On conjugacy classes in the Lie group E8

George Lusztig (M.I.T.)

Classification of simple Lie algebras (or Lie groups)/$\mathbf{C}$

(Killing 1879, a glory of 19-th century mathematics):

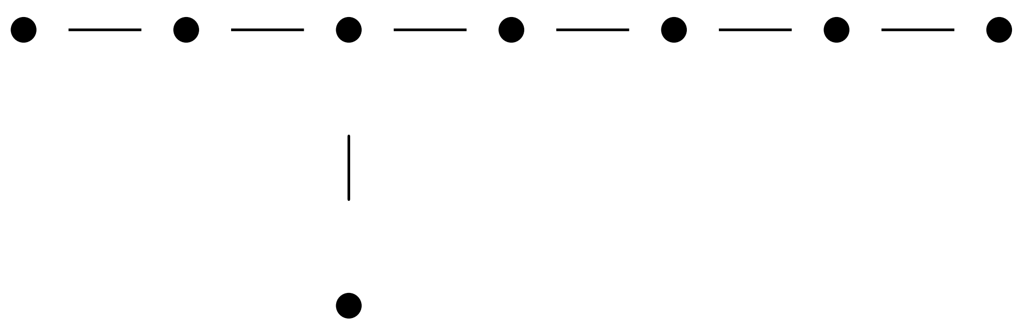
4 ∞ series and 5 exceptional groups of which the

largest one, E_8, has dimension 248. It is the most

noncommutative of all simple Lie groups: $\frac{\dim(G)}{\mathrm{rk}(G)^2} = \frac{248}{8^2}$

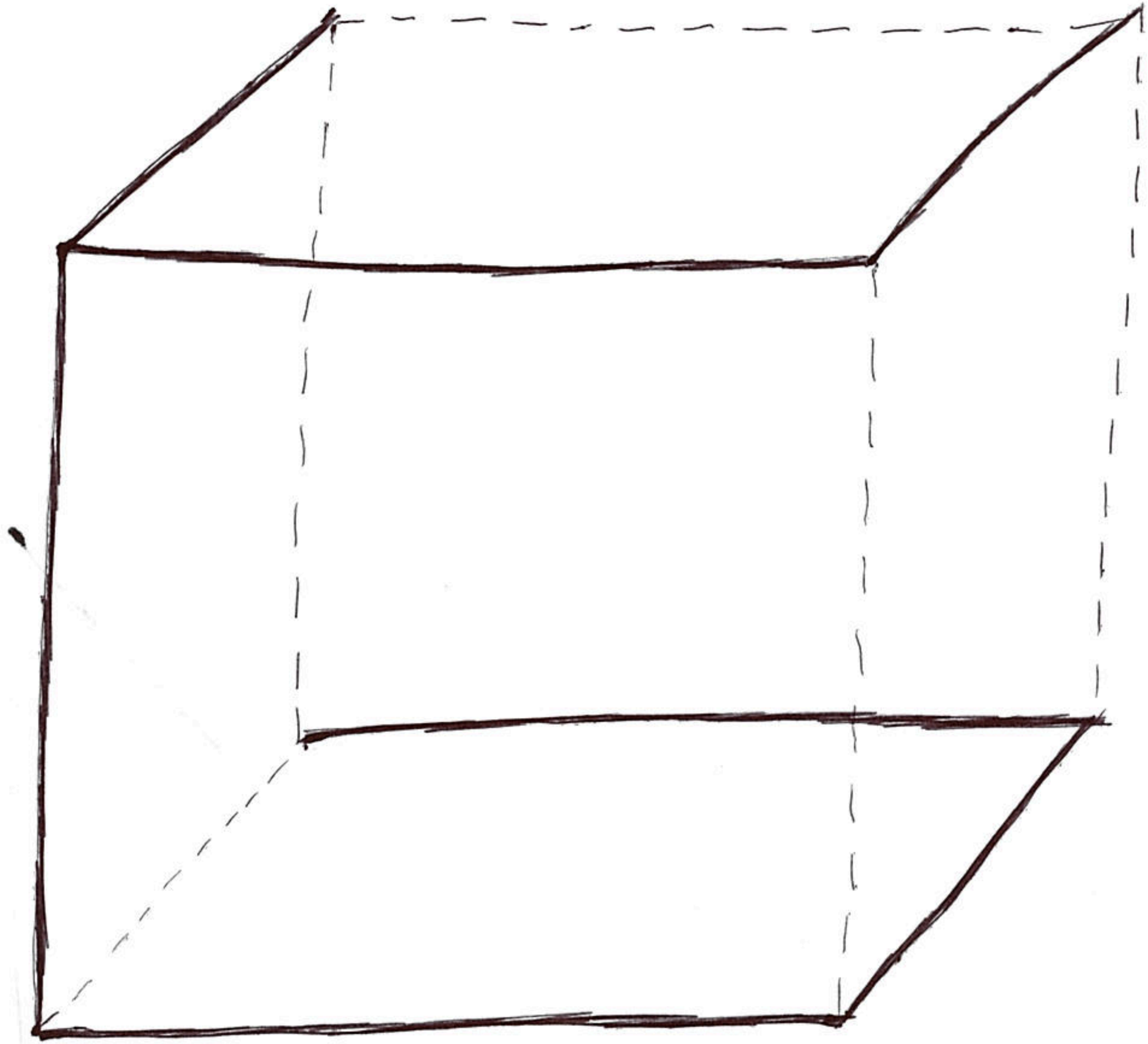
is maximal among simple Lie groups ($\cong 4$).

The Lie group $E8$ can be obtained from the graph $E8$:



by a method of Chevalley (1955), simplified using theory of

"canonical bases" (1990).



I=set of vertices of the graph $E8$.

V=free $\mathbf{Z}$-module with basis $\{\alpha_i; i \in I\}$ and bilinear form:

$(\alpha_i, \alpha_i) = 2$; $(\alpha_i, \alpha_j) = -\sharp$(edges joining i, j) if $i \neq j$.

This bilinear form defined by Korkine-Zolotarev (1873) before Killing and nonconstructibly earlier by Smith (1867). Roots:

$R = \{\alpha \in V, (\alpha, \alpha) = 2\}$; $\sharp R = 240$.

M=$\mathbf{C}$-vector with basis $X_\alpha(\alpha \in R)$, $t_i(i \in I)$.

For $i \in I$, $\epsilon = 1, -1$, define $E_{i,\epsilon} : M \to M$ by

$E_{i,\epsilon} X_\alpha = X_{\alpha+\epsilon\alpha_i}$ if $\alpha \in R, \alpha + \epsilon\alpha_i \in R$,

$E_{i,\epsilon} X_\alpha = 0$ if $\alpha \in R, \alpha + \epsilon\alpha_i \notin R \cup 0$,

$E_{i,\epsilon} X_{-\epsilon\alpha_i} = t_i$, $E_{i,\epsilon} t_j = |(\alpha_i, \alpha_j)| X_{\epsilon\alpha_i}$.

$R^+ = R \cap \sum_{i \in I} \mathbf{N}\alpha_i$, $M^+ = \sum_{\alpha \in R^+} \mathbf{C} X_\alpha \subset M$.

$\dim M = 248$, $\dim M^+ = 120$.

$G = E8(\mathbf{C}) =$ subgroup of $GL(M)$ generated by $\exp(\lambda E_{i,\epsilon})$, $i \in I$,

$\epsilon = 1, -1$, $\lambda \in \mathbf{C}^*$. (The Lie group $E8$.) Replacing

$\mathbf{C}$ by F_q leads to a finite group $E8(F_q)$ which, by Chevalley

(1955), is simple of order $q^{248}+$ lower powers of q.

The Weyl group is $W = E8(F_1) = \{g \in Aut(V); gR = R\}$. Note:

$\sharp(E8(F_1)) = \lim_{q\to 1} \frac{\sharp(E8(F_q))}{(q-1)^8} = 4!6!8!$.

For a group H, the *conjugacy classes* of H are the orbits of

the action $x : g \mapsto xgx^{-1}$ of H on itself. They form a set $cl(H)$.

Main guiding principle of this talk:

The conjugacy classes of $G = E8(\mathbf{C})$ should be organized

according to the conjugacy classes of $W = E8(F_1)$.

For $w \in W$ let $l(w) = \sharp(R^+ \cap w(R - R^+)) \in \mathbf{N}$ (length).

For $C \in cl(W)$ let

$C_{min} = \{w \in C; l : C \to \mathbf{N}$ reaches minimum at $w\}$.

$C \in cl(W)$ is *elliptic* if $\{x \in V; wx = x\} = 0$ for some/any $w \in C$.

$cl(W)$ has been described by Carter (1972);

$\sharp cl(W) = 112$, $\sharp\{C \in cl(W); C$ elliptic$\} = 30$.

The G-orbit of M^+ in the Grassmannian of 120-dimensional subspaces of M is a closed smooth subvariety $\mathcal{B}$ of dimension 120, the *flag manifold*.

The diagonal G-action on $\mathcal{B} \times \mathcal{B}$ has orbits in canonical bijection

$$\mathcal{O}_w \leftrightarrow w$$

with W. (Bruhat 1954, Harish Chandra 1956).

For $w \in W$ let $G_w = \{g \in G; (B, g(B)) \in \mathcal{O}_w$ for some $B \in \mathcal{B}\}$.

For $C \in cl(W)$ let $G_C = G_w$ where $w \in C_{min}$; one shows (using Geck-Pfeiffer 1993): G_C is independent of the choice of w in C_{min}; also, $G_C \neq \emptyset$, G_C is a union of conjugacy classes. Let

$\delta_C = \min_{\gamma \in cl(G); \gamma \subset G_C} \dim \gamma$,

$\boxed{G_C} = \cup_{\gamma \in cl(G); \gamma \subset G_C, \dim \gamma = \delta_C} \gamma$.

$\boxed{G_C}$ is $\neq \emptyset$, a union of conjugacy classes of fixed dimension, δ_C.

Let $G^{un} = \{g \in G;$ all eigenvalues of $g : M \to M$ are $1\}$.

Theorem. *(a)* $\cup_{C\in cl(W)} \boxed{G_C} = G$.

(b) For any $C, C' \in cl(W)$, $\boxed{G_C}, \boxed{G_{C'}}$ *are either equal or disjoint.*

(c) For any $C \in cl(W)$, $\boxed{G_C} \cap G^{un}$ *is either empty or a single conjugacy class.*

(d) If $C \in cl(W)$ *is elliptic then* $\delta_C = 248 - l(w)$ *for any* $w \in C_{min}$ *and* $\boxed{G_C} \cap G^{un}$ *is a single conjugacy class.*

See arxiv:1305.7168.

Proof: uses representation theory of $E8(F_q)$ (a part of its

character table known since 1980's) and computer calculation.

(Help with programming in GAP was provided by Gongqin Li.)

The subsets $\boxed{G_C}$ partition G into the *strata* of G.

If $\gamma, \gamma' \in cl(G)$ we say: $\gamma \sim \gamma'$ if γ, γ' are contained in the

same stratum. If $C, C' \in cl(W)$ we say: $C \sim C'$ if $\boxed{G_C} = \boxed{G_{C'}}$.

These are equivalence relations on $cl(G) = cl(E8(\mathbf{C}))$,

$cl(W) = cl(E8(F_1))$; by the theorem we have canonically

$cl(E8(\mathbf{C}))/\sim \leftrightarrow cl(E8(F_1))/\sim$.

Examples. If $C = \{1\}$ then $\boxed{G_C} = \{1\}$.

If $C \in cl(W)$ contains all w in W of length 1 then $\boxed{G_C}$ is a single

conjugacy class (it has dimension 58.)

Let $C_{cox} = \{w \in W; w \text{ has order } 30\}$; it is a single conjugacy

class in W, Coxeter class. If $C = C_{cox}$ then $G_C = \boxed{G_C}$ is the

union of all conjugacy classes of dimension 240 (Steinberg 1965).

Let $cl_u(E8(\mathbf{C})) = \{\gamma \in cl(E8(\mathbf{C}); \gamma \subset G^{un}\}$; $cl_u(E8(\bar{F}_p)) =$

analogous set with $\mathbf{C}$ replaced by $\bar{F}_p$, p a prime number.

$\sharp(cl_u(E8(\mathbf{C}))) = 70$, (Dynkin, Kostant), $\sharp(cl_u(E8(\bar{F}_p))) = 70 + n$

where $n = 4, 1, 0, 0, \ldots$ for $p = 2, 3, 5, 7, \ldots$

We have a natural imbedding $j_p : cl_u(E8(\mathbf{C})) \to cl_u(E8(\bar{F}_p))$ and

$cl(E8(\mathbf{C}))/\sim \leftrightarrow \cup_{p \text{ prime}} cl_u(E8(\bar{F}_p))$ (union taken using j_p).

Hence number of strata is 75.

The definitions above extend to any simple Lie group G.

If $G = S0_5(\mathbf{C})$, then $cl(W) = \{C_4, C_4^2, C', C'', \{1\}\}$

where C_4 consists of the elements of order 4;

$\boxed{G_{C_4}}$ is the union of classes of dimension 8,

$\boxed{G_{C_4^2}}$ is the union of classes of dimension 6,

$\boxed{G_{C'}}, \boxed{G_{C''}}$ are the two conjugacy classes of dimension 4,

$\boxed{G_{\{1\}}} = \{1\}$.

$A_5 =$ alternating group in 5 letters. Let

$Y = \{\phi \in \mathrm{Hom}(A_5, G);$ centralizer of $\phi(A_5)$ in G is finite$\}$.

We have $Y \neq \emptyset$, by Borovik (1989). (Note: G is the simple group

of largest dimension for which the analogue of Y is $\neq \emptyset$.)

G acts on Y by conjugation.

Problem: What is the number of orbits of this action?

Theorem. *Y is a single G-orbit.*

Sketch of proof. A_5 has generators x_2, x_3, x_5 with relations

$x_2^2 = x_3^3 = x_5^5 = 1$, $x_2 x_3 x_5 = 1$ (Hamilton, 1856).

For $n = 2, 3, 5$ let $\mathbf{c}_n$ be the unique conjugacy class in G such

that $\text{codim}\mathbf{c}_n = 240/n$ and any element of $\mathbf{c}_n$ has order n.

One can show: $\mathbf{c}_n \subset \boxed{G_{C_{cox}^{30/n}}}$.

According to Frey (1998), Serre (1998), $\phi \in \text{Hom}(A_5, G)$ is

in Y if and only if $\phi(x_n) \in \mathbf{c}_n$ for $n = 2, 3, 5$.

Thus Y can be identified with

$$Y' = \{(g_2, g_3, g_5) \in \mathbf{c}_2 \times \mathbf{c}_3 \times \mathbf{c}_5; g_2 g_3 g_5 = 1\}$$

and the G-action on Y' (simultaneous conjugation) has finite stabilizers hence all its orbits have dimension equal to $\dim G$. It is enough to show that for the analogue of Y' over F_q with q large and prime to $2, 3, 5$ we have

$$\sharp Y'(F_q) = q^{248} + \text{lower powers of } q.$$

By Burnside (1911):

$$\sharp Y'(F_q) = \frac{\sharp \mathbf{c}_2(F_q)\sharp \mathbf{c}_3(F_q)\sharp \mathbf{c}_5(F_q)}{\sharp E8(F_q)} \sum_\rho \frac{\rho(a_2)\rho(a_3)\rho(a_5)}{\rho(1)}$$

where $a_n \in \mathbf{c}_n(F_q)$ and ρ runs over the irreducible characters of the finite group $E8(F_q)$. From the representation theory of $E8(F_q)$ we can evaluate the right hand side and get the desired estimate for $\sharp Y'(F_q)$.

$E8$ is the only simple Lie group G in which the group of connected components of the centralizer of some $g \in G$ is not solvable. (Such g is unipotent, contained in $\boxed{G_{C^5_{cox}}}$.)