CHARACTERISTIC VARIETIES FOR PARABOLIC CHARACTER SHEAVES

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Abstract.

1.1. In this paper all algebraic varieties are assumed to be over a fixed algebraically closed field of characteristic 0.

For any morphism $\alpha : X \longrightarrow Y$ and $x \in X$, we write $\alpha^* : T^*_{\alpha(x)}Y \longrightarrow T^*_x X$ to be the map induced by α. If moreover, X, Y are smooth irreducible varieties, $\alpha : X \longrightarrow Y$ is a locally trivial fibration with smooth connected fibres and Λ is a closed Lagrangian subvariety of the cotangent bundle T^*Y, then let $\alpha^\bigstar(\Lambda)$ be the subset of T^*X defined by the condition that, $\alpha^\bigstar(\Lambda) \cap T^*_x X = \alpha^* \Lambda \cap T^*_{\alpha(x)}Y$ for any $x \in X$. Then

(a) *$\alpha^\bigstar(\Lambda)$ is a closed Lagrangian subvariety of T^*X. Moreover, the set of irreducible components of Λ is naturally in bijection with the set of irreducible components of $\alpha^\bigstar(\Lambda)$.*

Let X, Y be smooth irreducible varieties and let $\alpha : X \longrightarrow Y$ be a principal P-bundle for a free action of a connected linear algebraic group P on X.

(b) *If Λ' is a closed Lagrangian subvariety of T^*X stable under the P-action then $\Lambda' = \alpha^\bigstar(\Lambda)$ for a unique closed Lagrangian subvariety Λ of T^*Y.*

Let X be a smooth irreducible variety and let $i : Y \longrightarrow X$ be the inclusion of a locally closed smooth irreducible subvariety. Let Λ be a closed Lagrangian subvariety of T^*Y. Let $i_\bigstar(\Lambda)$ be the subset of T^*X consisting of all $\xi \in T^*_x X$ such that $x \in Y$ and the image of ξ under the obvious surjective map $T^*_x X \longrightarrow T^*_x Y$ belongs to $\Lambda \cap T^*_x Y$. Note that

(c) *$i_\bigstar(\Lambda)$ is a locally closed Lagrangian subvariety of T^*X. Moreover, the set of irreducible components of $i_\bigstar(\Lambda)$ is naturally in bijection with the set of irreducible components of Λ.*

For an algebraic variety X we write $\mathcal{D}(X)$ for the bounded derived category of constructible $\bar{\mathbf{Q}}_l$-sheaves on X where l is a fixed prime number. For X smooth

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and $C \in \mathcal{D}(X)$, we denote by $SS(X)$ the characteristic variety of C (a closed Lagrangian subvariety of T^*X). Let A be a connected linear algebraic group acting on a smooth variety X and let B be a connected subgroup of A. Let $\mu_A : T^*X \longrightarrow \mathrm{Lie}(A)^*$ be the moment map of the A-action on X. Consider the diagram $X \xleftarrow{pr_1} A \times X \xrightarrow{pr_2} A \times_B X \xrightarrow{\pi} X$ where B acts on $A \times X$ by $b : (a, x) \mapsto (ab^{-1}, bx)$, $A \times_B X$ is the quotient space and $\pi(a, x) = ax$. Then for any B-equivariant perverse sheaf C on X there is a well defined perverse sheaf C' on $A \times_B X$ such that $pr_2^* C' = pr_1^* C$ up to a shift. We set $\Gamma_B^A(C) = \pi_* C' \in \mathcal{D}(X)$. By [MV, 1.2] we have

(d) $SS(\Gamma_B^A(C)) \subset \overline{A \cdot SS(C)}$.

On the other hand, we have

(e) $SS(C) \subset \mu_B^{-1}(0)$.

Indeed, if $p_1 : B \times X \longrightarrow X$ is the action and $p_2 : B \times X \longrightarrow X$ is the second projection we have $p_1^*(C) = p_2^*(C)$. Hence $SS(p_1^*(C)) = SS(p_2^*(C))$. Using [KS, 4.1.2] we can rewrite this as $p_1^\bigstar(SS(C)) = p_2^\bigstar(SS(C))$. Hence if $x \in X$ and $\xi \in T_x^*X \cap SS(C)$ then the image of ξ under the map $T_x^*X \longrightarrow T_1^*(B)$ induced by $B \longrightarrow X, b \mapsto bx$ is 0. This proves (e).

1.2. Let G be a connected reductive algebraic group. Let $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\mathcal{N}$ be the variety of nilpotent elements in $\mathfrak{g}^*$. Let B be a Borel subgroup of G. Let K be a closed connected subgroup of G. Then $B_K = B \cap K$ is a parabolic subgroup of K. Assume that G acts on a smooth variety X. Let C be a B_K-equivariant perverse sheaf on X; assume also that there exists a finite covering $a : \tilde{B} \longrightarrow B$ such that C is $\tilde{B}$-equivariant for the $\tilde{B}$-action $\tilde{b} : x \mapsto a(\tilde{b})x$ on X. By 1.1(e) we have $\mu_{\tilde{B}}(SS(C)) = 0$. Since $\mathrm{Lie}(\tilde{B}) = \mathrm{Lie}(B)$ we then have $\mu_B(SS(C)) = 0$. It follows that $\mu_G(SS(C))$ is contained in the kernel of the obvious map $\mathfrak{g}^* \longrightarrow \mathrm{Lie}(B)^*$ hence is contained in $\mathcal{N}$. Since $\mathcal{N}$ is stable under the coadjoint action we have $\mu_G(K \cdot SS(C)) = K\mu_G(SS(C)) \subset \mathcal{N}$. Using this together with 1.1(d) and the fact that $\mu_G^{-1}(\mathcal{N})$ is closed in T^*X we see that $SS(\Gamma_{B_K}^K(C)) \subset \mu_G^{-1}(\mathcal{N})$. Applying 1.1(e) to $\Gamma_{B_K}^K(C), K$ instead of C, B we see that $SS(\Gamma_{B_K}^K(C)) \subset \mu_K^{-1}(0) = \mu_G^{-1}(\mathrm{Lie}(K)^\perp)$ where $\mathrm{Lie}(K)^\perp \subset \mathfrak{g}^*$ is the annihilator of $\mathrm{Lie}(K) \subset \mathfrak{g}$. Thus we have

(a) $$SS(\Gamma_{B_K}^K(C)) \subset \mu_G^{-1}(\mathrm{Lie}(K)^\perp \cap \mathcal{N}).$$

1.3. We now replace $G, \mathfrak{g}, B, K, B_K, X, C$ by $G \times G, \mathfrak{g} \times \mathfrak{g}, B \times B, G_\Delta, B_\Delta, X', C'$ where $G_\Delta = \{(g, g') \in G \times G; g = g'\}, B_\Delta = \{(g, g') \in B \times B; g = g'\}$, X' is a smooth variety with a given action of $G \times G$ and C' is a B_Δ-equivariant perverse sheaf on X'; we assume that there exists a finite covering $a' : \tilde{B}' \longrightarrow B \times B$ such that C' is $\tilde{B}'$-equivariant for the $\tilde{B}'$-action $\tilde{b}' : x' \mapsto a'(\tilde{b}')x'$ on X'. We have the following special case of 1.2(a):

(a) $$SS(\Gamma_{B_\Delta}^{G_\Delta}(C')) \subset \mu_{G \times G}^{-1}(\mathcal{N}^-)$$

where

$$\mathcal{N}^- = \{(f, f') \in \mathfrak{g}^* \times \mathfrak{g}^*; f + f' = 0, f, f' \text{ nilpotent }\}.$$

1.4. Let $\mathbf{W}$ be the Weyl group of G and let $\mathbf{I}$ be the set of simple reflections in $\mathbf{W}$. Let G' be a possibly disconnected algebraic group with identity component G and with a given connected component D. Now $G \times G$ acts transitively on D by $(g_1, g_2) : g \mapsto g_1 g g_2^{-1}$. Hence the moment map $\mu_{G\times G} : T^*D \longrightarrow \mathfrak{g}^* \times \mathfrak{g}^*$ is well defined. In [L1, 4.5] a class of perverse sheaves (called character sheaves) on D is introduced. These appear as constituents of some perverse cohomology sheaf of $\Gamma_{B_\Delta}^{G_\Delta}(C')$ for some C' as in 1.3 (with $X' = D$). Hence from 1.3(a) we deduce:

(a) *If K is a parabolic character sheaf on D then $SS(K) \subset \mu_{G\times G}^{-1}(\mathcal{N}^-)$.*

In the case where $G' = G = D$ a statement close to (a) appears in [MV, 2.8] (where it is attributed to the second author) and in [Gi].

1.5. We preserve the setup of 1.4. For any $J \subset \mathbf{I}$ let $\mathcal{P}_J$ be the set of parabolic subgroups of G of type J. In particular $\mathcal{P}_\emptyset$ is the set of Borel subgroups of G. For $J \subset \mathbf{I}$ let $\mathbf{W}_J$ be the subgroup of $\mathbf{W}$ generated by J; let $\mathbf{W}^J$ (resp. ${}^J\mathbf{W}$) be the set of all $w \in \mathbf{W}$ such that w has minimal length among the elements in $\mathbf{W}_J w$ (resp. $w\mathbf{W}_J$). Let $\delta : \mathbf{W} \xrightarrow{\sim} \mathbf{W}$ be the isomorphism such that $\delta(\mathbf{I}) = \mathbf{I}$ and such that $J \subset \mathbf{I}, P \in \mathcal{P}_J, g \in D \implies gPg^{-1} \in \mathcal{P}_{\delta(J)}$. Following [L2, 8.18], for $J, J' \subset \mathbf{I}$ and $y \in {}^{J'}\mathbf{W} \cap \mathbf{W}^J$ such that $\mathrm{Ad}(y)(\delta(J)) = J'$ we set

$$Z_{J,y,\delta} = \{(P, P', gU_P); P \in \mathcal{P}_J, P' \in \mathcal{P}_{J'}, g \in D, \mathrm{pos}(P', gPg^{-1}) = y\}.$$

Now $G \times G$ acts (transitively) on $Z_{J,y,\delta}$ by

$$(g_1, g_2) : (P, P', gU_P) \mapsto (g_2 P g_2^{-1}, g_1 P' g_1^{-1}, g_1 g g_2^{-1}).$$

Hence the moment map $\mu_{G\times G} : T^* Z_{J,y,\delta} \longrightarrow \mathfrak{g}^* \times \mathfrak{g}^*$ is well defined. In [L2, Section 11] a class of perverse sheaves (called parabolic character sheaves) on $Z_{J,y,\delta}$ is introduced. These appear as constituents of some perverse cohomology sheaf of $\Gamma_{B_\Delta}^{G_\Delta}(C')$ for some C' as in 1.3 (with $X' = Z_{J,y,\delta}$). Hence from 1.3(a) we deduce:

(a) *If K is a parabolic character sheaf on $Z_{J,y,\delta}$ then $SS(K) \subset \mu_{G\times G}^{-1}(\mathcal{N}^-)$.*

When $J = \mathbf{I}$, this reduces to 1.4(a).

1.6. Assume that G is adjoint. Let $\bar{G}$ be the De Concini-Procesi compactification of G. Then $G \times G$ acts naturally on $\bar{G}$ extending continuously the action $(g_1, g_2) : g \mapsto g_1 g g_2^{-1}$ of $G \times G$ on G. Hence the moment map $\mu_{G\times G} : T^*\bar{G} \longrightarrow \mathfrak{g}^* \times \mathfrak{g}^*$ is well defined. In [L2] a class of perverse sheaves (called parabolic character sheaves) on $\bar{G}$ is introduced. It has been shown by He [H2] and by Springer (unpublished) that any parabolic character sheaf on $\bar{G}$ appears as a constituent of some perverse cohomology sheaf of $\Gamma_{B_\Delta}^{G_\Delta}(C')$ for some C' as in 1.3 (with $X' = \bar{G}$). Hence from 1.3(a) we deduce:

(a) *If K is a parabolic character sheaf on $\bar{G}$ then $SS(K) \subset \mu_{G\times G}^{-1}(\mathcal{N}^-)$.*

1.7. In the setup of 1.4 let $\Lambda(D) = \mu_{G\times G}^{-1}(\mathcal{N}^-)$. We want to describe the variety $\Lambda(D)$. For $g \in D$ let I_g be the isotropy group at g of the $G \times G$-action on D that is, $I_g = \{(g_1, g_2) \in G \times G; g_2 = g^{-1} g_1 g\}$. We have $\text{Lie}(I_g) = \{(y_1, y_2) \in \mathfrak{g} \times \mathfrak{g}; y_2 = \text{Ad}(g)^{-1}(y_1)\}$ and the annihilator of $\text{Lie}(I_g)$ in $\mathfrak{g}^* \times \mathfrak{g}^*$ is $\text{Lie}(I_g)^\perp = \{(z_1, z_2) \in \mathfrak{g}^* \times \mathfrak{g}^*; z_1 + \text{Ad}(g)(z_2) = 0\}$. This may be identified with the fibre of T^*D at g. Then

$$\begin{aligned}\Lambda(D) &= \{(g, z_1, z_2) \in D \times \mathfrak{g}^* \times \mathfrak{g}^*; z_1 + \text{Ad}(g)(z_2) = 0, z_1 + z_2 = 0, z_2 \in \mathcal{N}\} \\ &= \{(g, z, -z); g \in D, z \in \mathcal{N}, \text{Ad}(g)(z) = z\} = \sqcup_{\mathcal{O}} X_{\mathcal{O}}\end{aligned}$$

where $\mathcal{O}$ runs over the (finite) set of $\text{Ad}(G)$-orbits on $\mathcal{N}$ which are normalized by some element of D and $X_{\mathcal{O}} = \{(g, z, -z); g \in D, z \in \mathcal{O}, \text{Ad}(g)(z) = z\}$. We pick $\xi \in \mathcal{O}$ and let $\mathcal{Z}' = \{h \in G'; \text{Ad}(h)\xi = \xi\}$, $\mathcal{Z} = \{h \in G; \text{Ad}(h)\xi = \xi\}$. Let $\underline{\mathcal{Z}}'$ (resp. $\underline{\mathcal{Z}}$) be the group of connected components of $\mathcal{Z}'$ (resp. $\mathcal{Z}$). Then $\underline{\mathcal{Z}}'$ is a finite group and $\underline{\mathcal{Z}}$ is a subgroup of $\underline{\mathcal{Z}}'$. Let $\underline{\mathcal{Z}}_1$ be the set of connected components of $\mathcal{Z}'$ that are contained in D. Then $\underline{\mathcal{Z}}_1$ is a subset of $\mathcal{Z}'$; also, $\underline{\mathcal{Z}}$ acts on $\underline{\mathcal{Z}}_1$ by conjugation inside $\underline{\mathcal{Z}}'$. Let $F_{\mathcal{O}}^D$ be the set of orbits of this action. Note that $F_{\mathcal{O}}^D$ is independent (up to unique isomorphism) of the choice of ξ.

Let $\tilde{X} = \{(g, r) \in D \times G; r^{-1} g r \in \mathcal{Z}'\}$. Then $\mathcal{Z}$ acts freely on $\tilde{X}$ by $h : (g, r) \mapsto (g, r h^{-1})$ and we have an isomorphism $\mathcal{Z} \backslash \tilde{X} \xrightarrow{\sim} X_{\mathcal{O}}$, $(g, r) \mapsto (g, \text{Ad}(r)\xi)$. By the change of variable $(g, r) \mapsto (g', r)$, $g' = r^{-1} g r$, $\tilde{X}$ becomes $\{(g', r); g' \in \mathcal{Z}' \cap D, r \in G\}$. In the new coordinates, the free action of $\mathcal{Z}$ on $\tilde{X}$ is $h : (g', r) \mapsto (h g' h^{-1}, r h^{-1})$. We see that $\tilde{X}$ is smooth of pure dimension $\dim(\mathcal{Z} \times G)$ and its connected components are indexed naturally by $\underline{\mathcal{Z}}_1$ (the connected component containing (g', r) is indexed by the image of g' in $\underline{\mathcal{Z}}_1$). The action of $\mathcal{Z}$ on $\tilde{X}$ permutes the connected components of $\tilde{X}$ according to the action of $\underline{\mathcal{Z}}$ on $\underline{\mathcal{Z}}_1$ considered above. We see that $X_{\mathcal{O}} = \mathcal{Z} \backslash \tilde{X}$ is smooth of pure dimension $\dim G$ and its connected components are indexed naturally by the set $F_{\mathcal{O}}^D$.

We see that $\Lambda(D)$ can be partitioned into finitely many locally closed, irreducible, smooth subvarieties of dimension $\dim G$, indexed by the finite set $F(D) := \sqcup_{\mathcal{O}} F_{\mathcal{O}}^D$. In particular, $\Lambda(D)$ has pure dimension $\dim G$. More precisely, one checks that

(a) $\Lambda(D)$ *is a closed Lagrangian subvariety of* T^*D.

1.8. In the setup of 1.5 we set $\Lambda(Z_{J,y,\delta}) = \mu_{G \times G}^{-1}(\mathcal{N}^-)$. We want to describe the variety $\Lambda(Z_{J,y,\delta})$.

Following [L2, 8.18] we consider the partition $Z_{J,y,\delta} = \sqcup_{\mathbf{s}} Z_{J,y,\delta}^{\mathbf{s}}$ where $Z_{J,y,\delta}^{\mathbf{s}}$ are certain locally closed smooth irreducible G_Δ-stable subvarieties of $Z_{J,Y,\delta}$ indexed by the elements $\mathbf{s}$ of a finite set $S(J, \text{Ad}(y)\delta)$ which is in canonical bijection with ${}^{J'}\mathbf{W}$, see [L1, 2.5]. Note that each $\mathbf{s}$ is a sequence $(J_n, J_n', u_n)_{n \geq 0}$ where J_n, J_n' are subsets of $\mathbf{I}$ such that J_n, J_n' are independent of n for large n and $u_n \in \mathbf{W}$ is 1 for large n.

We wish to define a Lagrangian subvariety $\Lambda(Z_{J,y,\delta}^{\mathbf{s}})$ of $T^*(Z_{J,y,\delta}^{\mathbf{s}})$.

Assume first that $\mathbf{s}$ is such that $J_n = J, J_n' = J', u_n = 1$ for all n. In this case we have $J = J'$. Let $P \in \mathcal{P}_J$ and let L be a Levi subgroup of P. Then

$\mathbf{d_s} = \{g \in D; gLg^{-1} = L, \mathrm{pos}(P, gPg^{-1}) = y\}$ is a connected component of the algebraic group $N_{G'}(L)$ with identity component L. Hence $\Lambda(\mathbf{d_s}) \subset T^*\mathbf{d_s}$ is defined as in 1.7. We have a diagram $Z^{\mathbf{s}}_{J,y,\delta} \xleftarrow{\alpha} G \times \mathbf{d_s} \xrightarrow{pr_2} \mathbf{d_s}$ where $\alpha(h, g) = (hPh^{-1}, hPh^{-1}, U_{hPh^{-1}}hgh^{-1}U_{hPh^{-1}})$. Note that α is a principal P-bundle where P acts on $G \times \mathbf{d_s}$ by $p : (h, g) = (hp^{-1}, \bar{p}g\bar{p}^{-1})$ (we denote the canonical homomorphism $P \longrightarrow L$ by $\bar{p}$). Let $\Lambda' = pr_2^\star \Lambda(\mathbf{d_s}) \subset T^*(G \times \mathbf{d_s})$. By 1.1(a) and 1.7(a), Λ' is a closed Lagrangian subvariety of $T^*(G \times \mathbf{d_s})$. It is clearly stable under the natural action of P on $T^*(G \times \mathbf{d_s})$ (since $\Lambda(\mathbf{d_s})$ is L-stable). By 1.1(b) there is a unique Lagrangian subvariety Λ'' of $T^*(Z^{\mathbf{s}}_{J,y,\delta})$ such that $\alpha^\bigstar \Lambda'' = \Lambda'$. We set $\Lambda(Z^{\mathbf{s}}_{J,y,\delta}) = \Lambda''$.

We now consider a general $\mathbf{s} = (J_n, J'_n, u_n)_{n\in\mathbf{N}}$. For any $r \in \mathbf{N}$ let $\mathbf{s}_r = (J_n, J'_n, u_n)_{n\geq r}$, $y_r = u_{r-1}^{-1} \dots u_1^{-1} u_0^{-1} y$. Then $Z^{\mathbf{s}_r}_{J_r,y_r,\delta}$ is defined and we have a canonical map $f_r : Z^{\mathbf{s}}_{J,y,\delta} \longrightarrow Z^{\mathbf{s}_r}_{J_r,y_r,\delta}$ (a composition of affine space bundles, see [L2, 8.20(a)]). Moreover for sufficiently large r, $\mathbf{s}_r, J_r, y_r, f_r$ are independent of r; we write $\mathbf{s}_\infty, J_\infty, y_\infty, f_\infty$ instead of $\mathbf{s}_r, J_r, y_r, f_r$. Note also that $\mathbf{s}_\infty, J_\infty, y_\infty$ are of the type considered earlier, so that $\Lambda(Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta})$ is defined as above. We set $\Lambda(Z^{\mathbf{s}}_{J,y,\delta}) = f_\infty^\bigstar(\Lambda(Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta}))$.

We now define

$$\Lambda'(Z_{J,y,\delta}) = \sqcup_{\mathbf{s}\in S(J,\mathrm{Ad}(y)\delta)} (i_{\mathbf{s}})_\bigstar(\Lambda(Z^{\mathbf{s}}_{J,y,\delta}))$$

where $i_{\mathbf{s}} : Z^{\mathbf{s}}_{J,y,\delta} \longrightarrow Z_{J,y,\delta}$ is the inclusion. From 1.1(c) we see that $\Lambda'(Z_{J,y,\delta})$ is a finite union of locally closed Lagrangian subvarieties of $T^*(Z_{J,y,\delta})$.

We state the following result:

Proposition 1.9. *We have $\Lambda(Z_{J,y,\delta}) = \Lambda'(Z_{J,y,\delta})$. In particular, $\Lambda'(Z_{J,y,\delta})$ is closed in $T^*(Z_{J,y,\delta})$ and $\Lambda(Z_{J,y,\delta})$ is a Lagrangian subvariety of $T^*(Z_{J,y,\delta})$.*

Let $x \in Z^{\mathbf{s}}_{J,y,\delta}$. We will show that $\Lambda(Z_{J,y,\delta}) \cap T^*_x(Z_{J,y,\delta}) = \Lambda'(Z_{J,y,\delta}) \cap T^*_x(Z_{J,y,\delta})$.

We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via a G-invariant symmetric bilinear form. Choose an element g_0 in D that normalizes B and a maximal torus T of B. Let P_J be the unique element in $\mathcal{P}_J$ that contains B. For $w \in W$, we choose a representative $\dot{w}$ of w in $N(T)$. Set $h_{J,y,\delta} = (P_J, {}^{\dot{y}^{-1}}P_{J'}, U_{{}^{\dot{y}^{-1}}P_{J'}} g_0 U_{P_J}) \in Z_{J,y,\delta}$. The element $\mathbf{s} \in S(J, \mathrm{Ad}(y)\delta)$ corresponds to an element $w' \in {}^{J'}W$ under the bijection in [L1, 2.5]. Set $w = (w')^{-1}y \in w \in W^{\delta(J)}$. By [H1, 1.10],

$$Z^{\mathbf{s}}_{J,y,\delta} = G_\Delta(P_{J_\infty}\dot{w}, 1) \cdot h_{J,y,\delta} = G_\Delta(L_{J_\infty}\dot{w}, 1) \cdot h_{J,y,\delta}.$$

Note that $\Lambda(Z_{J,y,\delta})$ and $\Lambda'(Z_{J,y,\delta})$ are stable under the action of G_Δ. Then we may assume that $x = (l_1\dot{w}, 1) \cdot h_{J,y,\delta}$ for some $l_1 \in L_{J_\infty}$. By [H1, 1.10(2)], $(P_{J_\infty})_\Delta \cdot (L_{J_\infty}, 1)x = (P_{J_\infty}, 1) \cdot x$, where $(P_{J_\infty})_\Delta = \{(g, g') \in P_{J_\infty} \times P_{J_\infty}; g = g'\}$. For $u \in U_{P_{J_\infty}}$, $(u, 1) \cdot x = (u'l, u') \cdot x$ for some $u' \in U_{P_{J_\infty}}$ and $l \in L_{J_\infty}$. Note that $(U_{P_{J_\infty}}, U_{P_{J_\infty}}) \cdot (l, 1) \cdot x = (U_{P_{J_\infty}}l, 1) \cdot x$ and $(P_{J_\infty}, 1) \cdot x \cong U_{P_{J_\infty}} \times (L_{J_\infty}, 1) \cdot x$. Thus $(l, 1) \cdot x = x$ and $(U_{P_{J_\infty}}, 1) \cdot x \subset (P_{J_\infty})_\Delta \cdot x$. Therefore, for $(f, -f) \in$

$\mu_{G\times G}(T^*_x(Z_{J,y,\delta}))$, we have that $(f,-f)(\mathrm{Lie}(U_{P_{J_\infty}}),0)=(f,-f)(\mathrm{Lie}(P_{J_\infty})_\Delta)=0$, i. e., $f\in\mathrm{Lie}(P_{J_\infty})$. In particular, f is nilpotent if and only if the image of f under $\mathrm{Lie}(P_{J_\infty})\to\mathrm{Lie}(P_{J_\infty})/\mathrm{Lie}(U_{P_{J_\infty}})\cong\mathrm{Lie}(L_{J_\infty})$ is nilpotent.

Hence $\mu_{G\times G}(\Lambda(Z_{J,y,\delta})\cap T^*_x(Z_{J,y,\delta}))$ consists elements of the form $(u+l,-u-l)$ with $u\in\mathrm{Lie}(U_{P_{J_\infty}})$, l nilpotent in $\mathrm{Lie}(L_{J_\infty})$ and $(u,-u)I_x=(l,-l)I_x=0$, where I_x is the Lie subalgebra of the isotropic subgroup of $G\times G$ at point x.

Denote by N_x the stalk at point x of the conormal bundle $N^*_{Z^{\mathbf{s}}_{J,y,\delta}}(Z_{J,y,\delta})$. Since $Z^{\mathbf{s}}_{J,y,\delta}=G_\Delta(P_{J_\infty},1)h_{J,y,\delta}$, we have

$$\mu_{G\times G}(N_x)=\{(u,-u);u\in\mathrm{Lie}(U_{P_{J_\infty}}),(u,-u)I_x=0\}.$$

Let $p_x:T^*_x(Z_{J,y,\delta})\to T^*_x(Z^{\mathbf{s}}_{J,y,\delta})\cong T^*_x(Z_{J,y,\delta})/N_x$ be the obvious surjective map. Then $\Lambda(Z_{J,y,\delta})\cap T^*_x(Z_{J,y,\delta})=p_x^{-1}\big(p_x(\Lambda(Z_{J,y,\delta})\cap T^*_x(Z_{J,y,\delta}))\big)$. Note that

$$I_x=\{(u_1+\mathrm{Ad}(l_1\dot{w}g_0)l,u_2+l);u_1\in\mathrm{Ad}(l_1\dot{w}\dot{y}^{-1})\mathrm{Lie}(U_{P_{J'}}),u_2\in\mathrm{Lie}(U_{P_J}),l\in\mathrm{Lie}(L_J)\}.$$

Thus for $l\in\mathrm{Lie}(L_{J_\infty})$, $(l,-l)I_x=0$ if and only if $\mathrm{Ad}(l_1\dot{w}g_0)l=l$.

We identify $T^*_x(Z_{J,y,\delta})$ with $\mu_{G\times G}\big(T^*_x(Z_{J,y,\delta})\big)\subset\mathfrak{g}\times\mathfrak{g}$ and regard $T^*_x(Z^{\mathbf{s}}_{J,y,\delta})$ as a subspace of $(\mathfrak{g}\times\mathfrak{g})/N_x$. Set $M=\{(l,-l);l\in\mathrm{Lie}(L_{J_\infty}),\mathrm{Ad}(l_1\dot{w}g_0)l=l\}\subset\mathfrak{g}\times\mathfrak{g}$. Then

(1) $p_x(\Lambda(Z_{J,y,\delta})\cap T^*_x(Z_{J,y,\delta}))=(M+N_x)/N_x$.

Now consider the commuting diagram

$$\begin{tikzcd}
(P_{J_\infty})_\Delta(L_{J_\infty}\dot{w},1)\cdot h_{J,y,\delta} \arrow[r, "i_1"] \arrow[d, "f'_\infty"'] & Z^{\mathbf{s}}_{J,y,\delta} \arrow[d, "f_\infty"'] \\
(P_{J_\infty})_\Delta(L_{J_\infty}\dot{w},1)\cdot h_{J_\infty,y_\infty,\delta} \arrow[r, "i_2"] & Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta},
\end{tikzcd}$$

where i_1,i_2 are inclusions and f'_∞ is the restriction of f_∞. Let $x'=f_\infty(x)\in Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta}$. Since f'_∞ is $P_{J_\infty}\times P_{J_\infty}$-invariant, we have the following commuting diagram

$$\begin{tikzcd}
T^*_{x'}\big((P_{J_\infty})_\Delta(L_{J_\infty}\dot{w},1)\cdot h_{J_\infty,y_\infty,\delta}\big) \arrow[r, "{}^1\mu_{P_{J_\infty}\times P_{J_\infty}}"] \arrow[d, "(f'_\infty)^*"'] & \mathrm{Lie}(P_{J_\infty})^*\times\mathrm{Lie}(P_{J_\infty})^* \arrow[d, "id"'] \\
T^*_x\big((P_{J_\infty})_\Delta(L_{J_\infty}\dot{w},1)\cdot h_{J,y,\delta}\big) \arrow[r, "{}^2\mu_{P_{J_\infty}\times P_{J_\infty}}"] & \mathrm{Lie}(P_{J_\infty})^*\times\mathrm{Lie}(P_{J_\infty})^*,
\end{tikzcd}$$

where ${}^1\mu_{P_{J_\infty}\times P_{J_\infty}}$ and ${}^2\mu_{P_{J_\infty}\times P_{J_\infty}}$ are the moment maps. Since the actions of $P_{J_\infty}\times P_{J_\infty}$ on $(P_{J_\infty})_\Delta(L_{J_\infty}\dot{w},1)\cdot h_{J,y,\delta}$ and $(P_{J_\infty})_\Delta(L_{J_\infty}\dot{w},1)\cdot h_{J_\infty,y_\infty,\delta}$ are transitive, ${}^1\mu_{P_{J_\infty}\times P_{J_\infty}}$ and ${}^2\mu_{P_{J_\infty}\times P_{J_\infty}}$ are injective.

Set $\Lambda_{x'}(Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta}) = \Lambda(Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta}) \cap T^*_{x'}(Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta})$. Then

$$^2\mu_{P_{J_\infty}\times P_{J_\infty}}\big(i_2^*(\Lambda_{x'}(Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta}))\big) = {}^1\mu_{P_{J_\infty}\times P_{J_\infty}}\big((f'_\infty)^* i_2^*(\Lambda_{x'}(Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta}))\big) = M.$$

Here we identify $\mathrm{Lie}(P_{J_\infty})^*$ with $\mathrm{Lie}(P^-_{J_\infty})$ via the symmetric bilinear form. Moreover, $i_1^* f_\infty^* = (f'_\infty)^* i_2^*$ maps $\Lambda_{x'}(Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta})$ bijectively onto its image. Therefore $^1\mu_{P_{J_\infty}\times P_{J_\infty}}\big(i_1^*(\Lambda(Z^{\mathbf{s}}_{J,y,\delta}) \cap T^*_x(Z^{\mathbf{s}}_{J,y,\delta}))\big) = M$ and i_1^* maps $\Lambda(Z^{\mathbf{s}}_{J,y,\delta}) \cap T^*_x(Z^{\mathbf{s}}_{J,y,\delta})$ bijectively onto its image. In other words,

(2) $\Lambda(Z^{\mathbf{s}}_{J,y,\delta}) \cap T^*_x(Z^{\mathbf{s}}_{J,y,\delta}) = (M+N_x)/N_x$.

Combining (1) and (2), $p_x(\Lambda(Z_{J,y,\delta}) \cap T^*_x(Z_{J,y,\delta})) = \Lambda(Z^{\mathbf{s}}_{J,y,\delta}) \cap T^*_x(Z^{\mathbf{s}}_{J,y,\delta})$. The proposition is proved.

Corollary 1.10. *The set of irreducible components of $\Lambda(Z_{J,y,\delta})$ is in natural bijection with $\sqcup_{\mathbf{s}\in S(J,\mathrm{Ad}(y)\delta)} F(\mathbf{d}_{\mathbf{s}_\infty})$ (notation of 1.7, 1.8).*

1.11. In the setup of 1.6 let $\Lambda(\bar{G}) = \mu^{-1}_{G\times G}(\mathcal{N}^-)$. We want to describe the variety $\Lambda(Z_{J,y,\delta})$. As in [L2, 12.3], we have $\bar{G} = \sqcup_{J\subset\mathbf{I}} G_J$ where G_J are the various $G\times G$-orbits in $\bar{G}$; moreover we may identify $G_J = T_J\backslash Z_{J,y_J,1}$ where y_J is the longest element in $\mathbf{W}^J$ and T_J is a torus acting freely on $Z_{J,y_J,1}$. Let $a_J : Z_{J,y_J,1} \longrightarrow G_J$ be the canonical map.

For each J let $\mu_{G\times G;J} : G_J \longrightarrow \mathfrak{g}^* \times \mathfrak{g}^*$ be the moment map of the restriction of the $G\times G$-action on $\bar{G}$ to G_J. Let $i_J : G_J \longrightarrow \bar{G}$ be the inclusion. From the definitions, we have

(a) $\Lambda(\bar{G}) = \sqcup_{J\subset\mathbf{I}} (i_J)_\bigstar \mu^{-1}_{G\times G;J}(\mathcal{N}^-)$

and $\Lambda(Z_{J,y_J,1}) = a_J^\bigstar(\mu^{-1}_{G\times G;J}(\mathcal{N}^-))$.

Since $\Lambda(Z_{J,y_J,1})$ is a T_J-stable Lagrangian subvariety of $T^*(Z_{J,y_J,1})$ (see 1.9), it follows that $\mu^{-1}_{G\times G;J}(\mathcal{N}^-)$ is a Lagrangian subvariety of $T^*(G_J)$. Hence, using 1.1(c), we see that $(i_J)_\bigstar \mu^{-1}_{G\times G;J}(\mathcal{N}^-)$ is a Lagrangian subvariety of $T^*\bar{G}$. Using this and (a) we see that

(b) $\Lambda(\bar{G})$ *is a Lagrangian subvariety of* $T^*\bar{G}$.

¿From the previous proof we see that

(c) *The set of irreducible components of $\Lambda(\bar{G})$ is in natural bijection with $\sqcup_{J\subset\mathbf{I}} \sqcup_{\mathbf{s}\in S(J,\mathrm{Ad}(y_J))} F(\mathbf{d}_{\mathbf{s}_\infty})$ (notation of 1.7, 1.8).*

1.12. Let $X = Z_{J,y,\delta}$ or $\bar{G}$. There is a well-defined map from the irreducible components of $\Lambda(X)$ to the nilpotent conjugacy classes of $\mathfrak{g}^*$ which sends the irreducible component C of $\Lambda(X)$ to the nilpotent conjugacy class $\mathcal{O}$, where $\mu_{G\times G}(C) \cap \{(f,-f) \in \mathcal{O}\times\mathcal{O}\}$ is dense in $\mu_{G\times G}(C)$.

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