UNIPOTENT CLASSES AND SPECIAL WEYL GROUP REPRESENTATIONS

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Introduction

0.1. Let G be a simple adjoint algebraic group over $\mathbf{C}$ and let $\mathcal{X}$ be the set of unipotent conjugacy classes in G. Let $C \in \mathcal{X}$ and let $u \in C$. The following invariants of C are important in representation theory:

-the dimension $\mathbf{b}_C$ of the fixed point set of $\mathrm{Ad}(u)$ on the flag manifold of G;

-the number $\mathbf{z}_C$ of connected components of the centralizer of u in G;

-the number $\tilde{\mathbf{z}}_C$ of connected components of the centralizer of a unipotent element in the simply connected covering of G which projects to u;

-the irreducible representation ρ_C of the Weyl group $\mathbf{W}$ of G corresponding to C and the constant local system under the Springer correspondence [Sp].

Let $\tilde{\mathcal{S}}_{\mathbf{W}}$ be the set of isomorphism classes of irreducible representations of $\mathbf{W}$ of the form ρ_C for some $C \in \mathcal{X}$. It is known [Sp] that $C \mapsto \rho_C$ is a bijection $\mathcal{X} \xrightarrow{\sim} \tilde{\mathcal{S}}_{\mathbf{W}}$.

Note that the definition of each of $\mathbf{b}_C$, $\mathbf{z}_C$, $\tilde{\mathbf{z}}_C$ is based on considerations of algebraic geometry and in the case of $\tilde{\mathcal{S}}_{\mathbf{W}}$, also on considerations of étale cohomology.

In [L1, Sec.9] I conjectured that $\tilde{\mathcal{S}}_{\mathbf{W}}$, $C \mapsto \mathbf{b}_C$ and $C \mapsto \mathbf{z}_C$ can be determined purely in terms of data involving the Weyl group $\mathbf{W}$ (more precisely, the "special representations" of the "parahoric" subgroups of $\mathbf{W}$, see 1.1, 1.2). At that time I could only prove this conjecture for $\tilde{\mathcal{S}}_{\mathbf{W}}$ and for $C \mapsto \mathbf{b}_C$ assuming that G is of classical type (my proof was based on [S1]) and a little later for G of type F_4 (based on [S2]). In [AL] the conjecture for $\tilde{\mathcal{S}}_{\mathbf{W}}$ and $C \mapsto \mathbf{b}_C$ was established for G of type E_6, E_7, E_8. At the time [L4] was written, I proved the remaining conjecture of [L1] (concerning $C \mapsto \mathbf{z}_C$); this was stated in [L4, 13.3]. For classical groups the proof involved a new description (in terms of "symbols") of the Springer correspondence for classical groups (given in [L5]) while for exceptional groups this was a purely mechanical verification based on the tables [Al]. The conjecture of [L1] is restated and proved here as Theorem 1.5(a),(b1),(b2). At the same time we state and prove a complement to that conjecture, namely that $C \mapsto \tilde{\mathbf{z}}_C$ is determined purely in terms of data involving $\mathbf{W}$ (see Theorem 1.5(b3)). Note that for classical groups

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this involves some combinatorial considerations while for exceptional groups this involves only a purely mechanical verification based on the known tables.

Notation. For a finite set F let $|F|$ be the cardinal of F. For i,j in $\mathbf{Z}$ we set $[i,j]=\{n\in\mathbf{Z}; i\le n\le j\}$. For x,y in $\mathbf{Z}$ we write $x\ll y$ if $x\le y-2$.

Contents

1. Statement of the main result

1.1. Let W be a finite crystallographic Coxeter group. Let $\mathrm{Irr}(W)$ be the set of isomorphism classes of irreducible representations of W over $\mathbf{Q}$. If $E\in\mathrm{Irr}(W)$ and E' is a finite dimensional $\mathbf{Q}[W]$-module, let $[E:E']_W$ be the multiplicity of E in E'. Let S^i_W be the i-th symmetric power of the reflection representation of W. For any $E\in\mathrm{Irr}(W)$ we define integers $f_E\ge 1$, $a_E\ge 0$ by the requirement that the generic degree of the Hecke algebra representation corresponding to E is of the form $\frac{1}{f_E}\mathbf{q}^{a_E}$+higher powers of $\mathbf{q}$ ($\mathbf{q}$ is an indeterminate); let b_E be the smallest integer $i\ge 0$ such that $[E:S^i_W]_W\ge 1$. As observed in [L1, Sec.2], we have $a_E\le b_E$ for any $E\in\mathrm{Irr}(W)$; following [L1, Sec.2] we set $\mathcal{S}_W=\{E\in\mathrm{Irr}(W); a_E=b_E\}$; this is the set of "special representations" of W. Let $\mathrm{Irr}(W)^\dagger=\{E\in\mathrm{Irr}(W); [E:S^{b_E}_W]_W=1\}$. We have $\mathcal{S}_W\subset\mathrm{Irr}(W)^\dagger$.

1.2. In this paper we fix a root datum of finite type $\mathcal{R}=(Y,X,\check\alpha_i,\alpha_i(i\in I),\langle,\rangle)$. (Here Y,X are free abelian groups of finite rank, $\langle,\rangle:Y\times X\longrightarrow\mathbf{Z}$ is a perfect pairing, $\check\alpha_i\in Y$ are the simple coroots and $\alpha_i\in X$ are the simple roots.) We assume that $I\ne\emptyset$ and that $\mathcal{R}$ is of adjoint type that is, $\{\alpha_i;i\in I\}$ is a $\mathbf{Z}$-basis of X. Let $R\subset X$ (resp. $\check R\subset Y$) be the set of roots (resp. coroots); let $\check\alpha\leftrightarrow\alpha$ be the canonical bijection $\check R\leftrightarrow R$. We assume that $\mathcal{R}$ is irreducible that is, there is a unique $\alpha_0\in R$ such that $\check\alpha_0-\check\alpha_i\notin\check R$ for any $i\in I$. Let $\tilde I=I\sqcup\{0\}$. For $i\in\tilde I$ let $s_i:X\longrightarrow X$ be the reflection determined by $\alpha_i,\check\alpha_i$. Let $\mathbf{W}$ be the subgroup of $GL(X)$ generated by $\{s_i;i\in I\}$, a finite crystallographic Coxeter group containing s_0. The elements $s_i(i\in\tilde I)$ in $\mathbf{W}$ satisfy the relations of the affine Weyl group of type dual to that of $\mathcal{R}$. Let $\tilde{\mathcal{A}}=\{J;J\subsetneqq\tilde I\}$. For any $J\in\tilde{\mathcal{A}}$, let $\mathbf{W}_J$ be the subgroup of $\mathbf{W}$ generated by $\{s_i;i\in J\}$, a finite crystallographic Coxeter group with set of generators $\{s_i;i\in J\}$, said to be a *parahoric subgroup* of $\mathbf{W}$.

Let Ω be the (commutative) subgroup of $\mathbf{W}$ consisting of all $\omega\in\mathbf{W}$ such that $\omega(\alpha_i)=\alpha_{\underline{\omega}(i)}$ $(i\in\tilde I)$ for some (necessarily unique) permutation $\underline{\omega}:\tilde I\xrightarrow{\sim}\tilde I$.

1.3. If $J \in \tilde{\mathcal{A}}$ and $E_1 \in \mathrm{Irr}(\mathbf{W}_J)^\dagger$, there is a unique $E \in \mathrm{Irr}(\mathbf{W})$ such that $b_E = b_{E_1}$ and $[E : \mathrm{Ind}_{\mathbf{W}_J}^{\mathbf{W}}(E_1)]_{\mathbf{W}} \ge 1$. (Then $[E : \mathrm{Ind}_{\mathbf{W}_J}^{\mathbf{W}}(E_1)]_{\mathbf{W}} = 1$ and $E \in \mathrm{Irr}(\mathbf{W})^\dagger$.) We write $E = j_{\mathbf{W}_J}^{\mathbf{W}}(E_1)$. Let $E \in \mathrm{Irr}(\mathbf{W})$ and let

$$\mathcal{Z}_E = \{(J, E_1); J \in \tilde{\mathcal{A}}, E_1 \in \mathcal{S}_{\mathbf{W}_J}, E = j_{\mathbf{W}_J}^{\mathbf{W}}(E_1)\}.$$

Let

$$\bar{\mathcal{S}}_{\mathbf{W}} = \{E \in \mathrm{Irr}(\mathbf{W}); \mathcal{Z}_E \ne \emptyset\}.$$

Let $E \in \bar{\mathcal{S}}_{\mathbf{W}}$. We set

$$\mathfrak{a}_E = \max_{(J,E_1)\in\mathcal{Z}_E} f_{E_1}.$$

Let $\mathcal{Z}_E^\spadesuit = \{(J, E_1) \in \mathcal{Z}_E; f_{E_1} = \mathfrak{a}_E\}$. We have $\mathcal{Z}_E^\spadesuit \ne \emptyset$.

If $(J, E_1) \in \mathcal{Z}_E$ and $\omega \in \Omega$ then $\mathrm{Ad}(\omega) : \mathbf{W}_J \xrightarrow{\sim} \mathbf{W}_{\underline{\omega}(J)}$ carries E_1 to a representation ${}^\omega E_1 \in \mathcal{S}_{\mathbf{W}_{\underline{\omega}(J)}}$ such that $\mathrm{Ind}_{\mathbf{W}_J}^{\mathbf{W}}(E_1) = \mathrm{Ind}_{\mathbf{W}_{\underline{\omega}(J)}}^{\mathbf{W}}({}^\omega E_1)$, $b_{{}^\omega E_1} = b_{E_1}$ and $f_{{}^\omega E_1} = f_{E_1}$. It follows that $j_{\mathbf{W}_J}^{\mathbf{W}}(E_1) = j_{\mathbf{W}_{\underline{\omega}(J)}}^{\mathbf{W}}({}^\omega E_1)$. Thus $(\underline{\omega}(J), {}^\omega E_1) \in \mathcal{Z}_E$ and $\omega : (J, E_1) \mapsto (\underline{\omega}(J), {}^\omega E_1)$ is an action of Ω on $\mathcal{Z}_E$. This restricts to an action of Ω on $\mathcal{Z}_E^\spadesuit$. The stabilizer in Ω of $(J, E_1) \in \mathcal{Z}_E^\spadesuit$ for this action is denoted by Ω_{J,E_1}. We set

$$\mathfrak{c}_E = \max_{(J,E_1)\in\mathcal{Z}_E^\spadesuit} |\Omega_{J,E_1}|.$$

1.4. Let G be a semisimple (adjoint) algebraic group over $\mathbf{C}$ with root datum $\mathcal{R}$. Let $\mathcal{X}$, $C \mapsto \rho_C$, $C \mapsto \mathbf{b}_C$, $C \mapsto \mathbf{z}_C$, $C \mapsto \tilde{\mathbf{z}}_C$, $\tilde{\mathcal{S}}_{\mathbf{W}}$ be as in 0.1.

Theorem 1.5. *(a)* $\tilde{\mathcal{S}}_{\mathbf{W}} = \bar{\mathcal{S}}_{\mathbf{W}}$.

(b) Let $C \in \mathcal{X}$. *Set* $E = \rho_C \in \bar{\mathcal{S}}_{\mathbf{W}}$. *Then:*

(b1) $\mathbf{b}_C = b_E$;

(b2) $\mathbf{z}_C = \mathfrak{a}_E$;

(b3) $\tilde{\mathbf{z}}_C/\mathbf{z}_C = \mathfrak{c}_E$.

For exceptional types the proof of (a),(b1)-(b3) consists in examining the existing tables. Some relevant data is collected in §7. The proof for the classical types is given in §3-§6 after combinatorial preliminaries in 1.9-1.11 and §2.

1.6. Let G' be a connected reductive group over $\mathbf{C}$ such that G is the quotient of G' by its centre.

Note that 1.5(a) is closely connected to the definition of a unipotent support of a character sheaf on G' provided by [L6, 10.7]. In fact, [L6, 10.7(iii)] provides a proof of the inclusion $\bar{\mathcal{S}}_{\mathbf{W}} \subset \tilde{\mathcal{S}}_{\mathbf{W}}$ without case by case checking.

For any $g \in G'$ let g_u be the unipotent part of g. We now state an alternative conjectural definition of the unipotent support of a character sheaf on G'.

Conjecture 1.7. *Let A be a character sheaf on G'. There exists a unique unipotent class C in G' such that:*

(i) $A|_{\{g\}} \neq 0$ for some $g \in G'$ with $g_u \in C$;

(ii) if $g' \in G'$ satisfies $A|_{\{g'\}} \neq 0$ then the conjugacy class of g'_u in G' has dimension $< \dim(C)$.

1.8. Theorem 1.5 remains valid if $\mathbf{C}$ is replaced by an algebraically closed field whose characteristic is either 0 or a prime which is good for G and which (if G is of type A_{n-1}) does not divide n.

1.9. In the rest of this section we discuss some preliminaries to the proof of 1.5.

If $J, J' \in \tilde{\mathcal{A}}$, $J \subset J'$ and $E_1 \in \mathrm{Irr}(\mathbf{W}_J)^\dagger$, there is a unique $E'_1 \in \mathrm{Irr}(\mathbf{W}_{J'})$ such that $b_{E_1} = b_{E'_1}$ and $[E'_1 : \mathrm{Ind}_{\mathbf{W}_J}^{\mathbf{W}_{J'}}(E_1)]_{\mathbf{W}_{J'}} \geq 1$. (Then $[E'_1 : \mathrm{Ind}_{\mathbf{W}_J}^{\mathbf{W}_{J'}}(E_1)]_{\mathbf{W}_{J'}} = 1$ and $E'_1 \in \mathrm{Irr}(\mathbf{W}_{J'})^\dagger$.) We write $E'_1 = j_{\mathbf{W}_J}^{\mathbf{W}_{J'}}(E_1)$. Note that

(a) $j_{\mathbf{W}_J}^{\mathbf{W}}(E_1) = j_{\mathbf{W}_{J'}}^{\mathbf{W}}(j_{\mathbf{W}_J}^{\mathbf{W}_{J'}}(E_1))$;

(b) if, in addition, $E_1 \in \mathcal{S}_{\mathbf{W}_J}$, then $E'_1 \in \mathcal{S}_{\mathbf{W}_{J'}}$ and $f_{E_1} \leq f_{E'_1}$.

(See [L1, Sec.4].)

Let $\mathcal{P}'$ be the collection of parahoric subgroups W of $\mathbf{W}$ such that $W = \mathbf{W}_J$ for some $J \subset \tilde{I}$, $|J| = |\tilde{I}| - 1$. From (a),(b) we see that

(c) $\bar{\mathcal{S}}_{\mathbf{W}} = \{E \in \mathrm{Irr}(\mathbf{W}); E = j_W^{\mathbf{W}}(E_1)$ for some $W \in \mathcal{P}'$ and some $E_1 \in \mathcal{S}_W\}$,

(d) $$\mathfrak{a}_E = \max_{(J,E_1) \in \mathcal{Z}_E; |J| = |\tilde{I}| - 1} f_{E_1} \text{ for } E \in \bar{\mathcal{S}}_{\mathbf{W}}.$$

If $W = W_1 \times W_2$ where W_1, W_2 are finite crystallographic Coxeter groups and $E_1 \in \mathrm{Irr}(W_1)$, $E_2 \in \mathrm{Irr}(W_2)$ then $E := E_1 \boxtimes E_2 \in \mathrm{Irr}(W)$ belongs to $\mathcal{S}_W$ if and only if $E_1 \in \mathcal{S}_{W_1}$ and $E_2 \in \mathcal{S}_{W_2}$; in this case we have

(e) $a_E = a_{E_1} + a_{E_2}$, $f_E = f_{E_1} f_{E_2}$.

1.10. We show:

(a) *if $J, J' \in \tilde{\mathcal{A}}$ and $\mathbf{W}_J = \mathbf{W}_{J'} \neq \mathbf{W}$ then $J = J'$.*

It is enough to show that if $J, J' \in \tilde{\mathcal{A}}$ and $\mathbf{W}_J \subset \mathbf{W}_{J'} \neq \mathbf{W}$ then $J \subset J'$. To see this we may assume that J consists of a single element j. We have $s_j \in \mathbf{W}_{J'}$. Assume that $j \notin J'$. If $J' \cup \{j\} \neq \tilde{I}$ then $\mathbf{W}_{J' \cup \{j\}}$ is a Coxeter group on the generators $\{s_h; h \in J' \cup \{j\}\}$. In particular s_j is not contained in the subgroup $\mathbf{W}_{J'}$ generated by $\{s_h; h \in J'\}$, a contradiction. Thus we have $J' \cup \{j\} = \tilde{J}$. We see that $\mathbf{W}_{J'}$ contains $\{s_h; h \in J' \cup \{j\}\}$ which generates $\mathbf{W}$. Thus $\mathbf{W}_J = \mathbf{W}$ which is again a contradiction. This proves (a).

1.11. For a subgroup $\tilde{\Omega}$ of Ω let $\mathcal{P}^{\tilde{\Omega}}$ be the collection of parahoric subgroups W of $\mathbf{W}$ such that $W = \mathbf{W}_J$ for some $J \in \tilde{\mathcal{A}}$ where J is $\tilde{\Omega}$-stable and is maximal with this property. From the definitions we have

$$\mathfrak{c}_E = \max |\tilde{\Omega}|,$$

where the maximum is taken over all subgroups $\tilde{\Omega} \subset \Omega$ and all $(J, E_1) \in \mathcal{Z}_E^{\spadesuit}$ such that $\mathbf{W}_J \in \mathcal{P}^{\tilde{\Omega}}$, $\tilde{\Omega} \subset \Omega_{J,E_1}$.

2. Combinatorics

2.1. In this section we fix $m \in \mathbf{N}$.

Let $Z_m = \{z_* = (z_0, z_1, z_2, \dots, z_m) \in \mathbf{N}^{m+1}; z_0 < z_1 < \dots < z_m\}$. Let $z_*^0 = z_*^{0,m} = (0, 1, 2, \dots, m) \in Z_m$. For any $z_* \in Z_m$ we have $z_* - z_*^0 \in \mathbf{N}^{m+1}$. Hence

$\rho_0 : Z_m \longrightarrow \mathbf{N}$, $z_* \mapsto \sum_{i \in [0,m]} (z_i - z_i^0)$ and

$\beta_0 : Z_m \longrightarrow \mathbf{N}$, $z_* \mapsto \sum_{0 \le i < j \le m} (z_i - z_i^0)$

are well defined. For any $n \in \mathbf{N}$ we set $Z_m^n = \{z_* \in Z_m; \rho_0(z_*) = n\}$.

2.2. Let X_m be the set of all $x_* = (x_0, x_1, x_2, \dots, x_m) \in \mathbf{N}^{m+1}$ such that $x_i \le x_{i+1}$ for $i \in [0, m-1]$, $x_i < x_{i+2}$ for $i \in [0, m-2]$. For $x_* \in X_m$ let $\mathfrak{S}(x_*)$ be the set of all $i \in [0, m]$ such that $x_{i-1} < x_i < x_{i+1}$ (with the convention $x_{-1} = -\infty, x_{m+1} = \infty$). Note that

(a) $|\mathfrak{S}(x_*)| \cong m - 1 \mod 2$;

(b) $\mathfrak{S}(x_*) = \emptyset$ if and only if m is odd and $x_i = x_{i+1}$ for $i = 0, 1, \dots, (m-1)/2$.

2.3. Let Y_m be the set of all $y_* = (y_0, y_1, y_2, \dots, y_m) \in \mathbf{N}^{m+1}$ such that $y_i \le y_{i+1}$ for $i \in [0, m-1]$, $y_i \ll y_{i+2}$ for $i \in [0, m-2]$. For $y_* \in Y_m$ let $\mathfrak{I}(y_*)$ be the set of all intervals $[i, j] \subset [0, m]$ (with $i \le j$) such that

$y_{i-1} - (i-1) < y_i - i = y_{i+1} - i + 1 = \dots = y_j - j < y_{j+1} - (j+1)$

(with the convention $y_{-1} = -\infty, y_{m+1} = \infty$). We have

(a) $\mathfrak{I}(y_*) = \emptyset$ if and only if m is odd and $y_i = y_{i+1}$ for $i = 0, 1, \dots, (m-1)/2$.

Let

$\mathfrak{I}'(y_*) = \{\mathcal{I} \in \mathfrak{I}(y_*); \mathcal{I} = [i, j]$ with $|[i, j]| =$odd$\}$,

$\mathfrak{I}''(y_*) = \{\mathcal{I} \in \mathfrak{I}(y_*); \mathcal{I} = [i, j]$ with $|[i, j]| =$even$\}$.

We have

(b) $|\mathfrak{I}'(y_*)| \cong m - 1 \mod 2$.

Let $R(y_*)$ be the set of all $k \in [0, m]$ such that $k = i$ or $k = j$ for some (necessarily unique) $[i, j] \in \mathfrak{I}(y_*)$. Let $R_0(y_*)$ be the set of all $k \in [0, m]$ such that $k = i$ for some (necessarily unique) $[i, j] \in \mathfrak{I}(y_*)$ with $i = j$. Clearly,

(c) $|R(y_*)| + |R_0(y_*)| = 2|\mathfrak{I}(y_*)|$.

2.4. Let $x_*, x'_* \in X_m$ and let $y_* = x_* + x'_* \in \mathbf{N}^{m+1}$. Note that $y \in Y_m$. If $k \in \mathfrak{S}(x_*)$ then $x_{k-1} < x_k < x_{k+1}$, $x'_{k-1} \le x'_k \le x'_{k+1}$ (and at least one of the last two $\le$ is $<$). Hence $y_{k-1} < y_k < y_{k+1}$ (and at least one of the last two $<$ is $\ll$). Hence $k \in R(y_*)$. Thus $\mathfrak{S}(x_*) \subset R(y_*)$. Similarly, $\mathfrak{S}(x'_*) \subset R(y_*)$. We see that $\mathfrak{S}(x_*) \cup \mathfrak{S}(x'_*) \subset R(y_*)$. If $k \in \mathfrak{S}(x_*) \cap \mathfrak{S}(x'_*)$ then $x_{k-1} < x_k < x_{k+1}$, $x'_{k-1} < x'_k < x'_{k+1}$ hence $y_{k-1} \ll y_k \ll y_{k+1}$ so that $k \in R_0(y_*)$. Thus, $\mathfrak{S}(x_*) \cap \mathfrak{S}(x'_*) \subset R_0(y_*)$ and

$$|\mathfrak{S}(x_*)| + |\mathfrak{S}(x'_*)| = |\mathfrak{S}(x_*) \cup \mathfrak{S}(x'_*)| + |\mathfrak{S}(x_*) \cap \mathfrak{S}(x'_*)| \le |R(y_*)| + |R_0(y_*)|.$$

Using this and 2.3(c) we see that

(a) $$|\mathfrak{S}(x_*)| + |\mathfrak{S}(x'_*)| \le 2|\mathfrak{I}(y_*)|,$$

with equality if and only if $\mathfrak{S}(x_*) \cup \mathfrak{S}(x'_*) = R(y_*)$ and $\mathfrak{S}(x_*) \cap \mathfrak{S}(x'_*) = R_0(y_*)$.

2.5. Let $y_* \in Y_m$. We consider a partition $[0,m] = \mathcal{J}_0 \sqcup \mathcal{J}_1 \sqcup \dots \mathcal{J}_t$ where for each $s \in [0,t]$ we have $\mathcal{J}_s = [m_s, m'_{s+1}]$ with $m_s \le m'_{s+1}$, $m_0 = 0$, $m'_{t+1} = m$ and for each $s \in [1,t]$ we have $m_s = m'_s + 1$. We require that for $s \in [1,t]$ we have $y_{m'_s} \ll y_{m_s}$ and for any $s \in [0,t]$ we have either

(i) $|\mathcal{J}_s| = 2$ and $(y_{m_s}, y_{m'_{s+1}}) = (a_s, a_s)$, or

(ii) $(y_{m_s}, y_{m_s+1}, \dots, y_{m'_{s+1}}) = (a_s, a_s+1, a_s+2, \dots)$.

for some $a_s \in \mathbf{N}$. Such a partition exists and is unique. Let

$\mathcal{G}_1(y_*) = \{s \in [0,t]; s \text{ is as in (i)}\}$, $\mathcal{G}_2(y_*) = \{s \in [0,t]; s \text{ is as in (ii)}\}$.

We have

$\mathfrak{I}(y_*) = \{[i,j]; i = m_s, j = m'_{s+1} \text{ for some } s \in \mathcal{G}_2(y_*)\}$;

$R(y_*) = \{i \in [0,m]; i = m_s \text{ or } i = m'_{s+1} \text{ for some } s \in \mathcal{G}_2(y_*)\}$;

$R_0(y_*) = \{i \in [0,m]; i = m_s = m'_{s+1} \text{ for some } s \in \mathcal{G}_2(y_*)\}$.

2.6. Let $y_* \in Y_m$. Let $S'(y_*)$ be the set consisting of all pairs

$x_* = (x_0, x_1, \dots, x_m)$, $x'_* = (x'_0, x'_1, \dots, x'_m)$

in $\mathbf{N}^{m+1}$ which satisfy (i)-(iv) below (notation in 2.5):

(i) for any $s \in \mathcal{G}_1(y_*)$ we have $(x_{m_s}, x_{m'_{s+1}}) = (u_s, u_s)$, $(x'_{m_s}, x'_{m'_{s+1}}) = (u'_s, u'_s)$, $u_s + u'_s = a_s$;

(ii) for any $s \in \mathcal{G}_2(y_*)$ we have either

(ii1) $(x_{m_s}, x_{m_s+1}, \dots, x_{m'_{s+1}}) = (u_s, u_s+1, u_s+1, u_s+2, u_s+2, u_s+3, \dots)$, $(x'_{m_s}, x'_{m_s+1}, \dots, x'_{m'_{s+1}}) = (u'_s, u'_s, u'_s+1, u'_s+1, u'_s+2, u'_s+2, \dots)$, $u_s + u'_s = a_s$, or

(ii2) $(x_{m_s}, x_{m_s+1}, \dots, x_{m'_{s+1}}) = (u_s, u_s, u_s+1, u_s+1, u_s+2, u_s+2, \dots)$,

$(x'_{m_s}, x'_{m_s+1}, \dots, x'_{m'_{s+1}}) = (u'_s, u'_s+1, u'_s+1, u'_s+2, u'_s+2, u'_s+3, \dots)$, $u_s+u'_s = a_s$;

(iii) for any $s \in [1,t]$ we have $x_{m'_s} < x_{m_s}$, $x'_{m'_s} < x'_{m_s}$;

(iv) if $\mathfrak{I}'(y_*) = \emptyset$ then for any $s \in \mathcal{G}_2(y_*)$, $(x_{m_s}, x_{m_s+1}, \dots, x_{m'_{s+1}})$, $(x'_{m_s}, x'_{m_s+1}, \dots, x'_{m'_{s+1}})$ are as in (ii1).

An element (x_*, x'_*) of $S'(y_*)$ can be constructed by induction as follows. Assume that the entries x_i, x'_i have been already chosen for $i \in \mathcal{J}_0 \cup \mathcal{J}_1 \cup \dots \cup \mathcal{J}_{s-1}$ for some $s \in [0,t]$ so that (i)-(iii) hold as far as it makes sense. In the case where $s > 0$ let $\xi = x_{m'_s}, \xi' = x'_{m'_s}$; in the case where $s = 0$ let $\xi = \xi' = -\infty$. In any case we have $\xi + \xi' \le a_s - 2$ hence we can find u_s, u'_s in $\mathbf{N}$ such that $\xi < u_s$, $\xi' < u'_s$, $u_s + u'_s = a_s$. (The number of choices is $y_{m_s} - y_{m'_s} - 1$ if $s > 0$ and $y_0 + 1$ if $s = 0$.) Then we define

$(x_{m_s}, x_{m_s+1}, \dots, x_{m'_{s+1}})$, $(x'_{m_s}, x'_{m_s+1}, \dots, x'_{m'_{s+1}})$

by (i) if $s \in \mathcal{G}_1(y_*)$ and by (ii) if $s \in \mathcal{G}_2(y_*)$. This gives two choices for each $s \in \mathcal{G}_2(y_*)$ such that $|\mathcal{J}_s| > 1$, unless $\mathfrak{I}'(y_*) = \emptyset$ when there is only one choice. This completes the inductive definition of x_*, x'_*. We see that $S'(y_*) \ne \emptyset$.

Let $S(y_*)$ be the set of all $(x_*, x'_*) \in X_m \times X_m$ such that (v),(vi),(vii) below hold:

(v) $x_* + x'_* = y_*$,

(vi) $\mathfrak{S}(x_*) \cup \mathfrak{S}(x'_*) = R(y_*)$, $\mathfrak{S}(x_*) \cap \mathfrak{S}(x'_*) = R_0(y_*)$ (or equivalently $|\mathfrak{S}(x_*)| + |\mathfrak{S}(x'_*)| = 2|\mathfrak{I}(y_*)|$),

(vii) if $\mathfrak{I}'(y_*) = \emptyset$ (so that m is odd), then $\mathfrak{S}(x'_*) = \emptyset$.

From the definitions we see that $S(y_*) = S'(y_*)$. Hence

(a) $S(y_*) \neq \emptyset$.

From 2.4(a) we see that:

(b) if $\mathfrak{I}(y_*) = \emptyset$ and $(x_*, x'_*) \in S(y_*)$ then $\mathfrak{S}(x_*) = \emptyset$, $\mathfrak{S}(x'_*) = \emptyset$.

On the other hand,

(c) if $\mathfrak{I}'(y_*) \neq \emptyset$ and $(x_*, x'_*) \in S(y_*)$ then $\mathfrak{S}(x_*) \neq \emptyset$, $\mathfrak{S}(x'_*) \neq \emptyset$.

Indeed, let $[i,j] \in \mathfrak{I}'(y_*)$. Then we have either $i \in \mathfrak{S}(x_*), j \in \mathfrak{S}(x'_*)$ or $i \in \mathfrak{S}(x'_*), j \in \mathfrak{S}(x_*)$; in both cases the conclusion of (c) holds.

2.7. In this subsection we assume that m is even, ≥ 2. We set

$\tilde{X}_m = \{x_* \in X_m; x_0 = 0, x_1 \geq 1\}$, $\tilde{Y}_m = \{y_* \in Y_m; y_1 \geq 1\}$.

If $x_* \in X_m$, $x'_* \in \tilde{X}_m$, then $x_* + x'_* \in \tilde{Y}_m$.

Let $y_* \in \tilde{Y}_m$ be such that

(a) $y_0 = 0, y_1 = 1$.

(Thus $\mathfrak{I}(y_*)$ contains an interval of form $[0,\alpha]$ hence $\mathfrak{I}(y_*) \neq \emptyset$.) Let $\tilde{S}'(y_*)$ be the set consisting of all pairs $x_* = (x_0, x_1, \dots, x_m)$, $x'_* = (x'_0, x'_1, \dots, x'_m)$ in $\mathbf{N}^{m+1}$ which satisfy the conditions (i)-(iii) in 2.6 together with conditions (i),(ii) below (notation in 2.5):

(i) for $s = 0$ (necessarily in $\mathcal{G}_2(y_*)$) we have

$(x_0, x_1, \dots, x_{m'_1}) = (0, 0, 1, 1, 2, 2, \dots)$, $(x'_0, x'_1, \dots, x'_{m'_1}) = (0, 1, 1, 2, 2, 3, 3, \dots)$ (so that $0 \in \mathfrak{S}(x'_*)$);

(ii) if $\mathfrak{I}(y_*) = \{[0,\alpha]\} \cup \mathfrak{I}''(y_*)$ (so that $\mathfrak{I}'(y_*) = \{[0,\alpha]\}$) then for any $s \in \mathcal{G}_2(y_*) - \{0\}$, $(x_{m_s}, x_{m_s+1}, \dots, x_{m'_{s+1}})$, $(x'_{m_s}, x'_{m_s+1}, \dots, x'_{m'_{s+1}})$ are as in 2.6(ii1).

We can construct an element in $\tilde{S}'(y_*)$ by the same method as in 2.6. In particular, $\tilde{S}'(y_*) \neq \emptyset$.

Now let $\tilde{S}(y_*)$ be the set of all $(x_*, x'_*) \in X_m \times \tilde{X}_m$ such that

(iii) $x_* + x'_* = y_*$,

(iv) $\mathfrak{S}(x_*) \cup \mathfrak{S}(x'_*) = R(y_*)$, $\mathfrak{S}(x_*) \cap \mathfrak{S}(x'_*) = R_0(y_*)$ (or equivalently $|\mathfrak{S}(x_*)| + |\mathfrak{S}(x'_*)| = 2|\mathfrak{I}(y_*)|$),

(v) if $\mathfrak{I}(y_*) = \{[0,\alpha]\} \cup \mathfrak{I}''(y_*)$, then $\mathfrak{S}(x'_*) = \{0\}$.

From the definitions we see that $\tilde{S}(y_*) = \tilde{S}'(y_*)$. Hence

(b) $\tilde{S}(y) \neq \emptyset$.

Note that

(c) if $\mathfrak{I}(y_*) = \{[0,\alpha]\}$ and $(x_*, x'_*) \in \tilde{S}(y_*)$, then $\mathfrak{S}(x_*) = \{\alpha\}$, $\mathfrak{S}(x'_*) = \{0\}$.

Indeed from 2.4(a) we see that $|\mathfrak{S}(x_*)| + |\mathfrak{S}(x'_*)| \leq 2$. On the other hand, we have $0 \in \mathfrak{S}(x'_*)$ and $\alpha \in \mathfrak{S}(x_*)$ (see (i)) since in this case α is even; (c) follows. Note that

(d) if $\mathfrak{I}'(y_*)$ contains at least one interval $\neq [0,\alpha]$ and $(x_*, x'_*) \in \tilde{S}(y_*)$, then $|\mathfrak{S}(x'_*)| \geq 3$.

Indeed, let $[i,j] \in \mathfrak{I}'(y_*)$, $[i,j] \neq [0,\alpha]$. Then we have either $i \in \mathfrak{S}(x_*), j \in \mathfrak{S}(x'_*)$

or $i \in \mathfrak{S}(x'_*), j \in \mathfrak{S}(x_*)$. Since $0 \in \mathfrak{S}(x'_*)$ we see that $|\mathfrak{S}(x'_*)| \geq 2$. Since $|\mathfrak{S}(x'_*)|$ is odd we see that $|\mathfrak{S}(x'_*)| \geq 3$.

2.8. Let $x^0_* \in X_m$ be $(0, 0, 1, 1, \ldots, (n-1), (n-1), n)$ if $m = 2n$ and $(0, 0, 1, 1, \ldots, n, n)$ if $m = 2n+1$. For any $x_* \in X_m$ we have $x_i \geq x^0_i$ for all $i \in [0, m]$. Hence

$\rho : X_m \to \mathbf{N}$, $\xi_* \mapsto \sum_{i \in [0,m]}(x_i - x^0_i)$ and

$\beta : X_m \to \mathbf{N}$, $x_* \mapsto \sum_{0 \leq i < j \leq m}(x_i - x^0_i)$

are well defined.

Let $y^0_* \in Y_m$ be $(0, 0, 2, 2, \ldots, (m-2), (m-2), m)$ if m is even and $(0, 0, 2, 2, \ldots, (m-1), (m-1))$ if m is odd. For any $y_* \in Y_m$ we have $y_i \geq y^0_i$ for all $i \in [0, m]$. Hence

$\rho' : Y_m \to \mathbf{N}$, $y_* \mapsto \sum_{i \in [0,m]}(y_i - y^0_i)$ and

$\beta' : Y_m \to \mathbf{N}$, $y_* \mapsto \sum_{0 \leq i < j \leq m}(y_i - y^0_i)$

are well defined. Since $x^0_* + x^0_* = y^0_*$ we have

$\rho'(x_* + x'_*) = \rho(x_*) + \rho(x'_*)$, $\beta'(x_* + x'_*) = \beta(x_*) + \beta(x'_*)$

for any $x_*, x'_* \in X_m$. For any $n \in \mathbf{N}$ we set $X^n_m = \{x_* \in X_m; \rho(x_*) = n\}$, $Y^n_m = \{y_* \in Y_m; \rho'(y_*) = n\}$.

Assume that $m = 2k$, $k \geq 1$. Let $\tilde{x}^0_* \in \tilde{X}_m$ be $(0, 1, 1, \ldots, k, k)$. For any $x_* \in \tilde{X}_m$ we have $x_i \geq \tilde{x}^0_i$ for all i. Hence

$\tilde{\rho} : \tilde{X}_m \to \mathbf{N}$, $\xi_* \mapsto \sum_{i \in [0,m]}(x_i - \tilde{x}^0_i)$ and

$\tilde{\beta} : \tilde{X}_m \to \mathbf{N}$, $x_* \mapsto \sum_{0 \leq i < j \leq m}(x_i - \tilde{x}^0_i)$

are well defined. Let $\tilde{y}^0_* = (0, 1, 2, 3, \ldots, m) \in Y_m$. For any $y_* \in \tilde{Y}_m$ we have $y_i \geq \tilde{y}^0_i$ for all i. Hence

$\tilde{\rho}' : \tilde{Y}_m \to \mathbf{N}$, $y_* \mapsto \sum_{i \in [0,m]}(y_i - \tilde{y}^0_i)$ and

$\tilde{\beta}' : \tilde{Y}_m \to \mathbf{N}$, $y_* \mapsto \sum_{0 \leq i < j \leq m}(y_i - \tilde{y}^0_i)$

are well defined. Since $x^0_* + \tilde{x}^0_* = \tilde{y}^0_*$ we have

$\tilde{\rho}'(x_* + x'_*) = \rho(x_*) + \tilde{\rho}(x'_*)$, $\tilde{\beta}'(x_* + x'_*) = \beta(x_*) + \tilde{\beta}(x'_*)$

for any $x_* \in X_m$, $x'_* \in \tilde{X}_m$. For any $n \in \mathbf{N}$ we set

$\tilde{X}^n_m = \{x_* \in \tilde{X}_m; \tilde{\rho}(x_*) = n\}$, $\tilde{Y}^n_m = \{y_* \in \tilde{Y}_m; \tilde{\rho}'(y_*) = n\}$.

2.9. Let $\mathcal{E}_m$ be the set of all $e_* = (e_0, e_1, \ldots, e_m) \in \mathbf{N}^{m+1}$ such that $e_0 \leq e_1 \leq \cdots \leq e_m$. For any $n \in \mathbf{N}$ let $\mathcal{E}^n_m = \{e_* \in \mathcal{E}_m; \sum_i e_i = n\}$.

Let $x_* \in X_m$. We associate to x_* an element $\hat{x}_* \in X_m$ as follows. Let $i_0 < i_1 < \cdots < i_s$ be the elements of $\mathfrak{S}(x_*)$ in increasing order. Clearly, each of the sets $[0, i_0 - 1], [i_0 + 1, i_1 - 1], \ldots, [i_{s-1} + 1, i_s - 1], [i_s + 1, m]$ has even cardinal, say $2t_0, 2(t_1 - 1), \ldots, 2(t_s - 1), 2t_{s+1}$ (respectively). We define $\hat{x}_* = (\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_m) \in \mathbf{N}^{m+1}$ by

$$(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{i_0} - 1) = (0, 0, 1, 1, \ldots, t_0 - 1, t_0 - 1), h_{i_0} = t_0,$$

$$(\hat{x}_{i_0+1}, \hat{x}_{i_0+2}, \ldots, \hat{x}_{i_1-1})$$
$$= (t_0 + 1, t_0 + 1, t_0 + 2, t_0 + 2, \ldots, t_0 + t_1 - 1, t_0 + t_1 - 1), h_{i_1} = t_0 + t_1,$$

$(\hat{x}_{i_1+1}, \hat{x}_{i_1+2}, \dots, \hat{x}_{i_2-1}) =$
$(t_0+t_1+1, t_0+t_1+1, t_0+t_1+2, t_0+t_1+2, \dots, t_0+t_1+t_2-1, t_0+t_1+t_2-1)$,
$h_{i_2} = t_0+t_1+t_2$, $\blacksquare$

$$\dots$$

$$\begin{aligned}&(\hat{x}_{i_s+1}, \hat{x}_{i_s+2}, \dots, hx_m)\\ &= (t_0+t_1+\dots+t_s+1, t_0+t_1+\dots+t_s+1, t_0+t_1+\dots+t_s+2,\\ &t_0+t_1+\dots+t_s+2, \dots, t_0+t_1+\dots+t_{s+1}, t_0+t_1+\dots+t_{s+1}).\end{aligned}$$

Note that $\hat{x}_*$ depends only on $\mathfrak{S}(x_*)$, not on x_* itself. We have $\hat{x}_* \in X_m$, $\mathfrak{S}(\hat{x}_*) = \mathfrak{S}(x_*)$. Let $e_* = x_* - \hat{x}_*$. We have $e_* \in \mathcal{E}_m$. Moreover for any $i \in [0, m-1]$ such that $\hat{x}_i = \hat{x}_{i+1}$ we have $e_i = e_{i+1}$.

2.10. Let $x_* \in X_m$, $e_* \in \mathcal{E}_m$. Then $x_* + e_* \in X_m$ hence $y_* := x_* + e_* + x_* \in Y_m$. Assume that $\mathfrak{S}(e_* + x_*) = \mathfrak{S}(x_*)$ and $(x_*, e_* + x_*) \in S(y_*)$. Then

$\mathfrak{S}(x_*) \cup \mathfrak{S}(e_* + x_*) = R(y_*)$, $\mathfrak{S}(x_*) \cap \mathfrak{S}(e_* + x_*) = R_0(y_*)$

hence $\mathfrak{S}(x_*) = R(y_*) = R_0(y_*)$. It follows that for any $\mathcal{I} \in \mathfrak{I}(y_*)$ we have $|\mathcal{I}| = 1$.

2.11. Conversely, let $y_* \in Y_m^n$ be such that for any $\mathcal{I} \in \mathfrak{I}(y_*)$ we have $|\mathcal{I}| = 1$. By 2.6(a) we can find $(x_*, x'_*) \in S(y_*)$. We have $x_* + x'_* = y_*$, $\mathfrak{S}(x_*) \cup \mathfrak{S}(x'_*) = R(y_*)$, $\mathfrak{S}(x_*) \cap \mathfrak{S}(x'_*) = R_0(y_*)$. From our assumption we have $R_0(y_*) = R(y_*)$. Hence $\mathfrak{S}(x_*) \cup \mathfrak{S}(x'_*) = \mathfrak{S}(x_*) \cap \mathfrak{S}(x'_*)$ so that $\mathfrak{S}(x_*) = \mathfrak{S}(x'_*)$. By 2.9 we have $\hat{x}_* = \hat{x}'_* \in X_m$ and $e_* := x_* - \hat{x}_* \in \mathcal{E}_m, e'_* := x'_* - \hat{x}'_* \in \mathcal{E}_m$. Moreover, if $i \in [0, m-1]$ and $\hat{x}_i = \hat{x}_{i+1}$ then $e_i = e_{i+1}$ and $e'_i = e'_{i+1}$ hence $\tilde{e}_i = \tilde{e}_{i+1}$ where $\tilde{e}_* = e_* + e'_* \in \mathcal{E}_m$. It follows that $\mathfrak{S}(\tilde{e}_* + \hat{x}_*) = \mathfrak{S}(\hat{x}_*) = \mathfrak{S}(x_*) = R(y_*) = R_0(y_*)$. Since $|\mathfrak{S}(x_*)| + |\mathfrak{S}(x'_*)| = 2|\mathfrak{I}(y_*)|$ we have $|\mathfrak{S}(\hat{x}_*)| + |\mathfrak{S}(\tilde{e}_* + \hat{x}_*)| = 2|\mathfrak{I}(y_*)|$. Also, if $\mathfrak{I}'(y_*) = \emptyset$ then by 2.6(vii) we have $\mathfrak{S}(x'_*) = \emptyset$. Hence $\mathfrak{S}(\hat{x}_*) = \mathfrak{S}(\tilde{e}_* + \hat{x}_*) = \emptyset$. In any case we see that $y_* = \hat{x}_* + \tilde{e}_* + \hat{x}_*$, $(\hat{x}_*, \tilde{e}_* + \hat{x}_*) \in S(y_*)$, $\mathfrak{S}(\tilde{e}_* + \hat{x}_*) = \mathfrak{S}(\hat{x}_*)$.

3. Type A_{n-1}

3.1. For $n \in \mathbf{N}$ let S_n be the group of all permutations of $\{1, 2, \dots, n\}$. We have $S_0 = S_1 = \{1\}$; for $n \ge 2$ we regard S_n as a Coxeter group whose generators are the transpositions $(i, i+1)$ for $i \in [1, n-1]$. We have $\mathcal{S}_{S_n} = \mathrm{Irr}(S_n)$. If k is large (relative to n) we have a natural bijection $\mathrm{Irr}(S_n) \leftrightarrow Z_k^n$, $[z_*] \leftrightarrow z_*$, see [L4, 4.4]. For example, $[(0, 1, \dots, k-n, k-n+2, \dots, k, k+1)]$ is the sign representation of S_n. For any $z_* \in Z_k^n$ we have $\beta_0(z_*) = b_{[z_*]}$, see [L4, (4.4.2)].

Assume now that $n = n' + n''$ with n', n'' in $\mathbf{N}$. The set of permutations of $\{1, 2, \dots, n\}$ which leave stable each of the subsets $\{1, 2, \dots, n'\}$, $\{n'+1, n'+2, \dots, n\}$ is a standard parabolic subgroup of S_n which may be identified with $S_{n'} \times S_{n''}$.

For $z'_* \in Z_k^{n'}$, $z''_* \in Z_k^{n'}$ we have $z'_* + z''_* - z^0_* \in Z_k^n$ and from the definitions we have:

(a) $[[z'_* + z''_* - z^0_*] : \mathrm{Ind}_{S_{n'}\times S_{n''}}^{S_n}([z'_*]\boxtimes[z''_*])]_{S_n} = 1.$

Note also that $\beta_0(z'_*)+\beta_0(z''_*)=\beta_0(z'_*+z''_*-z^0_*)$ hence $b_{[z'_*]}+b_{[z''_*]}=b_{[z'_*+z''_*-z^0_*]}$, so that

(b) $[z'_* + z''_* - z^0_*] = j_{S_{n'}\times S_{n''}}^{S_n}([z'_*]\boxtimes[z''_*])$.

3.2. In this subsection we assume that G is of type A_{n-1} $(n\ge 2)$. In this case 1.5(a),(b1),(b2) are immediate. We prove 1.5(b3).

For $C\in\mathcal{X}$ let $E=\rho_C$. We have $E=[z_*]$ for a unique $z_*\in Z^n_k$. We have $\mathbf{z}_C=1$ and $\tilde{\mathbf{z}}_C=\text{g.c.d.}\{n, z_j-z^0_j (j\in[0,k])\}$ where g.c.d. denotes the greatest common divisor. We identify $\{1,2,\dots,n\}=\mathbf{Z}/n$ in the obvious way. We also identify $\mathbf{W}=S_n$ as Coxeter groups so that the reflections $s_i (i\in\tilde{I})$ are the transpositions $(i,i+1)$ with $i\in\mathbf{Z}/n$ (with $i+1$ computed in $\mathbf{Z}/n$.) Now Ω is a cyclic group of order n with generator $\omega: i\mapsto i+1$ for all $i\in\mathbf{Z}/n$. For any $d|n$ (divisor $d\ge 1$ of n) let Ω_d be the subgroup of Ω generated by $\omega^{n/d}$. For any coset P of Ω_d in Ω let S_n^P be the set of all permutations w of $\mathbf{Z}/n$ such that for any $r\in P$ the subset $\{r+1,r+2,\dots,r+(n/d)\}$ is w-stable. We may identify S_n^P with a product of d copies of $S_{n/d}$. Note that $\mathcal{P}^{\Omega_d}$ (see 1.11) consists of the subgroups S_n^P as above; each of these subgroups is stable under the conjugation action of Ω_d on $\mathbf{W}$. An irreducible representation $\boxtimes_{h=1}^d[\tilde{z}_*^{(h)}]$ (with $\tilde{z}_*^{(h)}\in Z_k^{n/d}$) of S_n^P (identified with $S_{n/d}^d$) is Ω_d-stable if and only if $\tilde{z}_*^{(h)}=\tilde{z}_*$ is independent of h; in this case we have

$$j_{S_n^P}^{S_n}(\boxtimes_{h=1}^d[\tilde{z}_*^{(h)}]) = [\sum_{h=1}^d \tilde{z}_*^{(h)} - (d-1)z^0_*] = [d\tilde{z}_* - (d-1)z^0_*]$$

as we see by applying $(d-1)$ times 3.1(b). Using this and 1.11 we see that

$$\mathfrak{c}_{[z_*]} = \max d$$

where max is taken over all divisors $d\ge 1$ of n such that $z_*-z^0_*=d(\tilde{z}_*-z^0_*)$ for some $\tilde{z}_*\in Z_k^{n/d}$. Equivalently, we have

$$\mathfrak{c}_{[z_*]} = \text{g.c.d.}\{n, z_j - z^0_j (j\in[0,k])\}.$$

Since this is equal to $\tilde{\mathbf{z}}_C/\mathbf{z}_C$ we see that 1.5(b3) is proved in our case.

4. Type B_n

4.1. For $n\in\mathbf{N}$ let W_n be the group of permutations of the set $\{1,2,\dots,n,n',\dots,2',1'\}$ which commute with the involution $i\mapsto i', i'\mapsto i (i\in[1,n])$. We have $W_0=\{1\}$; for $n\ge 1$ we regard W_n as a Coxeter group of type $B_n=C_n$ whose generators are the transposition (n,n') and the products of two transpositions $(i,i+1)((i+1)',i')$ for $i\in[1,n-1]$. By [L2, §2] we have $\mathrm{Irr}(W_n)=\mathrm{Irr}(W_n)^\dagger$.

4.2. In the remainder of this section we fix an even integer $m = 2k$ which is large relative to n.

Let $U_k^n = \{(z_*; z'_*) \in Z_k \times Z_{k-1}; \rho_0(z_*) + \rho_0(z'_*) = n\}$. As in [L4, 4.5] we have a bijection

(a) $\mathrm{Irr}(W_n) \leftrightarrow U_k^n$, $[z_*; z'_*] \leftrightarrow (z_*; z'_*)$.

(In *loc.cit.* the notation $\begin{pmatrix} z_* \\ z'_* \end{pmatrix}$ was used instead of $(z_*; z'_*)$.) By [L2, §2] we have

(b) $b_{[z_*;z'_*]} = 2\beta_0(z_*) + 2\beta_0(z'_*) + \rho_0(z'_*)$.

There is a unique bijection $\zeta_n : \mathcal{S}_{W_n} \xrightarrow{\sim} X_m^n$ under which $x_* \in X_m^n$ corresponds to$\{[z_*, z'_*]\}$ where $z_* = (x_0, x_2, x_4, \dots, x_m)$, $z'_* = (x_1, x_3, x_5, \dots, x_{m-1})$. This bijection has the following property: if $E \in \mathcal{S}_{W_n}, x_* = \zeta_n(E)$ then $b_E = \beta(x_*)$, $f_E = 2^{(|\mathfrak{S}(x_*)|-1)/2}$.

4.3. Let $u_* \in Z_m$. Define $\ddot{u}_* \in Z_k$, $\dot{u}_* \in Z_{k-1}$ by $\ddot{u}_i = u_{2i} - i$ for $i \in [0, k]$, $\dot{u}_i = u_{2i+1} - i - 1$ for $i \in [0, k-1]$.

4.4. Let $(p, q) \in \mathbf{N}^2$ be such that $p + q = n$. The group of all permutations of $\{1, 2, \dots, n, n', \dots, 2', 1'\}$ in W_n that leave stable each of the subsets

$$\{1, 2, \dots, p\}, \{p', \dots, 2', 1'\}, \{p+1, \dots, n-1, n\} \cup \{n', (n-1)', \dots, (p+1)'\}$$

is a standard parabolic subgroup of W_n which may be identified with $S_p \times W_q$ in an obvious way.

Let $(\tilde{z}_*; \tilde{z}'_*) \in U_k^q$, $u_* \in Z_m^p$. Let $v_* = \tilde{z}_* + \ddot{u}_* - z_*^{0,k}$, $v'_* = \tilde{z}'_* + \dot{u}_* - z_*^{0,k-1}$. Then $(v_*; v'_*) \in U_k^n$, $[u_*] \in \mathrm{Irr}(S_p)$, $[\tilde{z}_*; \tilde{z}'_*] \in \mathrm{Irr}(W_q)$, $[v_*; v'_*] \in \mathrm{Irr}(W_n)$. We show:

(a) $[v_*; v'_*] = j_{S_p \times W_q}^{W_n}([u_*] \boxtimes [\tilde{z}_*; \tilde{z}'_*])$.

We can assume that $p \geq 1$ and that the result holds for p replaced by $\tilde{p} < p$. In the case where $[u_*]$ is the sign representation of S_p, (a) can be proved along the lines of [L3, 2.7]. If $[u_*]$ is not the sign representation of S_p, we can find p', p'' in $\mathbf{N}_{>0}$ such that $p' + p'' = p$ and $u' \in Z_{2k}^{p'}$, $u' \in Z_{2k}^{p''}$ such that $u_* = u'_* + u''_* - z_*^{0,m}$. By 3.1(b), we have $[u_*] = j_{S_{p'} \times S_{p''}}^{S_p}([u'_*] \boxtimes [u''_*])$. Hence

$$[u_*] \boxtimes [\tilde{z}_*; \tilde{z}'_*] = j_{S_{p'} \times S_{p''} \times W_q}^{S_p \times W_q}([u'_*] \boxtimes [u''_*] \boxtimes [\tilde{z}_*; \tilde{z}'_*])$$

and

$$\begin{aligned} j_{S_p \times W_q}^{W_n}([u_*] \boxtimes [\tilde{z}_*; \tilde{z}'_*]) &= j_{S_p \times W_q}^{W_n} j_{S_{p'} \times S_{p''} \times W_q}^{S_p \times W_q}([u'_*] \boxtimes [u''_*] \boxtimes [\tilde{z}_*; \tilde{z}'_*]) \\ &= j_{S_{p'} \times S_{p''} \times W_q}^{W_n}([u'_*] \boxtimes [u''_*] \boxtimes [\tilde{z}_*; \tilde{z}'_*]) \\ &= j_{S_{p'} \times W_{p''+q}}^{W_n} j_{S_{p'} \times S_{p''} \times W_q}^{S_{p'} \times W_{p''+q}}([u'_*] \boxtimes [u''_*] \boxtimes [\tilde{z}_*; \tilde{z}'_*]) \\ &= j_{S_{p'} \times W_{p''+q}}^{W_n}([u'_*] \boxtimes [\tilde{z}_* + \ddot{u}''_* - z_*^{0,k}; \tilde{z}'_* + \dot{u}''_* - z_*^{0,k-1}]) \\ &= [\tilde{z}_* + \ddot{u}''_* + \ddot{u}'_* - 2z_*^{0,k}; \tilde{z}'_* + \dot{u}''_* + \dot{u}'_* - 2z_*^{0,k-1}] \\ &= [\tilde{z}_* + \ddot{u}_* - z_*^{0,k}; \tilde{z}'_* + \dot{u}_* - z_*^{0,k-1}]. \end{aligned}$$

(We have used the induction hypothesis for p replaced by p' or p''.) This proves (a).

4.5. In the remainder of this section we assume that G has type B_n $(n \geq 2)$. We identify $\mathbf{W} = W_n$ as Coxeter groups in the standard way. The reflections $s_j (j \in \tilde{I})$ are the transpositions (n, n'), $(1, 1')$ and the products of two transpositions $(i, i+1)(i', (i+1)')$ for $i \in [1, n-1]$. The group Ω has order 2 with generator given by the involution $i \mapsto (n+1-i)', i' \mapsto (n+1-i)$ for $i \in [1, n]$.

Let $(r, p, q) \in \mathbf{N}^3$ be such that $r + p + q = n$. The group of all permutations of $\{1, 2, \dots, n, n', \dots, 2', 1'\}$ in W_n that leave stable each of the subsets

$$\{1, 2, \dots, r\} \cup \{r', \dots, 2', 1'\}, \{r+1, r+2, \dots, r+p\},$$
$$\{(r+p)', \dots, (r+2)', (r+1)'\},$$
$$\{r+p+1, \dots, n-1, n\} \cup \{n', (n-1)', \dots, (r+p+1)'\}$$

is a parahoric subgroup of $\mathbf{W}$ which may be identified with $W_r \times S_p \times W_q$ in an obvious way.

Let $(z_*; z'_*) \in U_k^r$, $(\tilde{z}_*; \tilde{z}'_*) \in U_k^q$, $u_* \in Z_{2k}^p$, Define $\ddot{u}_* \in Z_k, \dot{u}_* \in Z_{k-1}$ as in 4.3. Let $w_* = z_* + \tilde{z}_* + \ddot{u}_* - 2z_*^{0,k}$, $w'_* = z'_* + \tilde{z}'_* + \dot{u}_* - 2z_*^{0,k-1}$. Then $(w_*, w'_*) \in U_k^n$, $[z_*; z'_*] \in \mathrm{Irr}(W_r)$, $[u_*] \in \mathrm{Irr}(S_p)$, $[\tilde{z}_*; \tilde{z}'_*] \in \mathrm{Irr}(W_q)$, $[w_*, w'_*] \in \mathrm{Irr}(W_n)$. We show:

(a) $[w_*; w'_*] = j_{W_r \times S_p \times W_q}^{W_n}([z_*; z'_*] \boxtimes [u_*] \boxtimes [\tilde{z}_*; \tilde{z}'_*])$. In particular,
$[[w_*; w'_*] : \mathrm{Ind}_{W_r \times S_p \times W_q}^{W_n}([z_*; z'_*] \boxtimes [u_*] \boxtimes [\tilde{z}_*; \tilde{z}'_*])]_{W_n} = 1$.

Assume first that $p = 0$. We have:

(b) $$[[z_* + \tilde{z}_* - z_*^{0,k}; z'_* + \tilde{z}'_* - z_*^{0,k-1}] : \mathrm{Ind}_{W_r \times W_q}^{W_n}([z_*; z'_*] \boxtimes [\tilde{z}_*; \tilde{z}'_*])]_{W_n} = 1.$$

Using the definitions this can be deduced from the analogous statement for S_n, see 3.1(a). Moreover we have $b_{[z_* + \tilde{z}_* - z_*^{0,k}; z'_* + \tilde{z}'_* - z_*^{0,k-1}]} = b_{[z_*; z'_*]} + b_{[\tilde{z}_*; \tilde{z}'_*]}$. It follows that

(c) $$[z_* + \tilde{z}_* - z_*^{0,k}; z'_* + \tilde{z}'_* - z_*^{0,k-1}] = j_{W_r \times W_q}^{W_n}([z_*; z'_*] \boxtimes [\tilde{z}_*; \tilde{z}'_*]).$$

Thus (a) holds in this special case.

In the general case we use 4.4(a) with n replaced by $p + q$ and (c) applied to $n, r, 0, p + q$ instead of n, r, p, q. We obtain

$$j_{W_r \times S_p \times W_q}^{W_n}([z_*; z'_*] \boxtimes [u_*] \boxtimes [\tilde{z}_*; \tilde{z}'_*])$$
$$= j_{W_r \times W_{p+q}}^{W_n}(j_{W_r \times S_p \times W_q}^{W_r \times W_{p+q}}([z_*; z'_*] \boxtimes [u_*] \boxtimes [\tilde{z}_*; \tilde{z}'_*]))$$
$$= j_{W_r \times W_{p+q}}^{W_n}([z_*; z'_*] \boxtimes [\tilde{z}_* + \ddot{u}_* - z_*^{0,k}; \tilde{z}'_* + \dot{u}_* - z_*^{0,k-1}]) = [w_*; w'_*].$$

This proves (a).

4.6. By [L5, §13], there is a unique bijection $\tau : \tilde{\mathcal{S}}_{\mathbf{W}} \to Y_m^n$ such that for any $y_* \in Y_m^n$, the fibre $\tau^{-1}(y_*)$ is $[z_*, z'_*]$ where $z_* = (y_0, y_2 - 1, y_4 - 2, \dots, y_m - m/2)$, $z'_* = (y_1, y_3 - 1, y_5 - 2, \dots, y_{m-1} - (m-2)/2)$. This bijection has the following property: if $C \in \mathcal{X}$ and $y_* = \tau(\rho_C)$, then $\mathbf{b}_C = \beta'(y_*)$, $\mathbf{z}_C = 2^{|\mathfrak{I}(y_*)|-1}$. From [L5, §14] we see that:

$\tilde{\mathbf{z}}_C / \mathbf{z}_C = 2$ if $|\mathcal{I}| = 1$ for any $\mathcal{I} \in \mathfrak{I}(y_*)$,
$\tilde{\mathbf{z}}_C / \mathbf{z}_C = 1$ if $|\mathcal{I}| > 1$ for some $\mathcal{I} \in \mathfrak{I}(y_*)$.

4.7. In the setup of 4.5 we assume that $[z_*; z'_*] \in \mathcal{S}_{W_r}$, $[\tilde{z}_*; \tilde{z}'_*] \in \mathcal{S}_{W_q}$. Define $x_* \in X^r_m$, $\tilde{x}_* \in X^q_m$ by $\zeta_r([z_*; z'_*]) = x_*$, $\zeta_q([\tilde{z}_*; \tilde{z}'_*]) = \tilde{x}_*$. Let $e_* = u_* - z^{0,m}_* \in \mathcal{E}_m$. We show:

(a) $[w_*, w'_*] \in \tilde{\mathcal{S}}_{\mathbf{W}}$ and $\tau([w_*, w'_*]) = x_* + e_* + \tilde{x}_*$.

We have $w_i = x_{2i} + \tilde{x}_{2i} + u_{2i} - i - 2i$ for $i \in [0, k]$, $w'_i = x_{2i+1} + \tilde{x} +_{2i+1} + u_{2i+1} - i - 1 - 2i$ for $i \in [0, k-1]$. Define $y_* \in \mathbf{N}^{m+1}$ by $w_i = y_{2i} - i$ for $i \in [0, k]$, $w'_i = y_{2i+1} - i$ for $i \in [0, k-1]$. Then $y_* = x_* + \tilde{x}_* + e_*$. Since $x_* \in X_m, \tilde{x}_* \in X_m, e_* \in \mathcal{E}_m$ we have $y_* \in Y_m$. More precisely, $y_* \in Y^n_m$. Using 4.6 we deduce that $[w_*, w'_*] \in \tilde{\mathcal{S}}_{\mathbf{W}}$ and (a) follows.

From (a) and 4.5(a) we see that for (r, p, q) as in 4.5, the assignment
$(E_1, E_2, \tilde{E}_1) \mapsto j^{W_n}_{W_r \times S_p \times W_q}(E_1 \boxtimes E_2 \boxtimes \tilde{E}_1)$
is a map $j : \mathcal{S}_{W_r} \times \mathcal{S}_{S_p} \times \mathcal{S}_{W_q} \longrightarrow \tilde{\mathcal{S}}_{\mathbf{W}}$ and we have a commutative diagram

$$\begin{array}{ccc} \mathcal{S}_{W_r} \times \mathcal{S}_{S_p} \times \mathcal{S}_{W_q} & \xrightarrow{\ j\ } & \tilde{\mathcal{S}}_{\mathbf{W}} \\ {\scriptstyle \zeta_r \times \xi_p \times \zeta_q}\Big\downarrow & & {\scriptstyle \tau}\Big\downarrow \\ X^r_m \times \mathcal{E}^p_m \times X^q_m & \xrightarrow{\ h\ } & Y^n_m \end{array}$$

where h is given by $(x_*, e_*, \tilde{x}_*) \mapsto x_* + e_* + \tilde{x}_*$ and $\xi_p : \mathcal{S}_{S_p} \longrightarrow \mathcal{E}^p_m$ is the bijection $[e_* + z^{0,m}_*] \leftrightarrow e_*$.

4.8. Note that $\mathcal{P}'$ (see 1.9) is exactly the collection of parahoric subgroups $W_r \times S_0 \times W_q$ of W_n with (r, p, q) as in 4.5 and $p = 0$. By 4.7, $j^{W_n}_{W_r \times S_0 \times W_q}$ carries $\mathcal{S}_{W_r} \times \mathcal{S}_{S_0} \times \mathcal{S}_{W_q}$ into $\tilde{\mathcal{S}}_{\mathbf{W}}$. Hence $\bar{\mathcal{S}}_{\mathbf{W}} \subset \tilde{\mathcal{S}}_{\mathbf{W}}$.

Conversely, let $E \in \tilde{\mathcal{S}}_{\mathbf{W}}$. With τ as in 4.6, let $y_* = \tau(E) \in Y^n_m$. By 2.6(a) we can find $(x_*, \tilde{x}_*) \in S(y_*)$. Define r, q in $\mathbf{N}$ by $x_* \in X^r_m, \tilde{x}_* \in X^q_m$. We must have $r + q = n$. Let $e_* = (0, 0, \dots, 0) \in \mathcal{E}^0_m$. In the commutative diagram in 4.7 (with $p = 0$) we have $h(x_*, e_*, \tilde{x}_*) = y_*$, $(x_*, e_*, \tilde{x}_*) = (\zeta_r(E_1), \xi_p(\mathbf{Q}), \zeta_q(\tilde{E}_1))$ where $E_1 \in \mathcal{S}_{W_r}$, $\tilde{E}_1 \in \mathcal{S}_{W_q}$ (recall that ζ_r, ζ_q are bijections) and $\tau(j(E_1, \mathbf{Q}, \tilde{E}_1)) = \tau(E)$. Since τ is bijective we deduce that $E = j(E_1, \mathbf{Q}, \tilde{E}_1)$. Thus, $E \in \bar{\mathcal{S}}_{\mathbf{W}}$. Thus, $\tilde{\mathcal{S}}_{\mathbf{W}} \subset \bar{\mathcal{S}}_{\mathbf{W}}$. We see that $\tilde{\mathcal{S}}_{\mathbf{W}} = \bar{\mathcal{S}}_{\mathbf{W}}$. This proves 1.5(a) in our case.

4.9. In the remainder of this section we fix $C \in \mathcal{X}$ and we set $E = \rho_C \in \tilde{\mathcal{S}}_{\mathbf{W}}$, $y_* = \tau(E) \in Y^n_m$ (τ as in 4.6).

Let $(r, q) \in \mathbf{N}^2$, $E_1 \in \mathcal{S}_{W_r}$, $\tilde{E}_1 \in \mathcal{S}_{W_q}$ be such that $r + q = n$,
$E = j^{W_n}_{W_r \times S_0 \times W_q}(E_1 \boxtimes \mathbf{Q} \boxtimes \tilde{E}_1)$.
(These exist since $E \in \bar{\mathcal{S}}_{\mathbf{W}}$.) We set $x_* = \zeta_r(E_1) \in X^r_m$, $\tilde{x}_* = \zeta_q(\tilde{E}_1) \in X^q_m$. From the commutative diagram in 4.7 we see that $x_* + \tilde{x}_* = y_*$. By 4.6 we have $\mathbf{b}_C = \beta'(y_*)$. Since $\beta'(x_* + \tilde{x}_*) = \beta(x_*) + \beta(\tilde{x}_*)$, we have $\mathbf{b}_C = \beta(x_*) + \beta(\tilde{x}_*)$. Since $\beta(x_*) = b_{E_1}$, $\beta(\tilde{x}_*) = b_{\tilde{E}_1}$, we have $\mathbf{b}_C = b_{E_1} + b_{\tilde{E}_1}$ hence $\mathbf{b}_C = b_{E_1 \boxtimes \mathbf{Q} \boxtimes \tilde{E}_1}$. Since $E = j^{W_n}_{W_r \times S_0 \times W_q}(E_1 \boxtimes \mathbf{Q} \boxtimes \tilde{E}_1)$ we have $b_{E_1 \boxtimes \mathbf{Q} \boxtimes \tilde{E}_1} = b_E$ hence $\mathbf{b}_C = b_E$, proving 1.5(b1) in our case.

Next we note that $f_{E_1} = 2^{(|\mathfrak{S}(x_*)|-1)/2}$, $f_{\tilde{E}_1} = 2^{(|\mathfrak{S}(\tilde{x}_*)|-1)/2}$, $\mathbf{z}_C = 2^{|\mathfrak{I}(y_*)|-1}$, $|\mathfrak{S}(x_*)| + |\mathfrak{S}(\tilde{x}_*)| \le 2|\mathfrak{I}(y_*)|$. Hence

$$f_{E_1 \boxtimes \mathbf{Q} \boxtimes \tilde{E}_1} = 2^{(|\mathfrak{S}(x_*)|+|\mathfrak{S}(\tilde{x}_*)|-2)/2} \le 2^{|\mathfrak{I}(y_*)|-1} = \mathbf{z}_C.$$

Taking maximum over all $r, q, E_1, \tilde{E}_1$ as above we obtain $\mathfrak{a}_E \le \mathbf{z}_C$.

Using again 2.6(a) we can find $(x_*, \tilde{x}_*) \in S(y_*)$. Define r, q in $\mathbf{N}$ by $x_* \in X^r_m$, $\tilde{x}_* \in X^q_m$. We must have $r + q = n$. Define $E_1 \in \mathcal{S}_{W_r}$, $\tilde{E}_1 \in \mathcal{S}_{W_q}$ by $x_* = \zeta_r(E_1), \tilde{x}_* = \zeta_q(\tilde{E}_1)$. As earlier in the proof we have $E = j^{W_n}_{W_r \times \mathbf{Q} \times W_q}(E_1 \boxtimes \mathbf{Q} \boxtimes \tilde{E}_1)$. We have

$$f_{E_1 \boxtimes \mathbf{Q} \boxtimes \tilde{E}_1} = 2^{(|\mathfrak{S}(x_*)|+|\mathfrak{S}(\tilde{x}_*)|-2)/2} = 2^{|\mathfrak{I}(y_*)|-1} = \mathbf{z}_C.$$

It follows that $\mathfrak{a}_E = \mathbf{z}_C$, proving 1.5(b2) in our case.

4.10. Assume now that $\tilde{\mathbf{z}}_C/\mathbf{z}_C = 2$. By 4.6, for any $\mathcal{I} \in \mathfrak{I}(y_*)$ we have $|\mathcal{I}| = 1$. By 2.11 we can find (r, p, q) as in 4.5 with $q = r$ and $x_* \in X^r_m$, $e_* \in \mathcal{E}^p_m$ such that $y_* = x_* + e_* + x_*$, $(x_*, e_* + x_*) \in S(y_*)$, $\mathfrak{S}(e_* + x_*) = \mathfrak{S}(x_*)$. Define $E_1 \in \mathcal{S}_{W_r}$, $E_2 \in \mathcal{S}_{S_p}$ by $x_* = \zeta_r(E_1)$, $e_* = \xi_p(E_2)$. Using the commutative diagram in 4.7 we see that $E = j^{W_n}_{W_r \times S_p \times W_r}(E_1 \boxtimes E_2 \boxtimes E_1)$. Moreover,

$$f_{E_1 \boxtimes E_2 \boxtimes E_1} = 2^{(|\mathfrak{S}(x_*)|+|\mathfrak{S}(x_*)|-2)/2} = 2^{(|\mathfrak{S}(x_*)|+|\mathfrak{S}(e_*+x_*)|-2)/2} = 2^{|\mathfrak{I}(y_*)|-1} = \mathbf{z}_C.$$

We have $W_r \times S_p \times W_r = \mathbf{W}_J$ for a unique J which is Ω-stable. Moreover, $E_1 \boxtimes E_2 \boxtimes E_1$ is Ω-stable. We see that $\mathfrak{c}_E = 2$.

4.11. Conversely, assume that $\mathfrak{c}_E = 2$. Using 1.11 we see that there exist (r, p, q) as in 4.5 with $q = r$ and $E_1 \in \mathcal{S}_{W_r}$, $E_2 \in \mathcal{S}_{S_p}$ such that $E = j^{W_n}_{W_r \times S_p \times W_r}(E_1 \boxtimes E_2 \boxtimes E_1)$, $f_{E_1 \boxtimes E_2 \boxtimes E_1} = \mathbf{z}_C$. We set $x_* = \zeta_r(E_1) \in X^r_m$, $e_* = \xi_p(E_2)$. We have $y_* = x_* + e_* + x_*$ and

$$2^{(|\mathfrak{S}(x_*)|+|\mathfrak{S}(x_*)|-2)/2} = 2^{|\mathfrak{I}(y_*)|-1};$$

hence $|\mathfrak{S}(x_*)| + |\mathfrak{S}(x_*)| = |\mathfrak{I}(y_*)|$. Let $E'_1 = j^{W_{p+r}}_{S_p \times W_r}(E_2 \boxtimes E_1) \in \mathcal{S}_{W_{p+r}}$. Then $E = j^{W_n}_{W_r \times W_{p+r}}(E_1 \boxtimes E'_1)$. Using 1.5(b2) and the definition we have $f_{E_1 \boxtimes E'_1} \le \mathfrak{a}_E = \mathbf{z}_C$. By 1.9(b) we have $f_{E_2 \boxtimes E_1} \le f_{E'_1}$. Hence $\mathbf{z}_C = f_{E_1 \boxtimes E_2 \boxtimes E_1} \le f_{E_1 \boxtimes E'_1} \le \mathbf{z}_C$; this forces $f_{E_2 \boxtimes E_1} = f_{E'_1}$. The last equality can be rewritten as

$$2^{(|\mathfrak{S}(x_*)|-1)/2} = 2^{(|\mathfrak{S}(e_*+x_*)|-1)/2}$$

since $e_* + x_* = \zeta_{p+r}(E'_1)$ (a consequence of 4.4(a)). Hence $|\mathfrak{S}(e_* + x_*)| = |\mathfrak{S}(x_*)|$ and $|\mathfrak{S}(x_*)| + |\mathfrak{S}(e_* + x_*)| = 2|\mathfrak{I}(y_*)|$. Thus, $(x_*, e_* + x_*) \in S(y_*)$. Using 2.10 we see that for any $\mathcal{I} \in \mathfrak{I}(y_*)$ we have $|\mathcal{I}| = 1$. By 4.6 we have $\tilde{\mathbf{z}}_C/\mathbf{z}_C = 2$.

4.12. From 4.10, 4.11, we see that $\tilde{\mathbf{z}}_C/\mathbf{z}_C = 2$ if and only if $\mathfrak{c}_E = 2$. Since $\mathfrak{c}_E \in [1, 2]$ and $\tilde{\mathbf{z}}_C/\mathbf{z}_C \in [1, 2]$ we see that $\tilde{\mathbf{z}}_C/\mathbf{z}_C = \mathfrak{c}_E$; this proves 1.5(b3) in our case.

5. Type C_n

5.1. For $n \in \mathbf{N}$ let W'_n be the set of all elements in W_n which are even permutations of $\{1, 2, \dots, n, n', \dots, 2', 1'\}$. We have $W'_0 = W'_1 = \{1\}$. For $n \geq 2$ we regard W'_n as a Coxeter group of type D_n whose generators are the products of two transpositions $(i, i+1)((i+1)', i')$ for $i \in [1, n-1]$ and $(n-1, n')(n, (n-1)')$.

5.2. In this subsection we fix an integer k which is large relative to n.

Let V_k^n be the set of unordered pairs (z_*, z'_*) in $Z_{k-1} \times Z_{k-1}$ such that $\rho_0(z^*) + \rho_0(z'_*) = n$. If $n \geq 2$ we have as in [L4, 4.5] a map $\iota : \mathrm{Irr}(W'_n) \longrightarrow V_k^n$. (In *loc.cit.* the notation $\binom{z_*}{z'_*}$ was used instead of (z_*, z'_*).) Now ι is also defined when $n \in \{0, 1\}$; it is the unique map between two sets of cardinal 1.

Let ${}^\dagger V_k^n$ be the set of *ordered* pairs $(z_*; z'_*)$ in $Z_{k-1} \times Z_{k-1}$ such that $\rho_0(z^*) + \rho_0(z'_*) = n$ and either $\rho_0(z_*) > \rho_0(z'_*)$ or $z_* = z'_*$. We regard ${}^\dagger V_k^n$ as a subset of V_k^n by forgetting the order of a pair. We define a partition ${}^\dagger V_k^n = {}' V_k^n \sqcup {}'' V_k^n$ by

${}'' V_k^n = \{(z_*; z'_*) \in {}^\dagger V_k^n; z_* = z'_*\}$ if $n \geq 2$, ${}'' V_k^n = \emptyset$ if $n \leq 1$,

${}' V_k^n = \{(z_*; z'_*) \in {}^\dagger V_k^n; z_* \neq z'_*\}$ if $n \geq 1$, ${}' V_k^n = {}^\dagger V_k^n$ if $n = 0$.

By [L2, §2] we have $\mathrm{Irr}(W'_n)^\dagger = \iota^{-1}({}^\dagger V_k^n)$. For $(z_*; z'_*) \in^\dagger V_k^n$ and $\kappa \in \{0, 1\}$ we define $[z_*, z'_*]^\kappa \in \mathrm{Irr}(W'_n)^\dagger$ by the following requirements: if $(z_*; z'_*) \in {}' V_k^n$, then $\iota^{-1}(z_*; z'_*)$ has a single element $[z_*; z'_*]^0 = [z_*; z'_*]^1$; if $(z_*; z'_*) \in {}'' V_k^n$, then $\iota^{-1}(z_*; z'_*)$ consists of two elements $[z_*; z'_*]^0, [z_*; z'_*]^1$.

By [L2, §2], if $(z_*; z'_*) \in {}^\dagger V_k^n$ then $b_{[z_*; z'_*]^\kappa} = 2\beta_0(z_*) + 2\beta_0(z'_*) + \rho_0(z'_*)$.

There is a unique map $\zeta'_n : \mathcal{S}_{W'_n} \longrightarrow X_{2k-1}^n$ such that for any $x_* \in X_{2k-1}^n$, ${\zeta'_n}^{-1}(x_*)$ is $\{[z_*; z'_*]^0 = [z_*; z'_*]^1\}$ (if $\mathfrak{S}(x_*) \neq \emptyset$ or if $n = 0$) and is $\{[z_*; z'_*]^0, [z_*; z'_*]^1\}$ (if $\mathfrak{S}(x_*) = \emptyset$ and $n \geq 2$) where

$z_* = (x_1, x_3, x_5, \dots, x_{2k-1})$, $z'_* = (x_0, x_2, x_4, \dots, x_{2k-2})$.

This map has the following property: if $E \in \mathcal{S}_{W'_n}, x_* = \zeta'_n(E)$ then $b_E = \beta(x_*)$, $f_E = 2^{\max((|\mathfrak{S}(x_*)|-2)/2, 0)}$.

There is a unique map $\tilde{\zeta}_n : \mathcal{S}_{W'_n} \longrightarrow \tilde{X}_{2k}^n$ such that for any $x_* \in \tilde{X}_{2k}^n$, $\tilde{\zeta}_n^{-1}(x_*)$ is $\{[z_*; z'_*]^0 = [z_*, z'_*]^1\}$ (if $\mathfrak{S}(x_*) \neq \{0\}$ or if $n = 0$) and is $\{[z_*; z'_*]^0, [z_*, z'_*]^1\}$ (if $\mathfrak{S}(x_*) = \{0\}$ and $n \geq 2$) where $z_* = (x_2 - 1, x_4 - 1, \dots, x_{2k} - 1)$, $z'_* = (x_1 - 1, x_3 - 1, x_5 - 1, \dots, x_{2k-1} - 1)$.

This map has the following property: if $E \in \mathcal{S}_{W'_n}, x_* = \tilde{\zeta}_n(E)$, then $b_E = \beta(x_*)$, $f_E = 2^{\max((|\mathfrak{S}(x_*)|-3)/2, 0)}$.

5.3. In the remainder of this section we assume that G is of type C_n $(n \geq 3)$ and we identify $\mathbf{W} = W_n$ as Coxeter groups in the standard way; we also fix an even integer $m = 2k$ which is large relative to n. The reflections $s_j (j \in \tilde{I})$ are the transposition $(1, 1')$ and the products of two transpositions $(i, i+1)(i', (i+1)')$ for $i \in [1, n-1]$ and $(n-1, n')(n, (n-1)')$. The group Ω has order 2 with generator given by the transposition (n, n').

Let $(r, q) \in \mathbf{N}^2$ be such that $r + q = n$. The group of all permutations of $\{1, 2, \dots, n, n', \dots, 2', 1'\}$ in W_n that leave stable the subset $\{1, 2, \dots, r\} \cup \{r', \dots, 2', 1'\}$ and which restrict to an even permutation of $\{r+1, \dots, n-1, n\} \cup$

$\{n', (n-1)', \dots, (r+1)'\}$, is a parahoric subgroup of $\mathbf{W}$ which may be identified with $W_r \times W'_q$ in an obvious way. Let $(z_*; z'_*) \in U^r_k$, $(\tilde{z}_*; \tilde{z}'_*) \in {}^\dagger V^q_k$. Let

$\tilde{z}^!_* = (0, \tilde{z}_0 + 1, \tilde{z}_1 + 1, \dots, \tilde{z}_{k-1} + 1) \in Z_k$.

Let $w_* = z_* + \tilde{z}^!_* - z^{0,k}_*$, $w'_* = z'_* + \tilde{z}'_* - z^{0,k-1}_*$. Then $[z_*; z'_*] \in \mathrm{Irr}(W_r)$, $[\tilde{z}_*; \tilde{z}'_*]^\kappa \in \mathrm{Irr}(W'_q)^\dagger$ $(k = 0, 1)$, $[w_*; w'_*] \in \mathrm{Irr}(W_n)$ are well defined and we have

(a) $[[w_*; w'_*] : \mathrm{Ind}^{W_n}_{W_r \times W'_q}([z_*; z'_*] \boxtimes [\tilde{z}_*; \tilde{z}'_*]^\kappa)]_{W_n} = 1$.

(This can be deduced from the second sentence in 4.5(a) with $p = 0$.) Moreover, we have $b_{[w_*; w'_*]} = b_{[z_*; z'_*]} + b_{[\tilde{z}_*; \tilde{z}'_*]^\kappa}$. It follows that

(b) $[w_*; w'_*] = j^{W_n}_{W_r \times W'_q}([z_*; z'_*] \boxtimes [\tilde{z}_*; \tilde{z}'_*]^\kappa)$.

5.4. By [L5, §12], there is a unique bijection $\tilde{\tau} : \tilde{\mathcal{S}}_{\mathbf{W}} \to \tilde{Y}^n_m$ such that for any $y_* \in \tilde{Y}^n_m$, the fibre $\tilde{\tau}^{-1}(y_*)$ is $\{[z_*, z'_*]\}$ where $z_* = (y_0, y_2 - 1, y_4 - 2, \dots, y_m - m/2)$, $z'_* = (y_1 - 1, y_3 - 2, y_5 - 3, \dots, y_{m-1} - m/2)$. This bijection has the following property: if $C \in \mathcal{X}$ and $y_* = \tilde{\tau}(\rho_C)$ then $\mathbf{b}_C = \tilde{\beta}'(y_*)$, $\mathbf{z}_C = 2^{|\mathfrak{I}(y_*)| - 1 - \tilde{\delta}_{y_*})}$ where $\tilde{\delta}_{y_*} = 1$ if there exists $\mathcal{I} \in \mathfrak{I}'(y_*)$ such that $0 \notin \mathcal{I}$ and $\tilde{\delta}_{y_*} = 0$ if there is no $\mathcal{I} \in \mathfrak{I}'(y_*)$ such that $0 \notin \mathcal{I}$. Moreover, $\tilde{\mathbf{z}}_C = 2^{|\mathfrak{I}(y_*)| - 1)}$. Hence $\tilde{\mathbf{z}}_C / \mathbf{z}_C = 2^{\tilde{\delta}_{y_*}}$.

5.5. In the setup of 5.3 we assume that $[z_*; z'_*] \in \mathcal{S}_{W_r}$, $[\tilde{z}_*; \tilde{z}'_*]^k \in \mathcal{S}_{W'_q}$. We set $x_* = \zeta_r([z_*; z'_*]) \in X^r_m$, $\tilde{x}_* = \tilde{\zeta}_q([\tilde{z}_*; \tilde{z}'_*]^\kappa) \in \tilde{X}^q_m$. We show:

(a) $[w_*, w'_*] \in \tilde{\mathcal{S}}_{\mathbf{W}}$ and $\tilde{\tau}([w_*, w'_*]) = x_* + \tilde{x}_*$.

We have $w_i = x_{2i} + \tilde{x}_{2i} - i$, $w'_i = x_{2i+1} + \tilde{x}_{2i+1} - 1 - i$. Define $y_* \in \mathbf{N}^{m+1}$ by $y_{2i} = w_i + i$ for $i \in [0, k]$, $y_{2i+1} = w'_i + i + 1$ for $i \in [0, k-1]$. We have $y_* = x_* + \tilde{x}_*$. Since $x_* \in X_m, \tilde{x}_* \in X_m$, we have $y_* \in \tilde{Y}_m$. More precisely, $y_* \in \tilde{Y}^n_m$. Using 5.4 we deduce that $[w_*, w'_*] \in \tilde{\mathcal{S}}_{\mathbf{W}}$ and (a) follows.

From (a) and 5.3(b) we see that for (r, q) as in 5.3, the assignment $(E_1, \tilde{E}_1) \mapsto j^{W_n}_{W_r \times W'_q}(E_1 \boxtimes \tilde{E}_1)$ is a map $j : \mathcal{S}_{W_r} \times \mathcal{S}_{W'_q} \to \tilde{\mathcal{S}}_{\mathbf{W}}$ and we have a commutative diagram

$$\begin{array}{ccc} \mathcal{S}_{W_r} \times \mathcal{S}_{W'_q} & \xrightarrow{j} & \tilde{\mathcal{S}}_{\mathbf{W}} \\ {\scriptstyle \zeta_r \times \tilde{\zeta}_q}\big\downarrow & & {\scriptstyle \tilde{\tau}}\big\downarrow \\ X^r_m \times \tilde{X}^q_m & \xrightarrow{h} & \tilde{Y}^n_m \end{array}$$

where h is given by $(x_*, \tilde{x}_*) \mapsto x_* + \tilde{x}_*$.

5.6. Note that $\mathcal{P}'$ is exactly the collection of subgroups $W_r \times W'_q$ of W_n with (r, q) as in 5.3 and $q \neq 1$. (On the other hand $W_{n-1} \times W'_1$ is a maximal parabolic subgroup of the Coxeter group W_n.) By 5.5, $j^{W_n}_{W_r \times W'_q}$ carries $\mathcal{S}_{W_r} \times \mathcal{S}_{W'_q}$ into $\tilde{\mathcal{S}}_{\mathbf{W}}$. Hence $\bar{\mathcal{S}}_{\mathbf{W}} \subset \tilde{\mathcal{S}}_{\mathbf{W}}$.

Conversely, let $E \in \tilde{\mathcal{S}}_{\mathbf{W}}$. With $\tilde{\tau}$ as in 5.4, let $y_* = \tilde{\tau}(E) \in \tilde{Y}^n_m$. By 2.7(b) we can find $(x_*, \tilde{x}_*) \in \tilde{S}(y_*)$. (The assumption 2.7(a) is automatically satisfied since m is large relative to n.) Define r, q in $\mathbf{N}$ by $x_* \in X^r_m, \tilde{x}_* \in \tilde{X}^q_m$. We must have $r + q = n$. In the commutative diagram in 5.5 we have $h(x_*, \tilde{x}_*) = y_*$, $(x_*, \tilde{x}_*) =$

$(\zeta_r(E_1),\tilde\zeta_q(\tilde E_1))$ where $E_1\in\mathcal{S}_{W_r}$, $\tilde E_1\in\mathcal{S}_{W'_q}$ (recall that ζ_r, $\tilde\zeta_q$ are surjective) and $\tilde\tau(j(E_1,\tilde E_1))=\tilde\tau(E)$. Since $\tilde\tau$ is bijective we deduce that $E=j(E_1,\tilde E_1)$. Thus, $E\in\bar{\mathcal{S}}_{\mathbf{W}}$ and $\tilde{\mathcal{S}}_{\mathbf{W}}\subset\bar{\mathcal{S}}_{\mathbf{W}}$. We see that $\tilde{\mathcal{S}}_{\mathbf{W}}=\bar{\mathcal{S}}_{\mathbf{W}}$. This proves 1.5(a) in our case.

5.7. In the remainder of this section we fix $C\in\mathcal{X}$ and we set $E=\rho_C\in\tilde{\mathcal{S}}_{\mathbf{W}}$, $y_*=\tilde\tau(E)\in\tilde Y^n_m$ (with $\tilde\tau$ as in 5.4).

Let $(r,q)\in\mathbf{N}^2$, $E_1\in\mathcal{S}_{W_r}$, $\tilde E_1\in\mathcal{S}_{W'_q}$ be such that $r+q=n$, $E=j_{W_r\times W'_q}^{W_n}(E_1\boxtimes\tilde E_1)$. (These exist since $E\in\bar{\mathcal{S}}_{\mathbf{W}}$.) We set $x_*=\zeta_r(E_1)\in X^r_m$, $\tilde x_*=\tilde\zeta_q(\tilde E_1)\in\tilde X^q_m$. From the commutative diagram in 5.5 we see that $x_*+\tilde x_*=y_*$. By 5.4 we have $\mathbf{b}_C=\tilde\beta'(y_*)$. Since $\tilde\beta'(x_*+\tilde x_*)=\beta(x_*)+\tilde\beta(\tilde x_*)$ we have $\mathbf{b}_C=\beta(x_*)+\tilde\beta(\tilde x_*)$. Since $\beta(x_*)=b_{E_1}$, $\tilde\beta(\tilde x_*)=b_{\tilde E_1}$, we have $\mathbf{b}_C=b_{E_1}+b_{\tilde E_1}$ hence $\mathbf{b}_C=b_{E_1\boxtimes\tilde E_1}$. Since $E=j_{W_r\times W'_q}^{W_n}(E_1\boxtimes\tilde E_1)$ we have $b_{E_1\boxtimes\tilde E_1}=b_E$ hence $\mathbf{b}_C=b_E$, proving 1.5(b1) in our case.

If $|\mathfrak{S}(\tilde x_*)|\ge 3$ then

$$f_{E_1\boxtimes\tilde E_1}=f_{E_1}f_{\tilde E_1}=2^{(|\mathfrak{S}(x_*)|-1)/2}2^{(|\mathfrak{S}(\tilde x_*)|-3)/2}$$
$$=2^{(|\mathfrak{S}(x_*)|+|\mathfrak{S}(\tilde x_*)|-4)/2}\le 2^{|\mathfrak{I}(y_*)|-2}\le\mathbf{z}_C.$$

If $|\mathfrak{S}(\tilde x_*)|=1$ and $|\mathfrak{S}(x_*)|\le 2|\mathfrak{I}(y_*)|-3$ then

$$f_{E_1\boxtimes\tilde E_1}=f_{E_1}f_{\tilde E_1}=2^{(|\mathfrak{S}(x_*)|-1)/2}\le 2^{|\mathfrak{I}(y_*)|-2}\le\mathbf{z}_C.$$

If $|\mathfrak{S}(\tilde x_*)|=1$ (hence $\mathfrak{S}(\tilde x_*)=\{0\}$) and $|\mathfrak{S}(x_*)|=2|\mathfrak{I}(y_*)|-1$ then $\tilde\delta_{y_*}=0$ so that $\mathbf{z}_C=2^{|\mathfrak{I}(y_*)|-1}$ and

$$f_{E_1\boxtimes\tilde E_1}=f_{E_1}f_{\tilde E_1}=2^{(|\mathfrak{S}(x_*)|-1)/2}=2^{|\mathfrak{I}(y_*)|-1}=\mathbf{z}_C.$$

Thus in any case we have $f_{E_1\boxtimes\tilde E_1}\le\mathbf{z}_C$. Taking maximum over all $r,q,E_1,\tilde E_1$ as above we obtain $\mathfrak{a}_E\le\mathbf{z}_C$.

5.8. Assume now that $\tilde\delta_{y_*}=1$. Then $|\mathfrak{I}(y_*)|\ge 2$. By 2.7(b) we can find $(x_*,\tilde x_*)\in\tilde S(y_*)$. By 2.7(d) we have $|\mathfrak{S}(\tilde x_*)|\ge 3$. Define $(r,q)\in\mathbf{N}^2$ by $x_*\in X^r_m,\tilde x_*\in\tilde X^q_m$. We must have $r+q=n$. We can find $E_1\in\mathcal{S}_{W_r}$, $\tilde E_1\in\mathcal{S}_{W'_q}$ such that $x_*=\zeta_r(E_1),\tilde x_*=\tilde\zeta_q(\tilde E_1)$. As earlier in the proof, we have $E=j_{W_r\times W'_q}^{W_n}(E_1\boxtimes\tilde E_1)$ and

$$f_{E_1\boxtimes\tilde E_1}=f_{E_1}f_{\tilde E_1}=2^{(|\mathfrak{S}(x_*)|-1)/2+(|\mathfrak{S}(\tilde x_*)|-3)/2}=2^{|\mathfrak{I}(y_*)|-2}=\mathbf{z}_C.$$

5.9. Next we assume that $\tilde\delta_{y_*}=0$. By 2.7(b) we can find $(x_*,\tilde x_*)\in\tilde S(y_*)$. By 2.7(v) we have $\mathfrak{S}(\tilde x_*)=\{0\}$. Then $|\mathfrak{S}(x_*)|=2|\mathfrak{I}(y_*)|-1$. Define $(r,q)\in\mathbf{N}^2$ by $x_*\in X^r_m,\tilde x_*\in\tilde X^q_m$. We must have $r+q=n$. We can find $E_1\in\mathcal{S}_{W_r}$, $\tilde E_1\in\mathcal{S}_{W'_q}$ such that $x_*=\zeta_r(E_1)$, $\tilde x_*=\tilde\zeta_q(\tilde E_1)$. We have $E=j_{W_r\times W'_q}^{W_n}(E_1\boxtimes\tilde E_1)$ and

$$f_{E_1\boxtimes\tilde E_1}=f_{E_1}f_{\tilde E_1}=2^{(|\mathfrak{S}(x_*)|-1)/2}=2^{|\mathfrak{I}(y_*)|-1}=\mathbf{z}_C.$$

Using this and 5.8 we see that in any case, $\mathfrak{a}_E=\mathbf{z}_C$, proving 1.5(b2) in our case.

5.10. Assume first that $\delta_{y_*} = 1$. Let $r, q, x_*, \tilde{x}_*, E_1, \tilde{E}_1$ be as in 5.8. Then $|\mathfrak{S}(\tilde{x}_*)| \geq 3$ hence $q \geq 1$ (so that the unique J such that $\mathbf{W}_J = W_r \times W'_q$ is Ω-stable) and $\tilde{E}_1$ is Ω-stable. It follows that $\mathfrak{c}_E = 2$.

Conversely, assume that $\mathfrak{c}_E = 2$. Using 1.11 we see that there exist $(r,q) \in \mathbf{N}^2$ be such that $r + q = n$ with $q \geq 1$ and $E_1 \in \mathcal{S}_{W_r}$, $\tilde{E}_1 \in \mathcal{S}_{W'_q}$ such that $\tilde{E}_1$ is Ω-stable, $E = j_{W_r \times W'_q}^{W_n}(E_1 \boxtimes \tilde{E}_1)$, $f_{E_1 \boxtimes \tilde{E}_1} = \mathbf{z}_C$. We set $x_* = \zeta_r(E_1) \in X^r_m$, $\tilde{x}_* = \tilde{\zeta}_q(\tilde{E}_1) \in \tilde{X}^q_m$. We have $y_* = x_* + \tilde{x}_*$. Since $\tilde{E}_1$ is Ω-stable, we have $|\mathfrak{S}(\tilde{x}_*)| \geq 3$. Hence

$$2^{|\mathfrak{I}(y_*)|-2} \leq \mathbf{z}_C = f_{E_1 \boxtimes \tilde{E}_1} = 2^{(|\mathfrak{S}(x_*)|-1)/2 + (|\mathfrak{S}(\tilde{x}_*)|-3)/2} \leq 2^{|\mathfrak{I}(y_*)|-2}.$$

It follows that $2^{|\mathfrak{I}(y_*)|-2} = \mathbf{z}_C$ so that $\tilde{\delta}_{y_*} = 1$.

We see that $\tilde{\mathbf{z}}_C/\mathbf{z}_C = 2$ if and only if $\mathfrak{c}_E = 2$. Since $\mathfrak{c}_E \in [1,2]$ and $\tilde{\mathbf{z}}_C/\mathbf{z}_C \in [1,2]$ we see that $\tilde{\mathbf{z}}_C/\mathbf{z}_C = \mathfrak{c}_E$; this proves 1.5(b3) in our case.

6. Type D_n

6.1. In this section we assume that G is of type D_n ($n \geq 4$). We identify $\mathbf{W} = W'_n$ as Coxeter groups in the usual way. The reflections $s_j (j \in \tilde{I})$ are the products of two transpositions $(i, i+1)(i', (i+1)')$ for $i \in [1, n-1]$ and $(n-1, n')(n, (n-1)')$, $(1, 2')(2, 1')$. Define $\omega_1 \in W'_n$ by $i \mapsto (n+1-i)', i' \mapsto n+1-i$ for $i \in [1, n-1]$, $n \mapsto 1$, $n' \mapsto 1'$ (if n is even) and by $i \mapsto (n+1-i)', i' \mapsto n+1-i$ for $i \in [1, n]$ (if n is even). Define $\omega_2 \in W'_n$ by $i \mapsto i$ for $i \in [2, n-1]$, $1 \mapsto 1', 1' \mapsto 1, n \mapsto n', n' \mapsto n$. We have $\omega_1, \omega_2 \in \Omega$. If n is odd, Ω is cyclic of order 4 with generator ω_1 such that $\omega_1^2 = \omega_2$. If n is even, Ω is noncyclic of order 4 with generators ω_1, ω_2 of order 2.

6.2. In the remainder of this section we fix an odd integer $m = 2k - 1$ which is large relative to n.

Let $(p, q) \in \mathbf{N}^2$ be such that $p + q = n$, $q \geq 1$. The group of all permutations of $\{1, 2, \dots, n, n', \dots, 2', 1'\}$ in W_n that leave stable each of the subsets $\{1, 2, \dots, p\}$, $\{p', \dots, 2', 1'\}$ and induce an even permutation on the subset $\{p+1, \dots, n-1, n\} \cup \{n', (n-1)', \dots, (p+1)'\}$ is a standard parabolic subgroup of W'_n which may be identified with $S_p \times W'_q$ in an obvious way.

Let $(\tilde{z}_*; \tilde{z}'_*) \in {}'V^q_k$, $u_* \in Z^p_{2k-1}$. Define $\ddot{u}_* \in Z_{k-1}$, $\dot{u}_* \in Z_{k-1}$ by $\ddot{u}_i = u_{2i} - i$, $\dot{u}_i = u_{2i+1} - i - 1$ for $i \in [0, k-1]$. Let $v_* = \tilde{z}_* + \dot{u}_* - z^{0,k-1}_*$, $v'_* = \tilde{z}'_* + \ddot{u}_* - z^{0,k-1}_*$. Then $(v_*; v'_*) \in {}'V^n_k$, $[u_*] \in \mathrm{Irr}(S_p)$, $[\tilde{z}_*; \tilde{z}'_*] \in \mathrm{Irr}(W'_q)$, $[v_*; v'_*] \in \mathrm{Irr}(W'_n)$. We have:

(a) $[v_*; v'_*]^0 = j^{W'_n}_{S_p \times W'_q}([u_*] \boxtimes [\tilde{z}_*; \tilde{z}'_*]^0)$.

The proof is similar to that of 4.4(a).

6.3. Let $(r, p, q) \in \mathbf{N}^3$ be such that $r + p + q = n$. The group of all permutations of $\{1, 2, \dots, n, n', \dots, 2', 1'\}$ in W'_n that leave stable each of the subsets

$$\{r+1, r+2, \dots, r+p\}, \{(r+p)', \dots, (r+2)', (r+1)'\}$$

and induce an even permutation on each of the subsets

$$\{1,2,\dots,r\}\cup\{r',\dots,2',1'\},\{r+p+1,\dots,n-1,n\}\cup\{n',(n-1)',\dots,(r+p+1)'\}$$

is a parahoric subgroup of $\mathbf{W}$ which may be identified with $W'_r\times S_p^{(0)}\times W'_q$ in an obvious way. ($S_p^{(0)}$ is a copy of S_p.)

When $r=0,p\ge 2$, the group of all permutations of $\{1,2,\dots,n,n',\dots,2',1'\}$ in W'_n that leave stable each of the subsets

$$\{1',2,\dots,p\},\{p',\dots,2',1\},\{p+1,\dots,n-1,n\}\cup\{n',(n-1)',\dots,(p+1)'\}$$

is a parahoric subgroup of $\mathbf{W}$ which may be identified with $W'_r\times S_p^{(1)}\times W'_q$. ($S_p^{(1)}$ is a copy of S_p.)

When $p\ge 2,q=0$, the group of all permutations of $\{1,2,\dots,n,n',\dots,2',1'\}$ in W'_n that leave stable each of the subsets

$$\{r+1,r+2,\dots,n-1,n'\},\{n,(n-1)'\dots,(r+2)',(r+1)'\},\{1,2,\dots,r\}\cup\{r',\dots,2',1'\}$$

is a parahoric subgroup of $\mathbf{W}$ which may be identified with $W'_r\times S_p^{(2)}\times W'_q$. ($S_p^{(2)}$ is a copy of S_p.)

When $r=q=0$, the group of all permutations of $\{1,2,\dots,n,n',\dots,2',1'\}$ in W'_n that leave stable each of the subsets $\{1',2,3,\dots,n-1,n'\}$, $\{n,(n-1)'\dots,3',2',1\}$ is a parahoric subgroup of $\mathbf{W}$ which may be identified with $W'_r\times S_p^{(3)}\times W'_q$. ($S_p^{(3)}$ is a copy of S_p.)

Thus the parahoric subgroup $W'_r\times S_p^{(\lambda)}\times W'_q$ is defined in the following cases:

(a) $\lambda=0$; $p\ge 2,r=0,\lambda=1$; $p\ge 2,q=0,\lambda=2$; $r=q=0,\lambda=3$.

When $p=0$ we write also $W'_r\times W'_q$ instead of $W'_r\times S_p^{(0)}\times W'_q$.

Let $(z_*;z'_*)\in{}^\dagger V_k^r$, $(\tilde z_*;\tilde z'_*)\in{}^\dagger V_k^q$, $u_*\in Z_{2k-1}^p$, Define $\ddot u_*\in Z_{k-1}$, $\dot u_*\in Z_{k-1}$ by $\ddot u_i=u_{2i}-i$, $\dot u_i=u_{2i+1}-i-1$ for $i\in[0,k-1]$. Let $w_*=z_*+\tilde z_*+\dot u_*-2z_*^{0,k-1}$, $w'_*=z'_*+\tilde z'_*+\ddot u_*-2z_*^{0,k-1}$. Then $(w_*,w'_*)\in{}^\dagger V_k^n$.

For $\kappa,\tilde\kappa,\kappa'\in\{0,1\}$ we have $[z_*;z'_*]^\kappa\in\mathrm{Irr}(W'_r)^\dagger$, $[u_*]\in\mathrm{Irr}(S_p)$, $[\tilde z_*;\tilde z'_*]^{\tilde\kappa}\in\mathrm{Irr}(W'_q)^\dagger$, $[w_*,w'_*]^{\kappa'}\in\mathrm{Irr}(W'_n)^\dagger$. For λ as in (a) we have:

(b) $[w_*;w'_*]^{\kappa'}=j_{W'_r\times S_p^{(\lambda)}\times W'_q}^{W'_n}([z_*;z'_*]^\kappa\boxtimes[u_*]\boxtimes[\tilde z_*;\tilde z'_*]^{\tilde\kappa})$

with the following restriction on κ': if $z_*=z'_*,\tilde z_*=\tilde z'_*$, $\dot u_*=\ddot u_*$, then $w_*=w'_*$ and κ' in (b) is uniquely determined by $\kappa,\tilde\kappa,\lambda$; moreover, both $\kappa'=0$ and $\kappa'=1$ are obtained from some $(\kappa,\tilde\kappa,\lambda)$.

Now (b) can be proved in a way similar to 4.5(a); alternatively, from the second statement of 4.5(a) one can deduce that

$$[[w_*;w'_*]^{\kappa'}:\mathrm{Ind}_{W'_r\times S_p^{(\lambda)}\times W'_q}^{W'_n}([z_*;z'_*]^\kappa\boxtimes[u_*]\boxtimes[\tilde z_*;\tilde z'_*]^{\tilde\kappa})]_{W'_n}\ge 1;$$

we can also check directly that $b_{[w_*;w'_*]^{\kappa'}}=b_{[z_*;z'_*]^\kappa}+b_{[u_*]}+b_{[\tilde z_*;\tilde z'_*]^{\tilde\kappa}}$ and (b) follows.

6.4. By [L5, §13], we have $\tilde{\mathcal{S}}_{\mathbf{W}} \subset \mathrm{Irr}(W'_n)^\dagger$ and there is a unique map $\tau : \tilde{\mathcal{S}}_{\mathbf{W}} \to Y^n_m$ such that for $y_* \in Y^n_m$, $\tau^{-1}(y_*)$ consists of $[z_*; z'_*]^0 = [z_*; z'_*]^1$ (if $\mathfrak{I}(y_*) \ne \emptyset$) and consists of $[z_*; z'_*]^0, [z_*; z'_*]^1$ (if $\mathfrak{I}(y_*) = \emptyset$) where

$z_* = (y_1, y_3 - 1, y_5 - 2, \dots, y_m - (m-1)/2)$,

$z'_* = (y_0, y_2 - 1, y_4 - 2, \dots, y_{m-1} - (m-1)/2)$.

This map has the following property: if $C \in \mathcal{X}$ and $y_* = \tau(\rho_C)$, then $\mathbf{b}_C = \beta'(y_*)$, $\mathbf{z}_C = 2^{\max(|\mathfrak{I}(y_*)|-1-\delta_{y_*},0)}$ where $\delta_{y_*} = 1$ if $\mathfrak{I}'(y_*) \ne \emptyset$ and $\delta_{y_*} = 0$ if $\mathfrak{I}'(y_*) = \emptyset$. Moreover, $\tilde{\mathbf{z}}_C/\mathbf{z}_C$ is:

4 if $\delta_{y_*} = 1$ and $|\mathcal{I}| = 1$ for any $\mathcal{I} \in \mathfrak{I}(y_*)$,

2 if $\delta_{y_*} = 1$ and $|\mathcal{I}| > 1$ for some $\mathcal{I} \in \mathfrak{I}(y_*)$,

2 if $\mathfrak{I}(y_*) = \emptyset$,

1 if $\delta_{y_*} = 0$ and $|\mathcal{I}| > 1$ for some $\mathcal{I} \in \mathfrak{I}(y_*)$.

More precisely, let $\underline{G} \to G$ be a double covering which is a special orthogonal group and let $\underline{\mathbf{z}}_C$ be the number of connected components of the centralizer in $\underline{G}$ of a unipotent element of $\underline{G}$ which maps to an element of C. From [L5, §14] we see that:

$\tilde{\mathbf{z}}_C/\underline{\mathbf{z}}_C = 2$ if $|\mathcal{I}| = 1$ for any $\mathcal{I} \in \mathfrak{I}(y_*)$,

$\tilde{\mathbf{z}}_C/\mathbf{z}_C = 1$ if $|\mathcal{I}| > 1$ for some $\mathcal{I} \in \mathfrak{I}(y_*)$.

On the other hand, from $\underline{\mathbf{z}}_C = 2^{\max(|\mathfrak{I}(y_*)|-1,0)}$, $\mathbf{z}_C = 2^{\max(|\mathfrak{I}(y_*)|-1-\delta_{y_*},0)}$, we see that $\underline{\mathbf{z}}_C/\mathbf{z}_C = 2^{\delta_{y_*}}$.

6.5. In the setup of 6.3 we assume that $[z_*; z'_*]^\kappa \in \mathcal{S}_{W'_r}$, $[\tilde{z}_*; \tilde{z}'_*]^{\tilde{\kappa}} \in \mathcal{S}_{W'_q}$ and κ' is as in 6.3(b). Define $x_* \in X^r_m, \tilde{x}_* \in X^q_m$ by $\zeta'_r([z_*; z'_*]^\kappa) = x_*$, $\zeta'_q([\tilde{z}_*; \tilde{z}'_*]^{\tilde{\kappa}}) = \tilde{x}_*$. Let $e_* = u_* - z^{0,m}_* \in \mathcal{E}_m$. We show:

(a) $[w_*, w'_*]^{\kappa'} \in \tilde{\mathcal{S}}_{\mathbf{W}}$ and $\tau([w_*, w'_*]^{\kappa'}) = x_* + e_* + \tilde{x}_*$.

We have $w_i = x_{2i+1} + \tilde{x}_{2i+1} + u_{2i+1} - i - 2i - 1$, $w'_i = x_{2i} + \tilde{x}_{2i} + u_{2i} - i - 2i$ for $i \in [0, k-1]$. Define $y_* \in \mathbf{N}^{m+1}$ by $w_i = y_{2i+1} - i$, $w'_i = y_{2i} - i$ for $i \in [0, k-1]$. Then $y_* = x_* + \tilde{x}_* + e_*$. Since $x_* \in X_m, \tilde{x}_* \in X_m, e_* \in \mathcal{E}_m$ we have $y_* \in Y_m$. More precisely, $y_* \in Y^n_m$. Using 6.4 we deduce that $[w_*, w'_*]^{\kappa'} \in \tilde{\mathcal{S}}_{\mathbf{W}}$ and (a) follows.

From (a) and 6.3(b) we see that for λ as in 6.3(a), the assignment $(E_1, E_2, \tilde{E}_1) \mapsto j^{W_n}_{W_r \times S^{(\lambda)}_p \times W_q}(E_1 \boxtimes E_2 \boxtimes \tilde{E}_1)$ is a map $j : \mathcal{S}_{W'_r} \times \mathcal{S}_{S_p} \times \mathcal{S}_{W'_q} \to \tilde{\mathcal{S}}_{\mathbf{W}}$ and we have a commutative diagram

$$\begin{tikzcd} \mathcal{S}_{W'_r} \times \mathcal{S}_{S_p} \times \mathcal{S}_{W'_q} \arrow[r, "j"] \arrow[d, "\zeta'_r \times \xi_p \times \zeta'_q"'] & \tilde{\mathcal{S}}_{\mathbf{W}} \arrow[d, "\tau"'] \\ X^r_m \times \mathcal{E}^p_m \times X^q_m \arrow[r, "h"] & Y^n_m \end{tikzcd}$$

where h is given by $(x_*, e_*, \tilde{x}_*) \mapsto x_* + e_* + \tilde{x}_*$ and $\xi_p : \mathcal{S}_{S_p} \to \mathcal{E}^p_m$ is the bijection $[e_* + z^{0,m}_*] \leftrightarrow e_*$.

6.6. Note that $\mathcal{P}'$ is exactly the collection of parahoric subgroups $W'_r \times W'_q$ of W'_n with $(r, q) \in \mathbf{N}^2$ such that $r + q = n$, $r \ne 1$, $q \ne 1$. (On the other hand $W'_{n-1} \times W'_1$,

$W'_1 \times W'_{n-1}$ are maximal parabolic subgroup of the Coxeter group W'_n.) By 6.5, $j_{W'_r \times W'_q}^{W'_n}$ carries $\mathcal{S}_{W'_r} \times \mathcal{S}'_{W_q}$ into $\tilde{\mathcal{S}}_{\mathbf{W}}$. Hence $\bar{\mathcal{S}}_{\mathbf{W}} \subset \tilde{\mathcal{S}}_{\mathbf{W}}$.

Conversely, let $E \in \tilde{\mathcal{S}}_{\mathbf{W}}$, $y_* = \tau(E) \in Y_m^n$ (τ as in 6.4). By 2.6(a) we can find $(x_*, \tilde{x}_*) \in S(y_*)$. Define r, q in $\mathbf{N}$ by $x_* \in X_m^r$, $\tilde{x}_* \in X_m^q$. We must have $r + q = n$. In the commutative diagram in 6.5 we have $h(x_*, \tilde{x}_*) = y_*$, $x_* = \zeta'_r(E_1), \tilde{x}_* = \zeta'_q(\tilde{E}_1)$ where $E_1 \in \mathcal{S}_{W'_r}$, $\tilde{E}_1 \in \mathcal{S}_{W'_q}$ (recall that ζ'_r, ζ'_q are surjective) and $\tau(j(E_1, \tilde{E}_1) = \tau(E)$. Thus $j(E_1, \tilde{E}_1), E$ are in the same fibre of $\iota : {}^{\dagger}V_k^n \longrightarrow \mathrm{Irr}(W'_n)^{\dagger}$. Replacing E_1 or $\tilde{E}_1$ by an element in the same fibre of $\iota : {}^{\dagger}V_k^r \longrightarrow \mathrm{Irr}(W'_r)^{\dagger}$ or $\iota : {}^{\dagger}V_k^q \longrightarrow \mathrm{Irr}(W'_q)^{\dagger}$ we see that we can assume that $j(E_1, \tilde{E}_1) = E$. Thus, $E \in \bar{\mathcal{S}}_{\mathbf{W}}$. Thus, $\tilde{\mathcal{S}}_{\mathbf{W}} \subset \bar{\mathcal{S}}_{\mathbf{W}}$. We see that $\tilde{\mathcal{S}}_{\mathbf{W}} = \bar{\mathcal{S}}_{\mathbf{W}}$. This proves 1.5(a) in our case.

6.7. In the remainder of this section we fix $C \in \mathcal{X}$ and we set $E = \rho_C \in \tilde{\mathcal{S}}_{\mathbf{W}}$, $y_* = \tau(E) \in Y_m^n$ (with τ as in 6.4).

Let $(r, q) \in \mathbf{N}^2$, $E_1 \in \mathcal{S}_{W'_r}$, $\tilde{E}_1 \in \mathcal{S}_{W'_q}$ be such that $r+q = n$, $E = j_{W'_r \times W'_q}^{W'_n}(E_1 \boxtimes \tilde{E}_1)$. (These exist since $E \in \bar{\mathcal{S}}_{\mathbf{W}}$.) Define $x_* \in X_m^r, \tilde{x}_* \in X_m^q$ by $x_* = \zeta'_r(E_1), \tilde{x}_* = \zeta'_q(\tilde{E}_1)$. From the commutative diagram in 6.5 we see that $x_* + \tilde{x}_* = y_*$. By 6.4, we have $\mathbf{b}_C = \beta'(y_*)$. Since $\beta'(x_* + \tilde{x}_*) = \beta(x_*) + \beta(\tilde{x}_*)$ we have $\mathbf{b}_C = \beta(x_*) + \beta(\tilde{x}_*)$. Since $\beta(x_*) = b_{E_1}$, $\beta(\tilde{x}_*) = b_{\tilde{E}_1}$, we have $\mathbf{b}_C = b_{E_1} + b_{\tilde{E}_1}$ hence $\mathbf{b}_C = b_{E_1 \boxtimes \tilde{E}_1}$. Since $E = j_{W'_r \times W'_q}^{W'_n}(E_1 \boxtimes \tilde{E}_1)$ we have $b_{E_1 \boxtimes \tilde{E}_1} = b_E$ hence $\mathbf{b}_C = b_E$, proving 1.5(b1) in our case.

If $|\mathfrak{S}(x_*)| \geq 2$, $|\mathfrak{S}(\tilde{x}_*)| \geq 2$, then

$$\begin{aligned} f_{E_1 \boxtimes \tilde{E}_1} &= f_{E_1} f_{\tilde{E}_1} = 2^{(|\mathfrak{S}(x_*)|-2)/2} 2^{(|\mathfrak{S}(\tilde{x}_*)|-2)/2)} \\ &= 2^{(|\mathfrak{S}(x_*)|+|\mathfrak{S}(\tilde{x}_*)|-4)/2} \leq 2^{|\mathfrak{I}(y_*)|-2} \leq \mathbf{z}_C. \end{aligned}$$

If $|\mathfrak{S}(x_*)| = 0$, $2 \leq |\mathfrak{S}(\tilde{x}_*)| \leq 2|\mathfrak{I}(y_*)| - 2$ then

$$f_{E_1 \boxtimes \tilde{E}_1} = f_{E_1} f_{\tilde{E}_1} = 2^{(|\mathfrak{S}(\tilde{x}_*)|-2)/2)} \leq 2^{|\mathfrak{I}(y_*)|-2} \leq \mathbf{z}_C.$$

Similarly, if $2 \leq |\mathfrak{S}(x_*)| \leq 2|\mathfrak{I}(y_*)| - 2$, $|\mathfrak{S}(\tilde{x}_*)| = 0$, then $f_{E_1 \boxtimes \tilde{E}_1} \leq \mathbf{z}_C$. If $|\mathfrak{S}(x_*)| = 0$, $2 \leq |\mathfrak{S}(\tilde{x}_*)| = 2|\mathfrak{I}(y_*)|$ then $\mathfrak{I}'(y_*) = \emptyset$ and

$$f_{E_1 \boxtimes \tilde{E}_1} = f_{E_1} f_{\tilde{E}_1} = 2^{(|\mathfrak{S}(\tilde{x}_*)|-2)/2)} = 2^{|\mathfrak{I}(y_*)|-1} \leq \mathbf{z}_C.$$

Similarly, if $2 \leq |\mathfrak{S}(x_*)| = 2|\mathfrak{I}(y_*)|$, $|\mathfrak{S}(\tilde{x}_*)| = 0$ then $f_{E_1 \boxtimes \tilde{E}_1} \leq \mathbf{z}_C$. If $|\mathfrak{S}(x_*)| = 0$, $|\mathfrak{S}(\tilde{x}_*)| = 0$, then

$$f_{E_1 \boxtimes \tilde{E}_1} = f_{E_1} f_{\tilde{E}_1} = 1 = \mathbf{z}_C.$$

Thus in any case we have $f_{E_1 \boxtimes \tilde{E}_1} \leq \mathbf{z}_C$. Taking maximum over all $r, q, E_1, \tilde{E}_1$ as above we obtain $\mathfrak{a}_E \leq \mathbf{z}_C$.

6.8. Assume now that $\delta_{y_*}=1$. Then $|\mathfrak{I}(y_*)|\ge 2$. By 2.6(a) we can find $(x_*,\tilde{x}_*)\in S(y_*)$. By 2.6(c) we have $|\mathfrak{S}(x_*)|\ge 2$, $|\mathfrak{S}(\tilde{x}_*)|\ge 2$. Define $(r,q)\in\mathbf{N}^2$ by $x_*\in X^r_m,\tilde{x}_*\in X^q_m$. We must have $r+q=n$. We can find $E_1\in\mathcal{S}_{W'_r}$, $\tilde{E}_1\in\mathcal{S}_{W'_q}$ such that $x_*=\zeta'_r(E_1),\tilde{x}_*=\zeta'_q(\tilde{E}_1)$. As earlier in the proof we can assume that $E=j^{W'_n}_{W'_r\times W'_q}(E_1\boxtimes\tilde{E}_1)$ and we have

$$f_{E_1\boxtimes\tilde{E}_1}=f_{E_1}f_{\tilde{E}_1}=2^{(|\mathfrak{S}(x_*)|-2)/2+(|\mathfrak{S}(\tilde{x}_*)|-2)/2}=2^{|\mathfrak{I}(y_*)|-2}=\mathbf{z}_C.$$

6.9. Next we assume that $\mathfrak{I}(y_*)\ne\emptyset$ and $\delta_{y_*}=0$. By 2.6(a) we can find $(x_*,\tilde{x}_*)\in S(y_*)$. By 2.6(vii) we have $\mathfrak{S}(\tilde{x}_*)=\emptyset$. Then $|\mathfrak{S}(x_*)|=2|\mathfrak{I}(y_*)|$. Define $(r,q)\in\mathbf{N}^2$ by $x_*\in X^r_m,\tilde{x}_*\in X^q_m$. We must have $r+q=n$. We can find $E_1\in\mathcal{S}_{W'_r}$, $\tilde{E}_1\in\mathcal{S}_{W'_q}$ such that $x_*=\zeta'_r(E_1)$, $\tilde{x}_*=\zeta'_q(\tilde{E}_1)$. We can assume that $E=j^{W'_n}_{W'_r\times W'_q}(E_1\boxtimes\tilde{E}_1)$ and we have

$$f_{E_1\boxtimes\tilde{E}_1}=f_{\tilde{E}_1}=2^{(|\mathfrak{S}(\tilde{x}_*)|-2)/2}=2^{|\mathfrak{I}(y_*)|-1}=\mathbf{z}_C.$$

Now we assume that $\mathfrak{I}(y_*)=\emptyset$. By 2.6(a) we can find $(x_*,\tilde{x}_*)\in S(y_*)$. By 2.6(b) we have $\mathfrak{S}(x_*)=\emptyset$, $\mathfrak{S}(\tilde{x}_*)=\emptyset$. Define $(r,q)\in\mathbf{N}^2$ by $x_*\in X^r_m,\tilde{x}_*\in X^q_m$. We must have $r+q=n$. We can find $E_1\in\mathcal{S}_{W'_r}$, $\tilde{E}_1\in\mathcal{S}_{W'_q}$ such that $x_*=\zeta'_r(E_1),\tilde{x}_*=\zeta'_q(\tilde{E}_1)$. We can assume that $E=j^{W'_n}_{W'_r\times W'_q}(E_1\boxtimes\tilde{E}_1)$ and we have $f_{E_1\boxtimes\tilde{E}_1}=1=\mathbf{z}_C$.

We see that in any case, $\mathfrak{a}_E=\mathbf{z}_C$, proving 1.5(b2) in our case.

6.10. For $g\in\Omega$ let $\langle g\rangle$ be the subgroup of Ω generated by g.

When n is even the subgroups of Ω are $\{1\},\langle\omega_1\rangle,\langle\omega_2\rangle,\langle\omega_1\omega_2\rangle,\Omega$; when n is odd the subgroups of Ω are $\{1\},\langle\omega_2\rangle,\Omega$.

(a) The collection of subgroups $W'_r\times S^{(0)}_p\times W'_q$ (with $r=q\ge 1$) contains all subgroups in $\mathcal{P}^\Omega$.

(b) The collection of subgroups $W'_r\times W'_q$ contains all subgroups in $\mathcal{P}^{\langle\omega_2\rangle}$.

(c) For n even, the collection in (a) together with the subgroups $W'_0\times S^{(\lambda)}_p\times W'_0$ (with $\lambda=0$ or 3) contains all subgroups in $\mathcal{P}^{\langle\omega_1\rangle}$.

(d) For n even, the collection in (a) together with the subgroups $W'_0\times S^{(\lambda)}_p\times W'_0$ (with $\lambda=1$ or 2) contains all subgroups in $\mathcal{P}^{\langle\omega_1\omega_2\rangle}$.

6.11. Assume that $\tilde{\mathbf{z}}_C/\mathbf{z}_C=4$. Then $\delta_{y_*}=1$ and $|\mathcal{I}|=1$ for any $\mathcal{I}\in\mathfrak{I}(y_*)$. By 2.11 we can find r,p, $x_*\in X^r_m$, $e_*\in\mathcal{E}^p_m$ (with $r+p+r=n$) such that $y_*=x_*+e_*+x_*$, $(x_*,e_*+x_*)\in S(y_*)$, $\mathfrak{S}(x_*)=\mathfrak{S}(e_*+x_*)\ne\emptyset$. Note that $r\ge 1$. Define $E_1\in\mathcal{S}_{W'_r}$ by $\zeta'_r(E_1)=x_*$, $E_2\in\mathcal{S}_{S_p}$ by $\xi_p(E_2)=e_*$. We have $E=j^{W'_n}_{W'_r\times S^{(0)}_p\times W'_r}(E_1\boxtimes E_2\boxtimes E_1)$ and

$$\begin{aligned}f_{E_1\boxtimes E_2\boxtimes E_1}&=2^{(|\mathfrak{S}(x_*)|-2)/2}2^{(|\mathfrak{S}(x_*)|-2)/2}=2^{(|\mathfrak{S}(x_*)|-2)/2}2^{(|\mathfrak{S}(e_*+x_*)|-2)/2}\\&=2^{|\mathfrak{I}(y_*)|-2}=\mathbf{z}_C.\end{aligned}$$

We have $W'_r \times S_p^{0)} \times W'_r \in \mathcal{P}^\Omega$. Moreover, $E_1 \boxtimes E_2 \boxtimes E_1$ is Ω-stable. We see that $\mathfrak{c}_E = 4$.

6.12. Conversely, assume that $\mathfrak{c}_E = 4$. By 1.11 and 6.10(a), there exist (r, p, q) as in 6.3 with $q = r \ge 1$ and $E_1 \in \mathcal{S}_{W'_r}$, $E_2 \in \mathcal{S}_{S_p}$ such that $E = j_{W'_r \times S_p^{(0)} \times W'_r}^{W'_n}(E_1 \boxtimes E_2 \boxtimes E_1)$, $f_{E_1 \boxtimes E_2 \boxtimes E_1} = \mathbf{z}_C$ and such that E_1 extends to a W_r-module. We set $x_* = \zeta'_r(E_1) \in X^r_m$, $e_* = \xi_p(E_2)$. We have $y_* = x_* + e_* + x_*$. Since E_1 extends to a W_r-module we have $\mathfrak{S}(x_*) \ne \emptyset$, hence $\mathfrak{I}(y_*) \ne \emptyset$. Thus, $\mathbf{z}_C = 2^{|\mathfrak{I}(y_*)|-1-\delta_{y_*}}$, $2^{(|\mathfrak{S}(x_*)|-2+|\mathfrak{S}(x_*)|-2)/2} = 2^{|\mathfrak{I}(y_*)|-1-\delta_{y_*}}$ and $|\mathfrak{S}(x_*)| + |\mathfrak{S}(x_*)| = 2|\mathfrak{I}(y_*)| + 1 - \delta_{y_*}$. Since $|\mathfrak{S}(x_*)| + |\mathfrak{S}(x_*)| \le 2|\mathfrak{I}(y_*)|$, we have $1 - \delta_{y_*} \le 0$ hence $\delta_{y_*} = 1$ and $|\mathfrak{S}(x_*)| + |\mathfrak{S}(x_*)| = 2|\mathfrak{I}(y_*)|$.

Let $E'_1 = j_{S_p \times W'_r}^{W'_{p+r}}(E_2 \boxtimes E_1) \in \mathcal{S}_{W'_{p+r}}$. Then $E = j_{W'_r \times W'_{p+r}}^{W'_n}(E_1 \boxtimes E'_1)$. By 1.5(b2) we have $f_{E_1 \boxtimes E'_1} \le \mathbf{z}_C$. By 1.9(b) we have $f_{E_2 \boxtimes E_1} \le f_{E'_1}$. Hence $\mathbf{z}_C = f_{E_1 \boxtimes E_2 \boxtimes E_1} \le f_{E_1 \boxtimes E'_1} \le \mathbf{z}_C$; this forces $f_{E_2 \boxtimes E_1} = f_{E'_1}$. The last equality can be rewritten as

$$2^{(|\mathfrak{S}(x_*)|-2)/2} = 2^{(|\mathfrak{S}(e_*+x_*)|-2)/2}$$

since $e_* + x_* = \zeta'_{p+r}(E'_1)$ (a consequence of 6.2(a)). Hence $|\mathfrak{S}(e_* + x_*)| = |\mathfrak{S}(x_*)|$. We have also $(x_*, e_* + x_*) \in S(y_*)$. Using 2.10, we see that for any $\mathcal{I} \in \mathfrak{I}(y_*)$ we have $|\mathcal{I}| = 1$. Thus, $\tilde{\mathbf{z}}_C / \mathbf{z}_C = 4$.

Using this together with 6.11, we see that $\mathfrak{c}_E = 4$ if and only if $\tilde{\mathbf{z}}_C / \mathbf{z}_C = 4$.

6.13. Assume that $\mathfrak{I}(y_*) = \emptyset$. Then n is even. Define $e_* \in \mathbf{N}^{m+1}$ by $y_* = x^0_* + e_* + x^0_*$. We have $e_* \in \mathcal{E}^n_m$. Define $E_1 \in \mathcal{S}_{W'_0}$ by $\zeta'_0(E_1) = x^0_*$, $E_2 \in \mathcal{S}_{S_n}$ by $\xi_n(E_2) = e_*$. For some $\lambda \in [0, 3]$ we have $E = j_{W'_0 \times S_n^{(\lambda)} \times W'_0}^{W'_n}(E_1 \boxtimes E_2 \boxtimes E_1)$, see 6.3. We have $f_{E_1 \boxtimes E_2 \boxtimes E_1} = 1 = \mathbf{z}_C$. Note that $W'_0 \times S_n^{(\lambda)} \times W'_0 \in \mathcal{P}^{\Omega_1}$ where Ω_1 is $\langle \omega_1 \rangle$ or $\langle \omega_1 \omega_2 \rangle$; moreover $E_1 \boxtimes E_2 \boxtimes E_1$ is Ω_1-stable. We see that $\mathfrak{c}_E \ge 2$. By 6.12 we cannot have $\mathfrak{c}_E = 4$. Hence $\mathfrak{c}_E = 2$.

6.14. Assume that $\delta_{y_*} = 1$ and $|\mathcal{I}| > 1$ for some $\mathcal{I} \in \mathfrak{I}(y_*)$. We have $|\mathfrak{I}(y_*)| \ge 2$. By 2.6(a) we can find $(x_*, \tilde{x}_*) \in S(y_*)$. By 2.6(c) we have $|\mathfrak{S}(x_*)| \ge 2$, $|\mathfrak{S}(\tilde{x}_*)| \ge 2$. Define $(r, q) \in \mathbf{N}^2$ by $x_* \in X^r_m, \tilde{x}_* \in X^q_m$. We must have $r + q = n$ and $r \ge 1$, $q \ge 1$. We can find uniquely $E_1 \in \mathcal{S}_{W'_r}$, $\tilde{E}_1 \in \mathcal{S}_{W'_q}$ such that $x_* = \zeta'_r(E_1), \tilde{x}_* = \zeta'_q(\tilde{E}_1)$. We have $E = j_{W'_r \times W'_q}^{W'_n}(E_1 \boxtimes \tilde{E}_1)$ and

$$f_{E_1 \boxtimes \tilde{E}_1} = f_{E_1} f_{\tilde{E}_1} = 2^{(|\mathfrak{S}(x_*)|-2)/2 + (|\mathfrak{S}(\tilde{x}_*)|-2)/2} = 2^{|\mathfrak{I}(y_*)|-2} = \mathbf{z}_C.$$

We have $W'_r \times W'_q \in \mathcal{P}^{\langle \omega_2 \rangle}$ and $E_1 \boxtimes \tilde{E}_1$ is $\langle \omega_2 \rangle$-stable. We see that $\mathfrak{c}_E \ge 2$. By 6.12 we cannot have $\mathfrak{c}_E = 4$. Hence $\mathfrak{c}_E = 2$.

6.15. Assume that $\mathfrak{c}_E = 2$. By 1.11 and 6.10, either (i) or (ii) below holds.

(i) there exist (r, p, q) as in 6.3 with $q = r$, $\lambda \in [0, 3]$ (with $\lambda = 0$ unless $r = 0$) and $E_1 \in \mathcal{S}_{W'_r}$, $E_2 \in \mathcal{S}_{S_p}$ such that $E = j_{W'_r \times S_p^{(\lambda)} \times W'_r}^{W'_n}(E_1 \boxtimes E_2 \boxtimes E_1)$, $f_{E_1 \boxtimes E_2 \boxtimes E_1} = \mathbf{z}_C$;

(ii) there exist (r,q) with $r+q=n$ and $E_1\in\mathcal{S}_{W'_r}$, $\tilde E_1\in\mathcal{S}_{W'_q}$ such that E_1 extends to a W_r-module, $\tilde E_1$ extends to a W_q-module, $E=j_{W'_r\times W'_q}^{W'_n}(E_1\boxtimes\tilde E_1)$ and $f_{E_1\boxtimes\tilde E_1}=\mathbf{z}_C$.
Assume first that (i) holds. We set $x_*=\zeta'_r(E_1)\in X^r_m$. If $r\ge 1$ and E_1 extends to a W_r-module then $E_1\boxtimes E_2\boxtimes E_1$ is Ω-stable (note that $W'_r\times S_p^{(\lambda)}\times W'_r\in\mathcal{P}^\Omega$) so that $\mathfrak{c}_E=4$ contradicting $\mathfrak{c}_E=2$. Thus, either $r\ge 1$ and E_1 does not extend to a W_r-module or $r=0$. It follows that $\mathfrak{S}(x_*)=\emptyset$ and $f_{E_1}=1$ so that $\mathbf{z}_C=1$. Hence either $|\mathfrak{I}(y_*)|=0$ or $|\mathfrak{I}(y_*)|=2,\delta_{y_*}=1$. In the first case we have $\tilde{\mathbf{z}}_C/\mathbf{z}_C=2$. In the second case, using $\delta_{y_*}=1$ we see that $\tilde{\mathbf{z}}_C/\mathbf{z}_C\ge 2$; if we had $\tilde{\mathbf{z}}_C/\mathbf{z}_C=4$ we would have $\mathfrak{c}_E=4$, a contradiction. Thus in both cases we have $\tilde{\mathbf{z}}_C/\mathbf{z}_C=2$.

Next assume that (ii) holds. We set $x_*=\zeta'_r(E_1)\in X^r_m$, $\tilde x_*=\zeta'_q(\tilde E_1)\in\tilde X^q_m$. We have $y_*=x_*+\tilde x_*$. Since E_1 extends to a W_r-module and $\tilde E_1$ extends to a W_q-module we have $|\mathfrak{S}(x_*)|\ge 2$, $|\mathfrak{S}(\tilde x_*)|\ge 2$. Hence

$$2^{|\mathfrak{I}(y_*)|-2}\le\mathbf{z}_C=f_{E_1\boxtimes\tilde E_1}=f_{E_1}f_{\tilde E_1}=2^{(|\mathfrak{S}(x_*)|-2)/2+(|\mathfrak{S}(\tilde x_*)|-2)/2}\le 2^{|\mathfrak{I}(y_*)|-2}.$$

It follows that $2^{|\mathfrak{I}(y_*)|-2}=\mathbf{z}_C$ so that $\delta_{y_*}=1$. This implies that $\tilde{\mathbf{z}}_C/\mathbf{z}_C\ge 2$; if we had $\tilde{\mathbf{z}}_C/\mathbf{z}_C=4$ we would have $\mathfrak{c}_E=4$, a contradiction. Thus we have $\tilde{\mathbf{z}}_C/\mathbf{z}_C=2$.

Using this together with 6.13, 6.14, we see that $\mathfrak{c}_E=2$ if and only if $\tilde{\mathbf{z}}_C/\mathbf{z}_C=2$.

6.16. By 6.12, we have $\mathfrak{c}_E=4$ if and only if $\tilde{\mathbf{z}}_C/\mathbf{z}_C=4$. By 6.15, we have $\mathfrak{c}_E=2$ if and only if $\tilde{\mathbf{z}}_C/\mathbf{z}_C=2$. Since $\mathfrak{c}_E\in\{1,2,4\}$ and $\tilde{\mathbf{z}}_C/\mathbf{z}_C\in\{1,2,4\}$ we see that $\mathfrak{c}_E=\tilde{\mathbf{z}}_C/\mathbf{z}_C$; this proves 1.5(b3) in our case.

7. Exceptional types

7.1. In this section we assume that G is an exceptional group. For each type we give a table with rows indexed by the unipotent conjugacy classes in G in which the row corresponding to $C\in\mathcal{X}$ has four entries:

$$\rho_C\qquad\mathbf{b}_C\qquad a\times a'\quad(J,E_1)$$

where $a=\mathbf{z}_C$, $a'=\tilde{\mathbf{z}}_C/\mathbf{z}_C$ and (J,E_1) is an example of an element of $\mathcal{Z}_E$ ($E=\rho_C$) such that $f_{E_1}=\mathbf{z}_C$ and $|\Omega_{J,E_1}|=\tilde{\mathbf{z}}_C/\mathbf{z}_C$. (When $\Omega=\{1\}$ we have $a'=1$ and we write a instead of $a\times a'$). We specify an irreducible representation E_1 of a Weyl group either by using the notation of [L4, Ch.4] (for type E_6,E_7,E_8) or by specifying its degree. The representation is then determined by its b_{E_1} which equals $\mathbf{b}_C$ in the table or (in the case of G_2, F_4) by other information in the same row of the table. On the other hand, ϵ always denotes the sign representation. In a pair (J,E_1), J is any subset of $\tilde I$ such that $\mathbf{W}_J$ has the specified type; in addition, for type F_4, we denote by A_2 (resp. A'_2) a subset J of $\tilde I$ such that $\mathbf{W}_J$ is of type A_2 and is contained (resp. not contained) in a parahoric subgroup of type B_4).

The group Ω is $\{1\}$ for types G_2,F_4 and is a cyclic group of order $9-n$ for type $E_n(n=6,7,8)$.

Type G_2

ρ_C	$\mathbf{b}_C$	$a \times a'$	(J, E_1)
1	0	1	$(\emptyset, 1)$
2	1	6	$(G_2, 2)$
2	2	1	(A_1A_1, ϵ)
1	3	1	(A_2, ϵ)
1	6	1	(G_2, ϵ)

Type F_4

ρ_C	$\mathbf{b}_C$	$a \times a'$	(J, E_1)
1	0	1	$(\emptyset, 1)$
4	1	2	$(F_4, 4)$
9	2	2	$(F_4, 9)$
8	3	1	(A_2, ϵ)
8	3	1	(A_2', ϵ)
12	4	24	$(F_4, 12)$
16	5	2	$(C_3A_1, 3 \boxtimes \epsilon)$
9	6	2	$(B_4, 6)$
6	6	1	(A_2A_2', ϵ)
4	7	1	(A_3A_1, ϵ)
8	9	1	(C_3, ϵ)
8	9	2	$(B_4, 4)$
9	10	1	(C_3A_1, ϵ)
4	13	2	$(F_4, 4)$
2	16	1	(B_4, ϵ)
1	24	1	(F_4, ϵ)

Type E_6

ρ_C	$\mathbf{b}_C$	$a \times a'$	(J, E_1)
1_p	0	1×3	$(\emptyset, 1)$
6_p	1	1×3	$(D_4, 4)$
20_p	2	1×1	$(E_6, 20_p)$
30_p	3	2×3	$(D_4, 8)$

15_q	4	1×3	$(A_1A_1A_1A_1,\epsilon)$
64_p	4	1×1	$(E_6,64_p)$
60_p	5	1×1	$(E_6,60_p)$
24_p	6	1×1	$(E_6,24_p)$
81_p	6	1×1	$(E_6,81_p)$
80_s	7	6×1	$(E_6,80_s)$
60_s	8	1×1	$(A_3A_1A_1,\epsilon)$
10_s	9	1×3	$(A_2A_2A_2,\epsilon)$
$81'_p$	10	1×1	$(E_6,81'_p)$
$60'_p$	11	1×1	$(E_6,60'_p)$
$24'_p$	12	1×3	(D_4,ϵ)
$64'_p$	13	1×1	$(E_6,64'_p)$
$30'_p$	15	2×1	$(E_6,30'_p)$
$15'_q$	16	1×1	(A_5A_1,ϵ)
$20'_p$	20	1×1	$(E_6,20'_p)$
$6'_p$	25	1×1	$(E_6,6'_p)$
$1'_p$	36	1×1	(E_6,ϵ)

Type E_7

ρ_C	$\mathbf{b}_C$	$a\times a'$	(J,E_1)
1_a	0	1×2	$(\emptyset,1)$
$7'_a$	1	1×2	$(E_6,6_p)$
27_a	2	1×2	$(E_6,20_p)$
$56'_a$	3	2×2	$(E_6,30_p)$
$21'_b$	3	1×1	$(E_7,21'_b)$
120_a	4	2×1	$(E_7,120_a)$
35_b	4	1×2	$(A_7,14)$
$189'_b$	5	2×2	$(A_1D_4A_1,\epsilon\boxtimes 8\boxtimes\epsilon)$
105_b	6	1×1	$(E_7,105_b)$
210_a	6	1×2	$(A_7,35)$
168_a	6	1×2	$(A_7,56)$
$315'_a$	7	6×2	$(E_6,80_s)$
$189'_c$	7	1×1	$(E_7,189'_c)$

405_a	8	2×1	$(E_7, 405_a)$
280_b	8	1×2	$(A_7, 56)$
$70'_a$	9	1×2	$(A_2A_2A_2, \epsilon)$
$216'_a$	9	1×1	$(D_6A_1, 30\boxtimes\epsilon)$
$378'_a$	9	1×2	$(A_7, 70)$
420_a	10	2×1	$(E_7, 420_a)$
210_b	10	1×1	$(E_7, 210_b)$
$512'_a$	11	2×1	$(E_7, 512'_a)$
105_c	12	1×2	(D_4, ϵ)
84_a	12	1×2	$(A_7, 14)$
$420'_a$	13	2×1	$(D_6, 24)$
210_b	13	1×2	$(A_3A_3A_1, \epsilon)$
$378'_a$	14	2×1	$(D_6A_1, 24\boxtimes\epsilon)$
$105'_c$	15	1×1	$(A_5A_2, \epsilon\boxtimes 1)$
$405'_a$	15	2×2	$(E_6, 30'_p)$
216_a	16	1×2	$(A_7, 20)$
315_a	16	6×1	$(E_7, 315_a)$
$280'_b$	17	1×1	$(D_6A_1, 15\boxtimes\epsilon)$
70_a	18	1×1	(A_5A_2, ϵ)
189_c	20	1×2	$(E_6, 20'_p)$
$210'_a$	21	1×1	$(E_7, 210'_a)$
$168'_a$	21	1×1	$(E_7, 168'_a)$
$105'_b$	21	1×2	$(A_7, 7)$
189_b	22	1×1	$(E_7, 189_b)$
$120'_a$	25	2×1	$(E_7, 120'_a)$
15_a	28	1×2	(A_7, ϵ)
56_a	30	2×1	$(E_7, 56_a)$
$35'_b$	31	1×1	(D_6A_1, ϵ)
21_b	36	1×2	(E_6, ϵ)
$27'_a$	37	1×1	$(E_7, 27'_a)$
7_a	46	1×1	$(E_7, 7_a)$
$1'_a$	63	1×1	(E_7, ϵ)

Type E_8

ρ_C	$\mathbf{b}_C$	$a \times a'$	(J, E_1)
1_x	0	1	$(\emptyset, 1)$
8_z	1	1	$(E_8, 8_z)$
35_x	2	1	$(E_8, 35_x)$
112_z	3	2	$(E_8, 112_z)$
84_x	4	1	$(E_7A_1, 21'_b \boxtimes \epsilon)$
210_x	4	2	$(E_8, 210_x)$
560_z	5	2	$(E_7A_1, 120_a \boxtimes \epsilon)$
567_x	6	1	$(E_8, 567_x)$
700_x	6	2	$(E_8, 700_x)$
400_x	7	1	$(A_2A_1A_1A_1A_1, \epsilon)$
1400_z	7	6	$(E_6, 80_s)$
1400_x	8	6	$(E_8, 1400_x)$
1344_x	8	1	$(E_7A_1, 189'_c \boxtimes \epsilon)$
448_z	9	1	$(A_2A_2A_2, \epsilon)$
3240_z	9	2	$(E_7A_1, 405_a \boxtimes \epsilon)$
2240_x	10	6	$(E_6A_2, 80_s \boxtimes \epsilon)$
2268_x	10	2	$(E_8, 2268_x)$
4096_x	11	2	$(E_7, 512'_a)$
1400_z	11	1	$(E_7A_1, 210_b \boxtimes \epsilon)$
525_x	12	1	(D_4, ϵ)
4200_x	12	2	$(E_8, 4200_x)$
972_x	12	1	(A_3A_3, ϵ)
2800_z	13	2	$(E_8, 2800_z)$
4536_z	13	2	$(D_8, 560)$
6075_x	14	2	$(D_8, 280)$
2835_x	14	1	$(A_4A_2A_1, \epsilon)$
4200_z	15	1	(A_5, ϵ)
5600_z	15	2	$(E_6, 30'_p)$
4480_y	16	120	$(E_8, 4480_y)$
3200_x	16	1	(A_5A_1, ϵ)
7168_w	17	6	$(E_7A_1, 315_a \boxtimes \epsilon)$
4200_y	18	2	$(D_8, 252)$

3150_y	18	2	$(E_6A_2, 30'_p \boxtimes \epsilon)$
2016_w	19	1	$(A_5A_2A_1, \epsilon)$
1344_w	19	1	$(D_5A_3, 5 \boxtimes \epsilon)$
2100_y	20	1	(D_5, ϵ)
420_y	20	1	(A_4A_4, ϵ)
$5600'_z$	21	2	$(E_8, 5600'_z)$
$4200'_z$	21	2	$(D_8, 224)$
$3200'_x$	22	1	$(E_7A_1, 168_a \boxtimes \epsilon)$
$6075'_x$	22	1	$(E_8, 6075'_x)$
$2835'_x$	22	1	(A_6A_1, ϵ)
$4536'_z$	23	1	(D_5A_2, ϵ)
$4200'_x$	24	2	$(E_8, 4200'_x)$
$2800'_z$	25	2	$(E_7, 120'_a)$
$4096'_x$	26	2	$(E_8, 4096'_x)$
$840'_x$	26	1	(D_5A_3, ϵ)
$700'_x$	28	1	(A_7, ϵ)
$2240'_x$	28	2	$(E_8, 2240'_x)$
$1400'_z$	29	1	(A_7A_1, ϵ)
$2268'_x$	30	2	$(E_7, 56_a)$
$3240'_z$	31	2	$(E_7A_1, 56_a \boxtimes \epsilon)$
$1400'_x$	32	6	$(E_8, 1400'_x)$
$1050'_x$	34	1	$(D_8, 28)$
$525'_x$	36	1	$(E_6, \epsilon))$
$175'_x$	36	1	(A_8, ϵ)
$1400'_z$	37	6	$(E_8, 1400'_z)$
$1344'_x$	38	1	$(E_7A_1, 27'_a \boxtimes \epsilon)$
$448'_z$	39	1	(E_6A_2, ϵ)
$700'_x$	42	2	$(E_8, 700'_x)$
$400'_z$	43	1	$(D_8, 8)$
$567'_x$	46	1	$(E_7, 7_a)$
$560'_z$	47	1	$(E_7A_1, 7_a \boxtimes \epsilon)$
$210'_x$	52	2	$(E_8, 210'_x)$
$50'_x$	56	1	(D_8, ϵ)

$112'_z$	63	2	$(E_8, 112'_z)$
$84'_x$	64	1	(E_7A_1, ϵ)
$35'_x$	74	1	$(E_8, 35'_x)$
$8'_z$	91	1	$(E_8, 8'_z)$
$1'_x$	120	1	(E_8, ϵ)

Index

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