H-action on the point).

1.7. For any torus T' let $\mathcal{S}(T')$ be the category whose objects are the local systems of rank 1 on T' that are equivariant for the transitive T'-action $z : t \mapsto z^n t$ on T' for some $n \in \mathbf{Z}_{>0}$ invertible in $\mathbf{k}$.

1.8. Let $f : T' \longrightarrow T''$ be a morphism of tori and let $\mathcal{L} \in \mathcal{S}(T'')$. We show that the following two conditions are equivalent:

(i) $f^*\mathcal{L} \cong \bar{\mathbf{Q}}_l$;

(ii) $\mathcal{L}$ is equivariant for the T'-action $t' : t'' \mapsto f(t')t''$ on T''.

We can find $\kappa \in \mathrm{Hom}(T'', \mathbf{k}^*)$ and $\mathcal{E} \in \mathcal{S}(\mathbf{k}^*)$ such that $\mathcal{L} \cong \kappa^*(\mathcal{E})$. If the result holds for $\kappa f, \mathcal{E}$ instead of $f, \mathcal{L}$ then it also holds for $f, \mathcal{L}$. Thus we may assume that $T'' = \mathbf{k}^*$. We can assume that $T' \neq \{1\}$. Let T_0 be a codimension 1 subtorus of T' contained in $\ker f$. Then f induces a homomorphism $f' : T'/T_0 \longrightarrow T''$. If the result holds for $f', \mathcal{L}$ instead of $f, \mathcal{L}$ then it also holds for $f, \mathcal{L}$. Thus we may assume that $T' = T'' = \mathbf{k}^*$. In this case the result is immediate.

1.9. In this subsection we assume that $\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$. Let $F' : T \longrightarrow T$ be the Frobenius map for some $\mathbf{F}_q$-rational structure on the torus T. Let $T^{F'} = \{t \in T; F'(t) = t\}$. The following three sets coincide:

(i) the set of $\mathcal{L} \in \mathcal{S}(T)$ (up to isomorphism) such that $\mathcal{L}$ is T-equivariant for the T-action $t_0 : t \mapsto t_0 t F'(t_0)^{-1}$ on T;

(ii) the set of $\mathcal{L} \in \mathcal{S}(T)$ (up to isomorphism) such that $F'^*\mathcal{L} \cong \mathcal{L}$;

(iii) the set of $\mathcal{L} \in \mathcal{S}(T)$ (up to isomorphism) such that $\mathcal{L}$ is a direct summand of $L_!\bar{\mathbf{Q}}_l$ where $L : T \to T$ is $t \mapsto tF'(t^{-1})$;
moreover they are in a natural bijection with

(iv) the set of characters $\mathrm{Hom}(T^{F'}, \bar{\mathbf{Q}}_l^*)$.
For $\mathcal{L} \in \mathcal{S}(T)$ we have $L^*\mathcal{L} \cong \mathcal{L} \otimes F'^*(\check{\mathcal{L}})$. Hence $\mathcal{L}$ satisfies (ii) if and only if it satisfies $L^*\mathcal{L} \cong \bar{\mathbf{Q}}_l$. The last condition is clearly equivalent to the condition in (iii) and by 1.8 it is also equivalent to the condition in (i).

If L is as in (iii), $T^{F'}$ acts in an obvious way on $L_!\bar{\mathbf{Q}}_l$ and we have $L_!\bar{\mathbf{Q}}_l = \oplus_{\chi \in \mathrm{Hom}(T^{F'}, \bar{\mathbf{Q}}_l^*)} L_!^\chi \bar{\mathbf{Q}}_l$ where $L_!^\chi \bar{\mathbf{Q}}_l$ is the subsheaf on which $T^{F'}$ according to χ. It is clear that $\chi \mapsto L_!^\chi \bar{\mathbf{Q}}_l$ defines a bijection between the sets (iv) and (iii). The inverse of this bijection can be described as follows. Let $\mathcal{L}$ be as in (i). By restriction of the equivariant T-structure we obtain an equivariant $T^{F'}$-structure on $\mathcal{L}$. Since $T^{F'}$ acts trivially on T, it acts naturally on the stalk of $\mathcal{L}$ at 1; this action is via a character $\chi_{\mathcal{L}} : T'^F \to \bar{\mathbf{Q}}_l^*$. Now $\mathcal{L} \mapsto \chi_{\mathcal{L}}$ is the inverse of the bijection above.

1.10. Let $\mathcal{L} \in \mathcal{S}(T)$. Let $R_{\mathcal{L}} = \{\alpha \in R; \check{\alpha}^*\mathcal{L} \cong \bar{\mathbf{Q}}_l\}$. Then $R_{\mathcal{L}}$ is a root system and $R_{\mathcal{L}}^+ = R_{\mathcal{L}} \cap R^+$ is a set of positive roots for $R_{\mathcal{L}}$. Let $\Pi_{\mathcal{L}}$ be the unique set of simple roots of $R_{\mathcal{L}}$ such that $\Pi_{\mathcal{L}} \subset R_{\mathcal{L}}^+$. Let $W_{\mathcal{L}}$ be the subgroup of W generated by the reflections with respect to the roots in $R_{\mathcal{L}}$. Let $\mathbf{I}_{\mathcal{L}}$ be the set of reflections with respect to the roots in $\Pi_{\mathcal{L}}$. Then $(W_{\mathcal{L}}, \mathbf{I}_{\mathcal{L}})$ is a Coxeter group. Let $\check{R}_{\mathcal{L}} = \{\check{\alpha}; \alpha \in R_{\mathcal{L}}\}$.

2. The variety $\mathcal{Z}^{\mathbf{s}}$ and the local system $\bar{\mathcal{L}}$

2.1. In this and the next subsection we assume that $\mathbf{I}$ consists of a single element s. Let $a = \check{\alpha}_s : \mathbf{k}^* \to T$. To any $\mathcal{L} \in \mathcal{S}(T)$ such that

(a) $a^*\mathcal{L} \cong \bar{\mathbf{Q}}_l$
we will associate a local system $\underline{\mathcal{L}}$ of rank 1 on G.

Case 1. a is an imbedding. We have a diagram $T \xrightarrow{c} T/a(\mathbf{k}^*) \xrightarrow{d} G/G_{der} \xleftarrow{e} G$ where c, e are the obvious maps and d is induced by the inclusion $T \subset G$; note that d is an isomorphism. Now $\mathbf{k}^*$ acts on T by $x : t \mapsto a(x)t$ and on $T/a(\mathbf{k}^*)$ trivially; c is compatible with the $\mathbf{k}^*$-actions. From (a) we see that $\mathcal{L}$ is $\mathbf{k}^*$-equivariant. Since $\mathbf{k}^*$ acts freely on T there is a well defined local system $\mathcal{L}_1$ of rank 1 on $T/a(\mathbf{k}^*)$ such that $\mathcal{L} = c^*\mathcal{L}_1$. We set $\underline{\mathcal{L}} = e^*(d^{-1})^*\mathcal{L}_1$.

Case 2. a is not an imbedding. Then the centre $\mathcal{Z}$ of G is connected, the obvious homomorphism $G_{der} \times \mathcal{Z} \to G$ is an isomorphism and we can identify $G_{der} = PGL_2(\mathbf{k})$ compatibly with the standard épinglages. Thus we can identify $G = PGL_2(\mathbf{k}) \times \mathcal{Z}$. Let $G' = GL_2(\mathbf{k}) \times \mathcal{Z}$ and let $\pi : G' \to G$ be the obvious homomorphism. Let $K = \ker \pi$, a one dimensional torus. Let $T' = \pi^{-1}(T)$. Let $\pi_0 : T' \to T$ be the restriction of π. Let $a' : \mathbf{k}^* \to T'$ be the coroot of G' such that $\pi_0 a' = a$. Let $\mathcal{L}' = \pi_0^*\mathcal{L}$. Note that $a'^*\mathcal{L}' \cong \bar{\mathbf{Q}}_l$. Applying the construction in Case 1 to $G', T', \mathcal{L}'$ instead of $G, T, \mathcal{L}$ we obtain a local system $\underline{\mathcal{L}}'$ on G'. Now K acts on $T', G', G'/G'_{der}, T'/a'(\mathbf{k}^*)$ by translation, the analogues of c, d, e for G' are compatible with the K-action and $\mathcal{L}'$ is K-equivariant. Hence $\underline{\mathcal{L}}'$ is K-equivariant. Since K acts freely on G' there is a well defined local system $\underline{\mathcal{L}}$ of rank 1 on G such that $\pi^*\underline{\mathcal{L}} = \underline{\mathcal{L}}'$.

2.2. Define $f : G - B \to T$ by $f(y) = k(y)\dot{s}^{-1}$ and $f^1 : B \to T$ by $f^1(y) = k(y)$. We show:

(a) *we have canonically* $\underline{\mathcal{L}}|_{G-B} = f^*\mathcal{L}$ *and* $\underline{\mathcal{L}}|_B = f^{1*}\mathcal{L}$.

In the setup of 2.1, assume first that we are in case 1. Let $j : G - B \to G$, $h : B \to G$ be the inclusions. We must show that

$$j^*e^*(d^{-1})^*\mathcal{L}_1 = f^*c^*\mathcal{L}_1, h^*e^*(d^{-1})^*\mathcal{L}_1 = f^{1*}c^*\mathcal{L}_1.$$

It is enough to show that $d^{-1}ej = cf$, $d^{-1}eh = cf^1$ or that $ej = dcf$ (resp. $eh = dcf^1$). Both maps take $ut\dot{s}u'$ (resp. ut), where $u, u' \in U, t \in T$, to the image of t in G/G_{der}. (We use that $u, u', \dot{s} \in G_{der}$.)

Next we assume that we are in case 2. Define $\dot{s}' \in G'$ in terms of the unique épinglage of G' compatible under π with that of G in the same way that $\dot{s} \in G$ is defined in terms of the épinglage of G. Let $B' = \pi^{-1}(B)$. Define $f' : G' - B' \to T'$, $f'^1 : B' \to T'$ in terms of $G', B', T', \dot{s}'$ in the same way that f, f^1 are defined in terms of $G, B, T, \dot{s}$. Let $\pi_s : G' - B' \to G - B$, $\pi_1 : B' \to B$ be the restrictions of π. It is enough to show that

$$\pi_s^*(\underline{\mathcal{L}}|_{G-B}) = \pi_s^* f^*\mathcal{L}, \quad \pi_1^*(\underline{\mathcal{L}}|_B) = \pi_1^* f^{1*}\mathcal{L}.$$

We have

$$\pi_s^*(\underline{\mathcal{L}}|_{G-B}) = \underline{\mathcal{L}}'|_{G'-B'}, \pi_s^* f^*\mathcal{L} = f'^*\mathcal{L}', \pi_1^*(\underline{\mathcal{L}}|_B) = \underline{\mathcal{L}}'|_{B'}, \pi_1^* f^{1*}\mathcal{L} = f'^{1*}\mathcal{L}'.$$

Hence it is enough to show that $\underline{\mathcal{L}}'|_{G'-B'} = f'^*\mathcal{L}'$, $\underline{\mathcal{L}}'|_{B'} = f'^{1*}\mathcal{L}'$. But these are known from Case 1 applied to $G', \mathcal{L}'$ instead of $G, \mathcal{L}$. This proves (a).

2.3. We return to the general case. Let $s \in \mathbf{I}$. Let $P = P_s$. Let $\pi_P : P \to P/U_P$ be the obvious map. Note that P/U_P inherits an épinglage from G and that T, identified with its image under π_P is a maximal torus of P/U_P. To any $\mathcal{L} \in \mathcal{S}(T)$ such that $\check{\alpha}_s^*\mathcal{L} \cong \bar{\mathbf{Q}}_l$ we associate a local system of rank 1 on P, namely the inverse image of the local system $\underline{\mathcal{L}}$ on P/U_P (see 2.1) under π_P; this local system on P is denoted again by $\underline{\mathcal{L}}$.

Define $f_s : P - B \to T$ by $f_s(y) = k(y)\dot{s}^{-1}$ and $f_s^1 : B \to T$ by $f_s^1(y) = k(y)$. From 2.2(a) we deduce by taking inverse image under π_P:

(a) *we have canonically* $\underline{\mathcal{L}}|_{P-B} = f_s^*\mathcal{L}$ *and* $\underline{\mathcal{L}}|_B = f_s^{1*}\mathcal{L}$,

(as local systems over subsets of P.)

2.4. Let $\mathbf{s} = (s_1, s_2, \dots, s_r)$ be a sequence in $\mathbf{I}$. Let $\mathcal{L} \in \mathcal{S}(T)$. Let

$$\mathcal{I}_\mathbf{s} = \{i \in [1, r]; s_1 s_2 \dots s_i \dots s_2 s_1 \in W_\mathcal{L}\}.$$

Let
$$\mathcal{Y}=\{(y_i)\in G^{[1,r]}; y_i\in P_{s_i}(i\in\mathcal{I}_{\mathbf{s}}), y_i\in P_{s_i}-B(i\in[1,r]-\mathcal{I}_{\mathbf{s}})\}.$$
For $i\in[1,r]$ we define $f_{s_i}:P_{s_i}-B\to T$ by $f_{s_i}(y)=k(y)\dot{s}_i^{-1}$ and $f^1_{s_i}:B\to T$ by $f^1_{s_i}(y)=k(y)$. We have obvious projections $p_i:\mathcal{Y}\to P_{s_i}(i\in\mathcal{I}_{\mathbf{s}})$, $p_i:\mathcal{Y}\to P_{s_i}-B(i\in[1,r]-\mathcal{I}_{\mathbf{s}})$. Let
$$\underline{\underline{\mathcal{L}}}=\otimes_{i\in[1,r]}\mathcal{F}_i$$
with
$$\mathcal{F}_i=p_i^*\underline{s^*_{i-1}\dots s^*_2s^*_1\mathcal{L}} \text{ for } i\in\mathcal{I}_{\mathbf{s}}, \mathcal{F}_i=p_i^*f^*_{s_i}s^*_{i-1}\dots s^*_2s^*_1\mathcal{L} \text{ for } i\in[1,r]-\mathcal{I}_{\mathbf{s}}.$$
Here $\mathcal{F}_i,\underline{\underline{\mathcal{L}}}$ are local systems on $\mathcal{Y}$. Note that if $i\in\mathcal{I}_{\mathbf{s}}$ then the local system $\underline{s^*_{i-1}\dots s^*_2s^*_1\mathcal{L}}$ on P_{s_i} is well defined (see 2.1) since $\check{\alpha}^*_{s_i}(s^*_{i-1}\dots s^*_2s^*_1\mathcal{L})\cong\bar{\mathbf{Q}}_l$.

For any $\mathcal{J}\subset\mathcal{I}_{\mathbf{s}}$, let
$$\mathcal{Y}^{\mathcal{J}}=\{(y_i)\in G^{[1,r]}; y_i\in P_{s_i}-B(i\in[1,r]-\mathcal{J}), y_i\in B(i\in\mathcal{J})\}$$
and let $\mathbf{s}_{\mathcal{J}}=(s'_1,s'_2,\dots,s'_r)$ where $s'_i=s_i$ if $i\in[1,r]-\mathcal{J}$, $s'_i=1$ if $i\in\mathcal{J}$. Define $f^{\mathcal{J}}:\mathcal{Y}^{\mathcal{J}}\to T$ by $(y_i)\mapsto k(y_1)k(y_2)\dots k(y_r)[\mathbf{s}_{\mathcal{J}}]^{\bullet-1}$. We show:

(a) *We have canonically* $\underline{\underline{\mathcal{L}}}|_{\mathcal{Y}^{\mathcal{J}}}=(f^{\mathcal{J}})^*\mathcal{L}$.

We have obvious projections $p'_i:\mathcal{Y}^{\mathcal{J}}\to P_{s_i}-B(i\in[1,r]-\mathcal{J})$, $p'_i:\mathcal{Y}^{\mathcal{J}}\to B(i\in\mathcal{J})$. Using 2.3(a) we have canonically $\underline{\underline{\mathcal{L}}}|_{\mathcal{Y}^{\mathcal{J}}}=\otimes_{i\in[1,r]}\mathcal{F}'_i$ where
$$\mathcal{F}'_i=p'^*_if^*_{s_i}s^*_{i-1}\dots s^*_2s^*_1\mathcal{L} \text{ for } i\in[1,r]-\mathcal{J}, \mathcal{F}'_i=p'^*_if^{1*}_{s_i}s^*_{i-1}\dots s^*_2s^*_1\mathcal{L} \text{ for } i\in\mathcal{J}.$$
We define $\tilde{f}:\mathcal{Y}^{\mathcal{J}}\to T^{[1,r]}$ by $\tilde{f}=(\tilde{f}_i)$ where for $i\in[1,r]$, $\tilde{f}_i:\mathcal{Y}^{\mathcal{J}}\to T$ is given by
$$\tilde{f}_i=s_1s_2\dots s_{i-1}f_{s_i}p'_i \text{ for } i\in[1,r]-\mathcal{J}, \tilde{f}_i=s_1s_2\dots s_{i-1}f^1_{s_i}p'_i \text{ for } i\in\mathcal{J}.$$
Then $\mathcal{F}'_i=\tilde{f}_i^*\mathcal{L}$ for $i\in[1,r]$ and
$$\otimes_{i\in[1,r]}\mathcal{F}'_i=\tilde{f}^*(\mathcal{L}\boxtimes\mathcal{L}\boxtimes\dots\boxtimes\mathcal{L}).$$
For $y\in\mathcal{Y}^{\mathcal{J}})$ we have $f^{\mathcal{J}}(y)=\tilde{f}_1(y)\tilde{f}_2(y)\dots\tilde{f}_r(y)=m\tilde{f}(y)$ where $m:T^{[1,r]}\to T$ is multiplication. Hence $(f^{\mathcal{J}})^*\mathcal{L}=\tilde{f}^*m^*\mathcal{L}$. It is then enough to show that $m^*\mathcal{L}=\mathcal{L}\boxtimes\mathcal{L}\boxtimes\dots\boxtimes\mathcal{L}$; this is a known property of any local system in $\mathcal{S}(T)$. This proves (a).

Let
$$Y=\{(y_i)\in G^{[1,r]}; y_i\in P_{s_i}(i\in[1,r])\}.$$
Note that $\mathcal{Y}$ is an open dense subset of Y. Hence $IC(Y,\underline{\underline{\mathcal{L}}})$ is well defined. We show:

(b) $$IC(Y,\underline{\underline{\mathcal{L}}})|_{Y-\mathcal{Y}}=0.$$

For any $j \in [1,r]-\mathcal{I}_{\mathbf{s}}$ let

$$\Delta_j = \{\{(y_i) \in G^{[1,r]}; y_i \in P_{s_i}(i \in [1,r]-\{j\}), y_j \in B\}.$$

Clearly, $\{\Delta_j, j \in [1,r]-\mathcal{I}_{\mathbf{s}}\}$ are smooth divisors with normal crossings in the smooth variety Y. Using [L2, I, 1.6] we see that it suffices to prove the following statement.

(c) *For $j \in [1,r]-\mathcal{I}_{\mathbf{s}}$, the monodromy of $\underline{\underline{\mathcal{L}}}$ around the divisor Δ_j is non-trivial.*
We define a cross-section $\xi : \mathbf{k} \to Y$ to Δ_j in Y by

$$\xi(a) = (\dot{s}_1, \dots, \dot{s}_{j-1}, y_{s_j}(-a), \dot{s}_{j+1}, \dots, \dot{s}_r).$$

We have $\xi(0) \in \Delta_j$, $\xi(a) \in \mathcal{Y}$ for $a \in \mathbf{k}^*$. Let $\xi' : \mathbf{k}^* \to \mathcal{Y}$ be the restriction of ξ. It is enough to show that $\xi'^*\underline{\underline{\mathcal{L}}} \not\cong \bar{\mathbf{Q}}_l$ or, with notation in 2.4, that

$$\xi'^* p_j^* f_{s_j}^* s_{j-1}^* \dots s_2^* s_1^* \mathcal{L} \not\cong \bar{\mathbf{Q}}_l$$

or that

$$(s_1 s_2 \dots s_{j-1} f_{s_j} p_j \xi')^* \mathcal{L} \not\cong \bar{\mathbf{Q}}_l, (s_1 s_2 \dots s_{j-1} \check{\alpha}_{s_j})^* \mathcal{L} \not\cong \bar{\mathbf{Q}}_l.$$

(We have $f_{s_j} p_j \xi'(a) = k(y_{s_j}(-a))\dot{s}_j^{-1} = \check{\alpha}_{s_j}(a)$.) This follows from the fact that $j \notin \mathcal{I}_{\mathbf{s}}$. This proves (c) and hence (b).

(A similar result with a similar proof appears in [L6, VI, 28.10(b)].)

In the remainder of this paper we assume that $\mathbf{k}$ *is an algebraic closure of* $\mathbf{F}_q$*, a finite field with q elements and that we are given a fixed* $\mathbf{F}_q$*-rational structure on G such that B and T are defined over* $\mathbf{F}_q$. Let $F : G \to G$ be the corresponding Frobenius map. We set

$$\Gamma = \{g \in G; F(g) = g\}.$$

Now $F : G \to G$ induces an isomorphism $F : T \to T$. For $w \in W$ we write $\mathcal{L} \in \mathcal{S}(T)^{wF}$ instead of "$\mathcal{L} \in \mathcal{S}(T)$ and $(wF)^*\mathcal{L} \cong \mathcal{L}$".

Define $F_0 : T \to T$ by $t \mapsto t^q$. For any $t \in T$ we have $F(t) = F_0(\mathbf{c}(t)) = \mathbf{c}(F_0(t))$ where $\mathbf{c} : T \to T$ is a well defined automorphism. Then $w \mapsto \mathbf{c}w\mathbf{c}^{-1}$ is an automorphism of W denoted also by $w \mapsto \mathbf{c}(w)$. This restricts to a bijection $\mathbf{I} \xrightarrow{\sim} \mathbf{I}$. Let

$$\underline{B} := \{(b_i) \in B^{[0,r]}; k(b_r^{-1}F(b_0)) = 1\}.$$

We show:

(d) *If $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$ then the local system $\underline{\underline{\mathcal{L}}}$ is equivariant for the $\underline{B}$-action*

$$(b_0, b_1, \dots, b_r) : (y_1, y_2, \dots y_r) \mapsto (b_0 y_1 b_1^{-1}, b_1 y_2 b_2^{-1}, \dots, b_{r-1} y_r b_r^{-1})$$

on $\mathcal{Y}$.
By (a), the restriction of $\underline{\underline{\mathcal{L}}}$ to the $\underline{B}$-stable open dense subset $\mathcal{Y}^\emptyset$ of $\mathcal{Y}$ is $(f^\emptyset)^*\mathcal{L}$. Since $\mathcal{Y}$ is smooth, it is enough to show that the local system $(f^\emptyset)^*\mathcal{L}$ on $\mathcal{Y}^\emptyset$ is $\underline{B}$-equivariant. Now $\underline{B}$ acts on T by $(b_0, b_1, \dots, b_r) : t \mapsto k(b_0)^{-1}t([\mathbf{s}]F(b_0))$ and $f^\emptyset : \mathcal{Y}^\emptyset \to T$ is compatible with the $\underline{B}$-actions. Hence it is enough to show that $\mathcal{L}$ is $\underline{B}$-equivariant. An equivalent statement is that $\mathcal{L}$ is T-equivariant for the T-action $t_0 : t \mapsto t_0^{-1}t([\mathbf{s}]F(t_0))$. This follows from our assumption on $\mathcal{L}$; (d) is proved.

2.5. Let $\mathbf{w}=(w_1,w_2,\dots,w_r)$ be a sequence in W. Let

$$Z^{\mathbf{w}}=\{(B_i)\in\mathcal{B}^{[0,r]};\operatorname{pos}(B_{i-1},B_i)=w_i(i\in[1,r]),B_r=F(B_0)\},$$
$$\dot Z=\{(g_iU)\in(G/U)^{[0,r]};k(g_{i-1}^{-1}g_i)=\dot w_i(i\in[1,r]),g_r^{-1}F(g_0)\in U\}.$$

Let

$$\mathfrak{T}=\{(t_i)\in T^{[0,r]};t_i=w_i^{-1}(t_{i-1})(i\in[1,r]),t_r=F(t_0)\},$$

a finite subgroup of $T^{[0,r]}$ which may be identified via $(t_i)\mapsto t_0$ with $T^{F'}$ where $F':T\to T$ is $t\mapsto[\mathbf{w}]F(t)$. The free $\mathfrak{T}$-action $(t_i):(g_iU)\mapsto(g_it_i^{-1}U)$ on $\dot Z$ makes $\dot Z$ into a principal $\mathfrak{T}$-bundle over $Z^{\mathbf{w}}$ via the map $f:\dot Z\to Z^{\mathbf{w}}$, $(g_iU)\mapsto(g_iBg_i^{-1})$. Now $f_!\bar{\mathbf{Q}}_l$ is a local system on $Z^{\mathbf{w}}$ with a free action of $\mathfrak{T}=T^{F'}$ on each stalk. We have $f_!\bar{\mathbf{Q}}_l=\oplus_{\chi\in\operatorname{Hom}(T^{F'},\bar{\mathbf{Q}}_l^*)}f_!^\chi\bar{\mathbf{Q}}_l$ where $f_!^\chi\bar{\mathbf{Q}}_l$ is the subsheaf of $f_!\bar{\mathbf{Q}}_l$ on which $T^{F'}$ acts according to χ.

Now Γ acts on $Z^{\mathbf{w}}$ by $g:(B_i)\mapsto(gB_ig^{-1})$, and on $\dot Z$ by $g:(g_iU)\mapsto(gg_iU)$. This last action commutes with the $\mathfrak{T}$-action. Hence $f_!\bar{\mathbf{Q}}_l$ has a natural Γ-equivariant structure and each $f_!^\chi\bar{\mathbf{Q}}_l$ inherits a Γ-equivariant structure from $f_!\bar{\mathbf{Q}}_l$.

We now give an alternative construction of the local systems $f_!^\chi\bar{\mathbf{Q}}_l$. Let

$$\underline{Z}=\{(g_iU)\in(G/U)^{[0,r]};g_{i-1}^{-1}g_i\in B\dot w_iB(i\in[1,r]),g_r^{-1}F(g_0)\in U\}.$$

Define $\gamma:\underline{Z}\to Z^{\mathbf{w}}$ by $(g_iU)\mapsto(g_iBg_i^{-1})$. Define $\pi_{\mathbf{w}}:\underline{Z}\to T$ by

$$(g_iU)\mapsto k(g_0^{-1}g_1)k(g_1^{-1}g_2)\dots k(g_{r-1}^{-1}g_r)[\mathbf{w}]^{\bullet-1}.$$

The torus $\underline{T}:=\{(t_i)\in T^{[0,r]};t_r=F(t_0)\}$ acts on $\underline{Z}$ by $(t_i):(g_iU)\mapsto(g_it_i^{-1}U)$ and on T by $(t_i):t\mapsto t_0t([\mathbf{w}]F(t_0^{-1}))$. These actions are compatible with $\pi_{\mathbf{w}}$. Let $\mathcal{L}\in\mathcal{S}(T)^{[\mathbf{w}]F}$. By 1.8, $\mathcal{L}$ is equivariant for the $\underline{T}$-action on T. Hence $\pi_{\mathbf{w}}^*\mathcal{L}$ is equivariant for the (free) $\underline{T}$-action on $\underline{Z}$. Hence $\pi_{\mathbf{w}}^*\mathcal{L}=\gamma^*\mathcal{L}_{\mathbf{w}}$ for a well defined local system $\mathcal{L}_{\mathbf{w}}$ on $Z^{\mathbf{w}}$.

Now Γ acts on $\underline{Z}$ by $g:(g_iU)\mapsto(gg_iU)$ and on T trivially. Also, $\mathcal{L}$ has a natural Γ-equivariant structure in which Γ acts trivially on each stalk of $\mathcal{L}$. Since $\pi_{\mathbf{w}}$ is compatible with the Γ-actions it follows that $\pi_{\mathbf{w}}^*\mathcal{L}$ has a natural Γ-equivariant structure. Since γ is compatible with the Γ-actions it follows that $\mathcal{L}_{\mathbf{w}}$ has a natural Γ-equivariant structure.

Now assume that $\mathcal{L}$ and $\chi\in\operatorname{Hom}(T^{F'},\bar{\mathbf{Q}}_l^*)$ correspond to each other as in 1.9. Thus we assume that $\mathcal{L}=L_!^\chi\bar{\mathbf{Q}}_l$ where $L:T\to T$ is as in 1.9. We show

(a) $$\mathcal{L}_{\mathbf{w}}=f_!^\chi\bar{\mathbf{Q}}_l.$$

Since γ is smooth with connected fibres it is enough to show that $\pi_{\mathbf{w}}^*\mathcal{L}=\gamma^*f_!^\chi\bar{\mathbf{Q}}_l$. Let

$$\begin{aligned}\mathfrak{P}=&\{(g_iU,\tau_i)\in(G/U\times T)^{[0,r]};k(\tau_{i-1}g_{i-1}^{-1}g_i\tau_i^{-1})=\dot w_i(i\in[1,r]),\\&\tau_r=F(\tau_0),g_r^{-1}F(g_0)\in U\}.\end{aligned}$$

Define $f' : \mathfrak{P} \longrightarrow \underline{Z}$ by $(g_iU, \tau_i) \mapsto (g_iU)$ and $\gamma' : \mathfrak{P} \longrightarrow \dot{Z}$ by $(g_iU, \tau_i) \mapsto (g_i\tau_i^{-1}U)$. Define $\pi' : \mathfrak{P} \longrightarrow T$ by $(g_iU, \tau_i) \mapsto \tau_0$. Now $\mathfrak{T}$ acts on $\mathfrak{P}$ by $(t_i) : (g_iU, \tau_i) \mapsto (g_iU, t_i\tau_i)$, making f' into a principal $\mathfrak{T}$-bundle. We have a cartesian diagram of principal $\mathfrak{T} = T^{F'}$ bundles:

$$\begin{tikzcd}
\dot{Z} \arrow[d, "f"'] & \mathfrak{P} \arrow[l, "\gamma'"'] \arrow[r, "\pi'"] \arrow[d, "f'"'] & T \arrow[d, "L"'] \\
Z^{\mathbf{w}} & \underline{Z} \arrow[l, "\gamma"'] \arrow[r, "\pi_{\mathbf{w}}"] & T
\end{tikzcd}$$

It follows that $\gamma^*(f_!\bar{\mathbf{Q}}_l) = f'_!\bar{\mathbf{Q}}_l = \pi_{\mathbf{w}}^*(L_!\bar{\mathbf{Q}}_l)$, and taking χ-eigenspaces: $\gamma^*(f_!^{\chi}\bar{\mathbf{Q}}_l) = f_!'^{\chi}\bar{\mathbf{Q}}_l = \pi_{\mathbf{w}}^*(L_!^{\chi}\bar{\mathbf{Q}}_l)$. Thus, $\gamma^*(f_!^{\chi}\bar{\mathbf{Q}}_l) = \pi_{\mathbf{w}}^*\mathcal{L}$, as required.

From the definitions we see that (a) is compatible with the Γ-equivariant structures.

When $\mathbf{w} = (w)$ is a one term sequence with $w \in W$ we can identify $Z^{\mathbf{w}}$ with

$$\mathcal{B}_w = \{B' \in \mathcal{B}; \mathrm{pos}(B', F(B')) = w\}$$

via $B' \leftrightarrow (B', F(B'))$. Note that $\mathcal{B}_w$ is stable under conjugation by Γ. For $\mathcal{L} \in \mathcal{S}(T)^{wF}$, the local system $\mathcal{L}_{(w)}$ on $Z^{\mathbf{w}}$ can be then identified with a local system $\mathcal{L}_w$ on $\mathcal{B}_w$. The subvarieties $\mathcal{B}_w$ of $\mathcal{B}$ and the local systems $\mathcal{L}_w$ were introduced in [DL].

2.6. In the remainder of this section we fix a sequence $\mathbf{s} = (s_1, s_2, \dots, s_r)$ in $\mathbf{I}$ and $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$. Let $\mathcal{I}_{\mathbf{s}}$ be as in 2.4. Let

$$\bar{Z}^{\mathbf{s}} = \{(B_i) \in \mathcal{B}^{[0,r]}; \mathrm{pos}(B_{i-1}, B_i) \in \{1, s_i\}(i \in [1, r]), B_r = F(B_0)\},$$

$$\begin{aligned}\mathcal{Z}^{\mathbf{s}} = \{&(B_i) \in \mathcal{B}^{[0,r]}; \mathrm{pos}(B_{i-1}, B_i) \in \{1, s_i\}(i \in \mathcal{I}_{\mathbf{s}})\\ &\mathrm{pos}(B_{i-1}, B_i) = s_i(i \in [1, r] - \mathcal{I}_{\mathbf{s}}), B_r = F(B_0)\}.\end{aligned}$$

The variety $\bar{Z}^{\mathbf{s}}$ was introduced in [DL] (in the case where $l([\mathbf{s}]) = r$). For $\mathcal{J} \subset \mathcal{I}_{\mathbf{s}}$ the variety $Z^{\mathbf{s}\mathcal{J}}$ (as in 2.5) can be also described as

$$\begin{aligned}Z^{\mathbf{s}\mathcal{J}} = \{&(B_i) \in \mathcal{B}^{[0,r]}; B_{i-1} = B_i(i \in \mathcal{J}), \mathrm{pos}(B_{i-1}, B_i) = s_i(i \in [1, r] - \mathcal{J}),\\ &B_r = F(B_0)\} \subset \mathcal{Z}^{\mathbf{s}}.\end{aligned}$$

Consider the commutative diagram

$$\begin{tikzcd}
\bar{Z} & \bar{Z}_0 \arrow[l, "\delta_0"'] & \bar{Z}_1 \arrow[l, "\delta_1"'] \arrow[r, "\delta_2"] & \bar{Z}_2 \\
Z \arrow[u, "\epsilon"] & Z_0 \arrow[l, "d_0"'] \arrow[u, "\epsilon_0"] & Z_1 \arrow[l, "d_1"'] \arrow[r, "d_2"] \arrow[u, "\epsilon_1"] & Z_2 \arrow[u, "\epsilon_2"] \\
Z^{\mathcal{J}} \arrow[u, "e"] & Z_0^{\mathcal{J}} \arrow[l, "d'_0"'] \arrow[u, "e_0"] \arrow[d, "f_0^{\mathcal{J}}"'] & Z_1^{\mathcal{J}} \arrow[l, "d'_1"'] \arrow[r, "d'_2"] \arrow[u, "e_1"] \arrow[d, "f_1^{\mathcal{J}}"'] & Z_2^{\mathcal{J}} \arrow[u, "e_2"] \arrow[d, "f_2^{\mathcal{J}}"'] \\
 & T & T \arrow[l, "="'] \arrow[r, "="] & T
\end{tikzcd}$$

where the following notation is used.

$\bar{Z} = \bar{Z}^{\mathbf{s}}, Z = \mathcal{Z}^{\mathbf{s}}, Z^{\mathcal{J}} = Z^{\mathbf{s}_{\mathcal{J}}}$.

$\bar{Z}_0$ is the set of all $(g_0U, g_1U, \dots, g_rU) \in (G/U)^{[0,r]}$ such that $g_{i-1}^{-1}g_i \in P_{s_i}$ for $i \in [1,r]$ and $g_r^{-1}F(g_0) \in U$.

Z_0 is the set of all $(g_0U, g_1U, \dots, g_rU) \in (G/U)^{[0,r]}$ such that $g_{i-1}^{-1}g_i \in P_{s_i}$ for $i \in \mathcal{I}_{\mathbf{s}}$, $g_{i-1}^{-1}g_i \in P_{s_i} - B$ for $i \in [1,r] - \mathcal{I}_{\mathbf{s}}$ and $g_r^{-1}F(g_0) \in U$.

$Z_0^{\mathcal{J}}$ is the set of all $(g_0U, g_1U, \dots, g_rU) \in (G/U)^{[0,r]}$ such that $g_{i-1}^{-1}g_i \in P_{s_i} - B$ for $i \in [1,r] - \mathcal{J}$, $g_{i-1}^{-1}g_i \in B$ for $i \in \mathcal{J}$ and $g_r^{-1}F(g_0) \in U$.

d_0, δ_0 are given by $(g_0U, g_1U, \dots, g_rU) \mapsto (g_0Bg_0^{-1}, g_1Bg_1^{-1}, \dots, g_rBg_r^{-1})$.

$f_0^{\mathcal{J}}$ is given by $(g_0U, g_1U, \dots, g_rU) \mapsto k(g_0^{-1}g_1)\dots k(g_{r-1}^{-1}g_r)[\mathbf{s}_{\mathcal{J}}]^{\bullet -1}$.

$\bar{Z}_1$ is the set of all $(y_0, y_1, \dots, y_r) \in G^{[0,r]}$ such that $y_i \in P_{s_i}(i \in [1,r])$, $y_0^{-1}F(y_0) \in y_1y_2\dots y_rU$.

Z_1 is the set of all $(y_0, y_1, \dots, y_r) \in G^{[0,r]}$ such that $y_i \in P_{s_i}(i \in \mathcal{I}_{\mathbf{s}})$, $y_i \in P_{s_i} - B(i \in [1,r] - \mathcal{I}_{\mathbf{s}})$, $y_0^{-1}F(y_0) \in y_1y_2\dots y_rU$.

$Z_1^{\mathcal{J}}$ is the set of all $(y_0, y_1, \dots, y_r) \in G^{[0,r]}$ such that $y_i \in P_{s_i} - B(i \in [1,r] - \mathcal{J})$, $y_i \in B(i \in \mathcal{J})$, $y_0^{-1}F(y_0) \in y_1y_2\dots y_rU$.

d_1, δ_1 are given by $(y_0, y_1, \dots, y_r) \mapsto (y_0U, y_0y_1U, \dots, y_0y_1\dots y_rU)$.

$f_1^{\mathcal{J}}$ is $(y_0, y_1, \dots, y_r) \mapsto k(y_1)k(y_2)\dots k(y_r)[\mathbf{s}_{\mathcal{J}}]^{\bullet -1}$.

$\bar{Z}_2 = Y, Z_2 = \mathcal{Y}, Z_2^{\mathcal{J}} = \mathcal{Y}^{\mathcal{J}}$. (See 2.4.)

d_2, δ_2 are given by $(y_0, y_1, \dots, y_r) \mapsto (y_1, \dots, y_r)$.

$f_2^{\mathcal{J}}$ is $(y_1, \dots, y_r) \mapsto k(y_1)k(y_2)\dots k(y_r)[\mathbf{s}_{\mathcal{J}}]^{\bullet -1}$.

The maps $e, e_i, \epsilon_i (i \in [0,2])$ are the obvious imbeddings. For $i \in [0,2]$ the map d'_i is the restriction of d_i. From the definitions we have:

(a) *In our commutative diagram, all squares that do not involve T are cartesian.*

2.7. Γ acts:

on Z and $\bar{Z}$ by $g : (B_0, B_1, \dots, B_r) \mapsto (gB_0g^{-1}, gB_1g^{-1}, \dots, gB_rg^{-1})$;

on Z_0 and $\bar{Z}_0$ by $g : (g_0U, g_1U, \dots, g_rU) \mapsto (gg_0U, gg_1U, \dots, gg_rU)$;

on Z_1 and $\bar{Z}_1$ by $g : (y_0, y_1, \dots, y_r) \mapsto (gy_0, y_1, y_2, \dots, y_r)$;

on Z_2, $\bar{Z}_2$ trivially.

The subsets $Z^{\mathcal{J}}, Z_i^{\mathcal{J}}$ of $Z, Z_i(i \in [0,2])$ are stable under the Γ-action. The maps $d_i, d'_i, \delta_i, f_i^{\mathcal{J}}$ are compatible with the Γ-actions.

$\underline{B}$ (see 2.4) acts:

on Z trivially;

on Z_0 by $(b_0, b_1, \dots, b_r) : (g_0U, g_1U, \dots, g_rU) \mapsto (g_0b_0^{-1}U, g_1b_1^{-1}U, \dots, g_rb_r^{-1}U)$;

on Z_1 by

$(b_0, b_1, \dots, b_r) : (y_0, y_1, \dots, y_r) \mapsto (y_0b_0^{-1}, b_0y_1b_1^{-1}, b_1y_2b_2^{-1}, \dots, b_{r-1}y_rb_r^{-1})$;

on Z_2 by $(b_0, b_1, \dots, b_r) : (y_1, \dots, y_r) \mapsto (b_0y_1b_1^{-1}, b_1y_2b_2^{-1}, \dots, b_{r-1}y_rb_r^{-1})$.

The maps d_i are compatible with the $\underline{B}$-actions.

2.8. Now

(a) d_0, δ_0, d'_0 are principal $\underline{B}/U^{[0,r]}$-bundles.

(b) d_1, δ_1, d'_1 are principal $U^{[0,r]}$-bundles. (The action of $U^{[0,r]}$ on $\bar{Z}_1, Z_1, Z_1^{\mathcal{J}}$ is by restriction of the $\underline{B}$-action.)

(c) Each of d_2, δ_2, d'_2 is a composition of a principal Γ-bundle with a principal U-bundle.

2.9. We show for $i \in [0,2]$ that

(a) *Z_i is smooth of pure dimension say $\mathfrak{d}_i$ and it is open dense in $\bar{Z}_i$.*

Let P_i be the property expressed by (a). It is obvious that P_2 holds. Using P_2, 2.6(a) and 2.8(c) we see that P_1 holds. Using P_1, 2.6(a) and 2.8(b) we see that P_0 holds. Thus (a) holds. Using P_0, 2.6(a) and 2.8(a) we see that

(b) *Z is smooth of pure dimension say $\mathfrak{d}$ and it is open dense in $\bar{Z}$.*

We show:

(c) $\mathfrak{d} = r$.

From the definitions we see that $\mathfrak{d}_2 = r(\dim B + 1)$. From the arguments above we see successively that $\mathfrak{d}_1 = r(\dim B + 1) + \dim U$, $\mathfrak{d}_0 = \mathfrak{d}_1 - (r+1)\dim U$ $\mathfrak{d} = \mathfrak{d}_0 - r\dim T$; (c) follows.

We show:

(d) *The natural Γ-action on the set of connected components of Z_i ($i \in [0,2]$) or of Z is transitive.*

For Z_2 this is clear since Z_2 is connected. This also implies the result for Z_1 (see 2.8(c)). Using 2.8(b),(a) we deduce that the result also holds for Z_0 and for Z.

2.10. Let $\mathcal{J} \subset \mathcal{I}_{\mathbf{s}}$. For $i \in [0,2]$ we set $\mathcal{L}_i^{\mathcal{J}} = (f_i^{\mathcal{J}})^*\mathcal{L}$, a local system of rank 1 on $Z_i^{\mathcal{J}}$. Since $\mathcal{L}$ has a natural Γ-equivariant structure (with Γ acting trivially on each stalk) and $f_i^{\mathcal{J}}$ is compatible with the Γ-actions we see that $\mathcal{L}_i^{\mathcal{J}}$ has a natural Γ-equivariant structure. From the definitions we have isomorphisms compatible with the Γ-equivariant structures as follows:

(a) $$d_1'^*\mathcal{L}_1^{\mathcal{J}} \cong \mathcal{L}_0{}^{\mathcal{J}}; d_2'^*\mathcal{L}_2^{\mathcal{J}} \cong \mathcal{L}_1^{\mathcal{J}}.$$

From the definitions we see that

(b) Γ *acts trivially on any stalk of* $\mathcal{L}_2^{\mathcal{J}}$.
Let $\mathcal{L}^{\mathcal{J}}$ be the local system on $Z^{\mathcal{J}} = Z^{\mathbf{s}_{\mathcal{J}}}$ denoted in 2.5 by $\mathcal{L}_{\mathbf{w}}$ where $\mathbf{w} = \mathbf{s}_{\mathcal{J}}$. This is well defined since for $\mathcal{J} \subset \mathcal{I}_{\mathbf{s}}$ we have $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}_{\mathcal{J}}]F}$. (We use that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$ and $(s_1 s_2 \dots s_j \dots s_2 s_1)^* \mathcal{L} \cong \mathcal{L}$ for any $j \in \mathcal{J}$.) As in 2.5, $\mathcal{L}^{\mathcal{J}}$ has a natural Γ-equivariant structure. From the definitions we have ${d'_0}^* \mathcal{L}^{\mathcal{J}} \cong \mathcal{L}_0^{\mathcal{J}}$ compatibly with the Γ-equivariant structures.

2.11. For $i \in [1,2]$ we define a local system $\bar{\mathcal{L}}_i$ on Z_i by $\bar{\mathcal{L}}_2 = \underline{\underline{\mathcal{L}}}$, $\bar{\mathcal{L}}_1 = d_2^* \bar{\mathcal{L}}_2$ where $\underline{\underline{\mathcal{L}}}$ is as in 2.4. From 2.4(b) and the results in 2.7 we see that $\bar{\mathcal{L}}_i$ is $\underline{B}$-equivariant. Since $d_0 d_1 : Z_1 \longrightarrow Z$ is a principal $\underline{B}$-bundle we see that there is a well defined local system $\bar{\mathcal{L}}$ on Z such that $(d_0 d_1)^* \bar{\mathcal{L}} = \bar{\mathcal{L}}_1$. Let $\bar{\mathcal{L}}_0 = d_0^* \bar{\mathcal{L}}$. Then $\bar{\mathcal{L}}_1 = d_1^* \bar{\mathcal{L}}_0$. We regard $\bar{\mathcal{L}}_2$ as a Γ-equivariant local system on Z_2 with Γ acting trivially on each stalk. Since each d_i is compatible with the Γ-actions we see that $\bar{\mathcal{L}}_i$ ($i \in [0,2]$) and $\bar{\mathcal{L}}$ have natural Γ-equivariant structures which are compatible with d_i^*.

2.12. We show:
(a) *For any* $\mathcal{J} \subset \mathcal{I}_{\mathbf{s}}$ *we have* $\bar{\mathcal{L}}|_{Z^{\mathcal{J}}} \cong \mathcal{L}^{\mathcal{J}}$ *compatibly with the* Γ*-equivariant structures.*
(b) *For any* $i \in [0,2]$ *and* $\mathcal{J} \subset \mathcal{I}_{\mathbf{s}}$ *we have* $\bar{\mathcal{L}}_i|_{Z_i^{\mathcal{J}}} \cong \mathcal{L}_i^{\mathcal{J}}$ *compatibly with the* Γ*-equivariant structures.*
Note that (b) holds for $i = 2$ by 2.4(a) (the compatibility with the Γ-equivariant structures is automatic since Γ acts trivially on each stalk of the local systems involved). From this we get (using 2.10, 2.11) that (b) holds for $i = 1$, then for $i = 0$, and then that (a) holds.

2.13. We show:
(a) *We have* $IC(\bar{Z}, \bar{\mathcal{L}})|_{\bar{Z}-Z} = 0$.
(b) *For* $i \in [0,2]$ *we have* $IC(\bar{Z}_i, \bar{\mathcal{L}}_i)|_{\bar{Z}_i - Z_i} = 0$.
Note that the IC complexes in (a),(b) are well defined by 2.9(a),(b). Now (b) holds for $i = 2$ by 2.4(b). From this we get (using 2.8, 2.11) that (b) holds for $i = 1$, then for $i = 0$, and then that (a) holds.

2.14. Assume that $r \geq 2, h \in [2,r] \cap \mathcal{I}_{\mathbf{s}}, s_{h-1} = s_h$. We set $\mathbf{s}' := (s_1, \dots, s_{h-1}, s_{h+1}, \dots, s_r)$. Then $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}']F}$ so that $Z' := \mathcal{Z}^{\mathbf{s}'}$ is defined as in 2.6. We have a commutative diagram

$$\begin{array}{ccccccc}
Z & \xleftarrow{d_0} & Z_0 & \xleftarrow{d_1} & Z_1 & \xrightarrow{d_2} & Z_2 \\
{\scriptstyle\beta}\downarrow & & {\scriptstyle\beta_0}\downarrow & & {\scriptstyle\beta_1}\downarrow & & {\scriptstyle\beta_2}\downarrow \\
Z' & \xleftarrow{d'_0} & Z'_0 & \xleftarrow{d'_1} & Z'_1 & \xrightarrow{d'_2} & Z'_2
\end{array}$$

where
the upper row is as in 2.6,
the lower row is defined analogously in terms of $\mathbf{s}', \mathcal{L}$ instead of $\mathbf{s}, \mathcal{L}$,
β is $(B_0, B_1, \dots, B_r) \mapsto (B_0, B_1, \dots, B_{h-2}, B_h, \dots, B_r)$,

β_0 is $(g_0U, g_1U, \dots, g_rU) \mapsto (g_0U, g_1U, \dots, g_{h-2}U, g_hU, g_rU)$,
β_1 is $(y_0, y_1, \dots, y_r) \mapsto (y_0, y_1, \dots, y_{h-2}, y_{h-1}y_h, y_{h+1}, \dots, y_r)$,
β_2 is $(y_1, \dots, y_r) \mapsto (y_1, \dots, y_{h-2}, y_{h-1}y_h, y_{h+1}, \dots, y_r)$.
Let $\bar{\mathcal{L}}, \bar{\mathcal{L}}_i (i \in [0,2])$ be the local systems on Z, Z_i defined in 2.11; let $\bar{\mathcal{L}}', \bar{\mathcal{L}}'_i$ be the analogous local systems on Z', Z'_i. We show:

(a) $$\bar{\mathcal{L}} \cong \beta^*(\bar{\mathcal{L}}').$$

It is enough to show that $\delta_1^*\delta_0^*\bar{\mathcal{L}} \cong \delta_1^*\delta_0^*\beta^*(\bar{\mathcal{L}}')$ or equivalently that $\bar{\mathcal{L}}_1 \cong \beta_1^*\bar{\mathcal{L}}'_1$. Hence it is enough to show that $\delta_2^*\bar{\mathcal{L}}_2 \cong \beta_1^*\delta_2'^*\bar{\mathcal{L}}'_2$ or equivalently that $\delta_2^*\bar{\mathcal{L}}_2 \cong \delta_2^*\beta_2{}^*\bar{\mathcal{L}}'_2$. It is enough to show that $\bar{\mathcal{L}}_2 \cong \beta_2{}^*\bar{\mathcal{L}}'_2$. From the definition of $\underline{\mathcal{L}}$ in 2.4 and with the notation in 2.4 we see that it is enough to show that $m^*{}'\underline{\mathcal{L}} \cong {}'\underline{\mathcal{L}} \boxtimes {}'\underline{\mathcal{L}}$ (local systems on P_{s_h}) where ${}'\mathcal{L} = s_{h-1}^* \dots s_2^*s_1^*\mathcal{L} \cong s_h^*s_{h-1}^* \dots s_2^*s_1^*\mathcal{L}$ and $m : P_{s_h} \times P_{s_h} \longrightarrow P_{s_h}$ is multiplication. It is enough to show that, in the setup of 2.1 we have $m'^*\underline{\mathcal{L}} \cong \underline{\mathcal{L}} \boxtimes \underline{\mathcal{L}}$ (local systems on G) where $m' : G \times G \longrightarrow G$ is multiplication. This follows from the definitions in 2.1 using the isomorphism $m_1^*\mathcal{L} \cong \mathcal{L} \boxtimes \mathcal{L}$ (local systems on T) where $m_1 : T \times T \longrightarrow T$ is multiplication.

2.15. Assume that $r \geq 2, h \in [2,r], h \notin \mathcal{I}_{\mathbf{s}}, s_{h-1} = s_h$. Let Z^1 be the open subset of Z defined by the condition $\mathrm{pos}(B_{h-2}, B_h) = s_i$. Define $\beta : Z^1 \longrightarrow \mathcal{B}^r$ by $(B_0, B_1, \dots, B_r) \mapsto (B_0, B_1, \dots, B_{h-2}, B_h, \dots, B_r)$. We show:

(a) $$\beta_!(\bar{\mathcal{L}}|_{Z^1}) = 0.$$

Let $p = (B_0, B_1, \dots, B_{h-2}, B_h, \dots, B_r) \in \mathcal{B}^r$ be such that $\Phi := \beta^{-1}(p) \neq \emptyset$. Then

$$\Phi = \{(B_0, B_1, \dots, B_{h-2}, \tilde{B}, B_h, \dots, B_r); \tilde{B} \in \mathcal{B}, \mathrm{pos}(B_{h-2}, \tilde{B}) = \mathrm{pos}(\tilde{B}, B_h) = s_h\}. \blacksquare$$

It is enough to show that $H_c^*(\Phi, \bar{\mathcal{L}}) = 0$. Let

$$\begin{aligned}\Phi' =&\{(B_0, B_1, \dots, B_{h-2}, \tilde{B}, B_h, \dots, B_r); \tilde{B} \in \mathcal{B}, \mathrm{pos}(B_{h-2}, \tilde{B}) = s_h,\\ &\mathrm{pos}(\tilde{B}, B_h) \in \{1, s_h\}\}.\end{aligned}$$

Then Φ' is an affine line which is a cross section in $\bar{Z}$ to the divisor Δ_h (see 2.4) and $\Phi' \cap \Delta_h$ is the point $p' = (B_0, B_1, \dots, B_{h-2}, B_h, B_h, \dots, B_r)$. Moreover, $\Phi = \Phi' - \{p'\}$. The vanishing $H_c^*(\Phi, \bar{\mathcal{L}}) = 0$ follows from 2.4(c).

Now let $Z^2 = Z - Z^1$, that is the closed subset of Z defined by the condition $B_{h-2} = B_h$. Let $\mathbf{s}' := (s_1, \dots, s_{h-2}, s_{h+1}, \dots, s_r)$. We have $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}']F}$ so that $Z' := \mathcal{Z}^{\mathbf{s}'}$ is defined as in 2.6. Define $\beta' : Z^2 \longrightarrow Z'$ by $(B_0, B_1, \dots, B_r) \mapsto (B_0, B_1, \dots, B_{h-2}, B_{h+1}, \dots, B_r)$, an affine line bundle. Let $\bar{\mathcal{L}}$ be the local systems on Z defined in 2.11; let $\bar{\mathcal{L}}'$ be the analogous local system on Z'. From the definitions we have:

(b) $$\bar{\mathcal{L}}|_{Z^2} = \beta'^*(\bar{\mathcal{L}}').$$

3. The class $\mathbb{S}'(\mathcal{P}_J)$ of simple objects in $\mathcal{M}_\Gamma(\mathcal{P}_J)$

3.1. Let $J \subset \mathbf{I}$. We view $\mathcal{P}_J$ as a variety with Γ-action (conjugation). Hence $\mathcal{M}_\Gamma(\mathcal{P}_J)$ is well defined.

3.2. Let $\mathcal{L} \in \mathcal{S}(T)$. If $\mathbf{w}$ is as in 2.5 and $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{w}]F}$ then the local system $\mathcal{L}_\mathbf{w}$ on $Z^\mathbf{w}$ has a natural Γ-equivariant structure (see 2.5). The map $\Pi^\mathbf{w} : Z^\mathbf{w} \longrightarrow \mathcal{P}_J$, $(B_0, B_1, \dots, B_r) \mapsto P_{B_0,J}$, commutes with the Γ-actions. Hence for any $j \in \mathbf{Z}$, ${}^pH^j(\Pi^\mathbf{w}_! \mathcal{L}_\mathbf{w})$ is an object of $\mathcal{M}_\Gamma(\mathcal{P}_J)$.

If $\mathbf{s}$ is as in 2.6 and $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$ then the local system $\bar{\mathcal{L}}$ on $\mathcal{Z}^\mathbf{s}$ has a natural Γ-equivariant structure (see 2.6, 2.11). Hence $\bar{\mathcal{L}}^\sharp = IC(\bar{Z}^\mathbf{s}, \bar{\mathcal{L}})$ (see 2.6, 2.11, 2.13) has a natural Γ-equivariant structure. Define $\Upsilon^\mathbf{s} : \mathcal{Z}^\mathbf{s} \longrightarrow \mathcal{P}_J$ and $\bar{\Upsilon}^\mathbf{s} : \bar{Z}^\mathbf{s} \longrightarrow \mathcal{P}_J$ by $(B_0, B_1, \dots, B_r) \mapsto P_{B_0,J}$. These maps commute with the Γ-actions. Hence for any $j \in \mathbf{Z}$,

(a) $$ {}^pH^j(\Upsilon^\mathbf{s}_! \bar{\mathcal{L}}) = {}^pH^j(\bar{\Upsilon}^\mathbf{s}_! \bar{\mathcal{L}}^\sharp) $$

is an object of $\mathcal{M}_\Gamma(\mathcal{P}_J)$. (The equality in (a) follows from by 2.13.)

In 3.3-3.7 we will show that the following conditions for a simple object K in $\mathcal{M}_\Gamma(\mathcal{P}_J)$ are equivalent:

(i) $K \dashv_\Gamma {}^pH^\cdot(\Pi^\mathbf{w}_! \mathcal{L}_\mathbf{w})$ for some one term sequence $\mathbf{w}$ in W such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{w}]F}$.

(ii) $K \dashv_\Gamma {}^pH^\cdot(\Pi^\mathbf{w}_! \mathcal{L}_\mathbf{w})$ for some sequence $\mathbf{w}$ in W such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{w}]F}$.

(iii) $K \dashv_\Gamma {}^pH^\cdot(\Pi^\mathbf{w}_! \mathcal{L}_\mathbf{w})$ for some sequence $\mathbf{w}$ in $\mathbf{I} \cup \{1\}$ such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{w}]F}$.

(iv) $K \dashv_\Gamma {}^pH^\cdot(\Pi^\mathbf{s}_! \mathcal{L}_\mathbf{s})$ for some sequence $\mathbf{s}$ in $\mathbf{I}$ such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$.

(v) $K \dashv_\Gamma {}^pH^\cdot(\Upsilon^\mathbf{s}_! \bar{\mathcal{L}})$ for some sequence $\mathbf{s}$ in $\mathbf{I}$ such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$.

(vi) $K \dashv_\Gamma {}^pH^\cdot(\bar{\Upsilon}^\mathbf{s}_! \bar{\mathcal{L}}^\sharp)$ for some sequence $\mathbf{s}$ in $\mathbf{I}$ such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$.

3.3. Let $\mathbf{w} = (w_1, \dots, w_r)$ be a sequence in W such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{w}]F}$. Assume that for some $i \in [1, r]$, $w'_i, w''_i \in W$ satisfy $w_i = w'_i w''_i$ and $l(w_i) = l(w'_i) + l(w''_i)$. Let $\mathbf{w}' = (w_1, \dots, w_{i-1}, w'_i, w''_i, w_{i+1}, \dots, w_r)$. Define an isomorphism $Z^{\mathbf{w}'} \xrightarrow{\sim} Z^\mathbf{w}$ by

$$ (B_0, B_1, \dots, B_{r+1}) \mapsto (B_0, B_1, \dots, B_{i-1}, B_{i+1}, \dots, B_{r+1}). $$

This isomorphism is compatible with the Γ-actions, with the maps $\Pi^{\mathbf{w}'}, \Pi^\mathbf{w}$ and with the local systems $\mathcal{L}_{\mathbf{w}'}, \mathcal{L}_\mathbf{w}$. Hence for any j we have

(a) $$ {}^pH^j(\Pi^\mathbf{w}_! \mathcal{L}_\mathbf{w}) = {}^pH^j(\Pi^{\mathbf{w}'}_! \mathcal{L}_{\mathbf{w}'}) $$

(as objects of $\mathcal{M}_\Gamma(\mathcal{P}_J)$). Applying (a) repeatedly we see that conditions 3.2(ii), 3.2(iii), 3.2(iv) are equivalent.

3.4. We prove the equivalence of conditions 3.2(iii), 3.2(v).

Let $\mathbf{s} = (s_1, \dots, s_r)$ be a sequence in $\mathbf{I}$ such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$. Define a sequence ${}^0\mathcal{Z} \supset {}^1\mathcal{Z} \supset \dots$ of closed subsets of $\mathcal{Z}^\mathbf{s}$ by ${}^i\mathcal{Z} = \cup_{\mathcal{J} \subset \mathcal{I}_\mathbf{s}; |\mathcal{J}| \ge i} Z^{\mathbf{s}_\mathcal{J}}$ (notation of

2.6). Let $f^i : {}^i\mathcal{Z} \longrightarrow \mathcal{Z}^{\mathbf{s}}$, $f'^i : {}^i\mathcal{Z} - {}^{i+1}\mathcal{Z} \longrightarrow \mathcal{Z}^{\mathbf{s}}$ be the inclusions. The natural distinguished triangle

$$(\Upsilon^{\mathbf{s}}_! f'^i_! f'^{i*}\bar{\mathcal{L}}, \Upsilon^{\mathbf{s}}_! f^i_! f^{i*}\bar{\mathcal{L}}, \Upsilon^{\mathbf{s}}_! f^{i+1}_!(f^{i+1*}\bar{\mathcal{L}}))$$

gives rise for any $i \ge 0$ to a long exact sequence in $\mathcal{M}_\Gamma(\mathcal{P}_J)$:

(a)
$$\dots \longrightarrow {}^pH^{j-1}(\Upsilon^{\mathbf{s}}_! f^{i+1}_! f^{i+1*}\bar{\mathcal{L}}) \longrightarrow \oplus_{\mathcal{J}\subset\mathcal{I}_{\mathbf{s}};|\mathcal{J}|=i}{}^pH^j(\Pi^{\mathbf{s}\mathcal{J}}_!\mathcal{L}_{\mathbf{s}_\mathcal{J}}) \longrightarrow {}^pH^j(\Upsilon^{\mathbf{s}}_! f^i_! f^{i*}\bar{\mathcal{L}})$$
$$\longrightarrow {}^pH^j(\Upsilon^{\mathbf{s}}_! f^{i+1}_! f^{i+1*}\bar{\mathcal{L}}) \longrightarrow \oplus_{\mathcal{J}\subset\mathcal{I}_{\mathbf{s}};|\mathcal{J}|=i}{}^pH^{j+1}(\Pi^{\mathbf{s}\mathcal{J}}_!\mathcal{L}_{\mathbf{s}_\mathcal{J}}) \longrightarrow \dots.$$

Here we have used the equality

$$\Upsilon^{\mathbf{s}}_! f'^i_! f'^{i*}\bar{\mathcal{L}} = \oplus_{\mathcal{J}\subset\mathcal{I}_{\mathbf{s}};|\mathcal{J}|=i}\Pi^{\mathbf{s}\mathcal{J}}_!\mathcal{L}_{\mathbf{s}_\mathcal{J}}.$$

(See 2.12(a).) Note that $\Upsilon^{\mathbf{s}}_! f^0_! f^{0*}\bar{\mathcal{L}} = \Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}}$ and $\Upsilon^{\mathbf{s}}_! f^i_! f^{i*}\bar{\mathcal{L}} = 0$ for i large.

If K does not satisfy 3.2(iii) then from (b) we see that for any $i \ge 0$ we have $K \dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_! f^i_! f^{i*}\bar{\mathcal{L}})$ if and only if $K \dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_! f^{i+1}_! f^{i+1*}\bar{\mathcal{L}})$. Since $K \not\dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_! f^i_! f^{i*}\bar{\mathcal{L}})$ with large i it follows that $K \not\dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_! f^0_! f^{0*}\bar{\mathcal{L}})$ that is $K \not\dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}})$. Thus, K does not satisfy 3.2(v).

Assume now that K satisfies 3.2(iii). We may assume that $K \dashv_\Gamma {}^pH^\cdot(\Pi^{\mathbf{s}}_!\mathcal{L}_{\mathbf{s}})$ where $\mathbf{s}$ as in 3.2(iii) has a minimum possible number of terms in $\mathbf{I}$. By the equivalence of 3.2(ii), 3.2(iii) we see that we may assume that all terms of $\mathbf{s}$ are in $\mathbf{I}$ and that $K \not\dashv_\Gamma {}^pH^\cdot(\Pi^{\mathbf{s}\mathcal{J}}_!\mathcal{L}_{\mathbf{s}_\mathcal{J}})$ for and any $\mathcal{J}$ such that $|\mathcal{J}| > 0$. Then from (b) we see that for any $i > 0$, $K \dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_! f^i_! f^{i*}\bar{\mathcal{L}})$ if and only if $K \dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_! f^{i+1}_! f^{i+1*}\bar{\mathcal{L}})$. Since $K \not\dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_! f^i_! f^{i*}\bar{\mathcal{L}})$ with large i it follows that $K \not\dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_! f^1_! f^{1*}\bar{\mathcal{L}})$. Using again (b) (with $i = 0$) we see that $K \dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_! f^0_! f^{0*}\bar{\mathcal{L}})$ hence $K \dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}})$. Thus, K satisfies 3.2(v). The equivalence of 3.2(iii), 3.2(v) is proved.

3.5. Let $\mathbf{s} = (s_1, s_2, \dots, s_r)$ be a sequence in $\mathbf{I}$ such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$. Assume that $r \ge 2, h \in [2,r]\cap\mathcal{I}_{\mathbf{s}}$, $s_{h-1} = s_h$. Let $\mathbf{s}', \bar{\mathcal{L}}', \beta$ be as in 2.14. We have $\Upsilon^{\mathbf{s}} = \Upsilon^{\mathbf{s}'}\beta$ and using 2.14(a) we have $\Upsilon^{\mathbf{s}}_!(\bar{\mathcal{L}}) = \Upsilon^{\mathbf{s}'}_!\beta_!\beta^*\bar{\mathcal{L}}'$. Since β is a projective line bundle we have an exact sequence in $\mathcal{M}_\Gamma(\mathcal{P}_J)$:

(a)
$$\dots \longrightarrow {}^pH^{j-2}(\Upsilon^{\mathbf{s}'}_!\bar{\mathcal{L}}')(-1) \longrightarrow {}^pH^j(\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}}) \longrightarrow {}^pH^j(\Upsilon^{\mathbf{s}'}_!\bar{\mathcal{L}}') \longrightarrow \dots$$

3.6. Let $\mathbf{s} = (s_1, s_2, \dots, s_r)$ be a sequence in $\mathbf{I}$ such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$. Assume that $r \ge 2, h \in [2,r], h \notin \mathcal{I}_{\mathbf{s}}$, $s_{h-1} = s_h$. Let $Z^1, Z^2, \beta, \beta', \mathbf{s}', \bar{\mathcal{L}}'$ be as in 2.15; let $f_1 : Z^1 \longrightarrow \mathcal{Z}^{\mathbf{s}}, f_2 : Z^2 \longrightarrow \mathcal{Z}^{\mathbf{s}}$ be the inclusions. We have a distinguished triangle

$$(\Upsilon^{\mathbf{s}}_! f_{1!}f_1^*\bar{\mathcal{L}}, \Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}}, \Upsilon^{\mathbf{s}}_! f_{2!}f_2^*\bar{\mathcal{L}}).$$

We have $\Upsilon^{\mathbf{s}}_! f_{1!} = e_!\beta_!$ where $e : \mathcal{B}^r \longrightarrow \mathcal{P}_J$ is $(B_0, B_1, \dots, B_{h-2}, B_h, \dots, B_r) \mapsto P_{B_0,J}$. Using 2.15(a) we have $\Upsilon^{\mathbf{s}}_! f_{1!}f_1^*\bar{\mathcal{L}} = e_!\beta_!(\bar{\mathcal{L}}|_{Z^1}) = 0$. Hence the distinguished triangle above yields $\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}} = \Upsilon^{\mathbf{s}}_! f_{2!} f_2^* \bar{\mathcal{L}}$. We have $\Upsilon^{\mathbf{s}} f_2 = \Upsilon^{\mathbf{s}'}\beta'$. Using 2.15(b) we have

$$\Upsilon^{\mathbf{s}}_! f_{2!} f_2^* \bar{\mathcal{L}} = \Upsilon^{\mathbf{s}'}_! \beta'_!(\bar{\mathcal{L}}|_{Z^2}) = \Upsilon^{\mathbf{s}'}_! \beta'_! \beta'^* \bar{\mathcal{L}}' = \Upsilon^{\mathbf{s}'}_! (\bar{\mathcal{L}}' \otimes \beta'_! \beta'^* \bar{\mathbf{Q}}_l).$$

We see that

(a) $$\Upsilon^{\mathbf{s}}_! \bar{\mathcal{L}} = \Upsilon^{\mathbf{s}'}_! \bar{\mathcal{L}}'[-2](-1).$$

Hence for any j we have

(b) $${}^pH^j(\Upsilon^{\mathbf{s}}_! \bar{\mathcal{L}}) = {}^pH^{j-2}(\Upsilon^{\mathbf{s}'}_! \bar{\mathcal{L}}')(-1)$$

in $\mathcal{M}_\Gamma(\mathcal{P}_J)$.

3.7. Assume that 3.2(iv) holds. We show that 3.2(i) holds.

We may assume that $K \dashv_\Gamma {}^pH^\cdot(\Pi^{\mathbf{s}}_! \mathcal{L}_{\mathbf{s}})$ for some sequence $\mathbf{s} = (s_1, s_2, \dots, s_r)$ in $\mathbf{I}$ such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$ and that r is minimum possible. From the proof in 3.4 we see that $K \dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_! \bar{\mathcal{L}})$ and that r is also minimal for this property.

Assume first that $l(s_1 s_2 \dots s_r) < r$. We can find $h \in [2, r]$ such that

$$l(s_h s_{h+1} \dots s_r) = r - h + 1, l(s_{h-1} s_h \dots s_r) < r - h + 2.$$

We can find $s'_h, s'_{h+1}, \dots, s'_r$ in $\mathbf{I}$ such that

$$s'_h s'_{h+1} \dots s'_r = s_h s_{h+1} \dots s_r = y$$

and $s'_h = s_{h-1}$. Let

$$\mathbf{u}' = (s_1, s_2, \dots, s_{h-1}, s'_h, s'_{h+1}, \dots, s'_r), \mathbf{u}'' = (s_1, s_2, \dots, s_{h-1}, y).$$

From 3.3(a) we see that $\Pi^{\mathbf{s}}_! \mathcal{L}_{\mathbf{s}} = \Pi^{\mathbf{u}'}_! \mathcal{L}_{\mathbf{u}'} = \Pi^{\mathbf{u}''}_! \mathcal{L}_{\mathbf{u}''}$. Hence we may assume that $s_h = s_{h-1}$.

If $h \in \mathcal{I}_{\mathbf{s}}$ then using 3.5(a) we see that $K \dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}'}_! \bar{\mathcal{L}}')$ (notation of 3.5); since $\mathbf{s}'$ has $r-1$ terms this is a contradiction. If $h \notin \mathcal{I}_{\mathbf{s}}$ then using 3.6 we see that $K \dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}'}_! \bar{\mathcal{L}}')$ (notation of 3.6); since $\mathbf{s}'$ has $r-2$ terms this is a contradiction.

We see that $l(s_1 s_2 \dots s_r) = r$. Using 3.3(a) repeatedly we see that ${}^pH^j(\Pi^{\mathbf{s}}_! \mathcal{L}_{\mathbf{s}}) = {}^pH^j(\Pi^{\mathbf{w}}_! \mathcal{L}_{\mathbf{w}})$ where $\mathbf{w} = (w_1)$, $w_1 = s_1 s_2 \dots s_r$. Thus, 3.2(i) holds.

Since the implication 3.2(i) $\implies$ 3.2(ii) is obvious and the equivalence of 3.2(v), 3.2(vi) follows from 3.2(a) we see that the equivalence of 3.2(i)-3.2(vi) is established.

For an object A of $\mathcal{M}_\Gamma(\mathcal{P}_J)$ we write $A \in \mathbb{S}'(\mathcal{P}_J)$ instead of "A satisfies the equivalent conditions of 3.2(i)-3.2(vi) for some $\mathcal{L} \in \mathcal{S}(T)$".

3.8. The results in this and the next subsection are not used in the subsequeny sections.

Let $\mathbf{s} = (s_1, s_2, \dots, s_r)$ be a sequence in $\mathbf{I}$ such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$, $s_1 \in J$. Let $\mathbf{s}' = (s'_1, s'_2, \dots, s'_r)$ where $s'_i = s_{i+1}$ for $i \in [1, r-1]$ and $s'_r = \mathbf{c}(s_1)$ where $\mathbf{c} : W \to W$ is as in 2.4. Let $\mathcal{L}' = s_1^* \mathcal{L}$. We have $\mathcal{L}' \in \mathcal{S}(T)^{[\mathbf{s}']F}$. Let $\bar{\mathcal{L}}$ be the local system on $\mathcal{Z}^{\mathbf{s}}$ defined in 2.11 and let $\bar{\mathcal{L}}'$ be the analogous local system on $\mathcal{Z}^{\mathbf{s}}$ defined in terms of $\mathcal{L}'$. We show:

(a) $$\Upsilon_!^{\mathbf{s}} \bar{\mathcal{L}} \cong \Upsilon_!^{\mathbf{s}'} \bar{\mathcal{L}}'.$$

Define $\mathcal{I}'_{\mathbf{s}'}$ in terms of $\mathbf{s}', \mathcal{L}'$ in the same way as $\mathcal{I}_{\mathbf{s}}$ was defined in 2.4 in terms of $\mathbf{s}, \mathcal{L}$. If $i \in [2, r]$ we have $i \in \mathcal{I}_{\mathbf{s}}$ if and only if $i - 1 \in \mathcal{I}'_{\mathbf{s}'}$. Moreover, we have $1 \in \mathcal{I}_{\mathbf{s}}$ if and only if $r \in \mathcal{I}_{\mathbf{s}'}$. It follows that $f : \mathcal{Z}^{\mathbf{s}} \to \mathcal{Z}^{\mathbf{s}'}$, $(B_0, B_1, \dots, B_r) \mapsto (B_1, B_2, \dots, B_r, F(B_1))$, is well defined. From the definitions we see that $f^* \bar{\mathcal{L}}' \cong \bar{\mathcal{L}}$. Hence $\Upsilon_!^{\mathbf{s}} \bar{\mathcal{L}} = \Upsilon_!^{\mathbf{s}} f^* \bar{\mathcal{L}}'$. It remains to show that $\Upsilon^{\mathbf{s}} f = \Upsilon^{\mathbf{s}'}$. The first (resp. second) map takes $(B_0, B_1, \dots, B_r)$ to $P_{B_1,J}$ (resp. $P_{B_0,J}$). It is enough to show that $P_{B_1,J} = P_{B_0,J}$. This follows from the fact that $\mathrm{pos}(B_0, B_1) \in J$.

3.9. Let $\mathbf{s} = (s_1, s_2, \dots, s_r)$ be a sequence in $\mathbf{I}$ such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$. Let $s \in \mathbf{I}$ be such that $s \notin W_{\mathcal{L}}$. Let $\mathbf{u} = (s, s_1, s_2, \dots, s_r, \mathbf{c}(s))$, $\mathcal{L}' = s^* \mathcal{L}$. Let $\mathbf{v} = (s, s, s_1, s_2, \dots, s_r)$. We have $\mathcal{L}' \in \mathcal{S}(T)^{[\mathbf{u}]F}$, $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{v}]F}$. Let $\bar{\mathcal{L}}$ be as in 2.11; let $\bar{\mathcal{L}}', \bar{\mathcal{L}}''$ be the analogous local systems on $\mathcal{Z}^{\mathbf{u}}$, $\mathcal{Z}^{\mathbf{v}}$ defined in terms of $\mathcal{L}', \mathcal{L}$. We show

(a) $$\Upsilon_!^{\mathbf{s}} \bar{\mathcal{L}}[-2](-1) = \Upsilon_!^{\mathbf{u}} \bar{\mathcal{L}}'.$$

From 3.8 we have $\Upsilon_!^{\mathbf{u}} \bar{\mathcal{L}}' = \Upsilon_!^{\mathbf{v}} \bar{\mathcal{L}}''$. From 3.6(a) we have $\Upsilon_!^{\mathbf{v}} \bar{\mathcal{L}}'' = \Upsilon_!^{\mathbf{s}} \bar{\mathcal{L}}[-2](-1)$ and (a) follows. We see that

(b) $${}^pH^j(\Upsilon_!^{\mathbf{u}} \bar{\mathcal{L}}') = {}^pH^{j-2}(\Upsilon_!^{\mathbf{s}} \bar{\mathcal{L}})(-1).$$

4. The class $\mathbb{S}(\mathcal{P}_J)$ of simple objects in $\mathcal{M}_\Gamma(\mathcal{P}_J)$

4.1. In this section we fix $J \subset \mathbf{I}$.

In 1977 the author generalized the partition $(\mathcal{B}_w)_{w \in W}$ of $\mathcal{B}$ (see 2.5) by defining a partition of $\mathcal{P}_J$ into finitely many pieces stable under conjugation by Γ, as follows. To any $P \in \mathcal{P}_J$ we associate a sequence $P^0 \supset P^1 \supset P^2 \supset \dots$ in $\mathcal{P}$ by

$$P^0 = P, \quad P^n = (P^{n-1})^{F(P^{n-1})} \text{ for } n \geq 1,$$

a sequence $J_0 \supset J_1 \supset J_2 \supset \dots$ of subsets of $\mathbf{I}$ by $P^n \in \mathcal{P}_{J_n}$ and a sequence $w_0, w_1, w_2, \dots$ in W by

$$w_n = \mathrm{pos}(P^n, F(P^n)).$$

We have

(a) $J_0 = J$,
(b) $J_n = J_{n-1} \cap w_{n-1}\mathbf{c}(J_{n-1})w_{n-1}^{-1}$ for $n \geq 1$.
(see 1.3(a)),
(c) $w_n \in {}^{J_n}W^{\mathbf{c}(J_n)}$ for $n \geq 0$.
Clearly, for $n \geq |\mathbf{I}|$ we have $P^n = P^{n+1} = \dots$ hence
(d) $w_n = w_{n+1} = \dots$ and $J_n = J_{n+1} = \dots$.
We set $P^\infty = P^n$ for $n \geq |\mathbf{I}|$, $w_\infty = w_n$ for $n \geq |\mathbf{I}|$, $J_\infty = J_n$ for $n \geq |\mathbf{I}|$.

For any $\mathbf{t} = (J_n, w_n)_{n \geq 0}$ where $J_0 \supset J_1 \supset J_2 \supset \dots$ are subsets of J satisfying (a) and $w_0, w_1, w_2, \dots$ are elements of W satisfying (b),(c) let $\mathcal{P}_J^{\mathbf{t}}$ be the set of all $P \in \mathcal{P}_J$ which give rise to $\mathbf{t}$ by the procedure above. Let $\mathcal{T}'(J, \mathbf{c})$ be the set of all sequences $\mathbf{t}$ as above such that $\mathcal{P}_J^{\mathbf{t}} \neq \emptyset$. From (d) we see that $\mathcal{T}'(J, \mathbf{c})$ is a finite set. From (a),(b) we see that for $(J_n, w_n)_{n \geq 0} \in \mathcal{T}'(J, \mathbf{c})$, the J_n are uniquely determined by the w_n. The locally closed subvarieties $\mathcal{P}_J^{\mathbf{t}}$, $\mathbf{t} \in \mathcal{T}'(J, \mathbf{c})$, form the desired partition of $\mathcal{P}_J$. (See [L5, I, 1.3, 1.4] for some examples in the classical groups.)

4.2. In this subsection we review some results in the R.Bédard's Ph.D. Thesis (M.I.T. 1983), see also [Be].

(a) *$\mathcal{T}'(J, \mathbf{c})$ is precisely the set of all $(J_n, w_n)_{n \geq 0}$ with $J_n \subset \mathbf{I}$, $w_n \in W$ such that 4.1(a),(b),(c) hold and $w_n \in W_{J_n} w_{n-1} W_{\mathbf{c}(J_{n-1})}$ for $n \geq 1$.*
(With notation in [L5, I, 2.2], we have $\mathcal{T}'(J, \mathbf{c}) = \mathcal{T}(\mathbf{c}(J), \mathbf{c}^{-1})$.)

(b) *The assignment $(J_n, w_n)_{n \geq 0} \mapsto w_\infty$ defines a bijection $\mathcal{T}'(J, \mathbf{c}) \xrightarrow{\sim} {}^J W$.*

(c) *Let $z \in {}^J W^{\mathbf{c}(J)}$, $J_1 = J \cap z\mathbf{c}(J)z^{-1}$. Let $V = \{P \in \mathcal{P}_J; \text{pos}(P, F(P)) = z$, $V' = \{Q \in \mathcal{P}_{J_1}; \text{pos}(Q, F(Q)) \in zW_{\mathbf{c}(J)}\}$. Then $f : V \to V'$, $P \mapsto P^1 := P^{F(P)}$ is an isomorphism.*
Define $V' \to \mathcal{P}_J$ by $Q \mapsto P$ where P is the unique parabolic in $\mathcal{P}_J$ such that $Q \subset P$. We have automatically $P \in V$ hence $Q \mapsto P$ is a map $f' : V' \to V$. Clearly $f'f = 1$. We show $ff' = 1$. It is enough to show that, if Q, P are as above, then $P^{F(P)} = Q$. We have $\text{pos}(Q, F(Q)) = zu$ where $u \in W_{\mathbf{c}(J)}$. We can find $B_0, B_1 \in \mathcal{B}$ such that $B_0 \subset Q$, $B_1 \subset F(Q)$, $\text{pos}(B_0, B_1) = zu$. Since $l(zu) = l(z) + l(u)$ we can find $B_2 \in \mathcal{B}$ such that $\text{pos}(B_0, B_2) = z, \text{pos}(B_2, B_1) = u$. Since $u \in W_{\mathbf{c}(J)}$ and $B_1 \subset F(P)$ we have $B_2 \subset F(P)$. Since $B_0 \subset P, B_2 \subset F(P)$, $\text{pos}(B_0, B_2) = z$, we have $B_0 \subset P^{F(P)}$. Since $Q, P^{F(P)}$ are in $\mathcal{P}_{J_1}$ and both contain B_0 we have $Q = P^{F(P)}$. This proves (c).

Let $\mathbf{t} = (J_n, w_n)_{n \geq 0} \in \mathcal{T}'(J, \mathbf{c})$. For $m \geq 0$ we set $\mathbf{t}_m = (J'_n, w'_n)_{n \geq 0}$ where $J'_n = J_{n+m}, w'_n = w_{n+m}$. We have $\mathbf{t}_m \in \mathcal{T}'(J_m, \mathbf{c})$. We set $\mathbf{t}_\infty = (J'_n, w'_n)_{n \geq 0}$ where $J'_n = J_\infty, w'_n = w_\infty$. We have $\mathbf{t}_\infty \in \mathcal{T}'(J_\infty, \mathbf{c})$. Clearly, $P \mapsto P^1$ is a map

$$\vartheta : \mathcal{P}_J^{\mathbf{t}} \to \mathcal{P}_{J_1}^{\mathbf{t}_1}.$$

(d) *The map $P \mapsto P^1$ is an isomorphism $\mathcal{P}_J^{\mathbf{t}} \xrightarrow{\sim} \mathcal{P}_{J_1}^{\mathbf{t}_1}$. The map $P \mapsto P^\infty$ is an isomorphism $\mathcal{P}_J^{\mathbf{t}} \xrightarrow{\sim} \mathcal{P}_{J_\infty}^{\mathbf{t}_\infty}$.*
The first assertion of (d) follows from (c). The second assertion follows using the first assertion repeatedly.

(e) *Let* $\mathbf{t} = (J_n, w_n)_{n\ge 0} \in \mathcal{T}'(J,\mathbf{c})$ *be such that* $J_n = J$ *and* $w_n = w$ *for all* $n \ge 0$ *where* $w \in W$. *We have* $\mathbf{c}(J) = w^{-1}Jw$, $w \in {}^JW^{\mathbf{c}(J)}$. *If* $P \in \mathcal{P}_J^{\mathbf{t}}$ *then* $P^n = P$ *for* $n \ge 0$ *and* $\mathrm{pos}(P, F(P)) = w$. *From* $P = P^{F(P)}$ *we see that* $P, F(P)$ *have a common Levi. We have* $\mathcal{P}_J^{\mathbf{t}} = \{P \in \mathcal{P}_J; \mathrm{pos}(P, F(P)) = w\}$.

(f) *Let* $(J_n, w_n)_{n\ge 0} \in \mathcal{T}'(J,\mathbf{c})$. *For* $n \ge 0$ *we have* $w_n = \min(W_J w_\infty W_{\mathbf{c}(J_n)})$.

4.3. In the setup of 4.2(e) we show:

(a)
$$\dot w^{-1} L_J \dot w = L_{\mathbf{c}(J)} = F(L_J).$$

From $\mathrm{pos}(P_J, \dot w P_{\mathbf{c}(J)} \dot w^{-1}) = w$ we see that $P_J, \dot w P_{\mathbf{c}(J)}\dot w^{-1}$ have a common Levi subgroup containing T which must be L_J and also $\dot w L_{\mathbf{c}(J)} \dot w^{-1}$.

Let
$$\tilde{\mathcal{P}}_J^{\mathbf{t}} = \{gU_{P_J} \in G/U_{P_J}; g^{-1}F(g) \in U_{P_J} \dot w U_{P_{\mathbf{c}(J)}}\}.$$

Define $F' : L_J \to L_J$ by $g \mapsto \dot w F(g) \dot w^{-1}$. This is the Frobenius map for an $\mathbf{F}_q$-rational structure on L_J. We set

$$L_J^{F'} = \{l \in L_J; F'(l) = l\}.$$

The finite group $L_J^{F'}$ acts freely on $\tilde{\mathcal{P}}_J^{\mathbf{t}}$ by $l : gU_{P_J} \mapsto gl^{-1}U_{P_J}$ and the map $f : \tilde{\mathcal{P}}_J^{\mathbf{t}} \to \mathcal{P}_J^{\mathbf{t}}$, $gU_{P_J} \mapsto gP_Jg^{-1}$ is constant on the orbits of this action. We show:

(b) f *is a principal* $L_J^{F'}$*-bundle.*

We only show this at the level of sets. If $P \in \mathcal{P}_J; \mathrm{pos}(P, F(P)) = w$, we have $P = gP_Jg^{-1}$ where $g \in G$ satisfies

$$\begin{aligned} w =& \mathrm{pos}(gP_Jg^{-1}, F(g)F(P_J)F(g^{-1})) = \mathrm{pos}(P_J, g^{-1}F(g)P_{\mathbf{c}(J)}F(g^{-1})g) \\ &= \mathrm{pos}(P_J, \dot w P_{\mathbf{c}(J)} \dot w^{-1}). \end{aligned}$$

Hence there exists $y \in P_J$ such that $g^{-1}F(g)P_{\mathbf{c}(J)}F(g^{-1})g = y\dot w P_{\mathbf{c}(J)} \dot w^{-1} y^{-1}$ hence $g^{-1}F(g) \in P_J \dot w P_{\mathbf{c}(J)}$ that is $g^{-1}F(g) \in l' U_{P_J} \dot w U_{P_{\mathbf{c}(J)}}$ for some $l' \in L_J$. (We use (a).) By Lang's theorem for F' we can find $l \in L_J$ such that $l^{-1}F'(l) = l'$. Then $gl^{-1}U_{P_J} \in \tilde{\mathcal{P}}_J^{\mathbf{t}}$. We see that f is surjective.

Assume that $gU_{P_J}, g'U_{P_J}$ in $\tilde{\mathcal{P}}_J^{\mathbf{t}}$ have the same image under f that is $gP_Jg^{-1} = g'P_Jg'^{-1}$. Then $g' = gp^{-1}$ where $p \in P_J$. We may assume that $g' = gl^{-1}$, $l \in L_J$. We have $g^{-1}F(g) \in P_J \dot w P_{\mathbf{c}(J)}$ and $(gl^{-1})^{-1}F(gl^{-1}) \in U_{P_J} \dot w U_{P_{\mathbf{c}(J)}}$ that is

$$g^{-1}F(g) = U_{P_J} l^{-1} \dot w F(l) U_{P_{\mathbf{c}(J)}} \text{ and } U_{P_J} l^{-1} F'(l) \dot w U_{P_{\mathbf{c}(J)}} = U_{P_J} \dot w U_{P_{\mathbf{c}(J)}}.$$

Using [L3, 3.2] we deduce $l^{-1}F'(l)\dot w = \dot w$ hence $l \in L_J^{F'}$.

Let

$${}'\tilde{\mathcal{P}}_J^{\mathbf{t}} = \{g(U_{P_J} \cap F^{-1}(\dot w^{-1} U_{P_J} \dot w)) \in G/(U_{P_J} \cap F^{-1}(\dot w^{-1} U_{P_J} \dot w)); g^{-1}F(g) \in U_{P_J} \dot w\}.$$

We show:

(c) *The map* $g(U_{P_J}\cap F^{-1}(\dot w^{-1}U_{P_J}\dot w))\mapsto gU_{P_J}$ *is an isomorphism* $\gamma:{}'\tilde{\mathcal{P}}_J^{\mathbf{t}}\xrightarrow{\sim}\tilde{\mathcal{P}}_J^{\mathbf{t}}$.

We only show this at the level of sets. Let $gU_{P_J}\in\tilde{\mathcal{P}}_J^{\mathbf{t}}$. We have $g^{-1}F(g)=u\dot wF(u')$ for some $u\in U_{P_J},u'\in U_{P_J}$. Then $(gu'^{-1})^{-1}F(gu'^{-1})=u'u\dot w$ so that $\gamma(gu'^{-1}(U_{P_J}\cap F^{-1}(\dot w^{-1}U_{P_J}\dot w)))=gU_{P_J}$. We see that γ is surjective. The injectivity is immediate.

Next we show:

(d) $\mathcal{P}_J^{\mathbf{t}}$ *is a smooth variety of pure dimension equal to* $\dim U_{P_J}-\dim(U_{P_J}\cap F^{-1}(\dot w^{-1}U_{P_J}\dot w))$ *and its connected components are permuted transitively by the* Γ*-action on* $\mathcal{P}_J^{\mathbf{t}}$.

By (b),(c) it is enough to show that ${}'\tilde{\mathcal{P}}_J^{\mathbf{t}}$ is smooth, of pure dimension equal to $\dim U_{P_J}-\dim(U_{P_J}\cap F^{-1}(\dot w^{-1}U_{P_J}\dot w))$ and its connected components are permuted transitively by the Γ-action

$g_0:g(U_{P_J}\cap F^{-1}(\dot w^{-1}U_{P_J}\dot w))\mapsto g_0g(U_{P_J}\cap F^{-1}(\dot w^{-1}U_{P_J}\dot w))$

on ${}'\tilde{\mathcal{P}}_J^{\mathbf{t}}$. This follows from the fact that $\{g\in G;g^{-1}F(g)\in U_{P_J}\dot w\}$ is smooth of dimension $\dim U_{P_J}$ and $U_{P_J}\dot w$ is connected.

4.4. We now consider a general $\mathbf{t}=(J_n,w_n)_{n\ge0}\in\mathcal{T}'(J,\mathbf{c})$. Let $\mathcal{P}_J^{\mathbf{t}}\xrightarrow{\sim}\mathcal{P}_{J_\infty}^{\mathbf{t}_\infty}$ be the isomorphism in 4.2(d). By 4.3 for $\mathbf{t}_\infty$ instead of $\mathbf{t}$, $\tilde{\mathcal{P}}_{J_\infty}^{\mathbf{t}_\infty}$ is defined. Let

$$\tilde{\mathcal{P}}_J^{\mathbf{t}}:=\tilde{\mathcal{P}}_{J_\infty}^{\mathbf{t}_\infty}=\{gU_{P_{J_\infty}}\in G/U_{P_{J_\infty}};g^{-1}F(g)\in U_{P_{J_\infty}}\dot wU_{P_{\mathbf{c}(J_\infty)}}\}.$$

Define $F':L_{J_\infty}\to L_{J_\infty}$ by $l\mapsto\dot w_\infty F(l)\dot w_\infty^{-1}$. (We have $\dot w_\infty^{-1}L_{J_\infty}\dot w_\infty=L_{\mathbf{c}(J_\infty)}=F(L_{J_\infty})$, see 4.3(a).) Let $\Lambda=L_{J_\infty}^{F'}$. The finite group $\Gamma\times\Lambda$ acts on $\tilde{\mathcal{P}}_J^{\mathbf{t}}$ by

$$(g_0,l):gU_{P_{J_\infty}}\mapsto g_0gl^{-1}U_{P_{J_\infty}}$$

and on $\mathcal{P}_J^{\mathbf{t}}$ by $(g_0,l):P\mapsto g_0Pg_0^{-1}$. From 4.3(b) we see that

(a) *The map* $f:\tilde{\mathcal{P}}_J^{\mathbf{t}}\to\mathcal{P}_J^{\mathbf{t}}$, $gU_{P_{J_\infty}}\mapsto gP_Jg^{-1}$ *(which is compatible with the* $\Gamma\times\Lambda$*-actions) is a principal* Λ*-bundle.*

From 4.3(d) we see that:

(b) $\mathcal{P}_J^{\mathbf{t}}$ *is a smooth variety of pure dimension equal to* $\dim U_{P_{J_\infty}}-\dim(U_{P_{J_\infty}}\cap F^{-1}(\dot w_\infty^{-1}U_{P_{J_\infty}}\dot w_\infty))$ *and its connected components are permuted transitively by the* Γ*-action on* $\mathcal{P}_J^{\mathbf{t}}$.

Let M be a finite dimensional Λ-irreducible module over $\bar{\mathbf{Q}}_l$. We view M as a $\Gamma\times\Lambda$-module with Γ acting trivially and we form the $\Gamma\times\Lambda$-equivariant local system $M_{\tilde{\mathcal{P}}_J^{\mathbf{t}}}$ on $\tilde{\mathcal{P}}_J^{\mathbf{t}}$ as in 1.6. Using (a) we see that there is a well defined $\Gamma\times\Lambda$-equivariant local system $\underline{M}$ on $\mathcal{P}_J^{\mathbf{t}}$ with trivial action of Λ such that $f^*\underline{M}=M_{\tilde{\mathcal{P}}_J^{\mathbf{t}}}$ as $\Gamma\times\Lambda$-equivariant local systems. We will regard $\underline{M}$ as a Γ-equivariant local system.

Let $d=\dim\mathcal{P}_J^{\mathbf{t}}$ (see (b)). We show:

(c) $\underline{M}[d]$ *(an object of* $\mathcal{M}_\Gamma(\mathcal{P}_J^{\mathbf{t}})$ *by (b)) is simple.*
Let $r : \tilde{C} \to C$ be a finite principal covering with finite group H. Assume that C is connected. There is an obvious functor $E \mapsto E'$ from H-modules of finite dimension over $\bar{\mathbf{Q}}_l$ to local systems on C which are direct summands of $r_!\bar{\mathbf{Q}}_l$. If E is irreducible then E' is irreducible as a local system.

We apply this statement in the case where C is a connected component of $\mathcal{P}_J^{\mathbf{t}}$, $\tilde{C} = f^{-1}(C)$ (f as in (a)), r is the restriction of f, $H = \Lambda$ and $E = M$. Note that $E' = \underline{M}|_C$. We see that the local system $\underline{M}|_C$ is irreducible. It remains to use the transitivity statement in (b).

For an object A of $\mathcal{M}_\Gamma(\mathcal{P}_J^{\mathbf{t}})$ we write $A \in \mathbb{S}(\mathcal{P}_J^{\mathbf{t}})$ instead of "A is isomorphic to $\underline{M}[d]$ for some M as above."

From (c) we see that:

(d) $\underline{M}^\sharp[d]$ *is a simple object of* $\mathcal{M}_\Gamma(\mathcal{P}_J)$.

For an object A of $\mathcal{M}_\Gamma(\mathcal{P}_J)$ we write $A \in \mathbb{S}(\mathcal{P}_J)$ instead of "A is isomorphic to $\underline{M}^\sharp[d] \in \mathcal{M}_\Gamma(\mathcal{P}_J)$ for some $\mathbf{t} \in \mathcal{T}'(J, \mathbf{c})$ and some M as above."

4.5. For $w \in W$ we identify $\mathcal{B}_w$ with $Z^{(w)}$ as in 2.5. We show:

(a) *Let $a_1, a_2, b \in W$. Let $\mathcal{L} \in \mathcal{S}(T)^{a_1 a_2 F}$. Let V be a locally closed Γ-stable subvariety of $\mathcal{B}_b$. Let $X_{a_1,a_2} = \{(B_0, B_1, B_2) \in Z^{(a_1,a_2)}; B_1 \in V\}$, (see 2.5). Define $\kappa : X_{a_1,a_2} \to V$ by $(B_0, B_1, B_2) \mapsto B_1$. Let $\mathcal{E} = \mathcal{L}_{(a_1,a_2)}|_{X_{a_1,a_2}}$. Let V' be an algebraic variety with a Γ-action and let $m : V \to V'$ be a morphism compatible with the Γ-actions. Let A' be a simple object of $\mathcal{M}_\Gamma(V')$ such that $A' \dashv_\Gamma {}^pH^\cdot((m\kappa)_!\mathcal{E})$. Then there exists $e \in W$ such that $(bF)^*(e^*\mathcal{L}) \cong e^*\mathcal{L}$ and $A' \dashv_\Gamma {}^pH^\cdot(m_!(e^*\mathcal{L})_b|_V)$.*

We argue by induction on $l(a_1)$. If $l(a_1) = 0$ then $a_2 = b$, κ is an isomorphism and the result is obvious (with $e = 1$). Assume now that $l(a_1) > 0$. We can find $s \in \mathbf{I}$ such that $l(a_1) > l(sa_1)$. Let $\mathcal{E}_1 = (s^*\mathcal{L})_{(sa_1,a_2\mathbf{c}(s))}|_{X_{sa_1,a_2\mathbf{c}(s)}}$.

Assume first that $l(a_2\mathbf{c}(s)) = l(a_2) + 1$. We have an isomorphism $\iota : X_{a_1,a_2} \to X_{sa_1,a_2\mathbf{c}(s)}$, $(B_0, B_1, B_2) \mapsto (B'_0, B_1, F(B'_0))$ where $B'_0 \in \mathcal{B}$ is defined by

(b) $$\mathrm{pos}(B_0, B'_0) = s, \mathrm{pos}(B'_0, B_1) = sa_1.$$

Define $\kappa' : X_{sa_1,a_2\mathbf{c}(s)} \to V$ by $(B_0, B_1, B_2) \mapsto B_1$. We have $\kappa = \kappa'\iota$, $\iota^*\mathcal{E}_1 = \mathcal{E}$ hence $\iota_!\mathcal{E} = \mathcal{E}_1$. Thus $(m\kappa)_!\mathcal{E} = (m\kappa')_!\mathcal{E}_1$ and $A' \dashv_\Gamma {}^pH^\cdot((m\kappa')_!\mathcal{E}_1)$. By the induction hypothesis there exists $e' \in W$ such that $(bF)^*(e'^*s^*\mathcal{L}) \cong e'^*s^*\mathcal{L}$ and $A' \dashv_\Gamma {}^pH^\cdot(m_!(e'^*s^*\mathcal{L})_b|_V)$. The result follows with $e = se'$.

Assume next that $l(a_2\mathbf{c}(s)) = l(a_2) - 1$. We have a partition $X_{a_1,a_2} = X' \cup X''$ where X' (resp. X'') is the open (resp. closed) subset of X_{a_1,a_2} defined by $\mathrm{pos}(B_1, F(B'_0)) = a_2$ (resp. $\mathrm{pos}(B_1, F(B'_0)) = a_2\mathbf{c}(s)$). Let $j' = \kappa|_{X'}$, $j'' = \kappa|_{X''}$. By general principles we have either

(c) $A' \dashv_\Gamma {}^pH^\cdot((mj')_!(\mathcal{E}|_{X'}))$ or

(d) $A' \dashv_\Gamma {}^pH^\cdot((mj'')_!(\mathcal{E}|_{X''}))$.

Assume that (d) holds. We have $j'' = \kappa''\iota''$ where $\kappa'' : X_{sa_1,a_2\mathbf{c}(s)} \to V$ is given by $(B_0, B_1, B_2) \mapsto B_1$ and $\iota'' : X'' \to X_{sa_1,a_2\mathbf{c}(s)}$ is $(B_0, B_1, B_2) \mapsto (B'_0, B_1, F(B'_0))$

with B'_0 as in (b). We have $\mathcal{E}|_{X''} = \iota''^*\mathcal{E}_1$. Now ι'' is an affine line bundle hence $\iota''_!(\mathcal{E}|_{X''}) = \mathcal{E}_1[-2](-1)$. Hence $A' \dashv_\Gamma {}^pH^\cdot((\mapsto \kappa'')_!\mathcal{E}_1)$. By the induction hypothesis there exists $e' \in W$ such that $(bF)^*(e'^*s^*\mathcal{L}) \cong e'^*s^*\mathcal{L}$ and $A' \dashv_\Gamma {}^pH^\cdot(m_!(e'^*s^*\mathcal{L})_b|_V)$. The result follows with $e = se'$.

Assume now that (c) holds. We have $j' = \kappa'\iota'$ where $\kappa' : X_{sa_1,a_2} \longrightarrow V$ is $(B_0, B_1, B_2) \mapsto B_1$ and $\iota' : X' \longrightarrow X_{sa_1,a_2}$ is $(B_0, B_1, B_2) \mapsto (B'_0, B_1, F(B'_0))$ with B'_0 as in (b). Note that ι' makes X' into the complement of a section of an affine line bundle over X_{sa_1,a_2}. If $s \notin W_\mathcal{L}$ then by an argument as in the proof of 2.15 we see that $\iota'_!(\mathcal{E}|_{X'}) = 0$ contradicting (c). Thus we may assume that $s \in W_\mathcal{L}$. Then $\mathcal{E}_2 = \mathcal{L}_{(sa_1,a_2)}|_{X_{sa_1,a_2}}$ is defined and $\mathcal{E}|_{X'} = \iota'^*\mathcal{E}_2$. Hence we have a distinguished triangle $(\iota'_!\mathcal{E}_{X'}, \mathcal{E}_2, \mathcal{E}_2[-2](-1))$ hence a distinguished triangle $(m_!j'_!(\mathcal{E}|_{X'}, m_!\kappa'_!\mathcal{E}_2, m_!\kappa'_!\mathcal{E}_2[-2](-1))$. It follows that $A' \dashv_\Gamma {}^pH^\cdot(m_!\kappa'_!\mathcal{E}_2)$. By the induction hypothesis there exists $e \in W$ such that $(bF)^*(e^*\mathcal{L}) \cong e^*\mathcal{L}$ and $A' \dashv_\Gamma {}^pH^\cdot(m_!(e^*\mathcal{L})_b|_V)$. This completes the proof of (a).

4.6. Let $\mathbf{t} = (J_n, w_n)_{n\ge 0} \in \mathcal{T}'(J, \mathbf{c})$. Let $\mathcal{B}_\mathbf{t} = \{B' \in \mathcal{B}; P_{B',J} \in \mathcal{P}^\mathbf{t}_J\}$. For $a \in W$ let $\mathcal{B}_{\mathbf{t},a} = \mathcal{B}_\mathbf{t} \cap \mathcal{B}_a$. Define $\xi_{\mathbf{t},a} : \mathcal{B}_{\mathbf{t},a} \longrightarrow \mathcal{P}^\mathbf{t}_J$ by $B' \mapsto P_{B',J}$. We show:

(a) *Let $\mathcal{L} \in \mathcal{S}(T)^{aF}$. Let A be a simple object of $\mathcal{M}_\Gamma(\mathcal{P}^\mathbf{t}_J)$ such that $A \dashv_\Gamma {}^pH^\cdot(\xi_{\mathbf{t},a!}(\mathcal{L}_a|_{\mathcal{B}_{\mathbf{t},a}}))$. Then there exist $b, e \in W$ such that $b^*e^*\mathcal{L} \cong e^*\mathcal{L}$ and $A \dashv_\Gamma {}^pH^\cdot(\vartheta^*\xi_{\mathbf{t}_1,b!}((e^*\mathcal{L})_b|_{\mathcal{B}_{\mathbf{t}_1,b}}))$.*

Since $\xi_{\mathbf{t},a!}(\mathcal{L}_a|_{\mathcal{B}_{\mathbf{t},a}}) \ne 0$ we have $\mathcal{B}_{\mathbf{t},a} \ne \emptyset$. Thus there exists $B' \in \mathcal{B}$ such that $\mathrm{pos}(B', F(B')) = a$, $P_{B',J} \in \mathcal{P}^\mathbf{t}_J$. We have $\mathrm{pos}(P_{B',J}, F(P_{B',J})) = w_0$. Since $B' \subset P_{B',J}$, $F(B') \subset F(P_{B',J})$, it follows that $a \in W_J w_0 W_{\mathbf{c}(J)}$ and $w_0 = \min(W_J a W_{\mathbf{c}(J)})$.

Define $\phi : \mathcal{B}_{\mathbf{t},a} \longrightarrow \mathcal{B}_{\mathbf{t}_1}$ by $\phi(B') = (P_{B',J})^{F(B')}$. (For $B' \in \mathcal{B}_\mathbf{t}$ we have $(P_{B',J})^{F(B')} \in \mathcal{B}_{\mathbf{t}_1}$ since $(P_{B',J})^{F(B')} \subset (P_{B',J})^{F(P_{B',J})} \in \mathcal{P}^{\mathbf{t}_1}_{J_1}$.) We have a partition $\mathcal{B}_{\mathbf{t}_1} = \sqcup_{b\in W}\mathcal{B}_{\mathbf{t}_1,b}$. Setting $\mathcal{B}_{\mathbf{t},a,b} = \phi^{-1}(\mathcal{B}_{\mathbf{t}_1,b})$ we get a partition $\mathcal{B}_{\mathbf{t},a} = \sqcup_{b\in W}\mathcal{B}_{\mathbf{t},a,b}$. Let $\xi_{\mathbf{t},a,b} : \mathcal{B}_{\mathbf{t},a,b} \longrightarrow \mathcal{P}^\mathbf{t}_J$ be the restriction of $\xi_{\mathbf{t},a}$. By general principles we have $A \dashv_\Gamma {}^pH^\cdot(\xi_{\mathbf{t},a,b!}(\mathcal{L}_a|_{\mathcal{B}_{\mathbf{t},a,b}}))$ for some $b \in W$. Let $\phi_b : \mathcal{B}_{\mathbf{t},a,b} \longrightarrow \mathcal{B}_{\mathbf{t}_1,b}$ be the restriction of ϕ. We have $\vartheta\xi_{\mathbf{t},a,b} = \xi_{\mathbf{t}_1,b}\phi_b$ (both compositions carry B' to $(P_{B',J})^{F(P_{B',J})}$.) Hence $\xi_{\mathbf{t},a,b} = \vartheta^{-1}\xi_{\mathbf{t}_1,b}\phi_b$. Thus, $A \dashv_\Gamma {}^pH^\cdot((\vartheta^{-1})_!\xi_{\mathbf{t}_1,b!}\phi_{b!}(\mathcal{L}_a|_{\mathcal{B}_{\mathbf{t},a,b}}))$ and

(b) $$\vartheta_!A \dashv_\Gamma {}^pH^\cdot(\xi_{\mathbf{t}_1,b!}\phi_{b!}(\mathcal{L}_a|_{\mathcal{B}_{\mathbf{t},a,b}})).$$

We can write uniquely $a = a_1a_2$ where $a_1 \in W_J, a_2 \in {}^JW$. We show that for any $\tilde{B} \in \mathcal{B}_{\mathbf{t}_1,b}$ we have

(c) $$\phi_b^{-1}(\tilde{B}) = \{B' \in \mathcal{B}; \mathrm{pos}(B', \tilde{B}) = a_1, \mathrm{pos}(\tilde{B}, F(B')) = a_2\}.$$

Assume first that $B' \in \phi_b^{-1}(\tilde{B})$. We know that $\mathrm{pos}(B', F(B')) = a$. We have $\mathrm{pos}(B', (P_{B',J})^{F(B')}) \in W_J$ since $B', (P_{B',J})^{F(B')}$ are two Borel subgroups of

$P_{B',J}$. From the definitions we have $\mathrm{pos}((P_{B',J})^{F(B')}, F(B')) \in {}^JW$. We have automatically $\mathrm{pos}(B', (P_{B',J})^{F(B')}) = a_1$, $\mathrm{pos}((P_{B',J})^{F(B')}, F(B')) = a_2$ that is $\mathrm{pos}(B', \tilde{B}) = a_1$, $\mathrm{pos}(\tilde{B}, F(B')) = a_2$.

Conversely, assume that B' belongs to the right hand side of (c). We have $l(a_1a_2) = l(a_1) + l(a_2)$ hence $\mathrm{pos}(B', F(B')) = a_1a_2 = a$. Since the properties $\mathrm{pos}(B', \tilde{B}) \in W_J$, $\mathrm{pos}(\tilde{B}, F(B')) \in {}^JW$ characterize $\tilde{B}$ and $(P_{B',J})^{F(B')}$ has the same properties, it follows that $P^{F(B')} = \tilde{B}$ where $P = P_{B',J}$. Since $B' \subset P, F(B') \subset F(P)$ we have $\mathrm{pos}(P, F(P)) = \min(W_J\mathrm{pos}(B', F(B'))W_{\mathbf{c}(J)}) = \min(W_JaW_{\mathbf{c}(J)}) = w_0$. It follows that $P^{F(P)} \in \mathcal{P}_{J\cap w_0\mathbf{c}(J)w_0^{-1}} = J_1$. Clearly, $\tilde{B} = P^{F(B')} \subset P^{F(P)}$. Hence $P^{F(P)} = P_{\tilde{B},J_1} \in \mathcal{P}^{\mathbf{t}_1}_{J_1}$. It follows that $P \in \mathcal{P}^{\mathbf{t}}_J$. Thus, $B' \in \mathcal{B}_{\mathbf{t},a}$ and $\phi(B') = \tilde{B}$. Since $\tilde{B} \in \mathcal{B}_{\mathbf{t}_1,b}$ we see that $B' \in \phi_b^{-1}(\tilde{B})$. This proves (c).

From (c) we see that $(B_0, B_1, B_2) \mapsto B_0$ is an isomorphism $X_{a_1,a_2} \to \mathcal{B}_{\mathbf{t},a,b}$ where X_{a_1,a_2} is defined as in 4.5(a) in terms of $V = \mathcal{B}_{\mathbf{t}_1,b}$. Under this isomorphism, ϕ_b corresponds to $X_{a_1,a_2} \to \mathcal{B}_{\mathbf{t}_1,b}$, $(B_0, B_1, B_2) \mapsto B_1$. Applying 4.5(a) with $V' = \mathcal{P}^{\mathbf{t}_1}_{J_1}$, $m = \xi_{\mathbf{t}_1,b}$, $A' = \vartheta_!A$ we see that there exists $e \in W$ such that $(bF)^*(e^*\mathcal{L}) \cong e^*\mathcal{L}$ and $\vartheta_!A \dashv_\Gamma {}^pH^\cdot(\xi_{\mathbf{t}_1,b!}((e^*\mathcal{L})_b|_{\mathcal{B}_{\mathbf{t}_1,b}}))$. (The assumption of 4.5(a) is verified by (b).) Since ϑ is an isomorphism, it follows that $A \dashv_\Gamma {}^pH^\cdot(\vartheta^*\xi_{\mathbf{t}_1,b!}((e^*\mathcal{L})_b|_{\mathcal{B}_{\mathbf{t}_1,b}}))$. This proves (a).

4.7. Let $\mathbf{t} = (J_n, w_n)_{n\geq 0} \in \mathcal{T}'(J, \mathbf{c})$. Let $d = \dim \mathcal{P}^{\mathbf{t}}_J$. Let $a \in W$. We show:

(a) *Let $\mathcal{L} \in \mathcal{S}(T)^{aF}$. Let A be a simple object of $\mathcal{M}_\Gamma(\mathcal{P}^{\mathbf{t}}_J)$ such that $A \dashv_\Gamma {}^pH^\cdot(\xi_{\mathbf{t},a!}(\mathcal{L}_a|_{\mathcal{B}_{\mathbf{t},a}}))$. Then $A \in \mathbb{S}(\mathcal{P}^{\mathbf{t}}_J)$.*

More generally we show that (a) holds when $J, \mathbf{t}$ are replaced by $J_n, \mathbf{t}_n$, $n \geq 0$. First we show:

(b) if the result holds for $n = 1$ then it holds for $n = 0$.

Let A be as in (a). By 4.6(a) there exist $b, e \in W$ such that $(bF)^*(e^*\mathcal{L}) \cong e^*\mathcal{L}$ and $A \dashv_\Gamma {}^pH^\cdot(\vartheta^*\xi_{\mathbf{t}_1,b!}((e^*\mathcal{L})_b|_{\mathcal{B}_{\mathbf{t}_1,b}}))$. Since ϑ is an isomorphism, there exists a simple object A' of $\mathcal{M}_\Gamma(\mathcal{P}^{\mathbf{t}_1}_{J_1})$ such that $A = \vartheta^*A'$. From our assumption we have $A' \dashv_\Gamma {}^pH^\cdot(\xi_{\mathbf{t}_1,b!}((e^*\mathcal{L})_b|_{\mathcal{B}_{\mathbf{t}_1,b}}))$. Since (b) holds for $n = 1$ we have $A' \cong \underline{M}[d]$ for some irreducible $L^{F'}_{J_\infty}$-module M. Hence A is of the same form. Thus (b) holds.

Similarly, if the result holds for some $n \geq 1$ then it holds for $n - 1$. In this way we see that it suffices to prove the result for n large. Thus in the remainder of this proof we assume, as we may, that $J_0 = J_1 = \cdots = J$ and $w_0 = w_1 = \cdots = w$. We can write uniquely $a = a_1a_2$ where $a_1 \in W_J, a_2 \in {}^JW$. Since $\xi_{\mathbf{t},a!}(\mathcal{L}_a|_{\mathcal{B}_{\mathbf{t},a}}) \neq 0$, we have $\mathcal{B}_{\mathbf{t},a} \neq \emptyset$. From this we deduce as in the proof of 4.6(a) that $a \in W_JwW_{\mathbf{c}(J)}$. Since $w\mathbf{c}(J)w^{-1} = J$ we must have $wW_{\mathbf{c}(J)} = W_Jw$ so that $a \in W_Jw$. Since $w \in {}^JW$ (see 4.2(b)) it follows that $w = a_2$. In particular we have $a_2 \in {}^JW^{\mathbf{c}(J)}$. Hence if $B'' \in \mathcal{B}_a$ then $\mathrm{pos}(P_{B'',J}, F(P_{B'',J})) = a_2 = w$. Since in our case $\mathcal{P}^{\mathbf{t}}_J = \{P \in \mathcal{P}_J; \mathrm{pos}(P, F(P)) = w\}$ (see 4.2(e)) we see that $P_{B'',J} \in \mathcal{P}^{\mathbf{t}}_J$. Thus we have $\mathcal{B}_a = \mathcal{B}_{\mathbf{t},a}$ and $A \dashv_\Gamma {}^pH^\cdot(\xi_{\mathbf{t},a!}(\mathcal{L}_a))$. Let $\tilde{\mathcal{B}}_a = \{gU \in G/U; g^{-1}F(g) \in U\dot{a}U\}$. Recall from 4.3(b) that $f_a : \tilde{\mathcal{B}}_a \to \mathcal{B}_a$, $gU \mapsto gBg^{-1}$ is a finite principal covering with group $\mathfrak{T} = \{t \in T; \dot{a}F(t)\dot{a}^{-1} = t\}$. From 2.5 we see that $\mathcal{L}_a$ is a direct summand of $f_{a!}\bar{\mathbf{Q}}_l$. Thus we have $A \dashv_\Gamma {}^pH^\cdot(\xi_{\mathbf{t},a!}f_{a!}\bar{\mathbf{Q}}_l)$.

By 4.3(a), $L_J = \dot{w}L_{\mathbf{c}(J)}\dot{w}^{-1}$ is a common Levi subgroup of $P_J, \dot{w}P_{\mathbf{c}(J)}\dot{w}^{-1}$. Let $F' : L_J \to L_J$ be as in 4.3. Let $\Lambda = L_J^{F'}$. Let $\mathcal{B}'$ be the variety of Borel subgroups of L_J. For β, β' in $\mathcal{B}'$ we have $\mathrm{pos}(\beta U_{P_J}, \beta' U_{P_J}) = y$ for a unique $y \in W_J$; we then also write $\mathrm{pos}'(\beta, \beta') = y$. Let $Y' = \{\beta \in \mathcal{B}'; \mathrm{pos}'(\beta, F'(\beta)) = a_1\}$. Now Λ acts on Y' by conjugation. Let $\tilde{\mathcal{P}}_J^{\mathbf{t}}$ be as in 4.3. Note that Λ acts (freely) on $\tilde{\mathcal{P}}_J^{\mathbf{t}} \times Y'$ by $l : (gU_{P_J}, \beta) \mapsto (gl^{-1}U_{P_J}, l\beta l^{-1})$ and we can form the orbit space $\Lambda\backslash(\tilde{\mathcal{P}}_J^{\mathbf{t}} \times Y')$.

We define $\psi : \tilde{\mathcal{P}}_J^{\mathbf{t}} \times Y' \to \mathcal{B}_a$ by $(gU_{P_J}, \beta) \mapsto g\beta U_{P_J} g^{-1}$. We show that ψ is well defined, that is $\mathrm{pos}(g\beta U_{P_J} g^{-1}, F(g)F(\beta)U_{P_{\mathbf{c}(J)}}F(g^{-1})) = a$ for $gU_{P_J} \in \tilde{\mathcal{P}}_J^{\mathbf{t}}$. Since $a = a_1 w$, $l(a) = l(a_1) + l(w)$, it is enough to show that for $(gU_{P_J}, \beta) \in \tilde{\mathcal{P}}_J^{\mathbf{t}} \times Y'$ we have

(c) $\mathrm{pos}(g\beta U_{P_J} g^{-1}, gF'(\beta)U_{P_J} g^{-1}) = a_1$,

(d) $\mathrm{pos}(gF'(\beta)U_{P_J} g^{-1}, F(g)F(\beta)U_{P_{\mathbf{c}(J)}}F(g^{-1})) = w$.

Now the left hand side of (c) is equal to $\mathrm{pos}(\beta U_{P_J}, F'(\beta)U_{P_J}) = \mathrm{pos}'(\beta, F'(\beta)) = a_1$, proving (c). We have $g^{-1}F(g) = u\dot{w}u'$ where $u \in U_{P_J}, u' \in U_{P_{\mathbf{c}(J)}}$. The left hand side of (d) is equal to

$$
\begin{aligned}
&\mathrm{pos}(F'(\beta)U_{P_J}, g^{-1}F(g)F(\beta)U_{P_{\mathbf{c}(J)}}F(g^{-1})g)\\
&= \mathrm{pos}(F'(\beta)U_{P_J}, u\dot{w}u'F(\beta)U_{P_{\mathbf{c}(J)}}u'^{-1}\dot{w}^{-1}u^{-1}) = \mathrm{pos}(F'(\beta)U_{P_J}, \dot{w}F(\beta)U_{P_{\mathbf{c}(J)}}\dot{w}^{-1})\\
&= \mathrm{pos}(F'(\beta)U_{P_J}, F'(\beta)U_{\dot{w}P_{\mathbf{c}(J)}\dot{w}^{-1}}) = \mathrm{pos}(P_J, \dot{w}P_{\mathbf{c}(J)}\dot{w}^{-1}).
\end{aligned}
$$

(The last equality follows from [L5, II, 8.3].) This equals w.

We see that ψ is well defined. We show:

(e) *The map* $\bar{\psi} : \Lambda\backslash(\tilde{\mathcal{P}}_J^{\mathbf{t}} \times Y') \to \mathcal{B}_a$ *induced by* ψ *is an isomorphism.*

Let $B' \in \mathcal{B}_a$. We can find uniquely $\tilde{B} \in \mathcal{B}$ such that $\mathrm{pos}(B', \tilde{B}) = a_1$, $\mathrm{pos}(\tilde{B}, F(B')) = w$. Then $P_{B',J} = P_{\tilde{B},J}$ and $\mathrm{pos}(P_{\tilde{B},J}, P_{F(B'),\mathbf{c}(J)}) = w$. Hence $\mathrm{pos}(P_{B',J}, F(P_{B',J})) = w$ that is, $P_{B',J} \in \mathcal{P}_J^{\mathbf{t}}$. Since $\tilde{\mathcal{P}}_J^{\mathbf{t}} \to \mathcal{P}_J^{\mathbf{t}}$ is surjective (see 4.3) we can find $gU_{P_J} \in \tilde{\mathcal{P}}_J^{\mathbf{t}}$ such that $gP_Jg^{-1} = P_{B',J}$; note that gU_{P_J} is unique up to the action of Λ. Since $B' \subset P_{B',J}$ we have $g^{-1}B'g \subset P_J$ hence there is a unique $\beta \in \mathcal{B}'$ such that $g^{-1}B'g = \beta U_{P_J}$. As above we see that

$$\mathrm{pos}(gF'(\beta)U_{P_J}g^{-1}, F(g)F(\beta)U_{P_{\mathbf{c}(J)}}F(g^{-1})) = w$$

that is $\mathrm{pos}(gF'(\beta)U_{P_J}g^{-1}, F(B')) = w$. Thus $gF'(\beta)U_{P_J}g^{-1}$ is a Borel subgroup of $P_{B',J}$ whose relative position with $F(B')$ is w. But there is only one such Borel subgroup. Therefore $gF'(\beta)U_{P_J}g^{-1} = \tilde{B}$. Since $\mathrm{pos}(B', \tilde{B}) = a_1$ we have $\mathrm{pos}(g\beta U_{P_J}g^{-1}, gF'(\beta)U_{P_J}g^{-1}) = a_1$ hence $\mathrm{pos}(\beta U_{P_J}, F'(\beta)U_{P_J}) = a_1$ that is, $\mathrm{pos}'(\beta, F'(\beta)) = a_1$. Thus to $B' \in \mathcal{B}_a$ we have associated $(gU_{P_J}, \beta) \in \tilde{\mathcal{P}}_J^{\mathbf{t}} \times Y'$; its Λ-orbit is well defined. Thus we have a well defined map $\mathcal{B}_a \to \Lambda\backslash(\tilde{\mathcal{P}}_J^{\mathbf{t}} \times Y')$. Clearly, this is the inverse of $\bar{\psi}$. This proves (e).

Now let $\beta' = B \cap L_J$, and let U' be the unipotent radical of β'. Let $U'' = U' \cap F'^{-1}(\dot{a}_1^{-1}U'\dot{a}_1)$. Let $\tilde{Y}' = \{lU'' \in L_J/U''; l^{-1}F'(l) \in U'\dot{a}_1\}$. As in 4.3(b),(c) the map $\rho : \tilde{Y}' \to Y'$, $lU'' \mapsto l\beta' l^{-1}$ is a finite principal covering with group $\{t \in T; \dot{a}_1 F'(t)\dot{a}_1^{-1} = t\} = \mathfrak{T}$. Now Λ acts freely on $\tilde{\mathcal{P}}^{\mathbf{t}}_J \times \tilde{Y}'$ by

$$l_0 : (gU_{P_J}, lU'') \mapsto (gl_0^{-1}U_{P_J}, l_0 lU'')$$

and we can form the orbit space $\Lambda\backslash(\tilde{\mathcal{P}}^{\mathbf{t}}_J \times \tilde{Y}')$.

The map $\tilde{\mathcal{P}}^{\mathbf{t}}_J \times \tilde{Y}' \to \tilde{\mathcal{B}}_a$, $(gU_{P_J}, lU'') \mapsto glU$ induces a map $\tilde{\psi} : \Lambda\backslash(\tilde{\mathcal{P}}^{\mathbf{t}}_J \times \tilde{Y}') \to \tilde{\mathcal{B}}_a$ which is easily seen to be an isomorphism. We have a commutative diagram

$$\begin{array}{ccc} \Lambda\backslash(\tilde{\mathcal{P}}^{\mathbf{t}}_J \times \tilde{Y}') & \xrightarrow{\tilde{\psi}} & \tilde{\mathcal{B}}_a \\ {\scriptstyle\xi}\downarrow & & {\scriptstyle f_a}\downarrow \\ \Lambda\backslash(\tilde{\mathcal{P}}^{\mathbf{t}}_J \times Y') & \xrightarrow{\bar{\psi}} & \mathcal{B}_a \end{array}$$

where ξ is induced by $(gU_{P_J}, lU'') \mapsto (gU_{P_J}, l\beta' l^{-1})$ (a principal $\mathfrak{T}$-bundle). Note that the horizontal maps in this diagram are isomorphisms.

Under the isomorphism (e), the map $\xi_{\mathbf{t},a}$ becomes the map $\xi' : \Lambda\backslash(\tilde{\mathcal{P}}^{\mathbf{t}}_J \times Y') \to \mathcal{P}^{\mathbf{t}}_J$ induced by $(gU_{P_J}, \beta) \mapsto gP_J g^{-1}$. It follows that $\xi_{\mathbf{t},a!} f_{a!}\bar{\mathbf{Q}}_l = (\xi'\xi)_!\bar{\mathbf{Q}}_l$ and $A \dashv_\Gamma {}^pH^{\cdot}((\xi'\xi)_!\bar{\mathbf{Q}}_l)$. We extend the natural Γ-actions on $\mathcal{P}^{\mathbf{t}}_J$ and on $\Lambda\backslash(\tilde{\mathcal{P}}^{\mathbf{t}}_J \times \tilde{Y}')$ to $\Gamma \times \Lambda$-actions with Λ acting trivially. Then $A, {}^pH^j((\xi'\xi)_!\bar{\mathbf{Q}}_l)$ are naturally objects of $\mathcal{M}_{\Gamma\times\Lambda}(\mathcal{P}^{\mathbf{t}}_J)$ and $A \dashv_{\Gamma\times\Lambda} {}^pH^{\cdot}((\xi'\xi)_!\bar{\mathbf{Q}}_l)$. Let $f : \tilde{\mathcal{P}}^{\mathbf{t}}_J \to \mathcal{P}^{\mathbf{t}}_J$ be as in 4.3. Now $\Gamma\times\Lambda$ acts on $\tilde{\mathcal{P}}^{\mathbf{t}}_J$ as in 4.4 (compatibly with f). Moreover f^*A, $f^*({}^pH^j((\xi'\xi)_!\bar{\mathbf{Q}}_l)) = {}^pH^j(f^*(\xi'\xi)_!\bar{\mathbf{Q}}_l)$ are objects of $\mathcal{M}_{\Gamma\times\Lambda}(\tilde{\mathcal{P}}^{\mathbf{t}}_J)$ and $f^*A \dashv_{\Gamma\times\Lambda} {}^pH^{\cdot}(f^*(\xi'\xi)_!\bar{\mathbf{Q}}_l)$. Let $p_1 : \tilde{\mathcal{P}}^{\mathbf{t}}_J \times \tilde{Y}' \to \tilde{\mathcal{P}}^{\mathbf{t}}_J$ be the first projection. Now $\Gamma \times \Lambda$ acts on $\tilde{\mathcal{P}}^{\mathbf{t}}_J \times \tilde{Y}'$ by $(g_0, l_0) : (gU_J, lU'') \mapsto (g_0 g l_0^{-1}U_J, l_0 lU'')$ and p_1 is compatible with the $\Gamma \times \Lambda$-actions. We have $f^*(\xi'\xi)_!\bar{\mathbf{Q}}_l = p_{1!}\bar{\mathbf{Q}}_l$. We see that $f^*A \dashv_{\Gamma\times\Lambda} {}^pH^{\cdot}(p_{1!}\bar{\mathbf{Q}}_l)$. Let $p_2 : \tilde{\mathcal{P}}^{\mathbf{t}}_J \times \tilde{Y}' \to \tilde{Y}'$ be the second projection. Now $\Gamma \times \Lambda$ acts on $\tilde{Y}'$ by $(g_0, l_0) : lU'' \mapsto l_0 lU''$ and p_2 is compatible with the $\Gamma \times \Lambda$-actions. The obvious maps $\tilde{\mathcal{P}}^{\mathbf{t}}_J \xrightarrow{e}$ point $\xleftarrow{e'} \tilde{Y}'$ are again compatible with the $\Gamma \times \Lambda$-actions (the action on the point is trivial). We have $p_{1!}\bar{\mathbf{Q}}_l = e^*e'_!\bar{\mathbf{Q}}_l$ hence $f^*A \dashv_{\Gamma\times\Lambda} {}^pH^{\cdot}(e^*e'_!\bar{\mathbf{Q}}_l)$. We have a spectral sequence in $\mathcal{M}_{\Gamma\times\Lambda}(\tilde{\mathcal{P}}^{\mathbf{t}}_J)$ with $E_2 = {}^pH^{\cdot}(e^{*p}H^{\cdot}(e'_!\bar{\mathbf{Q}}_l))$ and E_∞ is an associated graded of ${}^pH^{\cdot}(e^*e'_!\bar{\mathbf{Q}}_l)$. We have $f^*A \dashv_{\Gamma\times\Lambda} E_\infty$ hence $f^*A \dashv_{\Gamma\times\Lambda} E_2$. Now ${}^pH^{\cdot}(e'_!\bar{\mathbf{Q}}_l)$ is just a $\Gamma \times \Lambda$-module M with trivial action of Γ and $e^{*p}H^{\cdot}(e'_!\bar{\mathbf{Q}}_l)$ is the local system $f^*\underline{M}$ (notation of 4.4). Since $\tilde{\mathcal{P}}^{\mathbf{t}}_J$ is smooth of pure dimension d, ${}^pH^j(e^{*p}H^{\cdot}(e'_!\bar{\mathbf{Q}}_l))$ is 0 if $j \neq d$ and is $f^*\underline{M}[d]$ if $j = d$. Thus $E_2 = f^*\underline{M}[d]$. We see that $f^*A \dashv_{\Gamma\times\Lambda} f^*(\underline{M})[d]$. It follows that $A \dashv_\Gamma \underline{M}[d]$. This implies (a).

4.8. Let $\mathbf{t} = (J_n, w_n)_{n\geq 0} \in \mathcal{T}'(J, \mathbf{c})$. Let $d = \dim \mathcal{P}^{\mathbf{t}}_J$. Let $\mathcal{L} \in \mathcal{S}(T)$ and $\mathbf{s}$ be as in 2.6. Let $\bar{\mathcal{L}}$ be the local system on $\mathcal{Z}^{\mathbf{s}}$ as in 2.11. Let $\Upsilon^{\mathbf{s}} : \mathcal{Z}^{\mathbf{s}} \to \mathcal{P}_J$, $\bar{\Upsilon}^{\mathbf{s}} : \bar{Z}^{\mathbf{s}} \to \mathcal{P}_J$ be as in 3.2. We show:

(a) *Let A be a simple object of $\mathcal{M}_\Gamma(\mathcal{P}_J)$ such that $A \dashv_\Gamma {}^pH^{\cdot}(\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}})$. Let $h : \mathcal{P}^{\mathbf{t}}_J \to \mathcal{P}_J$ be the inclusion. Let A' be a simple object of $\mathcal{M}_\Gamma(\mathcal{P}^{\mathbf{t}}_J)$ such that $A' \dashv_\Gamma {}^pH^{\cdot}(h^*A)$. Then $A' \cong \underline{M}[d]$ for some irreducible $L^{F'}_{J_\infty}$-module M.*

Let $K = \bar{\Upsilon}^{\mathbf{s}}_!\bar{\mathcal{L}}^\sharp$. Using 2.13(a) we see that $\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}} = K$. Hence $A \dashv_\Gamma {}^pH^{\cdot}(K)$. We have a spectral sequence in $\mathcal{M}_\Gamma(\mathcal{P}^{\mathbf{t}}_J)$ with $E_2 = {}^pH^{\cdot}(h^*({}^pH^{\cdot}K))$ and E_∞ is an associated graded of ${}^pH^{\cdot}(h^*K)$. Since $\bar{\Upsilon}^{\mathbf{s}}$ is proper, we see from the decomposition theorem [BBD] that $K \cong \oplus_i {}^pH^i(K)[-i]$ and that each ${}^pH^i(K)$ is semisimple as an object of $\mathcal{M}(\mathcal{P}_J)$ hence also as an object of $\mathcal{M}_\Gamma(\mathcal{P}_J)$. It also follows that $h^*K \cong \oplus_i h^*({}^pH^i(K))[-i]$ hence ${}^pH^j(h^*K) \cong \oplus_i {}^pH^{j-i}(h^*({}^pH^i(K)))$. This shows that $E_2 \cong E_\infty$ as objects of $\mathcal{M}(\mathcal{P}^{\mathbf{t}}_J)$. Thus the spectral sequence above is degenerate when regarded in $\mathcal{M}(\mathcal{P}^{\mathbf{t}}_J)$. But then it is also degenerate in $\mathcal{M}_\Gamma(\mathcal{P}^{\mathbf{t}}_J)$. Using $A' \dashv_\Gamma {}^pH^{\cdot}(h^*A)$ and the fact that A is a direct summand of ${}^pH^{\cdot}(K)$ (in $\mathcal{M}_\Gamma(\mathcal{P}_J)$) we see that $A' \dashv_\Gamma E_2$. It follows that $A' \dashv_\Gamma E_\infty$ that is $A' \dashv_\Gamma {}^pH^{\cdot}(h^*\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}})$. As in the proof of the implication 3.2(v) $\implies$ 3.2(i) we deduce that $A' \dashv_\Gamma {}^pH^{\cdot}(h^*\Pi^{(a)}_!\mathcal{L}_{(a)})$ ($\Pi^{(a)}$ as in 3.2) for some $a \in W$. Now (a) follows from 4.7(a).

4.9. Let A be a simple object of $\mathcal{M}_\Gamma(\mathcal{P}_J)$ such that $A \dashv_\Gamma {}^pH^{\cdot}(\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}})$ with $\mathcal{L}, \mathbf{s}$ as in 2.6. We show:

(a) *There exists $\mathbf{t} = (J_n, w_n)_{n\ge 0} \in \mathcal{T}'(J, \mathbf{c})$ and an irreducible $L^{F'}_{J_\infty}$-module M such that $A \cong \underline{M}^\sharp[d]$ where $\underline{M}^\sharp$ is as in 4.4 and $d = \dim \mathcal{P}^{\mathbf{t}}_J$.*

Since $\mathcal{P}_J = \cup_{\mathbf{t}\in\mathcal{T}'(J,\mathbf{c})}\mathcal{P}^{\mathbf{t}}_J$, we can find $\mathbf{t} \in \mathcal{T}'(J, \mathbf{c})$ such that $\mathrm{supp}(A) \cap \mathcal{P}^{\mathbf{t}}_J$ is open dense in $\mathrm{supp}(A)$. Then, denoting by $h : \mathcal{P}^{\mathbf{t}}_J \to \mathcal{P}_J$ the inclusion, we see that h^*A is a simple object of $\mathcal{M}_\Gamma(\mathcal{P}^{\mathbf{t}}_J)$. As in 4.8(a), we have $h^*A \cong \underline{M}[d]$ for some irreducible $L^{F'}_{J_\infty}$-module M. It follows that A is of the required form.

4.10. Let $\mathbf{t} = (J_n, w_n)_{n\ge 0} \in \mathcal{T}'(J, \mathbf{c})$. We set $w = w_\infty \in {}^JW$. We show:

(a) *for any $b \in W_{J_\infty}$, $B' \mapsto P_{B',J}$ is a well defined map $\mathcal{B}_{bw} \to \mathcal{P}^{\mathbf{t}}_J$;*

(b) *for $b = 1$, the map $\mathcal{B}_w \to \mathcal{P}^{\mathbf{t}}_J$ in (a) is surjective.*

We prove (a). Let $B' \in \mathcal{B}_{bw}$. Let $P = P_{B',J}$. By 4.2(f) we have $\mathrm{pos}(P, F(P)) = \min(W_J w W_{\mathbf{c}(J)}) = w_0$. Define P^n in terms of P as in 4.1. We have $P^1 = P^{F(P)} \in \mathcal{P}_{J_1}$. As in the proof of 4.6(c) (with $a_1 = b, a_2 = w$) we see that $b = \mathrm{pos}(B', P^{F(B')})$, $w = \mathrm{pos}(P^{F(B')}, F(B'))$. Since $b \in W_{J_1}$ we have $\mathrm{pos}(B', P^{F(B')}) \in W_{J_1}$ hence $B' \subset P^1$. From the definitions we have $w_1 = \min(W_{J_1} w W_{\mathbf{c}(J_1)})$ hence

$$\mathrm{pos}(P^1, F(P^1)) = \min(W_{J_1}\mathrm{pos}(B', F(B'))W_{\mathbf{c}(J_1)}) = \min(W_{J_1} bw W_{\mathbf{c}(J_1)}) = w_1.$$

By the same argument applied to $B', P^1, \mathbf{t}_1$ instead of $B', P, \mathbf{t}$ we see that $P^2 \in \mathcal{P}_{J_2}$ and $\mathrm{pos}(P^2, F(P^2)) = w_2$. (We have $w \in {}^{J_1}W$ since $J_1 \subset J$.) Continuing in this way we see that $P^n \in \mathcal{P}_{J_n}$ and $\mathrm{pos}(P^n, F(P^n)) = w_n$ for all $n \ge 0$. Thus $P \in \mathcal{P}^{\mathbf{t}}_J$. This proves (a).

We prove (b). Let $P \in \mathcal{P}^{\mathbf{t}}_J$. Define P^∞ in terms of P as in 4.1. We have $\mathrm{pos}(P^\infty, F(P^\infty)) = w$. Hence

$$\begin{aligned} pr_2 :& \{(B', B'') \in \mathcal{B} \times \mathcal{B}; B' \subset P^\infty, B'' \subset F(P^\infty), \mathrm{pos}(B', B'') = w\} \\ &\to \{B'' \in \mathcal{B}; B'' \subset F(P^\infty)\} \end{aligned}$$

is a bijection with inverse $B'' \mapsto ((P^\infty)^{B''}, B'')$. The condition that (B', B'') in the domain of pr_2 satisfies $B'' = F(B')$ is that B'' is a fixed point of the map $B'' \mapsto F((P^\infty)^{B''})$ of the flag manifold of $F(P^\infty)$ into itself. This map may be identified with the map induced by $F' : L_J \to L_J$ (see 4.3) on the flag manifold of L_J hence it has at least one fixed point. Thus there exist $(B', B'') \in \mathcal{B} \times \mathcal{B}$ such that $B'' = F(B')$, $B' \subset P^\infty$, $\mathrm{pos}(B', B'') = w$. Then $B' \in \mathcal{B}_a$ and $B' \subset P$ (since $P^\infty \subset P$). This proves (b).

4.11. Let $\mathbf{t} = (J_n, w_n)_{n\ge 0} \in \mathcal{T}'(J, \mathbf{c})$. We set $w = w_\infty \in {}^J W$. We show:

(a) *Let $b \in W_{J_\infty}$. Let $\mathcal{L} \in \mathcal{S}(T)^{bwF}$. Let A be a simple object of $\mathcal{M}_\Gamma(\mathcal{P}_J^{\mathbf{t}_1})$ such that $A \dashv_\Gamma {}^pH^\cdot(\xi_{\mathbf{t}_1, w\mathbf{c}(b)!}(b^*\mathcal{L})_{w\mathbf{c}(b)})$. Then $\vartheta^* A \dashv_\Gamma {}^pH^\cdot(\xi_{\mathbf{t},bw!}\mathcal{L}_{bw})$.*

The result makes sense since $\mathcal{B}_{\mathbf{t},bw} = \mathcal{B}_{bw}$ by 4.10(a) and $\mathcal{B}_{\mathbf{t}_1, w\mathbf{c}(b)} = \mathcal{B}_{w\mathbf{c}(b)}$ (this follows from 4.10(a) applied to $\mathbf{t}_1, w, w\mathbf{c}(b)w^{-1}$ instead of $\mathbf{t}, w, b$; note that $w\mathbf{c}(J_\infty)w^{-1} = J_\infty$ by 4.2(e) hence $w\mathbf{c}(b)w^{-1} \in W_{J_\infty}$). This shows also that $l(w\mathbf{c}(b)) = l(w) + l(\mathbf{c}(b))$. (Since $w \in {}^J W$ we have $l((w\mathbf{c}(b)w^{-1})w) = l(w\mathbf{c}(b)w^{-1}) + l(w)$.) We define $h : \mathcal{B}_{bw} \to \mathcal{B}_{w\mathbf{c}(b)}$ by $B' \mapsto B''$ with B'' defined by $\mathrm{pos}(B', B'') = b, \mathrm{pos}(B'', F(B')) = w$. This is an isomorphism whose inverse $\mathcal{B}_{w\mathbf{c}(b)} \to \mathcal{B}_{bw}$ is given by $B'' \mapsto B'$ with B' defined by $\mathrm{pos}(B', B'') = b, \mathrm{pos}(B'', F(B')) = w$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{B}_{bw} & \xrightarrow{h} & \mathcal{B}_{w\mathbf{c}(b)} \\ {\scriptstyle \xi_{\mathbf{t},bw}}\downarrow & & \downarrow{\scriptstyle \xi_{\mathbf{t}_1, w\mathbf{c}(b)}} \\ \mathcal{P}_J^{\mathbf{t}} & \xrightarrow{\vartheta} & \mathcal{P}_{J_1}^{\mathbf{t}_1} \end{array}$$

where the horizontal maps are isomorphism. From the definitions we see that $h^*((b^*\mathcal{L})_{w\mathbf{c}(b)}) = \mathcal{L}_{bw}$. The result follows.

4.12. Let $\mathbf{t} = (J_n, w_n)_{n\ge 0} \in \mathcal{T}'(J, \mathbf{c})$. We set $w = w_\infty \in {}^J W$. Let $d = \dim \mathcal{P}_J^{\mathbf{t}}$. Let $\Lambda = L_{J_\infty}^{F'}$. Let M be a finite dimensional irreducible Λ-module over $\bar{\mathbf{Q}}_l$. Let $\underline{M}[d] \in \mathcal{M}_\Gamma(\mathcal{P}_J^{\mathbf{t}})$ be as in 4.4(c). We show:

(a) *there exists $b \in W_{J_\infty}$ and $\mathcal{L} \in \mathcal{S}(T)^{bwF}$ such that $\underline{M}[d] \dashv_\Gamma {}^pH^\cdot(\xi_{\mathbf{t},bw!}\mathcal{L}_{bw})$;*

(b) *there exists $b \in W_{J_\infty}$ and $\mathcal{L} \in \mathcal{S}(T)^{bwF}$ such that $\underline{M}^\sharp[d] \dashv_\Gamma {}^pH^\cdot(\Pi_!^{(bw)}\mathcal{L}_{(bw)})$.*

We prove (a). More generally we show that for any $n \ge 0$:

(c) *there exists $b_n \in W_{J_\infty}$ and $\mathcal{L} \in \mathcal{S}(T)^{b_n wF}$ such that $\underline{M}[d]$ (regarded as an object of $\mathcal{M}_\Gamma(\mathcal{P}_{J_n}^{\mathbf{t}_n})$) satisfies $\underline{M}[d] \dashv_\Gamma {}^pH^\cdot(\xi_{\mathbf{t}_n, b_n w!}\mathcal{L}_{b_n w})$.*

If (c) holds for $n = 1$ then, by 4.11(a) it holds for $n = 0$. Similarly, if (c) holds for some $n \ge 1$ then it holds for $n - 1$. Hence it is enough to prove (c) for large n. Thus we may assume that $J_0 = J_1 = \cdots = J$ and $w_0 = w_1 = \cdots = w$. By [DL, 7.7] applied to Λ, there exists $a_1 \in W_{J_\infty}$ such that $M \dashv_\Lambda {}^pH^\cdot(e'_!\bar{\mathbf{Q}}_l)$ (in $\mathcal{M}_\Lambda($ point$)$). (The reference to [DL] could be replaced by a selfcontained proof, see 7.9.) By definition, $f^*\underline{M}[d] = e^*M[d]$ with f as in 4.3, e as in 4.7. We have $e^*M[d] \dashv_{\Gamma\times\Lambda} e^*({}^pH^\cdot(e'_!\bar{\mathbf{Q}}_l))[d]$ hence $f^*\underline{M}[d] \dashv_{\Gamma\times\Lambda} e^*({}^pH^\cdot(e'_!\bar{\mathbf{Q}}_l))[d]$ in $\mathcal{M}_{\Gamma\times\Lambda}(\tilde{\mathcal{P}}_J^{\mathbf{t}})$. Since $\tilde{\mathcal{P}}_J^{\mathbf{t}}$ is smooth of pure dimension d we have $e^*({}^pH^i(e'_!\bar{\mathbf{Q}}_l))[d] = {}^pH^{i+d}(e^*e'_!\bar{\mathbf{Q}}_l)$. Since

$$e^*e'_!\bar{\mathbf{Q}}_l = p_{1!}\bar{\mathbf{Q}}_l = f^*(\xi'\xi)_!\bar{\mathbf{Q}}_l = f^*\xi_{\mathbf{t},a!}f_{a!}\bar{\mathbf{Q}}_l$$

(notation of 4.7 with $a = a_1 w$) we have $f^*\underline{M}[d] \dashv_{\Gamma\times\Lambda} {}^pH^{\cdot}(f^*\xi_{\mathbf{t},a!}f_{a!}\bar{\mathbf{Q}}_l)$. Since f is a finite principal covering we have also $f^*\underline{M}[d] \dashv_{\Gamma\times\Lambda} f^*({}^pH^{\cdot}(\xi_{\mathbf{t},a!}f_{a!}\bar{\mathbf{Q}}_l))$. It follows that $\underline{M}[d] \dashv_{\Gamma} {}^pH^{\cdot}(\xi_{\mathbf{t},a!}f_{a!}\bar{\mathbf{Q}}_l)$. Now $f_{a!}\bar{\mathbf{Q}}_l = \oplus_{\mathcal{L}}\mathcal{L}_a$ where $\mathcal{L}$ runs over the local systems in $\mathcal{S}(T)^{aF}$ (up to isomorphism). Hence for some such $\mathcal{L}$ we have $\underline{M}[d] \dashv_{\Gamma} {}^pH^{\cdot}(\xi_{\mathbf{t},a!}\mathcal{L}_a)$. This proves (a).

We prove (b). Let $b, \mathcal{L}$ be such that (a) holds. Let $K = \Pi_!^{(bw)}\mathcal{L}_{(bw)}$, $K' = \xi_{\mathbf{t},bw!}\mathcal{L}_{bw}$. We have $\underline{M}[d] \dashv_{\Gamma} {}^pH^{\cdot}(K')$. Let $\kappa : \mathcal{P}_J^{\mathbf{t}} \to \mathcal{P}_J$ be the inclusion. We have $K = \kappa_! K'$. Let C be the closure of $\mathcal{P}_J^{\mathbf{t}}$ in $\mathcal{P}_J$. Let $\kappa' : \mathcal{P}_J^{\mathbf{t}} \to C$, $\kappa'' : C \to \mathcal{P}_J$ be the inclusions. Let $\tilde{M} = IC(C, \underline{M})[d]$. Let $K_1 = \kappa'_! K$. We show:

(d) $$\tilde{M} \dashv_{\Gamma} {}^pH^{\cdot}(K_1).$$

We have $\kappa'^*K_1 = K'$ and $\kappa'^*({}^pH^j(K_1)) = {}^pH^j(K')$. Let $0 = X_0 \subset X_1 \subset \dots \subset X_m = {}^pH^j(K_1)$ be a composition series of ${}^pH^j(K_1)$ in $\mathcal{M}_\Gamma(C)$. Aplying κ'^* to the exact sequence $0 \to X_{i-1} \to X_i \to X_i/X_{i-1} \to 0$ (where $1 \le i \le m$) we get an exact sequence $0 \to \kappa'^*(X_{i-1}) \to \kappa'^*(X_i) \to \kappa'^*(X_i/X_{i-1}) \to 0$ in $\mathcal{M}_\Gamma(\mathcal{P}_\Gamma^{\mathbf{t}})$. Since X_i/X_{i-1} is simple and $\mathcal{P}_\Gamma^{\mathbf{t}}$ is open dense in C we see that $\kappa'^*(X_i/X_{i-1})$ is either simple or 0. It follows that any composition factor of $\kappa'^*(X_m)$ is isomorphic to $\kappa'^*(X_i/X_{i-1})$ for some i. In particular, $\underline{M}[d] \cong \kappa'^*(X_i/X_{i-1})$ for some i. It follows that $\tilde{M} \cong X_i/X_{i-1}$ for some i. Thus (d) holds.

Applying $\kappa''_!$ to (d) we obtain

$$\underline{M}^\sharp[d] = \kappa''_!\tilde{M} \dashv_{\Gamma} \kappa''_!{}^pH^{\cdot}(K_1) = {}^pH^{\cdot}(\kappa''_!K_1) = {}^pH^{\cdot}(K).$$

This proves (b).

Theorem 4.13. *Let K be a simple object of $\mathcal{M}_\Gamma(\mathcal{P}_J)$. Then $K \in \mathbb{S}'(\mathcal{P}_J)$ (see 3.7) if and only if $K \in \mathbb{S}(\mathcal{P}_J)$.*

If $K \in \mathbb{S}(\mathcal{P}_J)$ then by 4.12(b) it satisfies 3.2(i). If K satisfies 3.2(v) then by 4.9(a), $K \in \mathbb{S}(\mathcal{P}_J)$. This completes the proof.

Theorem 4.14. *Let $\mathbf{t} = (J_n, w_n)_{n\ge 0} \in \mathcal{T}'(J, \mathbf{c})$. Let $h : \mathcal{P}_J^{\mathbf{t}} \to \mathcal{P}_J$ be the inclusion. Let $A \in \mathbb{S}(\mathcal{P}_J)$. Let A' be a simple object of $\mathcal{M}_\Gamma(\mathcal{P}_J^{\mathbf{t}})$ such that $A' \dashv_{\Gamma} {}^pH^{\cdot}(h^*A)$. Then $A' \in \mathbb{S}(\mathcal{P}_J^{\mathbf{t}})$.*

By assumption, A is as in 4.8(a). The result now follows from 4.8(a).

5. Some computations in the Weyl group

5.1. Let $\mathcal{L} \in \mathcal{S}(T)$. Define $\tilde{l} : W \to \mathbf{N}$ by $\tilde{l}(w) = |(\alpha \in R_{\mathcal{L}}^+; w(\alpha) \in R^-\}|$.

Let $\mathbf{s} = (s_1, \dots s_r)$ be a sequence in $\mathbf{I} \cup \{1\}$. Let

$$\mathcal{I}_{\mathbf{s}} = \{i \in [1, r]; s_i \ne 1, s_1 s_2 \dots s_i \dots s_2 s_1 \in W_{\mathcal{L}}\}.$$

This agrees with the definition in 2.4.

Lemma 5.2. *We have $|\mathcal{I}_{\mathbf{s}}| \geq \tilde{l}(s_r \dots s_1)$ with equality if $l(s_r \dots s_1) = l(s_1) + \dots + l(s_r)$.*

Let

$$X = \{\alpha \in R^+_{\mathcal{L}}; (s_r \dots s_1)(\alpha) \in R^-\},$$
$$X' = \{\alpha \in R^+_{\mathcal{L}}; \alpha = s_1 s_2 \dots s_{i-1}(\alpha_{s_i}) \text{ for some } i \in [1,r] \text{ such that } s_i \neq 1\}.$$

We have $X \subset X'$. We have $|X| = \tilde{l}(s_r \dots s_1)$ hence $\tilde{l}(s_r \dots s_1) \leq |X'|$. Define $f : \mathcal{I}_{\mathbf{s}} \to R_{\mathcal{L}}$ by $f(i) = s_1 s_2 \dots s_{i-1}(\alpha_{s_i})$; then $X' = f(\mathcal{I}_{\mathbf{s}}) \cap R^+_{\mathcal{L}}$. Hence $|X'| \leq |f(\mathcal{I}_{\mathbf{s}})| \leq |\mathcal{I}_{\mathbf{s}}|$ and the desired inequality is proved. Assume now that $l(s_r \dots s_1) = l(s_1) + \dots + l(s_r)$. Then $s_1 s_2 \dots s_{i-1}(\alpha_{s_i}) (i \in [1,r], s_i \neq 1)$ are distinct in R^+. Hence for $i \in \mathcal{I}_{\mathbf{s}}$, $s_1 s_2 \dots s_{i-1}(\alpha_{s_i})$ are distinct elements of X. Thus $|\mathcal{I}_{\mathbf{s}}| \leq |X|$. It follows that $|\mathcal{I}_{\mathbf{s}}| = |X|$.

Lemma 5.3. *Let $j \in \mathcal{I}_{\mathbf{s}}$. Let $\mathbf{s}(j) = (s'_1, s'_2, \dots, s'_r)$ with $s'_i = s_i$ for $i \neq j$, $s'_j = 1$. We have $\mathcal{I}_{\mathbf{s}(j)} = \mathcal{I}_{\mathbf{s}} - \{j\}$.*

Let $h \in \mathcal{I}_{\mathbf{s}} - \{j\}$. We have $s_1 s_2 \dots s_h \dots s_2 s_1 \in W_{\mathcal{L}}$. Hence if $j < h$, we have

$$\begin{aligned} &s_1 s_2 \dots \hat{s}_j \dots s_h \dots \hat{s}_j \dots s_2 s_1 \\ &= (s_1 s_2 \dots s_j \dots s_2 s_1)(s_1 s_2 \dots s_h \dots s_2 s_1)(s_1 s_2 \dots s_j \dots s_2 s_1) \in W_{\mathcal{L}} \end{aligned}$$

so that $h \in \mathcal{I}_{\mathbf{s}(j)}$. ($\hat{}$ denotes an omitted symbol.) If $j > h$ then $h \in \mathcal{I}_{\mathbf{s}(j)}$ is obvious.

Conversely, assume that $h \in \mathcal{I}_{\mathbf{s}(j)}$. Clearly, $h \neq j$. Assume first that $j < h$. We have

$$\begin{aligned} &s_1 s_2 \dots s_h \dots s_2 s_1 \\ &= (s_1 s_2 \dots s_j \dots s_2 s_1)(s_1 s_2 \dots \hat{s}_j \dots s_h \dots \hat{s}_j \dots s_2 s_1)(s_1 s_2 \dots s_j \dots s_2 s_1) \in W_{\mathcal{L}}. \end{aligned}$$

Hence $h \in \mathcal{I}_{\mathbf{s}}$. If $j > h$ then it is clear that $h \in \mathcal{I}_{\mathbf{s}}$. The lemma is proved.

Lemma 5.4. *Let h be the smallest element of $\mathcal{I}_{\mathbf{s}}$. Then $s_1 s_2 \dots s_h \dots s_2 s_1 \in \mathbf{I}_{\mathcal{L}}$.*

Let $\mathbf{s}' = (s_1, s_2, \dots, s_h, \dots, s_2, s_1)$. We have
$s_1 s_2 \dots s_h \dots s_2 s_1 \in W_{\mathcal{L}}$,
$s_1 s_2 \dots s_{h-1} \dots s_2 s_1 \notin W_{\mathcal{L}}$ or $s_{h-1} = 1$,
$s_1 s_2 \dots s_{h-2} \dots s_2 s_1 \notin W_{\mathcal{L}}$ or $s_{h-2} = 1$,
$\dots$.
Hence the middle term in $\mathbf{s}'$ has index in $\mathcal{I}_{\mathbf{s}'}$ but all terms preceding it have index not in $\mathcal{I}_{\mathbf{s}'}$. We show that the term in $\mathbf{s}'$ immediately following the middle term has index not in $\mathcal{I}_{\mathbf{s}'}$. Otherwise it would be $\neq 1$ and

$$s_1 s_2 \dots s_{h-1} s_h s_{h-1} s_h s_{h-1} \dots s_2 s_1 \notin W_{\mathcal{L}}.$$

Multiplying on the left and right by $s_1s_2\dots s_h\dots s_2s_1$ we find

$$s_1s_2\dots s_{h-1}\dots s_2s_1\in W_{\mathcal{L}},$$

a contradiction. Similarly we see that all terms in $\mathbf{s}'$ following the middle term have index not in $\mathcal{I}_{\mathbf{s}'}$. Thus $|\mathcal{I}_{\mathbf{s}'}|=1$. By Lemma 5.2 we have

$$\tilde{l}(s_1s_2\dots s_h\dots s_2s_1)\le 1.$$

Since $s_1s_2\dots s_h\dots s_2s_1\in W_{\mathcal{L}}-\{1\}$, it follows that $s_1s_2\dots s_h\dots s_2s_1\in \mathbf{I}_{\mathcal{L}}$. The lemma is proved.

5.5. We write the elements of $\mathcal{I}_{\mathbf{s}}$ in ascending order: $i_1<i_2<\dots<i_b$. Define a sequence $\mathbf{S}=(S_1,S_2,\dots,S_b)$ in W by

$S_1=s_1s_2\dots s_{i_1}\dots s_2s_1$,

$S_2=s_1s_2\dots\hat{s}_{i_1}\dots s_{i_2}\dots\hat{s}_{i_1}\dots s_2s_1$,

$\dots,$

$S_b=s_1s_2\dots\hat{s}_{i_1}\dots\hat{s}_{i_2}\dots\hat{s}_{i_{b-1}}\dots s_{i_b}\dots\hat{s}_{i_{b-1}}\dots\hat{s}_{i_2}\dots\hat{s}_{i_1}\dots s_2s_1$,

$\omega=s_1\dots\hat{s}_{i_1}\dots\hat{s}_{i_2}\dots\hat{s}_{i_b}\dots s_r$.

Lemma 5.6. *(a) $S_1,S_2,\dots,S_b$ belong to $\mathbf{I}_{\mathcal{L}}$.*

(b) $\tilde{l}(\omega^{-1})=0$.

(c) $s_1s_2\dots s_r=S_1S_2\dots S_b\omega$.

We use induction on b. Assume first that $b=0$. By 5.2 we have $\tilde{l}(s_r\dots s_1)=0$ that is $\tilde{l}(\omega^{-1})=0$ and the lemma is clear. Assume now that $b\ge 1$ and that the lemma holds for $b-1$. Let $h=i_1$. By 5.3 we have $\mathcal{I}_{\mathbf{s}(h)}=\mathcal{I}_{\mathbf{s}}-\{h\}$. By the induction hypothesis, $S_2,S_3,\dots,S_b$ belong to $\mathbf{I}_{\mathcal{L}}$, $s_1\dots\hat{s}_{i_1}\dots s_r=S_2S_3\dots S_b\omega$, $\tilde{l}(\omega^{-1})=0$. By 5.4 we have $S_1\in\mathbf{I}_{\mathcal{L}}$. It follows that

$$s_1s_2\dots s_r=(s_1\dots s_{i_1}\dots s_1)(s_1\dots\hat{s}_{i_1}\dots s_r)=S_1S_2\dots S_b\omega.$$

The lemma is proved.

5.7. Let $W'_{\mathcal{L}}=\{w\in W;w^*\mathcal{L}\cong\mathcal{L}\}$, a subgroup of W. Let $W'^0_{\mathcal{L}}=\{w\in W'_{\mathcal{L}};w(R^+_{\mathcal{L}})=R^+_{\mathcal{L}}\}$, a subgroup of $W'_{\mathcal{L}}$. Note that $W'_{\mathcal{L}}=W'^0_{\mathcal{L}}W_{\mathcal{L}}$ (semidirect product with $W_{\mathcal{L}}$ normal).

In the remainder of this section we fix an automorphism c of finite order of T that induces a permutation of R, one of $\check{R}$ and one of R^+.

Lemma 5.8. *Assume that $F_0^*(s_1s_2\dots s_rc)^*\mathcal{L}\cong\mathcal{L}$. Let $c'=\omega c$ so that $s_1s_2\dots s_rc=S_1S_2\dots S_bc'$ and $F_0^*(S_1S_2\dots S_bc')^*\mathcal{L}\cong\mathcal{L}$. Then c' is an automorphism of finite order of T that induces a permutation of $R_{\mathcal{L}}$, one of $\check{R}_{\mathcal{L}}$ and one of $R^+_{\mathcal{L}}$.*

We first show:

(i) $$c'(R_{\mathcal{L}})=R_{\mathcal{L}}.$$

It is enough to show that for any coroot $\kappa : \mathbf{k}^* \longrightarrow T$ such that $\kappa^*(\mathcal{L}) \cong \bar{\mathbf{Q}}_l$ we have $(c'\kappa)^*\mathcal{L} \cong \bar{\mathbf{Q}}_l$. From $F_0^*(S_1S_2\dots S_bc')^*\mathcal{L} \cong \mathcal{L}$ and $S_1S_2\dots S_r \in W_{\mathcal{L}}$ we deduce $F_0^*c'^*\mathcal{L} \cong \mathcal{L}$. Hence $\kappa^*F_0^*c'^*\mathcal{L} \cong \kappa^*\mathcal{L} \cong \bar{\mathbf{Q}}_l$. Now $F_0\kappa = \kappa F_0'$ where $F_0' : \mathbf{k}^* \longrightarrow \mathbf{k}^*$ is $x \mapsto x^q$. Hence $F_0'^*\kappa^*c'^*\mathcal{L} = \bar{\mathbf{Q}}_l = F_0'^*\bar{\mathbf{Q}}_l$ and $\kappa^*c'^*\mathcal{L} = \bar{\mathbf{Q}}_l$. This proves (i).

By 5.6(b) we have $\omega^{-1}(R^+_{\mathcal{L}}) \subset R^+$. Moreover, $c^{-1}(R^+) = R^+$ hence $c^{-1}\omega^{-1}(R^+_{\mathcal{L}}) \subset R^+$. Using (i) we see that $c'^{-1}(R^+_{\mathcal{L}}) \subset R^+_{\mathcal{L}}$ hence $c'(R^+_{\mathcal{L}}) = R^+_{\mathcal{L}}$. The lemma is proved.

5.9. Let $\tilde{\mathbf{s}} = (\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_{\tilde{r}})$ be a second sequence in $\mathbf{I}$. Then the subset $\mathcal{I}_{\tilde{\mathbf{s}}}$ is defined in terms of $\tilde{\mathbf{s}}, \mathcal{L}$ in the same way as $\mathcal{I}_{\mathbf{s}}$ is defined in terms of $\mathbf{s}, \mathcal{L}$.

We write the elements of $\mathcal{I}_{\tilde{\mathbf{s}}}$ in ascending order: $j_1 < j_2 < \dots < j_{\tilde{b}}$. Define $\tilde{\mathbf{S}} = (\tilde{S}_1, \tilde{S}_2, \dots, \tilde{S}_{\tilde{b}}), \tilde{\omega}$ in terms of $\tilde{\mathbf{s}}, \mathcal{L}$ in the same way that $\mathbf{S}, \omega$ are defined in 5.5 in terms of $\mathbf{s}, \mathcal{L}$. As in 5.6 we have $\tilde{S}_i \in \mathbf{I}_{\mathcal{L}}$ for $i \in [1, \tilde{b}]$ and $\tilde{s}_1\tilde{s}_2\dots\tilde{s}_{\tilde{r}} = \tilde{S}_1\tilde{S}_2\dots\tilde{S}_{\tilde{b}}\tilde{\omega}$. We assume that

$$F_0^*([\mathbf{s}]c)^*\mathcal{L} \cong \mathcal{L}, F_0^*([\tilde{\mathbf{s}}]c)^*\mathcal{L} \cong \mathcal{L}.$$

From 5.8 we see that ωc and $\tilde{\omega}c$ are automorphisms of finite order of T that induce a permutation of $R_{\mathcal{L}}$, one of $\check{R}_{\mathcal{L}}$ and one of $R^+_{\mathcal{L}}$. Moreover, we have

$$F_0^*(S_1S_2\dots S_b\omega c)^*\mathcal{L} \cong \mathcal{L}, F_0^*(\tilde{S}_1\tilde{S}_2\dots\tilde{S}_{\tilde{b}}\tilde{\omega}c)^*\mathcal{L} \cong \mathcal{L},$$

or equivalently $F_0^*(\omega c)^*\mathcal{L} \cong \mathcal{L}, F_0^*(\tilde{\omega}c)^*\mathcal{L} \cong \mathcal{L}$. Let

$$\mathfrak{F} = \{f \in W'^0_{\mathcal{L}}; f^{-1}\omega c f = \tilde{\omega}c\}.$$

Now for $f \in \mathfrak{F}$ we have $f\mathbf{I}_{\mathcal{L}}f^{-1} = \mathbf{I}_{\mathcal{L}}$; we set

$${}^f\tilde{\mathbf{S}} = (f\tilde{S}_1f^{-1}, f\tilde{S}_2f^{-1}, \dots f\tilde{S}_{\tilde{b}}f^{-1}),$$

a sequence in $\mathbf{I}_{\mathcal{L}}$. Since $F_0^*(\omega c)^*\mathcal{L} \cong \mathcal{L}$, we have

$$F_0^*((f\tilde{S}_1f^{-1})(f\tilde{S}_2f^{-1})\dots(f\tilde{S}_{\tilde{b}}f^{-1})\omega c)^*\mathcal{L} \cong \mathcal{L}.$$

We set $\rho = r + \tilde{r}$. Let $\mathcal{A}(W, c, \mathcal{L}, \mathbf{s}, \tilde{\mathbf{s}})$ be the set of all sequences $(a_0, a_1, \dots, a_\rho)$ in W such that

$a_{j-1}^{-1}a_j \in \{1, \tilde{s}_j\}$ for $j \in \mathcal{I}_{\tilde{\mathbf{s}}}$;
$a_{j-1}^{-1}a_j = \tilde{s}_j$ for $j \in [1, \tilde{r}] - \mathcal{I}_{\tilde{\mathbf{s}}}$;
$a_{\tilde{r}+i}a_{\tilde{r}+i-1}^{-1} \in \{1, s_i\}$ for $i \in \mathcal{I}_{\mathbf{s}}$;
$a_{\tilde{r}+i}a_{\tilde{r}+i-1}^{-1} = s_i$ for $i \in [1, r] - \mathcal{I}_{\mathbf{s}}$;
$a_\rho = ca_0c^{-1}$;
$a_0^*\mathcal{L} \cong \mathcal{L}$.

Replacing here $W, c, \mathcal{L}, \mathbf{s}, \tilde{\mathbf{s}}$ by $W_{\mathcal{L}}, \omega c, \bar{\mathbf{Q}}_l, \mathbf{S}, {}^f\tilde{\mathbf{S}}$, we obtain for any $f \in \mathfrak{F}$ a set $\mathcal{A}(W_{\mathcal{L}}, \omega c, \bar{\mathbf{Q}}_l, \mathbf{S}, {}^f\tilde{\mathbf{S}})$. From the definition, $\mathcal{A}(W_{\mathcal{L}}, \omega c, \bar{\mathbf{Q}}_l, \mathbf{S}, {}^f\tilde{\mathbf{S}})$ consists of all sequences $(A_0, A_1, \dots, A_{b+\tilde{b}})$ in $W_{\mathcal{L}}$ such that

$A_{j-1}^{-1}A_j \in \{1, f\tilde{S}_jf^{-1}\}$ for $j \in [1, \tilde{b}]$;
$A_{\tilde{b}+i}A_{\tilde{b}+i-1}^{-1} \in \{1, S_i\}$ for $i \in [1, b]$;
$A_{b+\tilde{b}} = (\omega c)A_0(\omega c)^{-1}$.

We now state the following result.

Proposition 5.10. *There is a canonical bijection*

$$\Psi : \mathcal{A}(W, c, \mathcal{L}, \mathbf{s}, \tilde{\mathbf{s}}) \xrightarrow{\sim} \sqcup_{f \in \mathfrak{F}} \mathcal{A}(W_{\mathcal{L}}, \omega c, \bar{\mathbf{Q}}_l, \mathbf{S}, {}^f\tilde{\mathbf{S}}).$$

Let $(a_0, a_1, \dots, a_\rho) \in \mathcal{A}(W, c, \mathcal{L}, \mathbf{s}, \tilde{\mathbf{s}})$. Consider the product

$$a_{\tilde{r}}(a_{\tilde{r}+i_1}^{-1} a_{\tilde{r}+i_1-1})a_{\tilde{r}}^{-1} = (a_{\tilde{r}} a_{\tilde{r}+1}^{-1})(a_{\tilde{r}+1}\alpha_{\tilde{r}+2}^{-1}) \dots (a_{\tilde{r}+i_1-2} a_{\tilde{r}+i_1-1}^{-1})$$
$$\times (a_{\tilde{r}+i_1-1} a_{\tilde{r}+i_1}^{-1})(a_{\tilde{r}+i_1-1} a_{\tilde{r}+i_1-2}^{-1}) \dots (a_{\tilde{r}+2} a_{\tilde{r}+1}^{-1})(a_{\tilde{r}+1} a_{\tilde{r}}^{-1}).$$

The right hand side is equal to $s_1 s_2 \dots s_{i_1-1} x s_{i_1-1} \dots s_2 s_1$ where x is either s_{i_1} or 1. Hence it is equal to $s_1 s_2 \dots s_{i_1-1} s_{i_1} s_{i_1-1} \dots s_2 s_1$ or to 1. Thus we have

$$a_{\tilde{r}}(a_{\tilde{r}+i_1}^{-1} a_{\tilde{r}+i_1-1})a_{\tilde{r}}^{-1} \in \{S_1, 1\}.$$

Similarly,

$$a_{\tilde{r}}(a_{\tilde{r}+i_1-1}^{-1} a_{\tilde{r}+i_1})(a_{\tilde{r}+i_2-1}^{-1} a_{\tilde{r}+i_2}) \dots (a_{\tilde{r}+i_{e-1}-1}^{-1} a_{\tilde{r}+i_{e-1}})(a_{\tilde{r}+i_e}^{-1} a_{\tilde{r}+i_e-1})$$
$$\times (a_{\tilde{r}+i_{e-1}}^{-1} a_{\tilde{r}+i_{e-1}-1}) \dots (a_{\tilde{r}+i_2}^{-1} a_{\tilde{r}+i_2-1})(a_{\tilde{r}+i_1}^{-1} a_{\tilde{r}+i_1-1})a_{\tilde{r}}^{-1} \in \{S_e, 1\}$$

for $e \in [1, b]$,

$$a_0^{-1}(a_{j_1-1} a_{j_1}^{-1})(a_{j_2-1} a_{j_2}^{-1}) \dots (a_{j_{e-1}-1} a_{j_{e-1}}^{-1})(a_{j_e} a_{j_e-1}^{-1})$$
$$(a_{j_{e-1}} a_{j_{e-1}-1}^{-1}) \dots (a_{j_2} a_{j_2-1}^{-1})(a_{j_1} a_{j_1-1}^{-1}) \in \{\tilde{S}_e, 1\}$$

for $e \in [1, \tilde{b}]$,

$$\omega = a_{\tilde{r}}(a_{\tilde{r}+i_1-1}^{-1} a_{\tilde{r}+i_1})(a_{\tilde{r}+i_2-1}^{-1} a_{\tilde{r}+i_2}) \dots (a_{\tilde{r}+i_b-1}^{-1} a_{\tilde{r}+i_b})a_\rho^{-1},$$

$$\tilde{\omega} = a_0^{-1}(a_{j_1-1} a_{j_1}^{-1})(a_{j_2-1} a_{j_2}^{-1}) \dots (a_{j_{\tilde{b}-1}} a_{j_{\tilde{b}}}^{-1})a_{\tilde{r}}.$$

Setting

(a) $\quad \hat{a}_e = a_{j_e} a_{j_e-1}^{-1}$ for $e \in [1, \tilde{b}]$, $\quad \hat{a}_{\tilde{b}+e} = a_{\tilde{r}} a_{\tilde{r}+i_e}^{-1} a_{\tilde{r}+i_e-1} a_{\tilde{r}}^{-1}$ for $e \in [1, b]$

we see that

(b) $\quad \hat{a}_{\tilde{b}+1}^{-1} \hat{a}_{\tilde{b}+2}^{-1} \dots \hat{a}_{\tilde{b}+e-1}^{-1} \hat{a}_{\tilde{b}+e} \hat{a}_{\tilde{b}+e-1} \dots \hat{a}_{\tilde{b}+2} \hat{a}_{\tilde{b}+1} \in \{S_e, 1\}$ for $e \in [1, b]$,

(b) $\quad a_0^{-1} \hat{a}_1^{-1} \hat{a}_2^{-1} \dots \hat{a}_{e-1}^{-1} \hat{a}_e \hat{a}_{e-1} \dots \hat{a}_2 \hat{a}_1 a_0 \in \{\tilde{S}_e, 1\}$ for $e \in [1, \tilde{b}]$,

(c) $$\omega = \hat{a}_{\tilde{b}+1}^{-1} \hat{a}_{\tilde{b}+2}^{-1} \dots \hat{a}_{\tilde{b}+b}^{-1} a_{\tilde{r}} a_\rho^{-1},$$

(d) $$\tilde{\omega} = a_0^{-1}\hat{a}_1^{-1}\hat{a}_2^{-1}\dots\hat{a}_{\tilde{b}}^{-1}a_{\tilde{r}}.$$

Since $S_e = S_e^{-1} \in W_{\mathcal{L}}$, $\tilde{S}_e = \tilde{S}_e^{-1} \in W_{\mathcal{L}}$ it follows by induction on e that

(e) $$\hat{a}_{\tilde{b}+e} = \hat{a}_{\tilde{b}+e}^{-1} \in W_{\mathcal{L}} \text{ for } e \in [1,b],$$

$$a_0^{-1}\hat{a}_e a_0 = a_0^{-1}\hat{a}_e^{-1}a_0 \in W_{\mathcal{L}} \text{ for } e \in [1,\tilde{b}].$$

Since a_0 normalizes $W_{\mathcal{L}}$ it follows that

(f) $$\hat{a}_e = \hat{a}_e^{-1} \in W_{\mathcal{L}} \text{ for } e \in [1,\tilde{b}].$$

Since $a_0^*\mathcal{L} \cong \mathcal{L}$ we can write uniquely

(g) $$a_0 = A_0 f \text{ with } A_0 \in W_{\mathcal{L}}, f \in W'^0_{\mathcal{L}}.$$

We set

(h) $$A_e = \hat{a}_e \dots \hat{a}_2\hat{a}_1 a_0 f^{-1} \quad (e \in [1,\tilde{b}]),$$

(i) $$A_{\tilde{b}+e} = \hat{a}_{\tilde{b}+1}\hat{a}_{\tilde{b}+2}\dots\hat{a}_{\tilde{b}+e}\hat{a}_{\tilde{b}}\dots\hat{a}_2\hat{a}_1 a_0 f^{-1} \quad (e \in [1,b]).$$

From (e),(f),(g) we see that $A_e \in W_{\mathcal{L}}$ for $e \in [0, b+\tilde{b}]$. From the definitions we have

$$A_{e-1}^{-1}A_e = fS_e f^{-1} \quad (e \in [1,\tilde{b}]),$$

$$A_{\tilde{b}+e-1}A_{\tilde{b}+e}^{-1} = S_e \quad (e \in [1,b]).$$

We have

$$\begin{aligned} A_{\tilde{b}+b} &= \hat{a}_{\tilde{b}+1}\hat{a}_{\tilde{b}+2}\dots\hat{a}_{\tilde{b}+b}\hat{a}_{\tilde{b}}\dots\hat{a}_2\hat{a}_1 a_0 f^{-1} = \omega a_\rho \tilde{\omega}^{-1} f^{-1} \\ &= (\omega c)(c^{-1}a_\rho c)(\tilde{\omega}c)^{-1}f^{-1} = (\omega c)a_0(\tilde{\omega}c)^{-1}f^{-1} = (\omega c)A_0 f(\tilde{\omega}c)^{-1}f^{-1}. \end{aligned}$$

In particular, $(\omega c)wf(\tilde{\omega}c)^{-1}f^{-1} = w'$ for some $w \in W_{\mathcal{L}}$. We have $(\omega c)w = w_1(\omega c)$ for some $w_1 \in W_{\mathcal{L}}$ hence $(\omega c)f(\tilde{\omega}c)^{-1}f^{-1} = w_1^{-1}w'$ belongs to $W_{\mathcal{L}} \cap W'^0_{\mathcal{L}} = \{1\}$. Thus $(\omega c)f(\tilde{\omega}c)^{-1}f^{-1} = 1$ that is $f \in \mathfrak{F}$. We also see that

$$A_{\tilde{b}+b} = (\omega c)A_0 f(\tilde{\omega}c)^{-1}f^{-1} = (\omega c)A_0(\omega c)^{-1}.$$

Thus to each $(a_0, a_1, \dots, a_\rho) \in \mathcal{A}(W, c, \mathcal{L}, \mathbf{s}, \tilde{\mathbf{s}})$ we have associated $f \in \mathfrak{F}$ and an element $(A_0, A_1, \dots, A_{b+\tilde{b}})$ in $\mathcal{A}(W_{\mathcal{L}}, \omega c, \bar{\mathbf{Q}}_l, \mathbf{S}, {}^f\tilde{\mathbf{S}})$. This defines the map Ψ in the proposition.

Conversely, assume that we are given $f \in \mathfrak{F}$ and an element $(A_0, A_1, \dots, A_{b+\tilde b})$ in $\mathcal{A}(W_{\mathcal{L}}, \omega c, \bar{\mathbf{Q}}_l, \mathbf{S}, {}^f\tilde{\mathbf{S}})$. We will construct a sequence $(a_0, a_1, \dots, a_\rho)$ in W as follows.

We set $a_0 = A_0 f$. We define $\hat a_1, \hat a_2, \dots, \hat a_{\tilde b}$ inductively so that (h) holds. We define $\hat a_{\tilde b+1}, \hat a_{\tilde b+2}, \dots, \hat a_{\tilde b+b}$ inductively so that (i) holds. We define the elements $a_0, a_1, a_2, \dots, a_{\tilde r}$ by induction as follows. Note that a_0 is already defined. Assume that $a_0, a_1, \dots, a_{u-1}$ are already defined for some $u \in [1, \tilde r]$. If $u = j_e$ for some $e \in [1, \tilde b]$ we set $a_u = \hat a_e a_{u-1}$. If $u \notin \mathcal{I}_{\tilde{\mathbf{s}}}$ we set $a_u = a_{u-1}\tilde s_u$. This completes the definition of $a_0, a_1, a_2, \dots, a_{\tilde r}$. We define the elements $a_{\tilde r}, a_{\tilde r+1}, \dots, a_{\tilde r+r}$ by induction as follows. Note that $a_{\tilde r}$ is already defined. Assume that $a_{\tilde r}, a_{\tilde r+1}, \dots, a_{\tilde r+u-1}$ are already defined for some $u \in [1, r]$. If $u = i_e$ for some $e \in [1, b]$ we set $a_{\tilde r+u} = a_{\tilde r+u-1} a_{\tilde r}^{-1} \hat a_{\tilde b+e} a_{\tilde r}$. If $u \notin \mathcal{I}_{\mathbf{s}}$ we set $a_{\tilde r+u} = s_u a_{\tilde r+u-1}$. This completes the definition of $a_{\tilde r}, a_{\tilde r+1}, \dots, a_{\tilde r+r}$.

From the definitions we see that $(a_0, a_1, a_2, \dots, a_\rho) \in \mathcal{A}(W, c, \mathcal{L}, \mathbf{s}, \tilde{\mathbf{s}})$. We have thus constructed a map

$$\sqcup_{f \in \mathfrak{F}} \mathcal{A}(W_{\mathcal{L}}, \omega c, \bar{\mathbf{Q}}_l, \mathbf{S}, {}^f\tilde{\mathbf{S}}) \to \mathcal{A}(W, c, \mathcal{L}, \mathbf{s}, \tilde{\mathbf{s}}).$$

From the definitions we see that this is an inverse to Ψ. The proposition is proved.

6. A basis for a space of intertwining operators

6.1. Let $\mathcal{L}, \tilde{\mathcal{L}} \in \mathcal{S}(T)$. Let $\mathbf{s} = (s_1, s_2, \dots, s_r)$, $\tilde{\mathbf{s}} = (\tilde s_1, \tilde s_2, \dots, \tilde s_{\tilde r})$ be two sequences in $\mathbf{I}$ such that $([\mathbf{s}]F)^*\mathcal{L} \cong \mathcal{L}, ([\tilde{\mathbf{s}}]F)^*\tilde{\mathcal{L}} \cong \tilde{\mathcal{L}}$. Define $\mathcal{I}_{\mathbf{s}} \subset [1, r]$ in terms of $\mathbf{s}, \mathcal{L}$ as in 2.4 or 5.1; define $\mathcal{I}_{\tilde{\mathbf{s}}} \subset [1, \tilde r]$ similarly in terms of $\tilde{\mathbf{s}}, \tilde{\mathcal{L}}$.

We set $\rho = r + \tilde r$. Until the end of 6.13 we fix a sequence $\mathbf{a} = (a_0, a_1, \dots, a_\rho)$ in W such that

$a_{j-1}^{-1} a_j \in \{1, \tilde s_j\}$ for $j \in [1, \tilde r]$;

$a_{\tilde r+i} a_{\tilde r+i-1}^{-1} \in \{1, s_i\}$ for $i \in [1, r]$;

if $j \in [1, \tilde r] - \mathcal{I}_{\tilde{\mathbf{s}}}$, $a_{j-1}\tilde s_j > a_{j-1}$ then $a_{j-1}^{-1} a_j = \tilde s_j$;

if $i \in [1, r] - \mathcal{I}_{\mathbf{s}}$, $s_i a_{\tilde r+i-1} > a_{\tilde r+i-1}$ then $a_{\tilde r+i} a_{\tilde r+i-1}^{-1} = s_i$;

$a_\rho = \mathbf{c}(a_0)$.

We define representatives $\ddot a_i$ for a_i in $N(T)$ ($i \in [0, \rho]$) as follows. We set $\ddot a_0 = \dot a_0$. For $j \in [1, \tilde r]$ we define $\ddot a_j$ inductively by

$\ddot a_j = \ddot a_{j-1}$ if $a_j = a_{j-1}$,

$\ddot a_j = \ddot a_{j-1}\dot{\tilde s}_j$ if $a_j = a_{j-1}\tilde s_j$.

For $i \in [1, r]$ we define $\ddot a_{\tilde r+i}$ inductively by

$\ddot a_{\tilde r+i} = \ddot a_{\tilde r+i-1}$ if $a_{\tilde r+i-1} = a_{\tilde r+i}$,

$\ddot a_{\tilde r+i} = \dot s_i^{-1} \ddot a_{\tilde r+i-1}$ if $a_{\tilde r+i} = s_i a_{\tilde r+i-1}$.

Consider the commutative diagrams

$$\begin{tikzcd}
Z \times \tilde Z & Z_0 \times \tilde Z_0 \arrow[l, "d_0 \times \tilde d_0"'] & Z_1 \times \tilde Z_1 \arrow[l, "d_1 \times \tilde d_1"'] \arrow[r, "d_2 \times \tilde d_2"] & Z_2 \times \tilde Z_2 \arrow[r, "="] & Z_2 \times \tilde Z_2 \\
X \arrow[u, "e"] & X_0 \arrow[l, "b_0"'] \arrow[u, "e_0"] & X_1 \arrow[l, "b_1"'] \arrow[u, "e_1"] \arrow[r, "b_2"] & X_2 \arrow[u, "e_2"] \arrow[r, "b_3"] & X_3 \arrow[u, "e_3"]
\end{tikzcd}$$

$$\begin{tikzcd}
Z_2 \times \tilde{Z}_2 \arrow[r, "="] & Z_2 \times \tilde{Z}_2 \arrow[r, "="] & Z_2 \times \tilde{Z}_2 & Z_2 \times \tilde{Z}_2 \arrow[l, "="'] & X_8 \\
X_3 \arrow[u, "e_3"] & X_4 \arrow[l, "b_4"'] \arrow[u, "e_4"] \arrow[r, "b_5"] & X_5 \arrow[u, "e_5"] & X_6 \arrow[l, "b_6"'] \arrow[u, "e_6"] \arrow[r, "b_7"] & X_7 \arrow[u, "b_8"]
\end{tikzcd}$$

where the following notation is used.

Z, Z_i, d_i are as in 2.6, $\tilde{Z}, \tilde{Z}_i, \tilde{d}_i$ are the analogous objects defined in terms of $\tilde{\mathbf{s}}, \tilde{\mathcal{L}}$ instead of $\mathbf{s}, \mathcal{L}$.

X is the set of all $((B_0, B_1, \dots, B_r), (\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_{\tilde{r}})) \in \mathcal{B}^{[0,r]} \times \mathcal{B}^{[0,\tilde{r}]}$ such that $(A0) - (A5)$ below hold:

$(A0)$ $\mathrm{pos}(B_0, \tilde{B}_j) = a_j (j \in [0, \tilde{r}])$, $\mathrm{pos}(B_i, \tilde{B}_{\tilde{r}}) = a_{\tilde{r}+i} (i \in [0, r])$,
$(A1)$ $\mathrm{pos}(B_{i-1}, B_i) = s_i$ if $i \in [1, r] - \mathcal{I}_{\mathbf{s}}$,
$(A2)$ $\mathrm{pos}(B_{i-1}, B_i) \in \{1, s_i\}$ if $i \in \mathcal{I}_{\mathbf{s}}$,
$(A3)$ $\mathrm{pos}(\tilde{B}_{j-1}, \tilde{B}_j) = \tilde{s}_j$ if $j \in [1, \tilde{r}] - \mathcal{I}_{\tilde{\mathbf{s}}}$,
$(A4)$ $\mathrm{pos}(\tilde{B}_{j-1}, \tilde{B}_j) \in \{1, \tilde{s}_j\}$ if $j \in \mathcal{I}_{\tilde{\mathbf{s}}}$,
$(A5)$ $F(B_0) = B_r, F(\tilde{B}_0) = \tilde{B}_{\tilde{r}}$.

X_0 is the set of all

$$((g_0U, g_1U, \dots, g_rU), (\tilde{g}_0U, \tilde{g}_1U, \dots, \tilde{g}_{\tilde{r}}U)) \in (G/U)^{[0,r]} \times (G/U)^{[0,\tilde{r}]}$$

such that $(B0) - (B5)$ below hold:

$(B0)$ $k(g_0^{-1}\tilde{g}_j) = \ddot{a}_j (j \in [0, \tilde{r}]), k(g_i^{-1}\tilde{g}_{\tilde{r}}) = \ddot{a}_{\tilde{r}+i} (i \in [0, r])$,
$(B1)$ $g_{i-1}^{-1}g_i \in P_{s_i} - B$ if $i \in [1, r] - \mathcal{I}_{\mathbf{s}}$,
$(B2)$ $g_{i-1}^{-1}g_i \in P_{s_i}$ if $i \in \mathcal{I}_{\mathbf{s}}$,
$(B3)$ $\tilde{g}_{j-1}^{-1}\tilde{g}_j \in P_{\tilde{s}_j} - B$ if $j \in [1, \tilde{r}] - \mathcal{I}_{\tilde{\mathbf{s}}}$,
$(B4)$ $\tilde{g}_{j-1}^{-1}\tilde{g}_j \in P_{\tilde{s}_j}$ if $j \in \mathcal{I}_{\tilde{\mathbf{s}}}$,
$(B5)$ $g_r^{-1}F(g_0) \in U$, $\tilde{g}_{\tilde{r}}^{-1}F(\tilde{g}_0) \in U$.

b_0 is

$$\begin{aligned}
&((g_0U, g_1U, \dots, g_rU), (\tilde{g}_0U, \tilde{g}_1U, \dots, \tilde{g}_{\tilde{r}}U)) \mapsto \\
&((g_0Bg_0^{-1}, g_1Bg_1^{-1}, \dots, g_rBg_r^{-1}), (\tilde{g}_0B\tilde{g}_0^{-1}, \tilde{g}_1B\tilde{g}_1^{-1}, \dots, \tilde{g}_{\tilde{r}}B\tilde{g}_{\tilde{r}}^{-1})).
\end{aligned}$$

X_1 is the set of all $((g_0, g_1, \dots, g_r), (\tilde{g}_0, \tilde{g}_1, \dots, \tilde{g}_{\tilde{r}})) \in G^{[0,r]} \times G^{[0,\tilde{r}]}$ such that $(B0) - (B5)$ hold.

b_1 is $((g_0, g_1, \dots, g_r), (\tilde{g}_0, \tilde{g}_1, \dots, \tilde{g}_{\tilde{r}})) \mapsto ((g_0U, g_1U, \dots, g_rU)), (\tilde{g}_0U, \tilde{g}_1U, \dots, \tilde{g}_{\tilde{r}}U))$.

X_2 is the set of all $(x, x', u, u', y_0, y_1, \dots, y_\rho) \in G \times G \times U \times U \times G^{[0,\rho]}$ such that $(C0) - (C'5)$ below hold:

$(C0)$ $k(y_z) = \ddot{a}_z (z \in [0, \rho])$
$(C1)$ $y_{\tilde{r}+i-1}y_{\tilde{r}+i}^{-1} \in P_{s_i} - B$ if $i \in [1, r] - \mathcal{I}_{\mathbf{s}}$,
$(C2)$ $y_{\tilde{r}+i-1}y_{\tilde{r}+i}^{-1} \in P_{s_i}$ if $i \in \mathcal{I}_{\mathbf{s}}$,
$(C3)$ $y_{j-1}^{-1}y_j \in P_{\tilde{s}_j} - B$ if $j \in [1, \tilde{r}] - \mathcal{I}_{\tilde{\mathbf{s}}}$,

$(C4)$ $y_{j-1}^{-1}y_j \in P_{\tilde{s}_j}$ if $j \in \mathcal{I}_{\tilde{\mathbf{s}}}$,
$(C5)$ $uF(y_0) = y_\rho u'$,
$(C'5)$ $y_0 x' = xF(y_0)$, $y_\rho y_{\tilde{r}}^{-1} x = u$.
b_2 is $((g_0, g_1, \dots, g_r), (\tilde{g}_0, \tilde{g}_1, \dots, \tilde{g}_{\tilde{r}})) \mapsto (x, x', u, u', y_0, y_1, \dots, y_\rho)$
where

$$x = g_0^{-1}F(g_0), x' = \tilde{g}_0^{-1}F(\tilde{g}_0), u = g_r^{-1}F(g_0), u' = \tilde{g}_{\tilde{r}}^{-1}F(\tilde{g}_0),$$
$$y_j = g_0^{-1}\tilde{g}_j (j \in [0, \tilde{r}]), y_{\tilde{r}+i} = g_i^{-1}\tilde{g}_{\tilde{r}} (i \in [0, r]).$$

X_3 is the set of all $(u, u', y_0, y_1, \dots, y_\rho) \in U^2 \times G^{[0,\rho]}$ such that $(C0) - (C5)$ hold.

b_3 is $(x, x', u, u', y_0, y_1, \dots, y_\rho) \mapsto (u, u', y_0, y_1, \dots, y_\rho)$.

X_4 is the set of all $(u, u', v, v', \tilde{v}, \tilde{v}', y_0, y_1, \dots, y_\rho) \in U^6 \times G^{[0,\rho]}$ such that $(C0) - (C4)$ hold and

$$y_0 = v\ddot{a}_0 v', y_\rho = \tilde{v}F(\ddot{a}_0)\tilde{v}', uF(v)F(\ddot{a}_0)F(v') = \tilde{v}F(\ddot{a}_0)\tilde{v}'u'.$$

b_4 is $(u, u', v, v', \tilde{v}, \tilde{v}', y_0, y_1, \dots, y_\rho) \mapsto (u, u', y_0, y_1, \dots, y_\rho)$.

X_5 is the set of all $(v, v', y_0, y_1, \dots, y_\rho) \in U^2 \times G^{[0,\rho]}$ such that $(C0) - (C4)$ hold and $y_0 = v\ddot{a}_0 v'$.

b_5 is $(u, u', v, v', \tilde{v}, \tilde{v}', y_0, y_1, \dots, y_\rho) \mapsto (v, v', y_0, y_1, \dots, y_\rho)$.

X_6 is the set of all $((g_0, g_1, \dots, g_r), (\tilde{g}_0, \tilde{g}_1, \dots, \tilde{g}_{\tilde{r}})) \in G^{[0,r]} \times G^{[0,\tilde{r}]}$ such that $(B0) - (B4)$ hold and $g_0 \in U, \tilde{g}_0 \in \ddot{a}_0 U$.

b_6 is $((g_0, g_1, \dots, g_r), (\tilde{g}_0, \tilde{g}_1, \dots, \tilde{g}_{\tilde{r}})) \mapsto (v, v', y_0, y_1, \dots, y_\rho)$
where

$$v = g_0^{-1}, v' = \ddot{a}_0^{-1}\tilde{g}_0, y_j = g_0^{-1}\tilde{g}_j (j \in [0, \tilde{r}]), y_{\tilde{r}+i} = g_i^{-1}\tilde{g}_{\tilde{r}} (i \in [0, r]).$$

X_7 is the set of all

$$((g_0U, g_1U, \dots, g_rU), (\tilde{g}_0U, \tilde{g}_1U, \dots, \tilde{g}_{\tilde{r}}U)) \in (G/U)^{[0,r]} \times (G/U)^{[0,\tilde{r}]}$$

such that $(B0) - (B4)$ hold and $g_0 \in U, \tilde{g}_0 \in \ddot{a}_0 U$.

b_7 is given by the same formula as b_1.

X_8 is the set of all $((B_0, B_1, \dots, B_r), (\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_{\tilde{r}})) \in \mathcal{B}^{[0,r]} \times \mathcal{B}^{[0,\tilde{r}]}$ such that $(A0) - (A4)$ hold and $B_0 = B, \tilde{B}_0 = \ddot{a}_0 B \ddot{a}_0^{-1}$.

b_8 is given by the same formula as b_0.

e, e_0 are the obvious imbeddings.

e_1 is

$$((g_0, g_1, \dots, g_r), (\tilde{g}_0, \tilde{g}_1, \tilde{g}_{\tilde{r}})) \mapsto ((g_0, g_0 g_1^{-1}, \dots, g_{r-1}^{-1}g_r), (\tilde{g}_0, \tilde{g}_0\tilde{g}_1^{-1}, \dots, \tilde{g}_{\tilde{r}-1}^{-1}\tilde{g}_{\tilde{r}})).$$

e_2 is $(x, x', u, u', y_0, y_1, \dots, y_\rho) \mapsto \lambda(y_0, y_1, \dots, y_\rho)$ where

$$\lambda(y_0, y_1, \dots, y_\rho) = ((y_{\tilde{r}}y_{\tilde{r}+1}^{-1}, y_{\tilde{r}+1}y_{\tilde{r}+2}^{-1}, \dots, y_{\tilde{r}+r-1}y_{\tilde{r}+r}^{-1}), (y_0^{-1}y_1, y_1^{-1}y_2, \dots, y_{\tilde{r}-1}^{-1}y_{\tilde{r}})). \blacksquare$$

e_3 is $(u, u', y_0, y_1, \dots, y_\rho) \mapsto \lambda(y_0, y_1, \dots, y_\rho)$.

e_4 is $(u, u', v, v', \tilde{v}, \tilde{v}', y_0, y_1, \dots, y_\rho) \mapsto \lambda(y_0, y_1, \dots, y_\rho)$.

e_5 is $(v, v', y_0, y_1, \dots, y_\rho) \mapsto \lambda(y_0, y_1, \dots, y_\rho)$.

e_6 is

$$((g_0, g_1, \dots, g_r), (\tilde{g}_0, \tilde{g}_1, \dots, \tilde{g}_{\tilde{r}})) \mapsto$$
$$((g_0^{-1} g_1, g_1^{-1} g_2, \dots, g_{r-1}^{-1} g_r), (\tilde{g}_0^{-1} \tilde{g}_1, \tilde{g}_1^{-1} \tilde{g}_2, \dots, \tilde{g}_{r-1}^{-1} \tilde{g}_r)).$$

6.2. Let $\underline{B}, \bar{\mathcal{L}}, \bar{\mathcal{L}}_i$ be as in 2.4, 2.11. Define $\tilde{\underline{B}}, \bar{\tilde{\mathcal{L}}}, \bar{\tilde{\mathcal{L}}}_i$ similarly, in terms of $\tilde{\mathbf{s}}, \tilde{\mathcal{L}}$ instead of $\mathbf{s}, \mathcal{L}$. Note that $\Gamma, \underline{B}$ act on Z and $Z_i (i \in [0, 2])$ as in 2.7. Similarly, $\Gamma, \tilde{\underline{B}}$ act on $\tilde{Z}$ and $\tilde{Z}_i (i \in [0, 2])$. These actions give rise to (commuting) actions of $\Gamma \times \Gamma, \underline{B} \times \tilde{\underline{B}}$ on $Z \times \tilde{Z}$ and $Z_i \times \tilde{Z}_i (i \in [0, 2])$ hence to actions of $\Gamma \times \Gamma \times \underline{B} \times \tilde{\underline{B}}$. By 2.11, $\bar{\mathcal{L}}$ and $\bar{\mathcal{L}}_i$ are $\underline{B}$-equivariant local systems with natural Γ-equivariant structures. Similarly, $\bar{\tilde{\mathcal{L}}}$ and $\bar{\tilde{\mathcal{L}}}_i$ are $\tilde{\underline{B}}$-equivariant local systems with natural Γ-equivariant structures. Hence $\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}$ and $\bar{\mathcal{L}}_i \boxtimes \bar{\tilde{\mathcal{L}}}_i$ are $\underline{B} \times \tilde{\underline{B}}$-equivariant local systems on $Z \times \tilde{Z}$ and $Z_i \times \tilde{Z}_i$ with natural $\Gamma \times \Gamma$-equivariant structures. These structures give rise to $\Gamma \times \Gamma \times \underline{B} \times \tilde{\underline{B}}$-equivariant structures on $\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}$ and $\bar{\mathcal{L}}_i \boxtimes \bar{\tilde{\mathcal{L}}}_i$. From the definitions we have

$$(d_0 \times \tilde{d}_0)^* (\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}) = \bar{\mathcal{L}}_0 \boxtimes \bar{\tilde{\mathcal{L}}}_0, (d_1 \times \tilde{d}_1)^* (\bar{\mathcal{L}}_0 \boxtimes \bar{\tilde{\mathcal{L}}}_0) = \bar{\mathcal{L}}_1 \boxtimes \bar{\tilde{\mathcal{L}}}_1,$$
$$(d_2 \times \tilde{d}_2)^* (\bar{\mathcal{L}}_2 \boxtimes \bar{\tilde{\mathcal{L}}}_2) = \bar{\mathcal{L}}_1 \boxtimes \bar{\tilde{\mathcal{L}}}_1,$$

compatibly with the $\Gamma \times \Gamma \times \underline{B} \times \tilde{\underline{B}}$-equivariant structures.

6.3. Let

$$\mathcal{T} = \{((t_0, t_1, \dots, t_r), (\tilde{t}_0, \tilde{t}_1, \dots, \tilde{t}_{\tilde{r}})) \in T^{[0,r]} \times T^{[0,\tilde{r}]};$$
$$t_r = F(t_0), \tilde{t}_{\tilde{r}} = F(\tilde{t}_0), \tilde{t}_j = a_j^{-1}(t_0) \text{ for } j \in [0, \tilde{r}], t_i = a_{\tilde{r}+i}(\tilde{t}_{\tilde{r}}) \text{ for } i \in [0, r].$$

This is a subgroup of $T^{[0,r]} \times T^{[0,\tilde{r}]}$ isomorphic to the finite subgroup

$$\mathcal{T}_0 = \{t_0 \in T; a_{\tilde{r}}^{-1}(t_0) = F(a_0^{-1}(t_0))\}$$

of T under $((t_i), (\tilde{t}_j)) \mapsto t_0$. Hence $\mathcal{T}$ is finite.

$\Gamma \times \mathcal{T}$ acts on X by

$$(g, (t_i), (\tilde{t}_j)) : ((B_0, B_1, \dots, B_r), (\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_{\tilde{r}})) \mapsto$$
$$((g B_0 g^{-1}, g B_1 g^{-1}, \dots, g B_r g^{-1}), (g \tilde{B}_0 g^{-1}, g \tilde{B}_1 g^{-1}, \dots, g \tilde{B}_{\tilde{r}} g^{-1}));$$

on X_0 by

$$(g, (t_i), (\tilde{t}_j)) : ((g_0 U, g_1 U, \dots, g_r U), (\tilde{g}_0 U, \tilde{g}_1 U, \dots, \tilde{g}_{\tilde{r}} U)) \mapsto$$
$$((g g_0 t_0^{-1} U, g g_1 t_1^{-1} U, \dots, g g_r t_r^{-1} U), (g \tilde{g}_0 \tilde{t}_0^{-1} U, g \tilde{g}_1 \tilde{t}_1^{-1} U, \dots, g \tilde{g}_{\tilde{r}} \tilde{t}_{\tilde{r}}^{-1} U));$$

on X_1 by

$$(g,(t_i),(\tilde t_j)):((g_0,g_1,\dots,g_r),(\tilde g_0,\tilde g_1,\dots,\tilde g_{\tilde r}))\mapsto$$
$$((gg_0t_0^{-1},gg_1t_1^{-1},\dots,gg_rt_r^{-1}),(g\tilde g_0\tilde t_0^{-1},g\tilde g_1\tilde t_1^{-1},\dots,g\tilde g_{\tilde r}\tilde t_{\tilde r}^{-1}));$$

on X_2 by

$(g,(t_i),(\tilde t_j)):(x,x',u,u',y_0,y_1,\dots,y_\rho)\mapsto(t_0xF(t_0)^{-1},\tilde t_0x'F(\tilde t_0)^{-1},t_rut_r^{-1},\tilde t_{\tilde r}u'\tilde t_{\tilde r}^{-1},$
$t_0y_0\tilde t_0^{-1},t_0y_1\tilde t_1^{-1},\dots,t_0y_{\tilde r}\tilde t_{\tilde r}^{-1},t_1y_{\tilde r+1}\tilde t_{\tilde r}^{-1},\dots,t_ry_\rho\tilde t_{\tilde r}^{-1});$

on X_3 by

$$(g,(t_i),(\tilde t_j)):(u,u',y_0,y_1,\dots,y_\rho)\mapsto$$
$$(t_rut_r^{-1},\tilde t_{\tilde r}u'\tilde t_{\tilde r}^{-1},t_0y_0\tilde t_0^{-1},t_0y_1\tilde t_1^{-1},\dots,t_0y_{\tilde r}\tilde t_{\tilde r}^{-1},t_1y_{\tilde r+1}\tilde t_{\tilde r}^{-1},\dots,t_ry_\rho\tilde t_{\tilde r}^{-1});$$

on X_4 by

$(g,(t_i),(\tilde t_j)):(u,u',v,v',\tilde v,\tilde v',y_0,y_1,\dots,y_\rho)\mapsto$
$(t_rut_r^{-1},\tilde t_{\tilde r}u'\tilde t_{\tilde r}^{-1},t_0vt_0^{-1},\tilde t_0v'\tilde t_0^{-1},t_0y_0\tilde t_0^{-1},t_0y_1\tilde t_1^{-1},\dots,t_0y_{\tilde r}\tilde t_{\tilde r}^{-1},t_1y_{\tilde r+1}\tilde t_{\tilde r}^{-1},\dots,t_ry_\rho\tilde t_{\tilde r}^{-1});$

on X_5 by

$$(g,(t_i),(\tilde t_j)):(v,v',\tilde v,\tilde v',y_0,y_1,\dots,y_\rho)\mapsto$$
$$(t_0vt_0^{-1},\tilde t_0v'\tilde t_0^{-1},t_0y_0\tilde t_0^{-1},t_0y_1\tilde t_1^{-1},\dots,t_0y_{\tilde r}\tilde t_{\tilde r}^{-1},t_1y_{\tilde r+1}\tilde t_{\tilde r}^{-1},\dots,t_ry_\rho\tilde t_{\tilde r}^{-1});$$

on X_6 by

$$(g,(t_i),(\tilde t_j)):((g_0,g_1,\dots,g_r),(\tilde g_0,\tilde g_1,\dots,\tilde g_{\tilde r}))\mapsto$$
$$((t_0g_0t_0^{-1},t_0g_1t_1^{-1},\dots,t_0g_rt_r^{-1}),(t_0\tilde g_0\tilde t_0^{-1},t_0\tilde g_1\tilde t_1^{-1},\dots,t_0\tilde g_{\tilde r}\tilde t_{\tilde r}^{-1}));$$

on X_7 by

$$(g,(t_i),(\tilde t_j)):((g_0U,g_1U,\dots,g_rU),(\tilde g_0U,\tilde g_1U,\dots,\tilde g_{\tilde r}U))\mapsto$$
$$((t_0g_0t_0^{-1}U,t_0g_1t_1^{-1}U,\dots,t_0g_rt_r^{-1}U),(t_0\tilde g_0\tilde t_0^{-1}U,t_0\tilde g_1\tilde t_1^{-1}U,\dots,t_0\tilde g_{\tilde r}\tilde t_{\tilde r}^{-1}U));$$

on X_8 by

$$(g,(t_i),(\tilde t_j)):((B_0,B_1,\dots,B_r),(\tilde B_0,\tilde B_1,\dots,\tilde B_{\tilde r}))\mapsto$$
$$((t_0B_0t_0^{-1},t_0B_1t_0^{-1},\dots,t_0B_rt_0^{-1}),(t_0\tilde B_0t_0^{-1},t_0\tilde B_1t_0^{-1},\dots,t_0\tilde B_{\tilde r}t_0^{-1})).$$

The maps b_i are compatible with the $\Gamma\times\mathcal{T}$-actions. Let

$$\mathcal{G}=\{((b_0,b_1,\dots,b_r),(\tilde b_0,\tilde b_1,\dots,\tilde b_{\tilde r}))\in\underline{B}\times\underline{\tilde B};$$
$$((k(b_0),k(b_1),\dots,k(b_r)),(k(\tilde b_0),k(\tilde b_1),\dots,k(\tilde b_{\tilde r})))\in\mathcal{T}\},$$

a subgroup of $\underline{B}\times\underline{\tilde B}$. The $\Gamma\times\mathcal{T}$-action on X_6 extends to a $\Gamma\times\mathcal{G}$-action on X_6:
$(g,(b_i),(\tilde b_j)):((g_0,g_1,\dots,g_r),(\tilde g_0,\tilde g_1,\dots,\tilde g_{\tilde r}))\mapsto$
$((k(b_0)g_0b_0^{-1},k(b_0)g_1b_1^{-1},\dots,k(b_0)g_rb_r^{-1}),(k(b_0)\tilde g_0\tilde b_0^{-1},k(b_0)\tilde g_1\tilde b_1^{-1},\dots,k(b_0)\tilde g_{\tilde r}\tilde b_{\tilde r}^{-1})).$

6.4. Now

(a) b_0 is a principal $\mathcal{T}$-bundle
($\mathcal{T}$ acts on X_0 by restriction of the $\Gamma \times \mathcal{T}$-action) and induces an isomorphism $\mathcal{T}\backslash X_0 \xrightarrow{\sim} X$. (See [L4, 3.4].)

(b) b_1 is a principal $U^{[0,r]} \times U^{[0,\tilde{r}]}$-bundle.

(c) b_2 is a principal Γ-bundle
(Γ acts on X_1 by restriction of the $\Gamma \times \mathcal{T}$-action) and induces an isomorphism $\Gamma\backslash X_1 \xrightarrow{\sim} X_2$. (See [L4, 3.8].)

(d) b_3 is an isomorphism.

(e) b_4 is a quasi-vector bundle (see[L4, 3.2]) with fibres of dimension $2(\dim U - l(a_0))$.

(f) b_5 is a quasi-vector bundle with fibres of dimension $2(\dim U - l(a_0))$.

(g) b_6 is an isomorphism.

(h) b_7 is a principal $U^{[0,r]} \times U^{[0,\tilde{r}]}$-bundle.

(i) b_8 is an isomorphism. (See [L4, 3.24].)

6.5. Now $\mathcal{T}$ is naturally a subgroup of $\underline{B} \times \underline{\tilde{B}}$ and Γ is a subgroup of $\Gamma \times \Gamma$ (the diagonal) hence $\Gamma \times \mathcal{T}$ is a subgroup of $\Gamma \times \Gamma \times \underline{B} \times \underline{\tilde{B}}$. Hence the actions in 6.2 give by restriction actions of $\Gamma \times \mathcal{T}$ on $Z \times \tilde{Z}$ and $Z_i \times \tilde{Z}_i (i \in [0,2])$ and the equivariant structures on $\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}$ and $\bar{\mathcal{L}}_i \boxtimes \bar{\tilde{\mathcal{L}}}_i$ in 6.2 restrict to $\Gamma \times \mathcal{T}$-equivariant structures on these local systems. Since e and $e_i (i \in [0,6])$ are compatible with the $\Gamma \times \mathcal{T}$-actions we see that the local systems $\mathcal{E} = e^*(\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})$, $\mathcal{E}_i = e_i^*(\bar{\mathcal{L}}_i \boxtimes \bar{\tilde{\mathcal{L}}}_i)$, $(i \in [0,2])$ and $\mathcal{E}_i = e_i^*(\bar{\mathcal{L}}_2 \boxtimes \bar{\tilde{\mathcal{L}}}_2)$, $(i \in [3,6])$ have natural $\Gamma \times \mathcal{T}$-equivariant structures. Moreover, the $\Gamma \times \mathcal{T}$-equivariant structure on $\mathcal{E}_6$ extends to a $\Gamma \times \mathcal{G}$-equivariant structure since e_6 is compatible with the $\Gamma \times \mathcal{G}$ actions (see 6.3). Since the restriction of the $\Gamma \times \mathcal{G}$ action on X_6 to the subgroup $U^{[0,r]} \times U^{[0,\tilde{r}]}$ is the free action which makes X_6 a principal bundle over X_7 (see 6.4(h)) it follows that there is a well defined local system $\mathcal{E}_7$ on X_7 with a natural $\Gamma \times \mathcal{T}$-equivariant structure such that $b_7^*\mathcal{E}_7 = \mathcal{E}_6$. Since b_8 is an isomorphism, there is a well defined local system $\mathcal{E}_8$ on X_7 with a natural $\Gamma \times \mathcal{T}$-equivariant structure such that $b_8^*\mathcal{E}_8 = \mathcal{E}_7$. We have

$$\text{(a)} \quad \begin{array}{l} b_0^*\mathcal{E} = \mathcal{E}_0; \quad b_1^*\mathcal{E}_1 = \mathcal{E}_0; \quad b_2^*\mathcal{E}_2 = \mathcal{E}_1; b_3^*\mathcal{E}_3 = \mathcal{E}_2; \quad b_4^*\mathcal{E}_3 = \mathcal{E}_4; \quad b_5^*\mathcal{E}_5 = \mathcal{E}_4; \\ \quad b_6^*\mathcal{E}_5 = \mathcal{E}_6; b_7^*\mathcal{E}_7 = \mathcal{E}_6; \quad b_8^*\mathcal{E}_8 = \mathcal{E}_7; \end{array}$$

compatibly with the $\Gamma \times \mathcal{T}$-equivariant structures. We show:

(b) *$\mathcal{T}$ acts trivially on any stalk of $\mathcal{E}$.*

(c) *Γ acts trivially on any stalk of $\mathcal{E}_i$ (if $i \in [2,8]$).*

(b) follows from the fact that $\underline{B} \times \underline{\tilde{B}}$ acts trivially on $Z \times \tilde{Z}$ and, being connected, it acts trivially on any stalk of $\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}$. Now (c) follows from the fact that $\Gamma \times \Gamma$ acts trivially on $Z_2 \times \tilde{Z}_2$ and on any stalk of $\bar{\mathcal{L}}_2 \boxtimes \bar{\tilde{\mathcal{L}}}_2$.

6.6. For any $h \in [0, \tilde{r}]$ let $\mathfrak{Y}_h$ be the set of all $(\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_h) \in \mathcal{B}^{[0,h]}$ such that
$\mathrm{pos}(B, \tilde{B}_j) = a_j (j \in [0, h])$,
$\mathrm{pos}(\tilde{B}_{j-1}, \tilde{B}_j) = \tilde{s}_j$ if $j \in [1, h], j \notin \mathcal{I}_{\tilde{\mathbf{s}}}$,
$\mathrm{pos}(\tilde{B}_{j-1}, \tilde{B}_j) \in \{1, \tilde{s}_j\}$ if $j \in [1, h] \cap \mathcal{I}_{\tilde{\mathbf{s}}}$,
$\tilde{B}_0 = \ddot{a}_0 B \ddot{a}_0^{-1}$.
Note that $\mathfrak{Y}_0$ is a point. Moreover, if $h \in [1, \tilde{r}]$ then we have an obvious map $\mathfrak{Y}_h \longrightarrow \mathfrak{Y}_{h-1}$ which is
-an isomorphism if $a_h = a_{h-1}\tilde{s}_h < a_{h-1}$;
-an isomorphism if $h \in \mathcal{I}_{\tilde{\mathbf{s}}}$, $a_{h-1}\tilde{s}_h > a_{h-1} = a_h$;
-a line bundle minus the zero section if $h \notin \mathcal{I}_{\tilde{\mathbf{s}}}$, $a_{h-1}\tilde{s}_h < a_{h-1} = a_h$;
-a line bundle if $a_h = a_{h-1}\tilde{s}_h > a_{h-1}$;
-a line bundle if $h \in \mathcal{I}_{\tilde{\mathbf{s}}}$, $a_{h-1}\tilde{s}_h < a_{h-1} = a_h$.
For any $h \in [0, r]$ let $\mathfrak{Y}_{\tilde{r}+h}$ be the set of all

$$((B_0, B_1, \dots, B_h), (\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_{\tilde{r}})) \in \mathcal{B}^{[0,h]} \times \mathcal{B}^{[0,\tilde{r}]}$$

such that
$\mathrm{pos}(B_0, \tilde{B}_j) = a_j (j \in [0, \tilde{r}])$, $\mathrm{pos}(B_i, \tilde{B}_{\tilde{r}}) = a_{\tilde{r}+i} (i \in [0, h])$,
$\mathrm{pos}(B_{i-1}, B_i) = s_i$ if $i \in [1, h], j \notin \mathcal{I}_{\mathbf{s}}$,
$\mathrm{pos}(B_{i-1}, B_i) \in \{1, s_i\}$ if $i \in [1, h] \cap \mathcal{I}_{\mathbf{s}}$,
$\mathrm{pos}(\tilde{B}_{j-1}, \tilde{B}_j) = \tilde{s}_j$ if $j \in [1, \tilde{r}], j \notin \mathcal{I}_{\tilde{\mathbf{s}}}$,
$\mathrm{pos}(\tilde{B}_{j-1}, \tilde{B}_j) \in \{1, \tilde{s}_j\}$ if $j \in \mathcal{I}_{\tilde{\mathbf{s}}}$,
$B_0 = B, \tilde{B}_0 = \ddot{a}_0 B \ddot{a}_0^{-1}$.
Note that $\mathfrak{Y}_{\tilde{r}}$ in the last definition may be identified with $\mathfrak{Y}_{\tilde{r}}$ in the earlier one. Moreover, if $h \in [1, r]$ then we have an obvious map $\mathfrak{Y}_{\tilde{r}+h} \longrightarrow \mathfrak{Y}_{\tilde{r}+h-1}$ which is
-an isomorphism if $a_{\tilde{r}+h} = s_h a_{\tilde{r}+h-1} < a_{\tilde{r}+h-1}$;
-an isomorphism if $h \in \mathcal{I}_{\mathbf{s}}$, $s_h a_{\tilde{r}+h-1} > a_{\tilde{r}+h-1} = a_{\tilde{r}+h}$;
-a line bundle minus the zero section if $h \notin \mathcal{I}_{\mathbf{s}}$, $s_h a_{\tilde{r}+h-1} < a_{\tilde{r}+h-1} = a_{\tilde{r}+h}$;
-a line bundle if $a_{\tilde{r}+h} = s_h a_{\tilde{r}+h-1} > a_{\tilde{r}+h-1}$;
-a line bundle if $h \in \mathcal{I}_{\mathbf{s}}$, $s_h a_{\tilde{r}+h-1} < a_{\tilde{r}+h-1} = a_{\tilde{r}+h}$.

6.7. Assume that for some $h \in [1, r] - \mathcal{I}_{\mathbf{s}}$ we have $a := a_{\tilde{r}+h} = a_{\tilde{r}+h-1}$, $s := s_h$, $sa < a$. We show that
(a) $H_c^*(X_8, \mathcal{E}_8) = 0$.
We have an obvious map $\phi : X_8 \longrightarrow \mathfrak{Y}_{\tilde{r}+h-1}$. (See 6.6.) It is enough to show that for any $p = ((B_0, \dots, B_{h-1}), (\tilde{B}_0, \tilde{B}_1, \dots, tB_{\tilde{r}})) \in \mathfrak{Y}_{\tilde{r}+h-1}$ we have $H_c^*(\mathfrak{Y}', \mathcal{E}_8) = 0$ where $\mathfrak{Y}' = \phi^{-1}(p)$. We may identify $\mathfrak{Y}'$ with the set of all $(B_h, B_{h+1}, \dots, B_r) \in \mathcal{B}^{[h,r]}$ such that
$\mathrm{pos}(B_i, \tilde{B}_{\tilde{r}}) = a_{\tilde{r}+i} (i \in [h, r])$,
$\mathrm{pos}(B_{i-1}, B_i) \in \{s_i, 1\}$ if $i \in [h+1, r] \cap \mathcal{I}_{\mathbf{s}}$.
$\mathrm{pos}(B_{i-1}, B_i) = s_i$ if $i \in [h, r], i \notin \mathcal{I}_{\mathbf{s}}$.
Since $sa < a$, there is a unique $D \in \mathcal{B}$ such that $\mathrm{pos}(B_{h-1}, D) = s$, $\mathrm{pos}(D, \tilde{B}_{\tilde{r}}) = sa$. Pick $C \in \mathcal{B}$ such that $\mathrm{pos}(\tilde{B}_{\tilde{r}}, C) = a^{-1} w_{\mathbf{I}}$. Since $\mathrm{pos}(D, C) = s w_{\mathbf{I}}$, $V :=$

$U_D \cap U_C$ is a one dimensional connected unipotent group. Now $\tilde{B}_{\tilde{r}}$ is the unique Borel such that $\mathrm{pos}(D, \tilde{B}_{\tilde{r}}) = sa$, $\mathrm{pos}(\tilde{B}_{\tilde{r}}, C) = a^{-1}w_{\mathbf{I}}$. Hence it is normalized by V and $V \subset U_{\tilde{B}_{\tilde{r}}}$. Since $\mathrm{pos}(B_{h-1}, C) = w_{\mathbf{I}}$, we have $U_{B_{h-1}} \cap U_C = \{1\}$ hence $U_{B_{h-1}} \cap V = \{1\}$. Let

$$\begin{aligned}\Xi &= \{E \in \mathcal{B}; \mathrm{pos}(B_{h-1}, E) = s, \mathrm{pos}(E, \tilde{B}_{\tilde{r}}) = a\}\\ &= \{E \in \mathcal{B}; \mathrm{pos}(B_{h-1}, E) = s, \mathrm{pos}(E, D) = s\}.\end{aligned}$$

Since $V \subset U_D$, $V \cap U_{B_{h-1}} = \{1\}$, V acts simply transitively (by conjugation) on the affine line $\{E' \in \mathcal{B}; \mathrm{pos}(E', D) = s\} = \Xi \sqcup \{B_{h-1}\}$. Pick $B_h \in \xi$. Define $v_0 \in V - \{1\}$ by $v_0 B_h v_0^{-1} = B_{h-1}$. Let

$$\mathfrak{Y}'' = \{(B_{h+1}, \dots, B_r) \in \mathcal{B}^{[h+1,r]}; (B_h, B_{h+1}, \dots, B_r) \in \mathfrak{Y}'\}.$$

The map $\zeta : (V - \{v_0\}) \times \mathfrak{Y}'' \longrightarrow \mathfrak{Y}'$,

$$(v, (B_{h+1}, \dots, B_r)) \mapsto (vB_h v^{-1}, vB_{h+1}v^{-1}, \dots, vB_r v^{-1})$$

is an isomorphism. Hence it is enough to show that

$$H_c^*((V - \{v_0\}) \times \mathfrak{Y}'', \zeta^*\mathcal{E}_8) = 0.$$

Let $\pi'' : (V - \{v_0\}) \times \mathfrak{Y}'' \longrightarrow \mathfrak{Y}''$ be the projection. It is enough to show that for any $p' = (B_{h+1}, \dots, B_r) \in \mathfrak{Y}''$ we have $H_c^*(\pi''^{-1}(p'), \zeta^*\mathcal{E}_8) = 0$ or equivalently that $H_c^*(V - \{v_0\}, \zeta'^*\mathcal{E}_8) = 0$ where $\zeta' : V - \{v_0\} \longrightarrow X_8$ is

$$v \mapsto ((B_0, B_1, \dots, B_{h-1}, vB_h v^{-1}, vB_{h+1}v^{-1}, \dots, vB_r v^{-1}), (\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_{\tilde{r}})).$$

Since $\zeta'^*\mathcal{E}_8$ is a local system of rank 1 on $V - \{v_0\} \cong \mathbf{k}^*$ with monodromy of finite order invertible in $\mathbf{k}$, it is enough to show that $\zeta'^*\mathcal{E}_8 \not\cong \bar{\mathbf{Q}}_l$. We can find $\epsilon = ((g_0, g_1, \dots, g_r), (\tilde{g}_0, \tilde{g}_1, \dots, \tilde{g}_{\tilde{r}})) \in X_6$ such that

$$b_8(b_7(\epsilon)) = ((B_0, B_1, \dots, B_{h-1}, B_h, B_{h+1}, \dots, B_r), (\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_{\tilde{r}})),$$

$g_{h-1}^{-1}\tilde{g}_{\tilde{r}} = \ddot{a}$ and $V = g_{h-1}y_{s_h}(\mathbf{k})g_{h-1}^{-1}$. Define $\lambda_0 \in \mathbf{k}$ by $y_{s_h}(\lambda_0)g_{h-1}^{-1}g_h \in B$. Define $\tilde{\zeta} : \mathbf{k} - \{\lambda_0\} \longrightarrow X_6$ by

$$\lambda \mapsto ((g_0, g_1, \dots, g_{h-1}, g_{h-1}y_{s_h}(\lambda)g_{h-1}^{-1}g_h, \dots, g_{h-1}y_{s_h}(\lambda)g_{h-1}^{-1}g_r), (\tilde{g}_0, \tilde{g}_1, \dots, \tilde{g}_{\tilde{r}})).$$

We have $\zeta'^*\mathcal{E}_8 = \tilde{\zeta}^*\mathcal{E}_6$. It is enough to prove that $\tilde{\zeta}^*\mathcal{E}_6 \not\cong \bar{\mathbf{Q}}_l$ or that $\tilde{\zeta}^*e_6^*(\bar{\mathcal{L}}_2 \boxtimes \bar{\bar{\mathcal{L}}}_2) \not\cong \bar{\mathbf{Q}}_l$. Note that $e_6\tilde{\zeta} : \mathbf{k} - \{\lambda_0\} \longrightarrow Z_2 \times \tilde{Z}_2$ is

$$\lambda \mapsto ((y_1, y_2, y_{h-1}, y_{s_h}(\lambda)y_h, y_{h+1}, \dots, y_r), (\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{\tilde{r}}))$$

where

$$((y_1, y_2, y_{h-1}, y_h, y_{h+1}, \dots, y_r), (\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{\tilde{r}})) = e_6(\epsilon).$$

From this and the definition of $\bar{\mathcal{L}}_2 = \underline{\underline{\mathcal{L}}}$ (see 2.4) we see that $(e_6\tilde{\zeta})^*(\bar{\mathcal{L}}_2 \boxtimes \bar{\bar{\mathcal{L}}}_2)$ is isomorphic to the inverse image of $\mathcal{L}$ under a map $\mathbf{k} - \{\lambda_0\} \longrightarrow T$ of the form

$$\lambda \mapsto t'\dot{s}_1 \dots \dot{s}_{h-1}k(y_{s_h}(\lambda - \lambda_0))\dot{s}_h^{-1} \dots \dot{s}_1^{-1}$$

where t' is a fixed element of T. This is also of the form $\lambda \mapsto \beta(\lambda - \lambda_0)t'$ where $\beta : \mathbf{k}^* \longrightarrow T$ is one of the two coroots with associated reflection $s_1s_2 \dots s_h \dots s_2s_1$. The desired result follows from the fact that $\beta^*(\mathcal{L}) \not\cong \bar{\mathbf{Q}}_l$. (Recall that $h \notin \mathcal{I}_{\mathbf{s}}$).

6.8. Assume that for some $h \in [1,\tilde r]-\mathcal{I}_{\tilde{\mathbf{s}}}$ we have $a := a_h = a_{h-1}$, $s := \tilde s_h$, $as < a$. We have

(a) $H_c^*(X_8, \mathcal{E}_8) = 0$.

The proof is entirely similar to that of 6.7(a).

6.9. Let

$$N_{\mathbf{a}} = \\ |\{h \in [1,\tilde r]; a_{h-1} \le a_h \ge a_{h-1}\tilde s_h\}| + |\{h \in [1,r]; a_{\tilde r+h-1} \le a_{\tilde r+h} \ge s_h a_{\tilde r+h-1}\}|.$$

Consider the following condition on $\mathbf{a}$:

(a) *for any $h \in [1,r]-\mathcal{I}_{\mathbf{s}}$ we have $a_{\tilde r+h} = s_h a_{\tilde r+h-1}$; for any $h \in [1,\tilde r]-\mathcal{I}_{\tilde{\mathbf{s}}}$ we have $a_h = a_{h-1}\tilde s_h$.*

If $\mathbf{a}$ satisfies (a), we have the following results:

(b) *X_8 is isomorphic to an affine space of dimension $N_{\mathbf{a}}$.*

(c) *X has pure dimension $N_{\mathbf{a}}$.*

Indeed, from the results in 6.6 we see by induction on $h \in [0,\rho]$ that $\mathfrak{Y}_h$ is an affine space. (In this case each of the maps $\mathfrak{Y}_h \longrightarrow \mathfrak{Y}_{h-1}$, ($h \in [1,\rho]$) in 6.6 is either an isomorphism or an affine line bundle.) The same argument yields the dimension of each $\mathfrak{Y}_h$. This yields (b). Now (c) follows from (b) and the results in 6.4.

6.10. We assume that $\mathbf{a}$ satisfies 6.9(a). Define $x = ((g_i), \tilde g_j)) \in G^{[0,r]} \times G^{[0,\tilde r]}$ by $g_i = \ddot a_{\tilde r}\ddot a_{\tilde r+i}^{-1}$, $\tilde g_j = \ddot a_j$. Our assumption on $\mathbf{a}$ implies that $x \in X_6$. From the definitions we see that x is a fixed point of the $\mathcal{T}$-action 6.3 on X_6. Hence, in the $\Gamma \times \mathcal{T}$-structure of $\mathcal{E}_6$, $\mathcal{T}$ acts on the stalk $\mathcal{E}_{6,x}$ of $\mathcal{E}_6$ at x. We show:

(a) *the $\mathcal{T}$-action on $\mathcal{E}_{6,x}$ is trivial if and only if $\mathcal{L} \otimes (a_0^{-1})^*\tilde{\mathcal{L}} \cong \bar{\mathbf{Q}}_l$.*

Let

$$\mathcal{J} = \{i \in [1,r]; a_{\tilde r+i-1} = a_{\tilde r+i}\}, \tilde{\mathcal{J}} = \{j \in [1,\tilde r]; a_{j-1} = a_j\}.$$

From our assumption on $\mathbf{a}$ we see that $\mathcal{J} \subset \mathcal{I}_{\mathbf{s}}$, $\tilde{\mathcal{J}} \subset \mathcal{I}_{\tilde{\mathbf{s}}}$. Let $f^{\mathcal{J}} : \mathcal{Y}^{\mathcal{J}} \longrightarrow T$ be as in 2.4; let $\tilde f^{\tilde{\mathcal{J}}} : \tilde{\mathcal{Y}}^{\tilde{\mathcal{J}}} \longrightarrow T$ be the analogous map defined in terms of $\tilde{\mathbf{s}}, \tilde{\mathcal{L}}, \tilde{\mathcal{J}}$ instead of $\mathbf{s}, \mathcal{L}, \mathcal{J}$. Define sequences $\mathbf{s}_{\mathcal{J}} = (s'_1, s'_2, \dots, s'_r)$, $\tilde{\mathbf{s}}_{\tilde{\mathcal{J}}} = (\tilde s'_1, \tilde s'_2, \dots, \tilde s'_{\tilde r})$ by

$s'_i = 1$ if $i \in \mathcal{J}$, $s'_i = s_i$ if $i \in [1,r]-\mathcal{J}$, $\tilde s'_j = 1$ if $j \in \tilde{\mathcal{J}}$, $\tilde s'_j = \tilde s_j$ if $j \in [1,\tilde r]-\tilde{\mathcal{J}}$.

We have $e_6(x) = ((y_1,\dots,y_r),(\tilde y_1,\dots,\tilde y_{\tilde r}))$ where $y_i = \ddot a_{\tilde r+i-1}\ddot a_{\tilde r+i}^{-1}$ for $i \in [1,r]$ and $\tilde y_j = \ddot a_{j-1}^{-1}\ddot a_j$ for $j \in [1,\tilde r]$. Equivalently, $y_i = \dot s'_i$ for $i \in [1,r]$ and $\tilde y_j = \dot{\tilde s}'{}_j$ for $j \in [1,\tilde r]$. Thus we have $e_6(x) \in \mathcal{Y}^{\mathcal{J}} \times \tilde{\mathcal{Y}}^{\tilde{\mathcal{J}}}$. Moreover, we have $(f^{\mathcal{J}} \times \tilde f^{\tilde{\mathcal{J}}})e_6(x) = (1,1) \in T \times T$. It is enough to show that the $\mathcal{T}$-action on the stalk of $\bar{\mathcal{L}}_2 \boxtimes \bar{\tilde{\mathcal{L}}}_2$ at $e_6(x)$ is trivial if and only if $\mathcal{L} \otimes (a_0^{-1})^*\tilde{\mathcal{L}} \cong \bar{\mathbf{Q}}_l$. (The $\mathcal{T}$-equivariant structure on $\bar{\mathcal{L}}_2 \boxtimes \bar{\tilde{\mathcal{L}}}_2$ is obtained by restricting the $\underline{B} \times \underline{\tilde B}$-equivariant structure in 6.2.) By 2.4(a) we have canonically

$$(\bar{\mathcal{L}}_2 \boxtimes \bar{\tilde{\mathcal{L}}}_2)|_{\mathcal{Y}^{\mathcal{J}} \times \tilde{\mathcal{Y}}^{\tilde{\mathcal{J}}}} = (f^{\mathcal{J}} \times \tilde f^{\tilde{\mathcal{J}}})^*(\mathcal{L} \boxtimes \tilde{\mathcal{L}}).$$

Now $\mathcal{Y}^{\mathcal{J}} \times \tilde{\mathcal{Y}}^{\tilde{\mathcal{J}}}$ is stable under the $\underline{B} \times \underline{\tilde{B}}$-action on $Z_2 \times \tilde{Z}_2$ and the previous equality shows that $(f^{\mathcal{J}} \times \tilde{f}^{\tilde{\mathcal{J}}})^*(\mathcal{L} \boxtimes \tilde{\mathcal{L}})$ has a $\underline{B} \times \underline{\tilde{B}}$-equivariant structure; this structure is unique since $\underline{B} \times \underline{\tilde{B}}$ is connected. By restriction to the subgroup $\mathcal{T}$ of $\underline{B} \times \underline{\tilde{B}}$ we obtain a $\mathcal{T}$-equivariant structure on $(f^{\mathcal{J}} \times \tilde{f}^{\tilde{\mathcal{J}}})^*(\mathcal{L} \boxtimes \tilde{\mathcal{L}})$ and it is enough to show that the $\mathcal{T}$-action on the stalk of $(f^{\mathcal{J}} \times \tilde{f}^{\tilde{\mathcal{J}}})^*(\mathcal{L} \boxtimes \tilde{\mathcal{L}})$ at $e_6(x)$ is trivial if and only if $\mathcal{L} \otimes (a_0^{-1})^* \tilde{\mathcal{L}} \cong \bar{\mathbf{Q}}_l$. We define a $\underline{B} \times \underline{\tilde{B}}$-action on $T \times T$ by

$$((b_i), (\tilde{b}_j)) : (t, \tilde{t}) \mapsto (k(b_0)t([\mathbf{s}_{\mathcal{J}}]Fk(b_0^{-1})), k(\tilde{b}_0)\tilde{t}([\tilde{\mathbf{s}}_{\tilde{\mathcal{J}}}]Fk(\tilde{b}_0^{-1}))).$$

From the definitions we see that $f^{\mathcal{J}} \times \tilde{f}^{\tilde{\mathcal{J}}} : Z_2 \times \tilde{Z}_2 \longrightarrow T \times T$ is compatible with the $\underline{B} \times \underline{\tilde{B}}$-actions. Moreover, $\mathcal{L} \boxtimes \tilde{\mathcal{L}}$ is equivariant for the $\underline{B} \times \underline{\tilde{B}}$-action on $T \times T$ as above. (In fact it is equivariant for the action of the bigger group $B^{[0,r]} \times B^{[0,\tilde{r}]}$ given by the same formula; this follows from 1.8 using that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}_{\mathcal{J}}]F}$ and $\mathcal{L} \in \mathcal{S}(T)^{[\tilde{\mathbf{s}}_{\tilde{\mathcal{J}}}]F}$.) By restriction to the subgroup $\mathcal{T}$ of $\underline{B} \times \underline{\tilde{B}}$ we obtain a $\mathcal{T}$-equivariant structure on $\mathcal{L} \boxtimes \tilde{\mathcal{L}}$. We may identify the stalk of $(f^{\mathcal{J}} \times \tilde{f}^{\tilde{\mathcal{J}}})^*(\mathcal{L} \boxtimes \tilde{\mathcal{L}})$ at $e_6(x)$ with the stalk of $\mathcal{L} \boxtimes \tilde{\mathcal{L}}$ at $(1,1)$ as $\mathcal{T}$-modules. Let $\chi : \mathcal{T} \longrightarrow \bar{\mathbf{Q}}_l^*$ be the character by which $\mathcal{T}$ acts on the stalk of $\mathcal{L} \boxtimes \tilde{\mathcal{L}}$ at $(1,1)$. It is enough to show that $\chi = 1$ if and only if $\mathcal{L} \otimes (a_0^{-1})^* \tilde{\mathcal{L}} \cong \bar{\mathbf{Q}}_l$.

Now let T' be the subgroup of $B^{[0,r]} \times B^{[0,\tilde{r}]}$ consising of all elements $((t_0, t_1, \dots, t_r), (\tilde{t}_0, \tilde{t}_1, \dots, \tilde{t}_{\tilde{r}}))$ with coordinates in T such that $\tilde{t}_j = a_j^{-1}(t_0)$ for $j \in [0, \tilde{r}]$, $t_i = a_{\tilde{r}+i}(\tilde{t}_{\tilde{r}})$ for $i \in [0, r]$. We have $\mathcal{T} \subset T'$ and $((t_i), (\tilde{t}_j)) \mapsto t_0$ is an isomorphism $T' \xrightarrow{\sim} T$. Moreover $\mathcal{L} \boxtimes \tilde{\mathcal{L}}$ is T'-equivariant where T' acts on $T \times T$ by restriction of the $B^{[0,r]} \times B^{[0,\tilde{r}]}$ action. Now T' acts on T by $t_0 : t \mapsto t_0 t F'(t_0^{-1})$ where $F' : T \longrightarrow T$, $F'(t_0) = (a_{\tilde{r}} a_\rho^{-1} F)(t_0)$ is the Frobenius map for an $\mathbf{F}_q$-rational structure on T. The map $m : T \longrightarrow T \times T$, $t \mapsto (t, a_0^{-1}(t))$ is compatible with the T'-actions. Hence $m^*(\mathcal{L} \boxtimes \tilde{\mathcal{L}}) = \mathcal{L} \otimes (a_0^{-1})^* \tilde{\mathcal{L}}$ is a T'-equivariant local system on T and the natural action of $\mathcal{T}$ at any stalk of this local system is via χ. If we identify T' with T as above, $\mathcal{T}$ becomes the subgroup $T^{F'}$ of T. It remains to use the bijection (i)-(iv) in 1.9, with $\mathcal{L}$ replaced by $\mathcal{L} \otimes (a_0^{-1})^* \tilde{\mathcal{L}}$.

6.11. We assume that $\mathbf{a}$ satisfies 6.9(a). We show:

(a) $\mathcal{E}_6 \cong \bar{\mathbf{Q}}_l$.

We write $\bar{\mathcal{L}}_2 = \underline{\underline{\mathcal{L}}} = \otimes_{i \in [1,r]} \mathcal{F}_i$ as in 2.4. Similarly $\bar{\tilde{\mathcal{L}}}_2 = \otimes_{j \in [1,\tilde{r}]} \tilde{\mathcal{F}}_j$ where $\tilde{\mathcal{F}}_j$ are local systems on $\tilde{Z}_2$. From the definitions we have

$$\mathcal{E}_6 = \otimes_{i \in [1,r]} e_6^*(\mathcal{F}_i \boxtimes \bar{\mathbf{Q}}_l) \otimes \otimes_{j \in [1,\tilde{r}]} e_6^*(\bar{\mathbf{Q}}_l \boxtimes \tilde{\mathcal{F}}_j).$$

It is enough to show:

(b) $e_6^*(\mathcal{F}_i \boxtimes \bar{\mathbf{Q}}_l) \cong \bar{\mathbf{Q}}_l$ for any $i \in [1, r]$;

(c) $e_6^*(\bar{\mathbf{Q}}_l \boxtimes \tilde{\mathcal{F}}_j) \cong \bar{\mathbf{Q}}_l$ for any $j \in [1, \tilde{r}]$.

We prove (b). The general case can be reduced to the case where G has simply connected derived group, which we now assume. Let $\psi : Z_2 \times \tilde{Z}_2 \longrightarrow Z_2$ be the projection. Let p_i, f_{s_i} be as in 2.4. Let $\tilde{f}_i$ be the obvious map from P_{s_i} to the quotient of $P_{s_i}/U_{P_{s_i}}$ by its derived subgroup. Then $e_6^*(\mathcal{F}_i \otimes \bar{\mathbf{Q}}_l)$ is the inverse image of a local system of rank 1 under

$\tilde{f}_i p_i \psi e_6$ (if $i \in \mathcal{I}_{\mathbf{s}}$), or $f_{s_i} p_i \psi e_6$ (if $i \in [1,r] - \mathcal{I}_{\mathbf{s}}$).

It is enough to show that the image of $\tilde{f}_i p_i \psi e_6$ (if $i \in \mathcal{I}_{\mathbf{s}}$) or of $f_{s_i} p_i \psi e_6$ (if $i \in [1,r] - \mathcal{I}_{\mathbf{s}}$) is a point.

Let $\xi = ((g_0, g_1, \dots, g_r), (\tilde{g}_0, \tilde{g}_1, \dots, \tilde{g}_{\tilde{r}})) \in X_6$. We have $p_i \psi e_6(\xi) = g_{i-1}^{-1} g_i$. We have $g_i^{-1} \tilde{g}_{\tilde{r}} \in U \ddot{a}_{\tilde{r}+i} U$, $g_{i-1}^{-1} \tilde{g}_{\tilde{r}} \in U \ddot{a}_{\tilde{r}+i-1} U$, hence $g_{i-1}^{-1} g_i \in u_1 \ddot{a}_{\tilde{r}+i-1} u_2 \ddot{a}_{\tilde{r}+i}^{-1} u_3$ with u_1, u_2, u_3 in U. Moreover $g_{i-1}^{-1} g_i \in P_{s_i}$.

Case 1. Assume that $i \in \mathcal{I}_{\mathbf{s}}$ and $a_{\tilde{r}+i-1} = a_{\tilde{r}+i}$. Then $\ddot{a}_{\tilde{r}+i-1} = \ddot{a}_{\tilde{r}+i}^{-1}$ and $\ddot{a}_{\tilde{r}+i-1} u_2 \ddot{a}_{\tilde{r}+i}^{-1} = \ddot{a}_{\tilde{r}+i-1} u_2 \ddot{a}_{\tilde{r}+i-1}^{-1} \in P_{s_i}$ is unipotent. Hence $g_{i-1}^{-1} g_i$ is a product of three unipotent elements of P_{s_i} so that $\tilde{f}_i(g_{i-1}^{-1} g_i) = 1$.

Case 2. Assume that $i \in \mathcal{I}_{\mathbf{s}}$ and $a_{\tilde{r}+i-1} = s_i a_{\tilde{r}+i}$. Then $\ddot{a}_{\tilde{r}+i} = \dot{s}_i^{-1} \ddot{a}_{\tilde{r}+i-1}^{-1}$ and

$$\ddot{a}_{\tilde{r}+i-1} u_2 \ddot{a}_{\tilde{r}+i}^{-1} = \ddot{a}_{\tilde{r}+i-1} u_2 \ddot{a}_{\tilde{r}+i-1}^{-1} \dot{s}_i = u_2' \dot{s}_i$$

where $u_2' \in G$ is unipotent. Thus $g_{i-1}^{-1} g_i = u_1 u_2' \dot{s}_i u_3$. Since $u_1 u_2' \dot{s}_i u_3, u_1, \dot{s}_i, u_3$ belong to P_{s_i} we see that u_2' is a unipotent element of P_{s_i}. Note also that $\dot{s}_i$ belongs to the derived subgroup of P_{s_i}. We see that $\tilde{f}_i(g_{i-1}^{-1} g_i) = 1$.

Case 3. Assume that $i \notin \mathcal{I}_{\mathbf{s}}$. By our assumption we have $a_{\tilde{r}+i-1} = s_i a_{\tilde{r}+i}$. Then $\ddot{a}_{\tilde{r}+i} = \dot{s}_i^{-1} \ddot{a}_{\tilde{r}+i-1}^{-1}$ and as before we have $g_{i-1}^{-1} g_i = u_1 u_2' \dot{s}_i u_3$ where u_2' belongs to the unipotent group $P_{s_i} \cap \ddot{a}_{\tilde{r}+i-1} U \ddot{a}_{\tilde{r}+i-1}^{-1}$. This unipotent group is normalized by T. By [Bo, 14.4] we have $P_{s_i} \cap \ddot{a}_{\tilde{r}+i-1} U \ddot{a}_{\tilde{r}+i-1}^{-1} = U_1 U_2 \dots U_n$ where $U_1, U_2, \dots, U_n$ are the connected one dimensional unipotent subgroups of $P_{s_i} \cap \ddot{a}_{\tilde{r}+i-1} U \ddot{a}_{\tilde{r}+i-1}^{-1}$ normalized by T, in any order. Thus any of $U_1, U_2, \dots, U_n$ is either $y_{s_i}(\mathbf{k})$ or else is contained in U; moreover we can assume that $U_1, U_2, \dots, U_{n-1}$ are contained in U and $U_n = y_{s_i}(\mathbf{k})$. Thus $u_2' \in U y_{s_i}(\mathbf{k})$ and

$$g_{i-1}^{-1} g_i \in U y_{s_i}(\mathbf{k}) \dot{s}_i U = U \dot{s}_i x_{s_i}(\mathbf{k}) U = U \dot{s}_i U.$$

Note that $f_{s_i}(g_{i-1}^{-1} g_i) = 1$.

We see that in Case 1 and 2 we have $\tilde{f}_i p_i \psi e_6(\xi) = 1$ for any $\xi \in X_6$. In Case 3 we have $f_{s_i} p_i \psi e_6(\xi) = 1$ for any $\xi \in X_6$. This completes the proof of (b).

The proof of (c) is entirely similar.

6.12. We set $X_{-1} = X$, $\mathcal{E}_{-1} = \mathcal{E}$. Let $\delta_i = \dim X_i (i \in [-1, 8])$. For $i \in [-1, 8]$ we show:

(a) *Assume that* $\mathbf{a}$ *satisfies 6.9(a); if* $n \neq 2\delta_i$ *or if* $\tilde{\mathcal{L}} \not\cong a_0^* \check{\mathcal{L}}$ *then* $H_c^n(X_i, \mathcal{E}_i)^{\Gamma \times \mathcal{T}} = 0$*; if* $\tilde{\mathcal{L}} = w^* \check{\mathcal{L}}$ *where* $w \in W$ *and* $w^* \mathcal{L} \cong a_0^* \mathcal{L}$ *then* $H_c^{2\delta_i}(X_i, \mathcal{E}_i)^{\Gamma \times \mathcal{T}}(\delta_i) = \bar{\mathbf{Q}}_l$ *canonically. Assume that* $\mathbf{a}$ *does not satisfy 6.9(a); then* $H_c^n(X_i, \mathcal{E}_i)^{\Gamma \times \mathcal{T}} = 0$ *for all* n.

Here (δ_i) is a Tate twist, The upper index denotes $\Gamma \times \mathcal{T}$-invariants (the action of $\Gamma \times \mathcal{T}$ comes from the $\Gamma \times \mathcal{T}$-equivariant structure of $\mathcal{E}_i$).

Let P_i be the statement of (a). From 6.4 we see that the statements P_0, P_1 are equivalent and that the statements $P_2, P_3, \dots, P_8$ are equivalent. From 6.4 we see also that $H_c^n(X_2, \mathcal{E}_2) = H_c^n(X_1, \mathcal{E}_1)^\Gamma$ and that $H_c^n(X_{-1}, \mathcal{E}_{-1}) = H_c^n(X_0, \mathcal{E}_0)^\mathcal{T}$ for any n. Since Γ acts trivially on any stalk of $\mathcal{E}_2$ (see 6.5(c)) and $\mathcal{T}$ acts trivially on any stalk of $\mathcal{E}_{-1}$ (see 6.5(b)) it follows that $H_c^n(X_2, \mathcal{E}_2)^\Gamma = H_c^n(X_1, \mathcal{E}_1)^\Gamma$ and that $H_c^n(X_{-1}, \mathcal{E}_{-1})^\mathcal{T} = H_c^n(X_0, \mathcal{E}_0)^\mathcal{T}$ for any n. This shows that the statements P_1, P_2 are equivalent and the statements P_{-1}, P_0 are equivalent. We see that the statements $P_{-1}, P_0, \dots, P_8$ are all equivalent. Thus it is enough to show that P_8 holds.

Assume first that $\mathbf{a}$ does not satisfy 6.9(a). Then the result follows from 6.7(a) or 6.8(a).

Next we assume that $\mathbf{a}$ satisfies 6.9(a). Let $\bar{x} = b_8(b_7(x)) \in X_8$ where $x \in X_6$ is as in 6.10. We can write 6.11(a) in the form $b_7^* b_8^* \mathcal{E}_8 \cong \bar{\mathbf{Q}}_l$. Since b_8 is an isomorphism and b_7 is a principal $U^{[0,r]} \times U^{[0,\tilde{r}]}$-bundle, it follows that $\mathcal{E}_8 \cong \bar{\mathbf{Q}}_l$. Using this and 6.9(b) we see that

$$H_c^n(X_8, \mathcal{E}_8) = 0 \text{ for } i \neq 2\delta_8, H_c^{2\delta_8}(X_8, \mathcal{E}_8)(\delta_8) = \mathcal{E}_{8,\bar{x}},$$

where $\mathcal{E}_{8,\bar{x}}$ is the stalk of $\mathcal{E}_8$ at $\bar{x}$. Moreover the last equality is compatible with the natural $\mathcal{T}$-actions (coming from the $\mathcal{T}$-equivariant structure of $\mathcal{E}_8$). Now $\mathcal{E}_{8,\bar{x}}$ may be canonically identified with the stalk $\mathcal{E}_{6,x}$ of $\mathcal{E}_6$ at x. Thus we have $H_c^{2\delta_8}(X_8, \mathcal{E}_8)(\delta_8) = \mathcal{E}_{6,x}$, compatibly with the $\mathcal{T}$-actions. Using 6.10(a) we see that the $\mathcal{T}$-action on $H_c^{2\delta_8}(X_8, \mathcal{E}_8)$ is trivial if and only if $\mathcal{L} \otimes (a_0^{-1})^* \tilde{\mathcal{L}} \cong \bar{\mathbf{Q}}_l$. Taking $\mathcal{T}$-invariants we see that

$H_c^{2\delta_8}(X_8, \mathcal{E}_8)^\mathcal{T}(\delta_8) = \mathcal{E}_{6,x}$ if $\mathcal{L} \otimes (a_0^{-1})^* \tilde{\mathcal{L}} \cong \bar{\mathbf{Q}}_l$;

$H_c^{2\delta_8}(X_8, \mathcal{E}_8)^\mathcal{T} = 0$ if $\mathcal{L} \otimes (a_0^{-1})^* \tilde{\mathcal{L}} \ncong \bar{\mathbf{Q}}_l$.

By 6.5(c), Γ acts trivially on $H_c^{2\delta_8}(X_8, \mathcal{E}_8)$ hence

$H_c^{2\delta_8}(X_8, \mathcal{E}_8)^{\Gamma \times \mathcal{T}} = H_c^{2\delta_8}(X_8, \mathcal{E}_8)^\mathcal{T}$.

We see that the first assertion in (a) for $i = 8$ holds. It remains to prove that, if $\tilde{\mathcal{L}} = w^* \check{\mathcal{L}}$ where $w \in W$ and $w^* \mathcal{L} \cong a_0^* \mathcal{L}$, then $\mathcal{E}_{6,x} = \bar{\mathbf{Q}}_l$ canonically. By the proof of 6.10(a), $\mathcal{E}_{6,x}$ is canonically isomorphic to the stalk of $\mathcal{L} \otimes (a_0^{-1})^* \tilde{\mathcal{L}}$ at 1 that is to the stalk of $\mathcal{L} \otimes (w a_0^{-1})^* \check{\mathcal{L}}$ at 1. The stalk of $(w a_0^{-1})^* \check{\mathcal{L}}$ at 1 is the same as the stalk of $\check{\mathcal{L}}$ at $w a_0(1) = 1$. Thus $\mathcal{E}_{6,x}$ is canonically isomorphic to the stalk of $\mathcal{L} \otimes \check{\mathcal{L}} = \bar{\mathbf{Q}}_l$ at 1. Thus P_8 holds. We see that (a) holds for $i \in [-1, 8]$.

6.13. We now write $X_\mathbf{a}, \mathcal{E}_\mathbf{a}, e_\mathbf{a}$ instead of $X, \mathcal{E}, e$ in 6.1. We identify $X_\mathbf{a}$ with its image under the imbedding $e_\mathbf{a}$. Let $\bar{X}_\mathbf{a}$ be the closure of $X_\mathbf{a}$ in $Z \times \tilde{Z}$. Recall that $\mathcal{E}_\mathbf{a} = e_\mathbf{a}^*(\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})$. Let $\bar{e}_\mathbf{a} : \bar{X}_\mathbf{a} \to Z \times \tilde{Z}$ by the inclusion. By 6.9(c), $X_\mathbf{a}$ (hence also $\bar{X}_\mathbf{a}$) has pure dimension $N_\mathbf{a}$ and $\dim(\bar{X}_\mathbf{a} - X_\mathbf{a}) < N_\mathbf{a}$. Hence the natural map

$$\text{(a)} \qquad H_c^{2N_\mathbf{a}}(X_\mathbf{a}, e_\mathbf{a}^*(\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})) \to H_c^{2N_\mathbf{a}}(\bar{X}_\mathbf{a}, \bar{e}_\mathbf{a}^*(\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}))$$

(induced by the open imbedding $X_\mathbf{a} \subset \bar{X}_\mathbf{a}$) is an isomorphism. Let

$$\xi'_\mathbf{a} : H_c^{2N_\mathbf{a}}(Z \times \tilde{Z}, (\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})) \to H_c^{2N_\mathbf{a}}(X_\mathbf{a}, e_\mathbf{a}^*(\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}))$$

be the linear map obtained by composing the linear map

$$H_c^{2N_{\mathbf{a}}}(Z \times \tilde{Z}, (\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})) \to H_c^{2N_{\mathbf{a}}}(\bar{X}_{\mathbf{a}}, \bar{e}_{\mathbf{a}}^*(\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}))$$

with the inverse of (a). By taking Γ-invariants and applying a Tate twist we obtain from $\xi'_{\mathbf{a}}$ a linear map

(b) $$\xi_{\mathbf{a}} : H_c^{2N_{\mathbf{a}}}(Z \times \tilde{Z}, (\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}))^\Gamma(N_{\mathbf{a}}) \to H_c^{2N_{\mathbf{a}}}(X_{\mathbf{a}}, e_{\mathbf{a}}^*(\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}))^\Gamma(N_{\mathbf{a}}).$$

6.14. Let $\mathcal{A}$ be the set of all $\mathbf{a}$ as in 6.1. Note that the subvarieties $X_{\mathbf{a}}$ ($\mathbf{a} \in \mathcal{A}$) form a partition of $Z \times \tilde{Z}$.

(a) *For any $n \in \mathbf{Z}$, the linear map*

$$\xi_n : H_c^{2n}(Z \times \tilde{Z}, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})^\Gamma(n) \to \oplus_{\mathbf{a} \in \mathcal{A}; N_{\mathbf{a}}=n} H_c^{2n}(X_{\mathbf{a}}, e_{\mathbf{a}}^*(\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}))^\Gamma(n)$$

whose components are the maps $\xi_{\mathbf{a}}$ ($\mathbf{a} \in \mathcal{A}, N_{\mathbf{a}} = n$), is an isomorphism. Moreover, $H_c^{n'}(Z \times \tilde{Z}, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})^\Gamma = 0$ for any odd n'.

The proof is almost identical to that of [L4, 2.7]. (We use the fact that $H_c^m(X_{\mathbf{a}}, e_{\mathbf{a}}^*(\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}))^\Gamma = 0$ for any $\mathbf{a} \in \mathcal{A}$ and any $m \neq 2N_{\mathbf{a}}$; this follows from 6.12(a) for $i = -1$; note that, in view of 6.5(b), 6.12(a) for $i = -1$ remains valid if $\Gamma \times \mathcal{T}$-invariants are replaced by Γ-invariants.)

From (a) and 6.12(a) for $i = -1$ we see that the following holds.

(b) *If for any $w \in W$ we have $\tilde{\mathcal{L}} \not\cong w^*\check{\mathcal{L}}$, then $H_c^m(Z \times \tilde{Z}, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})^\Gamma = 0$ for any $m \in \mathbf{Z}$.*

Now assume that $\tilde{\mathcal{L}} = w^*\check{\mathcal{L}}$ for some $w \in W$. Let

$\mathcal{A}_w = \{\mathbf{a} = (a_0, a_1, \dots, a_\rho) \in \mathcal{A}; \mathbf{a}$ satisfies 6.9(a), $a_0^*\mathcal{L} \cong w^*\mathcal{L}.\}$.

Note that for $w = 1$ we have $\mathcal{A}_1 = \mathcal{A}(W, \mathbf{c}, \mathcal{L}, \mathbf{s}, \tilde{\mathbf{s}})$, see 5.9.

By 6.12(a) for $i = -1$, the summand in the target of ξ_n corresponding to $\mathbf{a}$ is canonically $\bar{\mathbf{Q}}_l$ if $\mathbf{a} \in \mathcal{A}_w$ and is 0 if $\mathbf{a} \notin \mathcal{A}_w$. Hence for any $\mathbf{a} \in \mathcal{A}_w$ with $N_{\mathbf{a}} = n$ there is a unique element $b_{\mathbf{a}}^w \in H_c^{2n}(Z \times \tilde{Z}, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})^\Gamma(n)$ such that $\xi_n(b_{\mathbf{a}}^w)$ is contained in the summand corresponding to $\mathbf{a}$ and, as an element of that summand, it corresponds to $1 \in \bar{\mathbf{Q}}_l$. Moreover,

(c) $\{b_{\mathbf{a}}^w; \mathbf{a} \in \mathcal{A}_w, N_{\mathbf{a}} = n\}$ *is a $\bar{\mathbf{Q}}_l$-basis of $H_c^{2n}(Z \times \tilde{Z}, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})^\Gamma(n)$.*

Taking $w = 1$ and taking direct sum over n we obtain

(d) $\{b_{\mathbf{a}}^1; \mathbf{a} \in \mathcal{A}(W, \mathbf{c}, \mathcal{L}, \mathbf{s}, \tilde{\mathbf{s}})\}$ *is a $\bar{\mathbf{Q}}_l$-basis of $\oplus_n H_c^{2n}(Z \times \tilde{Z}, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})^\Gamma(n)$.*

6.15. More generally, if J is as in 3.1, we set $\mathcal{A}_J = \{\mathbf{a} = (a_0, a_1, \dots, \alpha_\rho) \in \mathcal{A}; a_0 \in W_J\}$. Let

$$(Z \times \tilde{Z})_J = \{((B_0, B_1, \dots, B_r), (\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_{\tilde{r}})) \in Z \times \tilde{Z}; P_{B_0,J} = P_{\tilde{B}_0,J}\}.$$

Note that the subvarieties $X_{\mathbf{a}}$ ($\mathbf{a} \in \mathcal{A}_J$) form a partition of $(Z \times \tilde{Z})_J$. As in 6.14 we see that for any $n \in \mathbf{Z}$ we have an isomorphism

(a) $$H_c^{2n}((Z \times \tilde{Z})_J, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})^\Gamma(n) \xrightarrow{\sim} \oplus_{\mathbf{a} \in \mathcal{A}_J; N_{\mathbf{a}}=n} H_c^{2n}(X_{\mathbf{a}}, e_{\mathbf{a}}^*(\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}))^\Gamma(n).$$

Moreover,

(b) $$H_c^{n'}((Z \times \tilde{Z})_J, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})^{\Gamma} = 0 \text{ for any odd } n'.$$

As in 6.14 we see that the following holds:

(c) *If for any $w \in W_J$ we have $\tilde{\mathcal{L}} \not\cong w^*\check{\mathcal{L}}$, then $H_c^m((Z \times \tilde{Z})_J, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})^{\Gamma} = 0$ for any $m \in \mathbf{Z}$.*

Define $h : (Z \times \tilde{Z}) \longrightarrow \mathcal{P}_J$ by $((B_0, B_1, \dots, B_r), (\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_{\tilde{r}})) \mapsto P_{B_0,J} = P_{\tilde{B}_0,J}$. We have

$$H_c^m((Z \times \tilde{Z})_J, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}) = H_c^m(\mathcal{P}_J, h_!(\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})) = H_c^m(\mathcal{P}_J, (\Upsilon_!^{\mathbf{s}}\bar{\mathcal{L}}) \otimes (\Upsilon_!^{\tilde{\mathbf{s}}}\bar{\tilde{\mathcal{L}}})).$$

(Notation of 3.2.) From 2.13 we see that

(d) $$\Upsilon_!^{\mathbf{s}}\bar{\mathcal{L}} = \bar{\Upsilon}_!^{\mathbf{s}}\bar{\mathcal{L}}^{\sharp}$$

(notation of 3.2). By the decomposition theorem [BBD], $\bar{\Upsilon}_!^{\mathbf{s}}\bar{\mathcal{L}}^{\sharp} \cong \oplus_j {}^pH^j(\bar{\Upsilon}_!^{\mathbf{s}}\bar{\mathcal{L}}^{\sharp})[-j]$. Hence $\Upsilon_!^{\mathbf{s}}\bar{\mathcal{L}} \cong \oplus_j {}^pH^j(\Upsilon_!^{\mathbf{s}}\bar{\mathcal{L}})[-j]$. Similarly, $\Upsilon_!^{\tilde{\mathbf{s}}}\bar{\tilde{\mathcal{L}}} \cong \oplus_{\tilde{j}} {}^pH^{\tilde{j}}(\Upsilon_!^{\tilde{\mathbf{s}}}\bar{\tilde{\mathcal{L}}})[-\tilde{j}]$. We see that

(e) $$H_c^m((Z \times \tilde{Z})_J, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}) = \oplus_{j,j'} H_c^{m-j-\tilde{j}}(\mathcal{P}_J, {}^pH^j(\Upsilon_!^{\mathbf{s}}\bar{\mathcal{L}}) \otimes {}^pH^{\tilde{j}}(\Upsilon_!^{\tilde{\mathbf{s}}}\bar{\tilde{\mathcal{L}}}))$$

(compatibly with the Γ-actions).

Proposition 6.16. *Let $J \subset \mathbf{I}$. Let A be a simple object of $\mathcal{M}_{\Gamma}(\mathcal{P}_J)$. Assume that $A \dashv_{\Gamma} {}^pH^j(\Upsilon_!^{\mathbf{s}}\bar{\mathcal{L}})$ and $A \dashv_{\Gamma} {}^pH^{j'}(\Upsilon_!^{\tilde{\mathbf{s}}}\bar{\tilde{\mathcal{L}}})$ where $j, j' \in \mathbf{Z}$. Then $\tilde{\mathcal{L}} \cong w^*\check{\mathcal{L}}$ for some $w \in W_J$ and $j = j' \mod 2$.*

Using 6.15(d) and the fact that $\bar{\Upsilon}^{\mathbf{s}}$ is proper we see that

$$\mathfrak{D}(\Upsilon_!^{\mathbf{s}}\bar{\mathcal{L}}) = \bar{\Upsilon}_!^{\mathbf{s}}\mathfrak{D}(IC(\bar{Z}^{\mathbf{s}}, \bar{\mathcal{L}})) = \bar{\Upsilon}_!^{\mathbf{s}}(IC(\bar{Z}^{\mathbf{s}}, \bar{\check{\mathcal{L}}}))[2r] = \Upsilon_!^{\mathbf{s}}\bar{\check{\mathcal{L}}}[2r].$$

Hence $\mathfrak{D}({}^pH^j(\Upsilon_!^{\mathbf{s}}\bar{\mathcal{L}})) = {}^pH^{-j+2r}(\Upsilon_!^{\mathbf{s}}\bar{\check{\mathcal{L}}})$. We see that $\mathfrak{D}(A) \dashv_{\Gamma} {}^pH^{-j+2r}(\Upsilon_!^{\mathbf{s}}\bar{\check{\mathcal{L}}})$. By 1.6(a) we have $\dim H_c^0(\mathcal{P}_J, \mathfrak{D}(A) \otimes A)^{\Gamma} = 1$. Since $\mathfrak{D}(A)$ (resp. A) is a direct summand of ${}^pH^{-j+2r}(\Upsilon_!^{\mathbf{s}}\bar{\check{\mathcal{L}}})$ (resp. ${}^pH^{j'}(\Upsilon_!^{\tilde{\mathbf{s}}}\bar{\tilde{\mathcal{L}}})$) we see that

$$H_c^0(\mathcal{P}_J, {}^pH^{-j+2r}(\Upsilon_!^{\mathbf{s}}\bar{\check{\mathcal{L}}}) \otimes {}^pH^{j'}(\Upsilon_!^{\tilde{\mathbf{s}}}\bar{\tilde{\mathcal{L}}}))^{\Gamma} \neq 0.$$

Using 6.15(e) we see that $H_c^{-j+2r+j'}((Z \times \tilde{Z})_J, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})^{\Gamma} \neq 0$. Using 6.15(b),(c) we see that the proposition holds.

6.17. In the case where $J = \mathbf{I}$, the first assertion of 6.16 reduces to the disjointness theorem [DL, 6.2, 6.3] (we use also the equivalence of 3.2(i), 3.2(vi)).

6.18. In this subsection we assume that $\tilde{\mathcal{L}} = \check{\mathcal{L}}$ and

(a) any $s \in \mathbf{I}$ is in the $\mathbf{c}$-orbit of some $s_i (i \in [1, r])$.

Since $\dim(Z \times \tilde{Z}) = \rho$ we have $H_c^i(Z \times \tilde{Z}, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}) = 0$ for $i < 2\rho$. We show:

(b) $\dim H_c^{2\rho}(Z \times \tilde{Z}, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})^\Gamma = n_{\mathcal{L}}$ *where* $n_{\mathcal{L}}$ *is* 1 *if* $R = R_{\mathcal{L}}$ *and* 0 *if* $R \neq R_{\mathcal{L}}$.

By 6.14(c) we have $\dim H_c^{2\rho}(Z \times \tilde{Z}, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})^\Gamma = |\mathfrak{A}|$ where $\mathfrak{A} = \{\mathbf{a} \in \mathcal{A}_1; N_{\mathbf{a}} = \rho\}$. Note that $\mathfrak{A}$ is the set of all sequences $\mathbf{a} = (a_0, a_1, \dots, a_\rho)$ in W such that

$a_{j-1}^{-1} a_j \in \{1, \tilde{s}_j\}$ for $j \in \mathcal{I}_{\tilde{\mathbf{s}}}$;

$a_{j-1}^{-1} a_j = \tilde{s}_j$ for $j \in [1, \tilde{r}] - \mathcal{I}_{\tilde{\mathbf{s}}}$;

$a_{\tilde{r}+i} a_{\tilde{r}+i-1}^{-1} \in \{1, s_i\}$ for $i \in \mathcal{I}_{\mathbf{s}}$;

$a_{\tilde{r}+i} a_{\tilde{r}+i-1}^{-1} = s_i$ for $i \in [1, r] - \mathcal{I}_{\mathbf{s}}$;

if $h \in [1, \tilde{r}]$ then $a_{h-1} \le a_h \ge a_{h-1} \tilde{s}_h$;

if $h \in [1, r]$ then $a_{\tilde{r}+h-1} \le a_{\tilde{r}+h} \ge s_h a_{\tilde{r}+h-1}$;

$a_\rho = \mathbf{c}(a_0)$;

$a_0^* \mathcal{L} \cong \mathcal{L}$.

If $\mathbf{a} \in \mathfrak{A}$ then $l(a_0) \le l(a_1) \le \dots \le l(a_\rho) = l(a_0)$ hence $l(a_0) = l(a_1) = \dots = l(a_\rho)$ and since $a_0 \le a_1 \le \dots \le a_\rho$, we have $a_0 = a_1 = \dots = a_\rho$. This forces $\mathcal{I}_{\tilde{\mathbf{s}}} = [1, \tilde{r}]$. Thus, $s_1, s_1 s_2 s_1, \dots, s_1 s_2 \dots s_r \dots s_2 s_1$ are in $W_{\mathcal{L}}$ hence $s_i \in W_{\mathcal{L}}$ for $i \in [1, r]$. Using $([\mathbf{s}]F)^* \mathcal{L} \cong \mathcal{L}$ and $[\mathbf{s}] \in W_{\mathcal{L}}$ we deduce $F^* \mathcal{L} \cong \mathcal{L}$. Hence if $i \in [1, r]$ then $\mathbf{c}^n(s_i) \in W_{\mathcal{L}}$ for any $n \ge 1$. Using (a) we deduce that $\mathbf{I} \subset W_{\mathcal{L}}$ hence $W = W_{\mathcal{L}}$ and $R = R_{\mathcal{L}}$.

If $R = R_{\mathcal{L}}$ then $\mathfrak{A}$ is in bijection with the set

$\mathfrak{A}' = \{a_0 \in W; a_0 = \mathbf{c}(a_0), a_0 \tilde{s}_h < a_0 \text{ for } h \in [1, \tilde{r}], s_h a_0 < a_0 \text{ for } h \in [1, r]\}$.

We set $I' = \{s \in \mathbf{I}; s = s_h \text{ for some } h \in [1, r]\}$, $I'' = \{s \in \mathbf{I}; s = \tilde{s}_h \text{ for some } h \in [1, \tilde{r}]\}$. We see that

$\mathfrak{A}' = \{a_0 \in W; I' \subset L_{a_0}, I'' \subset R_{a_0}, a_0 = \mathbf{c}(a_0)\}$

where for $w \in W$ we set $L_w = \{s \in \mathbf{I}; sw < w\}$, $R_w = \{s \in \mathbf{I}; ws < w\}$.

For a_0 such that $a_0 = \mathbf{c}(a_0)$, the set L_{a_0} is $\mathbf{c}$-stable; hence the condition $I' \subset L_{a_0}$ is equivalent to $I' \cup \mathbf{c}(I') \cup \mathbf{c}^2(I') \cup \dots \subset L_{a_0}$ that is (by (a)) to $\mathbf{I} = L_{a_0}$. We see that $\mathfrak{A}'$ has exactly one element: $w_{\mathbf{I}}$. This proves (b).

7. The variety X

7.1. In this section we study the variety

$$X = \{(B', g) \in \mathcal{B} \times G; g^{-1} F(g) \in U_{B'}\}.$$

Note that X is an étale covering of the smooth connected variety $X' = \{(B', u) \in \mathcal{B} \times G; u \in U_{B'}\}$ of dimension via the map $\rho_1 : X \to X'$, $(B', g) \mapsto (B', g^{-1} F(g))$. Since $\dim X' = 2d$ where $d = \dim \mathcal{B}$ we see that X is smooth of pure dimension $2d$. Here is one of the main results of this section.

Proposition 7.2. *If G is simply connected then X is connected.*

The proof is given in 7.16. Note that X is not necessarily connected without the assumption that G is simply connected.

7.3. Let $w \in W$. Let $i_w : \mathcal{B}_w \longrightarrow \mathcal{B}$ be the inclusion. Let $A' \in \mathbb{S}(\mathcal{B}_w)$. Let A be a simple object of $\mathcal{M}_\Gamma(\mathcal{B})$ such that $A \dashv_\Gamma {}^pH^\cdot(i_{w!}A')$. We show:

(a) $$A \in \mathbb{S}(\mathcal{B}).$$

We argue by induction on $l(w)$. If $l(w) = 0$ then $i_{w!}A' \in \mathbb{S}(\mathcal{B})$ and the result is clear. We now assume that $l(w) > 0$. We have $A = \mathcal{L}_w[l(w)]$ where $\mathcal{L} \in \mathcal{S}(T)^{wF}$. Let $K = \mathcal{L}_w^\sharp[l(w)]$. We have $K \in \mathbb{S}(\mathcal{B})$. Assume first that $A \not\dashv_\Gamma {}^pH^\cdot(i_{w'!}i_{w'}^*K)$ for any $w' \in W$ such that $w' < w$. Then $A \not\dashv_\Gamma {}^pH^\cdot(u_!u^*K)$ where $u : \cup_{w';w'<w}\mathcal{B}_{w'} \longrightarrow \mathcal{B}$ is the inclusion. Let $u' : \cup_{w';w'\le w}\mathcal{B}_{w'} \longrightarrow \mathcal{B}$ be the inclusion. Since $i_{w!}i_w^*K = A'$ we see that $A \dashv_\Gamma {}^pH^\cdot(u'_!u'^*K)$. Thus $A \dashv_\Gamma K$ hence $A \cong K$ and $A \in \mathbb{S}(\mathcal{B})$.

Next we assume that $A \dashv_\Gamma {}^pH^\cdot(i_{w'!}i_{w'}^*K)$ for some $w' \in W$ such that $w' < w$. It follows that there exists j' such that $A \dashv_\Gamma {}^pH^\cdot(i_{w'!}({}^pH^{j'}(i_{w'}^*K)))$. Hence there exists a simple object A'' of $\mathcal{M}_\Gamma(\mathcal{B}_{w'})$ such that $A'' \dashv {}^pH^{j'}(i_{w'}^*K)$ and $A \dashv_\Gamma {}^pH^\cdot(i_{w'!}A'')$. From 4.14 we see that $A'' \in \mathbb{S}(\mathcal{B}_{w'})$. From the induction hypothesis we see that $A \in \mathbb{S}(\mathcal{B})$. This proves (a).

7.4. Let $w \in W$. Let $f_w : \tilde{\mathcal{B}}_w \longrightarrow \mathcal{B}_w$ be as in 4.7, a finite principal covering. Let $U_w = \{u \in U; \dot{w}F(u)\dot{w}^{-1} \in U\}$. Let $\hat{\mathcal{B}}_w = \{z \in G; z^{-1}F(z) \in U\dot{w}\}$. Define $a' : \hat{\mathcal{B}}_w \longrightarrow \tilde{\mathcal{B}}_w$ by $z \mapsto zU$, a principal U_w-bundle. Let

$X_w = \{(B', g) \in \mathcal{B} \times G; g^{-1}F(g) \in U_{B'}, \mathrm{pos}(B', F(B')) = w\}$,
$\tilde{X}_w = \{(zU_w, g) \in G/U_w \times G; z^{-1}F(z) \in U\dot{w}, g^{-1}F(g) \in zUz^{-1}\}$,
$\tilde{X}'_w = \{(z, y) \in G \times G; z^{-1}F(z) \in U\dot{w}, y^{-1}F(y) \in U\dot{w}\}$,
$\tilde{X}''_w = U_w\backslash\tilde{X}'_w$,

where U_w acts (freely) by $u : (z, y) \mapsto (zu^{-1}, yu^{-1})$. Define $\pi_w : X_w \longrightarrow \mathcal{B}_w$ by $(B', g) \mapsto B'$. Define $\pi' : \tilde{X}_w \longrightarrow \tilde{\mathcal{B}}_w$ by $(zU_w, g) \mapsto zU$. Under the identification $\tilde{X}_w = \tilde{X}''_w$, $(zU_w, g) \leftrightarrow (z, gz)U_w$, π' becomes $\pi'' : \tilde{X}''_w \longrightarrow \tilde{\mathcal{B}}_w$, $(z, y)U_w \mapsto zU$. Define $a : \tilde{X}'_w \longrightarrow \tilde{X}''_w$ by $(z, y) \mapsto (z, y)U_w$. Define $h : \tilde{X}'_w \longrightarrow \hat{\mathcal{B}}_w$ by $(z, y) \mapsto z$. Define $\gamma : \tilde{X}_w \longrightarrow X_w$ by $(zU_w, g) \mapsto (zBz^{-1}, g)$. Now Γ acts on $\tilde{X}''_w$ by $g_0 : (z, y) \mapsto (g_0z, y)$. We show:

(i) if $A \in \mathcal{M}_\Gamma(\hat{\mathcal{B}}_w)$ is simple and $A \dashv_\Gamma {}^pH^\cdot(h_!\bar{\mathbf{Q}}_l)$ then $A \cong \bar{\mathbf{Q}}_l[\dim U]$;
(ii) if $A \in \mathcal{M}_\Gamma(\tilde{\mathcal{B}}_w)$ is simple and $A \dashv_\Gamma {}^pH^\cdot(a'_!h_!\bar{\mathbf{Q}}_l)$ then $A \cong \bar{\mathbf{Q}}_l[l(w)]$;
(iii) if $A \in \mathcal{M}_\Gamma(\tilde{\mathcal{B}}_w)$ is simple and $A \dashv_\Gamma {}^pH^\cdot(\pi''_!a_!\bar{\mathbf{Q}}_l)$ then $A \cong \bar{\mathbf{Q}}_l[l(w)]$;
(iv) if $A \in \mathcal{M}_\Gamma(\tilde{\mathcal{B}}_w)$ is simple and $A \dashv_\Gamma {}^pH^\cdot(\pi'_!\bar{\mathbf{Q}}_l)$ then $A \cong \bar{\mathbf{Q}}_l[l(w)]$;
(v) if $A \in \mathcal{M}_\Gamma(\mathcal{B}_w)$ is simple and $A \dashv_\Gamma {}^pH^\cdot(\pi_{w!}\bar{\mathbf{Q}}_l))$ then $A \in \mathbb{S}(\mathcal{B}_w)$;
(vi) if $A \in \mathcal{M}_\Gamma(\mathcal{B})$ is simple and $A \dashv_\Gamma {}^pH^\cdot(i_{w!}\pi_{w!}\bar{\mathbf{Q}}_l)$ then $A \in \mathbb{S}(\mathcal{B})$.

Now (i) is obvious; (ii) follows from (i) using $a'_!\bar{\mathbf{Q}}_l = \bar{\mathbf{Q}}_l[-2\delta](-\delta)$ where $\delta = \dim U_w$; (iii) follows from (ii) using $a'h = \pi''a$; (iv) follows from (iii) using $a_!\bar{\mathbf{Q}}_l = \bar{\mathbf{Q}}_l[-2\delta](-\delta)$ and the identification $\pi' = \pi''$. Since the diagram formed by π', f_w, γ, π_w is cartesian we have $\pi'_!\bar{\mathbf{Q}}_l = \pi'_!\gamma^*\bar{\mathbf{Q}}_l = f_w^*\pi_{w!}\bar{\mathbf{Q}}_l$.

We prove (v). Let A be as in (v). Let A' be a simple object of $\mathcal{M}_\Gamma(\tilde{\mathcal{B}}_w)$ such that $A' \dashv_\Gamma f_w^*A$. We have

$$A' \dashv_\Gamma f_w^*({}^pH^\cdot(\pi_{w!}\bar{\mathbf{Q}}_l)) = {}^pH^\cdot(f_w^*\pi_{w!}\bar{\mathbf{Q}}_l) = {}^pH^\cdot(\pi'_!\bar{\mathbf{Q}}_l)$$

and using (iv) we see that $A' \cong \bar{\mathbf{Q}}_l[l(w)]$. Then there exists a non-zero morphism $f_w^* A \longrightarrow \bar{\mathbf{Q}}_l[l(w)]$ in $\mathcal{M}_\Gamma(\tilde{\mathcal{B}}_w)$. Hence there exists a non-zero morphism $A \longrightarrow f_{w!}\bar{\mathbf{Q}}_l[l(w)]$ in $\mathcal{M}_\Gamma(\mathcal{B}_w)$. Now $f_{w!}\bar{\mathbf{Q}}_l[l(w)] = \oplus_{\mathcal{L}} \mathcal{L}_w[l(w)]$ where $\mathcal{L}$ runs over the local systems in $\mathcal{S}(T)^{wF}$ (up to isomorphism). We must have $A \cong \mathcal{L}_w[l(w)]$ for some $\mathcal{L}$ as above. This proves (v).

We prove (vi). We have $A \dashv_\Gamma {}^pH^\cdot(i_{w!}({}^pH^{j'}(\pi_{w!}\bar{\mathbf{Q}}_l)))$ for some j'. Hence there exists a simple object A_1 of $\mathcal{M}_\Gamma(\mathcal{B}_w)$ such that $A_1 \dashv_\Gamma {}^pH^{j'}(\pi_{w!}\bar{\mathbf{Q}}_l)$ and $A \dashv_\Gamma {}^pH^\cdot(i_{w!}A_1)$. From (v) we see that $A_1 \in \mathbb{S}(\mathcal{B}_w)$. From 7.3 we see that $A \in \mathbb{S}(\mathcal{B})$.

7.5. Let $J \subset \mathbf{I}$. Define $\pi_J : X \longrightarrow \mathcal{P}_J$ by $(B', g) \mapsto P_{B',J}$. This is compatible with the Γ-actions where Γ acts on X by $g_0 : (B', g) \mapsto (g_0 B' g_0^{-1}, g g_0^{-1})$. Hence ${}^pH^j(\pi_{J!}\bar{\mathbf{Q}}_l)$ has a Γ-equivariant structure. We show:

Theorem 7.6. *(a) If A is a simple object of $\mathcal{M}_\Gamma(\mathcal{P}_J)$ such that $A \dashv_\Gamma {}^pH^\cdot(\pi_{J!}\bar{\mathbf{Q}}_l)$ then $A \in \mathbb{S}(\mathcal{P}_J)$.*

Note that π_J is a composition $\pi'\pi''$ where $\pi'' : X \longrightarrow \mathcal{B}$ is $(B', g) \mapsto B'$ and $\pi'' : \mathcal{B} \longrightarrow \mathcal{P}_J$ is $B' \mapsto P_{B',J}$. We have a spectral sequence in $\mathcal{M}_\Gamma(\mathcal{P}_J)$ with $E_2 = {}^pH^\cdot(\pi'_!({}^pH^\cdot(\pi''_!\bar{\mathbf{Q}}_l)))$ and E_∞ is an associated graded of ${}^pH^\cdot(\pi'_!\pi''_!\bar{\mathbf{Q}}_l)$. We have $A \dashv_\Gamma E_\infty$ hence $A \dashv_\Gamma E_2$. Hence we can find a simple object $A_1 \in \mathcal{M}_\Gamma(\mathcal{B})$ such that $A_1 \dashv_\Gamma {}^pH^\cdot(\pi''_!\bar{\mathbf{Q}}_l)$ and $A \dashv_\Gamma {}^pH^\cdot(\pi'_!A_1)$. Using the partition $\mathcal{B} = \sqcup_w \mathcal{B}_w$ we see that $A_1 \dashv_\Gamma {}^pH^\cdot(i_{w!}i_w^*\pi''_!\bar{\mathbf{Q}}_l)$ for some $w \in W$ (with i_w as in 7.3). Since $i_w^*\pi''_!\bar{\mathbf{Q}}_l = \pi_{w!}\bar{\mathbf{Q}}_l$ (with π_w as in 7.4) we see that $A_1 \dashv_\Gamma {}^pH^\cdot(i_{w!}\pi_{w!}\bar{\mathbf{Q}}_l)$. Using 7.4(vi) we see that $A_1 \in \mathbb{S}(\mathcal{B}_w)$. By 4.13 (for $\mathcal{B}$ instead of $\mathcal{P}_J$) we see that $A_1 \dashv_\Gamma {}^pH^\cdot({}^\emptyset\bar{\Upsilon}^{\mathbf{s}}_! \bar{\mathcal{L}}^\sharp)$ for some sequence $\mathbf{s}$ in $\mathbf{I}$ and some $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$ (here ${}^\emptyset\bar{\Upsilon}^{\mathbf{s}}$ is $\bar{\Upsilon}^{\mathbf{s}}$ of 3.2 with J replaced by $\emptyset$). We have a spectral sequence in $\mathcal{M}_\Gamma(\mathcal{P}_J)$ with $E_2 = {}^pH^\cdot(\pi'_!({}^pH^\cdot({}^\emptyset\bar{\Upsilon}^{\mathbf{s}}_!\bar{\mathcal{L}}^\sharp)))$ and E_∞ is an associated graded of

$${}^pH^\cdot(\pi'_!{}^\emptyset\bar{\Upsilon}^{\mathbf{s}}_!\bar{\mathcal{L}}^\sharp) = {}^pH^\cdot(\bar{\Upsilon}^{\mathbf{s}}_!\bar{\mathcal{L}}^\sharp)$$

(with $\bar{\Upsilon}^{\mathbf{s}}$ as in 3.2). Now A_1 is a direct summand of ${}^pH^\cdot({}^\emptyset\bar{\Upsilon}^{\mathbf{s}}_!\bar{\mathcal{L}}^\sharp)$. Hence ${}^pH^\cdot(\pi'_!A_1)$ is a direct summand of ${}^pH^\cdot(\pi'_!({}^pH^\cdot({}^\emptyset\bar{\Upsilon}^{\mathbf{s}}_!\bar{\mathcal{L}}^\sharp)))$. Hence $A \dashv_\Gamma E_2$. Our spectral sequence is degenerate (by an argument as in 4.8). Hence $A \dashv_\Gamma E_\infty$ and $A \dashv_\Gamma {}^pH^\cdot(\bar{\Upsilon}^{\mathbf{s}}_!\bar{\mathcal{L}}^\sharp)$. Using 4.13 we deduce that $A \in \mathbb{S}(\mathcal{P}_J)$. The theorem is proved.

7.7. We have a Γ-action $g_1 : (B', g) \mapsto (B', g_1 g)$ on X (this is different from the Γ-action on X in 7.5). This induces a Γ-module structure on $H_c^{2d}(X, \bar{\mathbf{Q}}_l)$ (d as in 7.1).

Proposition 7.8. *The Γ-module $H_c^{2d}(X, \bar{\mathbf{Q}}_l)$ contains a copy of Reg, the left regular representation of Γ.*

Let $\rho_1 : X \longrightarrow X'$ be as in 7.1. Let $X'' = \{u \in G; u \text{ unipotent}\}$. Define $\rho_2 : X' \longrightarrow X''$ by $(B', u) \mapsto u$. Let $\rho = \rho_2\rho_1 : X \longrightarrow X''$. It is well known that ρ_2 is a semismall morphism. Recall that ρ_1 is a finite étale covering. Hence ρ is a semismall morphism. Using this, we see that $\rho_!\bar{\mathbf{Q}}_l[2d]$ is a perverse sheaf on X' (recall that X is smooth of pure dimension $2d$ and ρ is proper). By the decomposition theorem [BBD], ${}^pH^\cdot(\rho_!\bar{\mathbf{Q}}_l)$ is a semisimple perverse sheaf. Hence

$\rho_!\bar{\mathbf{Q}}_l[2d]$ is a semisimple perverse sheaf. Now Γ acts on X'' trivially and ρ is compatible with the Γ-actions. Since $\bar{\mathbf{Q}}_l$ is naturally a Γ-equivariant local system on X we see that we have naturally $\rho_!\bar{\mathbf{Q}}_l[2d] \in \mathcal{M}_\Gamma(X'')$. Let $j : \{1\} \to X''$ be the inclusion of the unit element into X''. If E is an irreducible Γ-module over $\bar{\mathbf{Q}}_l$ we can regard E as a Γ-equivariant local system on $\{1\}$. Then $j_!E$ is a simple object of $\mathcal{M}_\Gamma(X'')$. Let n_E be the number of times $j_!E$ appears in a direct sum decomposition of $\rho_!\bar{\mathbf{Q}}_l[2d]$ (a semisimple object of $\mathcal{M}_\Gamma(X'')$) into a direct sum of simple objects. Since $\rho_!\bar{\mathbf{Q}}_l[2d]$ is selfdual, we have using 1.6(a):

$$\begin{aligned} n_E &= \dim H^0_c(X'', j_!E \otimes \rho_!\bar{\mathbf{Q}}_l[2d])^\Gamma = \dim H^{2d}_c(\{1\}, E \otimes j^*\rho_!\bar{\mathbf{Q}}_l)^\Gamma \\ &= \dim(H^{2d}_c(\mathcal{B} \times \Gamma, \bar{\mathbf{Q}}_l) \otimes E)^\Gamma = \dim(\mathrm{Reg}(-d) \otimes E)^\Gamma = \dim E. \end{aligned}$$

We see that $\rho_!\bar{\mathbf{Q}}_l[2d]$ contains as a direct summand the perverse sheaf $j_!\mathrm{Reg}$ where *Reg* is regarded as an object of $\mathcal{M}_\Gamma(\{1\})$. It follows that the Γ-module $H^0_c(\rho_!\bar{\mathbf{Q}}_l[2d])$ contains as a direct summand the Γ-module $H^0_c(j_!\mathrm{Reg}) = \mathrm{Reg}$. Equivalently the Γ-module $H^{2d}(X, \bar{\mathbf{Q}}_l)$ contains Reg as a direct summand. The proposition is proved.

Corollary 7.9. *Let E be an irreducible Γ-module over $\bar{\mathbf{Q}}_l$. There exists $w \in W$ such that E appears in the Γ-module $\oplus_i H^i_c(\tilde{\mathcal{B}}_w, \bar{\mathbf{Q}}_l)$. ($\Gamma$ acts on $\tilde{\mathcal{B}}_w$ by $g_1 : zU \to g_1zU$.)*

From 7.8 we see that E appears in the Γ-module $H^{2d}_c(X, \bar{\mathbf{Q}}_l)$. Using this and the partition $X = \sqcup_{w\in W} X_w$ (with X_w as in 7.4, Γ-stable) we see that there exists $w \in W$ such that E appears in the Γ-module $H^{2d}_c(X_w, \bar{\mathbf{Q}}_l)$.

Now Γ acts:
on $\tilde{X}_w$ by $g_1 : (zU_w, g) \mapsto (zU_w, g_1g)$,
on $\tilde{X}'_w$ by $g_1 : (z, y) \mapsto (z, g_1y)$,
on $\tilde{X}''_w$ by $g_1 : (z, y)U_w \mapsto (z, g_1y)U_w$,
on $\hat{\mathcal{B}}_w$ by $g_1 : z \mapsto g_1z$.
(Notation of 7.4.) Moreover the maps γ, a, a' in 7.4 are compatible with the Γ-actions. Since γ is a finite principal covering we see that E appears in the Γ-module $H^{2d}_c(\tilde{X}_w, \bar{\mathbf{Q}}_l)$. We identify $\tilde{X}_w = \tilde{X}''_w$ as in 7.4. We see that E appears in the Γ-module $H^{2d}_c(\tilde{X}''_w, \bar{\mathbf{Q}}_l)$. Since a is an affine space bundle we see that E appears in the Γ-module $H^{4d-2l(w)}_c(\tilde{X}'_w, \bar{\mathbf{Q}}_l)$. We have $\tilde{X}'_w = \hat{\mathcal{B}}_w \times \hat{\mathcal{B}}_w$ and $\oplus_i H^i_c(\tilde{X}'_w, \bar{\mathbf{Q}}_l) = \oplus_{i,i'} H^i_c(\hat{\mathcal{B}}_w, \bar{\mathbf{Q}}_l) \otimes H^{i'}_c(\hat{\mathcal{B}}_w, \bar{\mathbf{Q}}_l)$ with Γ acting only on the seond factor. It follows that E appears in the Γ-module $\oplus_i H^i_c(\hat{\mathcal{B}}_w, \bar{\mathbf{Q}}_l)$. Since a' is an affine space bundle we see that the corollary holds.

7.10. For $w \in W$ let $\mathfrak{T}_w = \{t \in T; \dot{w}F(t)\dot{w}^{-1} = t\}$. Now $\Gamma \times \Gamma \times \mathfrak{T}_w$ acts on X_w by $(g_0, g_1, t) : (B', g) \mapsto (g_0B'g_0^{-1}, g_1gg_0^{-1})$, on $\tilde{X}_w$ by $(g_0, g_1, t) : (zU_w, g) \mapsto (g_0zt^{-1}U_w, g_1gg_0^{-1})$, on $\tilde{X}'_w$ by $(g_0, g_1, t) : (z, y) \mapsto (g_0zt^{-1}, g_1yt^{-1})$, on $\tilde{X}''_w$ by $(g_0, g_1, t) : (z, y)U_w \mapsto (g_0zt^{-1}, g_1yt^{-1})U_w$. Moreover the maps γ, a in 7.4 are compatible with the $\Gamma \times \Gamma \times \mathfrak{T}_w$-actions. Also $\Gamma \times \mathfrak{T}_w$ acts on $\hat{\mathcal{B}}_w$ by $(g_1, t) : z \mapsto$

g_1zt^{-1}, on $\tilde{\mathcal{B}}_w$ by $(g_1,t): zU \mapsto g_1zt^{-1}U$; the map a' in 7.4 is compatible with the $\Gamma\times\mathfrak{T}_w$-actions.

For any $\theta\in\hat{\mathfrak{T}}_w := \mathrm{Hom}(\mathfrak{T}_w,\bar{\mathbf{Q}}_l^*)$ let $H_c^i(\tilde{\mathcal{B}}_w)_\theta$ be the subspace of $H_c^i(\tilde{\mathcal{B}}_w)$ on which $\mathfrak{T}_w$ acts via θ; this is naturaly a $\Gamma\times\mathfrak{T}_w$-module. Now $\Gamma\times\Gamma$ acts on X by $(g_0,g_1):(B',g)\mapsto(g_0B'g_0^{-1},g_1gg_0^{-1})$.

Let $\mathcal{G}(\Gamma)$ (resp. $\mathcal{G}(\Gamma\times\Gamma)$ or $\mathcal{G}(\Gamma\times\Gamma\times\mathfrak{T}_w)$) be the Grothendieck group of Γ-modules (resp $\Gamma\times\Gamma$-modules or $\Gamma\times\Gamma\times\mathfrak{T}_w$-modules) of finite dimension over $\bar{\mathbf{Q}}_l$. Let $\Pi:\mathcal{G}(\Gamma\times\Gamma\times\mathfrak{T}_w)\to\mathcal{G}(\Gamma\times\Gamma\times\mathfrak{T}_w)$ be the homomorphism which takes an irreducible $\Gamma\times\Gamma\times\mathfrak{T}_w$-module to the space of $\mathfrak{T}_w$-invariants (an irreducible $\Gamma\times\Gamma\times\mathfrak{T}_w$-module or 0). In the setup of 7.9 we show:

Proposition 7.11.

(a) $$\sum_i(-1)^iH_c^i(X,\bar{\mathbf{Q}}_l)=\sum_{\substack{w\in W\\ \theta\in\hat{\mathfrak{T}}_w\\ i,i'}}(-1)^{i+i'}H_c^i(\tilde{\mathcal{B}}_w,\bar{\mathbf{Q}}_l)_{\theta^{-1}}\otimes H_c^{i'}(\tilde{\mathcal{B}}_w,\bar{\mathbf{Q}}_l)_\theta.$$

equality in $\mathcal{G}(\Gamma\times\Gamma)$.

By the arguments in the proof of 7.9 we have

$$\sum_i(-1)^iH_c^i(X,\bar{\mathbf{Q}}_l)=\sum_{w\in W}\sum_i(-1)^iH_c^i(X_w,\bar{\mathbf{Q}}_l),$$

equality in $\mathcal{G}(\Gamma\times\Gamma)$ and

$$\begin{aligned}\sum_i(-1)^iH_c^i(X_w,\bar{\mathbf{Q}}_l)&=\Pi(\sum_i(-1)^iH_c^i(\tilde{X}_w,\bar{\mathbf{Q}}_l))=\Pi(\sum_i(-1)^iH_c^i(\tilde{X}''_w,\bar{\mathbf{Q}}_l))\\ &=\Pi(\sum_i(-1)^iH_c^i(\tilde{X}'_w,\bar{\mathbf{Q}}_l))=\Pi(\sum_{i,i'}(-1)^{i+i'}H_c^i(\hat{\mathcal{B}}_w,\bar{\mathbf{Q}}_l)\otimes H_c^{i'}(\hat{\mathcal{B}}_w,\bar{\mathbf{Q}}_l))\\ &=\sum_{\theta\in\hat{\mathfrak{T}}_w}\sum_{i,i'}(-1)^{i+i'}H_c^i(\hat{\mathcal{B}}_w,\bar{\mathbf{Q}}_l)_{\theta^{-1}}\otimes H_c^{i'}(\hat{\mathcal{B}}_w,\bar{\mathbf{Q}}_l)_\theta\\ &=\sum_{\theta\in\hat{\mathfrak{T}}_w}\sum_{i,i'}(-1)^{i+i'}H_c^i(\tilde{\mathcal{B}}_w,\bar{\mathbf{Q}}_l)_{\theta^{-1}}\otimes H_c^{i'}(\tilde{\mathcal{B}}_w,\bar{\mathbf{Q}}_l)_\theta,\end{aligned}$$

equalities in $\mathcal{G}(\Gamma\times\Gamma\times\mathfrak{T}_w)$. This proves (a).

7.12. Restricting the $\Gamma\times\Gamma$-action on the modules in 7.11(a) to Γ by $g_1\mapsto(1,g_1)$ gives

$$\sum_i(-1)^iH_c^i(X,\bar{\mathbf{Q}}_l)=\sum_{\substack{w\in W\\ \theta\in\hat{\mathfrak{T}}_w\\ i,i'}}(-1)^{i+i'}\dim(H_c^i(\tilde{\mathcal{B}}_w,\bar{\mathbf{Q}}_l)_{\theta^{-1}})H_c^{i'}(\tilde{\mathcal{B}}_w,\bar{\mathbf{Q}}_l)_\theta.$$

equality in $\mathcal{G}(\Gamma)$. From [DL, 7.1, (7.6.3)] we see that the right hand side is equal to $|W|\mathrm{Reg}$. Thus we have

Proposition 7.13.

(a) $$\sum_i (-1)^i H^i_c(X, \bar{\mathbf{Q}}_l) = |W| Reg \text{ in } \mathcal{G}(\Gamma).$$

Lemma 7.14. *Assume that G is simply connected. Let $\mathbf{s} = (s_1, \dots, s_r), \mathcal{L}$ be as in 2.6 and that 6.18(a) holds. Let $Z^{\mathbf{s}}$ be as in 2.5. Let $\dot{Z}$ be as in 2.5 (with $\mathbf{w} = \mathbf{s}$). Let $Z = \mathcal{Z}^{\mathbf{s}}$ be as in 2.6. Then:*

(a) $\dim H^{2r}_c(Z, \bar{\mathcal{L}}) = n_{\mathcal{L}}$*;*

(b) $\dim H^{2r}_c(Z^{\mathbf{s}}, \mathcal{L}_{\mathbf{s}}) = n_{\mathcal{L}}$*;*

(c) $\dim H^{2r}_c(\dot{Z}, \bar{\mathbf{Q}}_l) = 1$*.*

We prove (a). We shall use 6.18(b) assuming that $\tilde{\mathbf{s}} = \mathbf{s}$. Then $Z = Z'$ and $\dim Z = r$ hence $H^i_c(Z, \bar{\mathcal{L}}) = 0$ for $i < 2r$ and $H^i_c(Z, \bar{\tilde{\mathcal{L}}}) = 0$ for $i < 2r$. Hence $H^{4r}_(Z \times Z, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}) = H^{2r}_c(Z, \bar{\mathcal{L}}) \otimes H^{2r}_c(Z, \bar{\tilde{\mathcal{L}}})$. Since in this case $\rho = 2r$ we see from 6.18(b) that $\dim(H^{2r}_c(Z, \bar{\mathcal{L}}) \otimes H^{2r}_c(Z, \bar{\tilde{\mathcal{L}}}))^\Gamma = n_{\mathcal{L}}$. Using Poincaré duality on the smooth variety Z of pure dimension r (see 2.9) we deduce that $\dim(H^0(Z, \bar{\tilde{\mathcal{L}}}) \otimes H^0(Z, \bar{\mathcal{L}}))^\Gamma = n_{\mathcal{L}}$. Note that the Γ-modules $H^0(Z, \bar{\tilde{\mathcal{L}}})$, $H^0(Z, \bar{\mathcal{L}})$ are dual to each other. Hence the previous equality can be written as $\dim \mathrm{End}_\Gamma(H^0(Z, \bar{\mathcal{L}})) = n_{\mathcal{L}}$ where $\mathrm{End}_\Gamma()$ is the space of Γ-module endomorphisms. If $H^0(Z, \bar{\mathcal{L}}) = 0$, it follows that $n_\lambda = 0$; in this case we have also $H^0(Z, \bar{\tilde{\mathcal{L}}}) = 0$ and by Poicaré duality, $H^{2r}_c(Z, \bar{\mathcal{L}}) = 0$ and (a) holds. Thus we may assume that $H^0(Z, \bar{\mathcal{L}}) \neq 0$. Then $\dim \mathrm{End}_\Gamma(H^0(Z, \bar{\mathcal{L}})) \geq 1$ hence $\dim \mathrm{End}_\Gamma(H^0(Z, \bar{\mathcal{L}})) = n_{\mathcal{L}} = 1$. Then $R = R_{\mathcal{L}}$. Since G is simply connected, it follows that $\mathcal{L} \cong \bar{\mathbf{Q}}_l$. Then $H^0(Z, \bar{\mathcal{L}}) = H^0(Z, \bar{\mathbf{Q}}_l)$ may be identified with the permutation representation V of Γ on the set C of connected components of Z. Let Γ' be the isotropy group in Γ of some connected component of Z. Since Γ acts transitively on C (see 2.9) we see that $\dim V = |\Gamma/\Gamma'|$, $1 = \dim \mathrm{End}_G(V) = |\Gamma' \backslash \Gamma / \Gamma'|$. It follows that $\Gamma = \Gamma'$ hence $\dim V = 1$. Then we have $\dim H^0(Z, \bar{\tilde{\mathcal{L}}}) = 1$ and, by Poincaré duality, $\dim H^{2r}_c(Z, \bar{\mathcal{L}}) = 1$. This proves (a).

We prove (b). Note that $Z^{\mathbf{s}}$ is an open dense subset of Z (using the commutative diagram in 2.6 with $\mathcal{J} = \emptyset$ and 2.8, this statement is reduced to the statement that $Z_2^\emptyset$ in 2.6 is open dense in Z_2 which is clear). Note also that $\bar{\mathcal{L}}|_{Z^{\mathbf{s}}} = \mathcal{L}_{\mathbf{s}}$ (see 2.12) and that Z has pure dimension r (see 2.9). We see that (b) follows from (a).

We prove (c). Let $f^\chi_! \bar{\mathbf{Q}}_l$ be as in 2.5 with $\mathbf{w} = \mathbf{s}$. From the definitions we have $H^{2r}_c(\dot{Z}, \bar{\mathbf{Q}}_l) = \oplus_\chi H^{2r}_c(Z^{\mathbf{s}}, f^\chi_! \bar{\mathbf{Q}}_l)$ where χ runs over $\mathrm{Hom}(T^{F'}, \bar{\mathbf{Q}}_l^*)$ (as in 2.5). Using (b) and 2.5(a) we see that $\dim H^{2r}_c(Z^{\mathbf{s}}, f^\chi_! \bar{\mathbf{Q}}_l)$ is 1 if $\chi = 1$ and is 0 if $\chi \neq 1$. The result follows.

7.15. Assume that G is simply connected. Let d be as in 7.1. Let $w = w_{\mathbf{I}}$. We show:

(a) $\tilde{\mathcal{B}}_w$ *is connected;*

(b) *X_w is connected (notation of 7.4);*
Assume that $\mathbf{s}$ is a reduced expression for w. The associated variety $\dot{Z}$ (see 2.5) is connected by 7.14(c) (note that $\dot{Z}$ has pure dimension $d = l(w)$). But $\dot{Z}$ may be identified with $\tilde{\mathcal{B}}_w$. Hence (a) holds. Since $a' : \hat{\mathcal{B}}_w \longrightarrow \tilde{\mathcal{B}}_w$ is a principal U_w-bundle (as in 7.4) we see that $\hat{\mathcal{B}}_w$ is connected. Since $\tilde{X}'_w \cong \hat{\mathcal{B}}_w \times \hat{\mathcal{B}}_w$ (as in 7.4) we see that $\tilde{X}'_w$ is connected. Since $a : \tilde{X}'_w \longrightarrow \tilde{X}''_w$ (as in 7.4) is a principal U_w-bundle we see that $\tilde{X}''_w$ is connected. Since $\tilde{X}_w = \tilde{X}''_w$ (as in 7.4) we see that $\tilde{X}_w$ is connected. Since $\gamma : \tilde{X}_w \longrightarrow X_w$ (as in 7.4) is surjective, we see that X_w is connected. Hence (b) holds.

Note that (a),(b) above do not necessarily hold without the assumption that G is simply connected.

7.16. *Proof of Proposition 7.2.* Note that $X = \cup_{w' \in W} X_{w'}$, that X is of pure dimension $2d$, that X_w is an open subset of X and that for any $w' \in W - \{w\}$, $X_{w'}$ has pure dimension equal to $d + l(w') < 2d$. It follows that X is connected if and only if X_w is connected. Hence 7.2 is a consequence of 7.15(b).

8. A conjecture

8.1. In this section we assume that G has connected centre. Let $\mathcal{L} \in \mathcal{S}(T)$ be such that $(wF)^* \mathcal{L} \cong \mathcal{L}$ for some $w \in W$. Our assumption on G guarantees that
(a) $w \in W'_{\mathcal{L}} \implies w \in W_{\mathcal{L}}$.
(Notation of 5.7.) Let $\mathcal{X}_{\mathcal{L}}$ be the set of all sequences $\mathbf{s} = (s_1, \dots, s_r)$ in $\mathbf{I}$ such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$. Note that $\mathcal{X}_{\mathcal{L}} \neq \emptyset$.

To any $\mathbf{s} \in \mathcal{X}_{\mathcal{L}}$ we associate an element $\omega \in W$ as in 5.5. We show that ω is independent of the choice of $\mathbf{s}$. We must show that if $\tilde{\mathbf{s}}$ is another element of $\mathcal{X}_{\mathcal{L}}$ and $\tilde{\omega} \in W$ is associated to $\tilde{\mathbf{s}}$ in the same way as ω is associated to $\mathbf{s}$ then $\tilde{\omega} = \omega$. Using 5.6 we see that $F^* \omega^* \mathcal{L} \cong \mathcal{L}$, $F^* \tilde{\omega}^* \mathcal{L} \cong \mathcal{L}$. Hence $\omega^* \mathcal{L} \cong \tilde{\omega}^* \mathcal{L}$ so that $\tilde{\omega} \omega^{-1} \in W'_{\mathcal{L}}$ and, using (a), $\tilde{\omega} \omega^{-1} \in W_{\mathcal{L}}$. By the proof of 5.8 (with $c = \mathbf{c}$) we have $\omega \mathbf{c}(R^+_{\mathcal{L}}) = R^+_{\mathcal{L}}$ and similarly $\tilde{\omega} \mathbf{c}(R^+_{\mathcal{L}}) = R^+_{\mathcal{L}}$. Thus $\omega^{-1}(R^+_{\mathcal{L}}) = \tilde{\omega}^{-1}(R^+_{\mathcal{L}}$ and $\tilde{\omega} \omega^{-1}(R^+_{\mathcal{L}}) = R^+_{\mathcal{L}}$. This together with $\tilde{\omega} \omega^{-1} \in W_{\mathcal{L}}$ yields $\tilde{\omega}^{-1} \omega = 1$, as desired.

Using the previous paragraph and (a) we see that for any $\mathbf{s}, \tilde{\mathbf{s}} \in \mathcal{X}_{\mathcal{L}}$, the set $\mathfrak{F}$ defined as in 5.9 (with $c = \mathbf{c}$) is equal to $\{1\}$.

8.2. For $\mathbf{s}, \tilde{\mathbf{s}} \in \mathcal{X}_{\mathcal{L}}$ let $Z, \tilde{Z}, r, \tilde{r}$ be as in 6.1. Let $\bar{\mathcal{L}}$ be as in 6.2 and let $\bar{\tilde{\mathcal{L}}}$ be the analogous local system on $\tilde{Z}$ defined in terms of $\mathcal{L}$. Then $\check{\bar{\tilde{\mathcal{L}}}}$ (dual of $\bar{\tilde{\mathcal{L}}}$) is the analogous local system on $\tilde{Z}$ defined in terms of $\check{\mathcal{L}}$. Let

$$V_{\tilde{\mathbf{s}}, \mathbf{s}} = \mathrm{Hom}_{\Gamma}(\oplus_i H^i_c(\tilde{Z}, \bar{\tilde{\mathcal{L}}})(i/2), \oplus_{i'} H^{i'}_c(Z, \bar{\mathcal{L}})(i'/2)).$$

where Hom_{Γ} is the space of homomorphisms of Γ-modules. Using 2.13(a) we see that Poincaré duality holds on $\tilde{Z}$ in the form

$$\mathrm{Hom}(H^i_c(\tilde{Z}, \bar{\tilde{\mathcal{L}}})(i/2), \bar{\mathbf{Q}}_l) = H^{2\tilde{r}-i}_c(\tilde{Z}, \check{\bar{\tilde{\mathcal{L}}}})(\tilde{r} - i/2).$$

Hence

$$\mathrm{Hom}(\oplus_i H_c^i(\tilde{Z},\bar{\bar{\tilde{\mathcal{L}}}})(i/2),\bar{\mathbf{Q}}_l) = \oplus_i H_c^i(\tilde{Z},\check{\bar{\tilde{\mathcal{L}}}})(i/2),$$

$$V_{\mathbf{s},\tilde{\mathbf{s}}} = ((\oplus_{i'} H_c^{i'}(Z,\bar{\mathcal{L}})(i'/2)) \otimes (\oplus_i H_c^i(\tilde{Z},\check{\bar{\tilde{\mathcal{L}}}})(i/2)))^{\Gamma},$$

$$V_{\mathbf{s},\tilde{\mathbf{s}}} = (\oplus_n H_c^n(Z\times\tilde{Z},\bar{\mathcal{L}}\boxtimes\check{\bar{\tilde{\mathcal{L}}}})(n/2))^{\Gamma}.$$

By 6.14(d) and 6.15(b), the last vector space has a distinguished basis $\{b^1_{\mathbf{a}};\mathbf{a}\in\mathcal{A}_1\}$, with $\mathcal{A}_1$ as in 6.14.

Let $C_{\mathcal{L}}$ be the category whose objects are the elements of $\mathcal{X}_{\mathcal{L}}$ and in which the set of morphisms from $\tilde{\mathbf{s}}$ to $\mathbf{s}$ is the vector space $V_{\tilde{\mathbf{s}},\mathbf{s}}$. The composition of morphisms is given by composing linear maps.

8.3. We will view T as a maximally $\mathbf{F}_q$-split torus of a second connected reductive algebraic group G' over $\mathbf{F}_q$ in such a way that $R_{\mathcal{L}}$ is the set of roots of G' with respect to T' and $R^+_{\mathcal{L}}$ is the set of positive roots of G' with respect to T' and a Borel subgroup B' of G' which is defined over $\mathbf{F}_q$ and contains T'.

Replacing $G,T,B,\mathcal{L}$ by $G',T,B',\bar{\mathbf{Q}}_l$ in the definition of the set $\mathcal{X}_{\mathcal{L}}$ and of the category $C_{\mathcal{L}}$ in 8.2 we obtain a set $\mathcal{X}'_{\bar{\mathbf{Q}}_l}$ and a category $C'_{\bar{\mathbf{Q}}_l}$. Note that the objects of $C'_{\bar{\mathbf{Q}}_l}$ are the elements of $\mathcal{X}'_{\bar{\mathbf{Q}}_l}$ that is the sequences $\mathbf{S}=(S_1,S_2,\dots,S_b)$ in $W_{\mathcal{L}}$. For $\tilde{\mathbf{S}},\mathbf{S}$ in $\mathcal{X}'_{\bar{\mathbf{Q}}_l}$ we denote by $V'_{\tilde{\mathbf{S}},\mathbf{S}}$ the vector space of morphisms from $\tilde{\mathbf{S}}$ to $\mathbf{S}$ in $C'_{\bar{\mathbf{Q}}_l}$.

The following is conjecturally a functor $\Phi: C_{\mathcal{L}}\to C'_{\bar{\mathbf{Q}}_l}$. To an object $\mathbf{s}$ of $C_{\mathcal{L}}$, Φ associates the object $\mathbf{S}$ of $C'_{\bar{\mathbf{Q}}_l}$ defined as in 5.5. Given two objects $\tilde{\mathbf{s}},\mathbf{s}$ of $C_{\mathcal{L}}$ we set $\tilde{\mathbf{S}}=\Phi(\tilde{\mathbf{s}}),\mathbf{S}=\Phi(\mathbf{s})$ and we define a linear map $\Phi: V_{\tilde{\mathbf{s}},\mathbf{s}}\to V'_{\tilde{\mathbf{S}},\mathbf{S}}$ to be the isomorphism which maps the distinguished basis of $V_{\tilde{\mathbf{s}},\mathbf{s}}$ onto the analogous distinguished basis of $V'_{\tilde{\mathbf{S}},\mathbf{S}}$ according to the bijection $\mathcal{A}(W,\mathbf{c},\mathcal{L},\mathbf{s},\tilde{\mathbf{s}})\xrightarrow{\sim}\mathcal{A}(W_{\mathcal{L}},\omega\mathbf{c},\bar{\mathbf{Q}}_l,\mathbf{S},\tilde{\mathbf{S}})$ described in 5.10. (As pointed out in 8.1, the set $\mathfrak{F}$ which appears in 5.10 is in our case equal to $\{1\}$.) We expect that Φ is a functor and that moreover it is an equivalence of categories.

Index of Notation

References

[Be] R.Bédard, *On the Brauer liftings for modular representations*, J.Algebra **93** (1985), 332-353.
[BBD] A.Beilinson, J.Bernstein and P.Deligne, *Faisceaux pervers*, Astérisque **100** (1982), 5-171.
[Bo] A.Borel, *Linear algebraic groups*, Benjamin,, New York, Amsterdam, 1969.
[DL] P.Deligne and G.Lusztig, *Representations of reductive groups over finite fields*, Ann. Math. **103** (1976), 103-161.
[L1] G.Lusztig, *Characters of reductive groups over a finite field, Ann.Math.Studies 107*, Princeton U.Press, 1984.
[L2] G.Lusztig, *Character sheaves I*, Adv.Math. **56** (1985), 193-237; II, Adv.Math. **57** (1985), 226-265.
[L3] G.Lusztig, *Green functions and character sheaves*, Ann.Math. **131** (1990), 355-408.
[L4] G.Lusztig, *Homology bases arising from reductive groups over a finite field, in "Algebraic groups and their representations", ed. R.W.Carter et al.* (1998), Kluwer Acad. Publ., 53-72.
[L5] G.Lusztig, *Parabolic character sheaves I, II*, Moscow Math.J. **4** (2004), 153-179, 869-896.
[L6] G.Lusztig, *Character sheaves on disconnected groups, VI*, Represent.Th.(electronic) **8** (2004), 377-413.

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