GENERIC CHARACTER SHEAVES ON DISCONNECTED GROUPS AND CHARACTER VALUES

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Introduction

The theory of character sheaves [L3] on a reductive group G over an algebraically closed field and the theory of irreducible characters of G over a finite field are two parallel theories; the first one is geometric (involving intersection cohomology complexes on G), the second one involves functions on the group of rational points of G. In the case where G is connected, a bridge between the two theories was constructed in [L1] and strengthened in [L2], [S]. In this paper we begin the construction of the analogous bridge in the general case, extending the method of [L1]. Here we restrict ourselves to character sheaves which are "generic" (in particular their support is a full connected component of G) and show how such character sheaves are related to characters of representations (see Theorem 1.2). The same problem for certain non-generic character sheaves in the case where G is GL_n extended by an outer automorphism and the characteristic is odd is considered in [W].

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1. Statement of the Theorem

1.1. Let $\mathbf{k}$ be an algebraic closure of a finite field $\mathbf{F}_q$. Let G be a reductive algebraic group over $\mathbf{k}$ with identity component G^0 such that G/G^0 is cyclic, generated by a fixed connected component D. We assume that G has a fixed $\mathbf{F}_q$-rational structure with Frobenius map $F : G \to G$ such that $F(D) = D$. Let l be a prime number invertible in $\mathbf{k}$; let $\bar{\mathbf{Q}}_l$ be an algebraic closure of the l-adic numbers. All group representations are assumed to be finite dimensional over $\bar{\mathbf{Q}}_l$. We say "local system" instead of "$\bar{\mathbf{Q}}_l$-local system".

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Let $\mathcal{B}$ be the variety of Borel subgroups of G^0. Now $F: G \longrightarrow G$ induces a morphism $\mathcal{B} \longrightarrow \mathcal{B}$ denoted again by F. We fix $B^* \in \mathcal{B}$ and a maximal torus T of B^* such that $F(B^*) = B^*$, $F(T) = T$. Let U^* be the unipotent radical of B^*. Let NB^* (resp. NT) be the normalizer of B^* (resp. T) in G. Let $\tilde{T} = NT \cap NB^*$, a closed F-stable subgroup of G with identity component T. Let $\tilde{T}_D = \tilde{T} \cap D$.

Let $\mathcal{N} = NT \cap G^0$. Let $W = \mathcal{N}/T$ be the Weyl group. Let $\underline{D}: T \xrightarrow{\sim} T$, $\underline{D}: W \xrightarrow{\sim} W$ be the automorphisms induced by $\mathrm{Ad}(d): \mathcal{N} \longrightarrow \mathcal{N}$ where d is any element of $\tilde{T}_D$. Now $F: \mathcal{N} \longrightarrow \mathcal{N}$ induces an automorphism of W denoted again by F. For $w \in W$ let $[w]$ be the inverse image of w under the obvious map $\mathcal{N} \longrightarrow W$ and let $\underline{w}$ be the automorphism $\mathrm{Ad}(x): T \longrightarrow T$ for any $x \in [w]$. For $w \in W$ let $\mathcal{O}_w$ be the G^0-orbit in $\mathcal{B} \times \mathcal{B}$ (G^0 acting by simultaneous conjugation on both factors) that contains (B^*, xB^*x^{-1}) for some/any $x \in [w]$. Define the "length function" $l: W \longrightarrow \mathbf{N}$ by $l(w) = \dim \mathcal{O}_w - \dim \mathcal{B}$. For any $y \in G^0$ we define $k(y) \in \mathcal{N}$ by $y \in U^* k(y) U^*$. For $y \in G^0, \tau \in \tilde{T}$ we have $k(\tau y \tau^{-1}) = \tau k(y) \tau^{-1}$ and $F(k(y)) = k(F(y))$. For $x \in G^0$ we define $F_x: G \longrightarrow G$ by $F_x(g) = xF(g)x^{-1}$; this is the Frobenius map for an $\mathbf{F}_q$-rational structure on G. (Indeed if $y \in G^0$ is such that $x = y^{-1}F(y)$, then $\mathrm{Ad}(y): G \xrightarrow{\sim} G$ carries F_x to F.) If $w \in W$ satisfies $\underline{D}(w) = w$ and $x \in [w]$ then $T, \tilde{T}$ are F_x-stable; thus F_x is the Frobenius map for an $\mathbf{F}_q$-rational structure on $\tilde{T}$ whose group of rational points is $\tilde{T}^{F_x}$. Since $\tilde{T}_D^{F_x}$ is the set of rational points of $\tilde{T}_D$ (a homogeneous T-space under left translation) for the rational structure defined by $F_x: \tilde{T}_D \longrightarrow \tilde{T}_D$, we have $\tilde{T}_D^{F_x} \neq \emptyset$.

Let $Z_\emptyset = \{(B_0, g) \in \mathcal{B} \times D; gB_0g^{-1} = B_0\}$. Let $d \in \tilde{T}_D$. We set

$$\dot{Z}_{\emptyset,d} = \{(h_0U^*, g) \in (G^0/U^*) \times D; h_0^{-1} g h_0 d^{-1} \in B^*\}.$$

Define $a_\emptyset: \dot{Z}_{\emptyset,d} \longrightarrow Z_\emptyset$ by $(h_0U^*, g) \mapsto (h_0B^*h_0^{-1}, g)$. Now $a_\emptyset$ is a principal T-bundle where T acts (freely) on $\dot{Z}_{\emptyset,d}$ by $t_0: (h_0U^*, g) \mapsto (h_0t_0^{-1}, g)$. Define $p_\emptyset: Z_\emptyset \longrightarrow D$ by $(B_0, g) \mapsto g$. We define $b_\emptyset: \dot{Z}_{\emptyset,d} \longrightarrow T$ by $(h_0U^*, g) \mapsto k(h_0^{-1} g h_0 d^{-1})$. Note that $b_\emptyset$ commutes with the T-actions where T acts on T by

(a) $t_0: t \mapsto t_0 t \underline{D}(t_0^{-1})$.

Let $\mathcal{L}$ be a local system of rank 1 on T such that

(i) $\mathcal{L}^{\otimes n} \cong \bar{\mathbf{Q}}_l$ for some $n \geq 1$ invertible in $\mathbf{k}$;

(ii) $\underline{D}^* \mathcal{L} \cong \mathcal{L}$;

From (i),(ii) we see (using [L3, 28.2(a)]) that $\mathcal{L}$ is equivariant for the T-action (a) on T. Hence $b_\emptyset^* \mathcal{L}$ is a T-equivariant local system on $\dot{Z}_{\emptyset,d}$. Since $a_\emptyset$ is a principal T-bundle there is a well defined local system $\tilde{\mathcal{L}}_\emptyset$ on $Z_\emptyset$ such that $a_\emptyset^* \tilde{\mathcal{L}}_\emptyset = b_\emptyset^* \mathcal{L}$. Note that the isomorphism class of $\tilde{\mathcal{L}}_\emptyset$ is independent of the choice of d. Assume in addition that:

(iii) $\{w \in W; \underline{D}(w) = w, \underline{w}^* \mathcal{L} \cong \mathcal{L}\} = \{1\}$.

We show:

(b) $p_{\emptyset!} \tilde{\mathcal{L}}_\emptyset$ *is an irreducible intersection cohomology complex on* D.

We identify $Z_\emptyset$ with the variety $X = \{(g, xB^*) \in G \times G^0/B^*; x^{-1} g x \in NB^*\}$ (as in [L3, I, 5.4] with $P = B^*, L = T, S = \tilde{T}_D$) by $(g, xB^*) \leftrightarrow (xB^*x^{-1}, g)$. Then $\tilde{\mathcal{L}}_\emptyset$ becomes the local system $\bar{\mathcal{E}}$ on X defined as in [L3, I, 5.6] in terms of the local system $\mathcal{E} = j^*\mathcal{L}$ on $\tilde{T}_D$ where $j : \tilde{T}_D \longrightarrow T$ is $y \mapsto d^{-1}y$. (Note that $\mathcal{E}$ is equivariant for the conjugation action of T on $\tilde{T}_D$.) In our case we have $\bar{\mathcal{E}} = IC(X, \bar{\mathcal{E}})$ since X is smooth. Hence from [L3, I, 5.7] we see that $p_{\emptyset!}\bar{\mathcal{E}}$ is an intersection cohomology complex on D corresponding to a semisimple local system on an open dense subset of D which, by the results in [L3, II, 7.10], is irreducible if and only if the following condition is satisfied: if $w \in W, x \in [w]$ satisfy $\mathrm{Ad}(x)(\tilde{T}_D) = \tilde{T}_D$ and $\mathrm{Ad}(x)^*\mathcal{E} \cong \mathcal{E}$, then $w = 1$. This is clearly equivalent to condition (iii). This proves (b).

From (b) and the definitions we see that $p_{\emptyset!}\tilde{\mathcal{L}}_\emptyset[\dim D]$ is a character sheaf on D in the sense of [L3, VI]. A character sheaf on D of this form is said to be *generic*. We can state the following result.

Theorem 1.2. *Let A be a generic character sheaf on D such that $F^*A \cong A$ where $F : D \longrightarrow D$ is the restriction of $F : G \longrightarrow G$. Let $\psi : F^*A \longrightarrow A$ be an isomorphism. Define $\chi_\psi : D^F \longrightarrow \bar{\mathbf{Q}}_l$ by $g \mapsto \sum_{i\in\mathbf{Z}}(-1)^i\mathrm{tr}(\psi, \mathcal{H}^i_g(A))$ where $\mathcal{H}^i$ is the i-th cohomology sheaf and $\mathcal{H}^i_g$ is its stalk at g. There exists a G^F-module V and a scalar $\lambda \in \bar{\mathbf{Q}}_l^*$ such that $\chi_\psi(g) = \lambda\mathrm{tr}(g, V)$ for all $g \in D^F$.*

The proof is given in §3. We now make some preliminary observations. In the setup of 1.1 we have $A = p_{\emptyset!}\tilde{\mathcal{L}}_\emptyset[\dim D]$ where $\mathcal{L}$ satisfies 1.1(i),(ii),(iii) and $F^*(p_{\emptyset!}\tilde{\mathcal{L}}_\emptyset) \cong p_{\emptyset!}\tilde{\mathcal{L}}_\emptyset$. Hence we have $p_{\emptyset!}\widetilde{F^*\mathcal{L}}_\emptyset \cong p_{\emptyset!}\tilde{\mathcal{L}}_\emptyset$. By a computation in [L3, IV, 21.18] we deduce that there exists $w' \in W$ such that $\underline{D}(w') = w'$, $\underline{w}'^*F^*\mathcal{L} \cong \mathcal{L}$. Setting $w = F(w')$ we see that

(a) $\underline{D}(w) = w$, $F^*\underline{w}^*\mathcal{L} \cong \mathcal{L}$.

1.3. Let $\mathbf{w} = (w_1, w_2, \dots, w_r)$ be a sequence in W. Let $l_\mathbf{w} = l(w_1) + l(w_2) + \cdots + l(w_r)$. Let

$$Z_\mathbf{w} = \{(B_0, B_1, \dots, B_r, g) \in \mathcal{B}^{r+1} \times D; gB_0g^{-1} = B_r, (B_{i-1}, B_i) \in \mathcal{O}_{w_i}(i \in [1, r])\}.$$

This agrees with the definition in 1.1 when $r = 0$, that is $\mathbf{w} = \emptyset$. Let $d \in \tilde{T}_D$. We define $\dot{Z}_{\mathbf{w},d}$ as in 1.1 when $r = 0$ and by

$$\begin{aligned}\dot{Z}_{\mathbf{w},d} = \{&(h_0U^*, h_1B^*, \dots, h_{r-1}B^*, h_rU^*, g) \in \\ &(G^0/U^*) \times (G^0/B^*) \times \dots \times (G^0/B^*) \times (G^0/U^*) \times D; \\ &k(h_{i-1}^{-1}h_i) \in [w_i](i \in [1, r]), h_r^{-1}gh_0d^{-1} \in U^*\};\end{aligned}$$

when $r \geq 1$. Define $a_\mathbf{w} : \dot{Z}_{\mathbf{w},d} \longrightarrow Z_\mathbf{w}$ as in 1.1 when $r = 0$ and by

$$\begin{aligned}&(h_0U^*, h_1B^*, \dots, h_{r-1}B^*, h_rU^*, g) \mapsto \\ &(h_0B^*h_0^{-1}, h_1B^*h_1^{-1}, \dots, h_{r-1}B^*h_{r-1}, h_rB^*h_r^{-1}, g),\end{aligned}$$

when $r \geq 1$. Note that $a_{\mathbf{w}}$ is a principal T-bundle where T acts (freely) on $\dot{Z}_{\mathbf{w},d}$ as in 1.1 when $r = 0$ and by

$$t_0 : (h_0U^*, h_1B^*, \dots, h_{r-1}B^*, h_rU^*, g) \mapsto$$
$$(h_0t_0^{-1}U^*, h_1B^*, \dots, h_{r-1}B^*, h_rdt_0^{-1}d^{-1}U^*, g)$$

when $r \geq 1$. Define $p_{\mathbf{w}} : Z_{\mathbf{w}} \to D$ by $(B_0, B_1, \dots, B_r, g) \mapsto g$.

In the remainder of this subsection we assume that $w_1w_2 \dots w_r = 1$; this holds automatically when $r = 0$. We define $b_{\mathbf{w}} : \dot{Z}_{\mathbf{w},d} \to T$ as in 1.1 when $r = 0$ and by

$$(h_0U^*, h_1B^*, \dots, h_{r-1}B^*, h_rU^*, g) \mapsto k(h_0^{-1}h_1)k(h_1^{-1}h_2)\dots k(h_{r-1}^{-1}h_r)$$

when $r \geq 1$. Note that $b_{\mathbf{w}}$ commutes with the T-actions where T acts on T as in 1.1(a).

Let $\mathcal{L}$ be a local system of rank 1 on T such that 1.1(i),(ii) hold. As in 1.1, $\mathcal{L}$ is equivariant for the T-action 1.1(a) on T. Hence $b_{\mathbf{w}}^*\mathcal{L}$ is a T-equivariant local system on $\dot{Z}_{\mathbf{w},d}$. Since $a_{\mathbf{w}}$ is a principal T-bundle there is a well defined local system $\tilde{\mathcal{L}}_{\mathbf{w}}$ on $Z_{\mathbf{w}}$ such that $a_{\mathbf{w}}^*\tilde{\mathcal{L}}_{\mathbf{w}} = b_{\mathbf{w}}^*\mathcal{L}$.

Lemma 1.4. *Assume that $w_1w_2 \dots w_r = 1$ and that $\mathcal{L}$ (as in 1.3) satisfies*
(i) $\check{\alpha}^*\mathcal{L} \not\cong \bar{\mathbf{Q}}_l$ for any coroot $\check{\alpha} : \mathbf{k}^* \to T$.
Then $p_{\mathbf{w}!}\tilde{\mathcal{L}}_{\mathbf{w}}[l_{\mathbf{w}}](l_{\mathbf{w}}/2) \cong p_{\emptyset!}\tilde{\mathcal{L}}_{\emptyset}$. (Note that $l_{\mathbf{w}}$ is even.)

Assume first that for some $i \in [1, r]$ we have $w_i = w_i'w_i''$ where w_i', w_i'' in W satisfy $l(w_i'w_i'') = l(w_i') + l(w_i'')$. Let

$$\mathbf{w}' = (w_1, w_2, \dots, w_{i-1}, w_i', w_i'', w_{i+1}, \dots, w_n).$$

The map $(B_0, B_1, \dots, B_{r+1}, g) \mapsto (B_0, B_1, B_{i-1}, B_{i+1}, \dots, B_{r+1}, g)$ defines an isomorphism $Z_{\mathbf{w}'} \to Z_{\mathbf{w}}$ compatible with the maps $p_{\mathbf{w}'}, p_{\mathbf{w}}$ and with the local systems $\tilde{\mathcal{L}}_{\mathbf{w}'}, \tilde{\mathcal{L}}_{\mathbf{w}}$. Since $l_{\mathbf{w}'} = l_{\mathbf{w}}$ we have

(a) $p_{\mathbf{w}!}\tilde{\mathcal{L}}_{\mathbf{w}}[l_{\mathbf{w}}](l_{\mathbf{w}}/2) \cong p_{\mathbf{w}'!}\tilde{\mathcal{L}}_{\mathbf{w}'}[l_{\mathbf{w}'}](l_{\mathbf{w}'}/2)$.

Using (a) repeatedly we can assume that $l(w_i) = 1$ for all $i \in [1, r]$. We will prove the result in this case by induction on r. Note that r is even. When $r = 0$ the result is obvious. We now assume that $r \geq 2$. Since $w_1w_2 \dots w_r = 1$, we can find $j \in [1, r-1]$ such that $l(w_1w_2 \dots w_j) = j$, $l(w_1w_2 \dots w_{j+1}) = j - 1$. We can find a sequence $\mathbf{w}' = (w_1', w_2', \dots, w_r')$ in W such that $l(w_i') = 1$ for all $i \in [1, r]$, $w_1'w_2' \dots w_j' = w_1w_2 \dots w_j$, $w_j' = w_{j+1}'$, $w_i' = w_i$ for $i \in [j + 1, r]$. Let

$\mathbf{u} = (w_1w_2 \dots w_j, w_{j+1}, \dots, w_r) = (w_1'w_2' \dots w_j', w_{j+1}', \dots, w_r')$.

Using (a) repeatedly we see that

$$p_{\mathbf{w}!}\tilde{\mathcal{L}}_{\mathbf{w}}[l_{\mathbf{w}}](l_{\mathbf{w}}/2) \cong p_{\mathbf{u}!}\tilde{\mathcal{L}}_{\mathbf{u}}[l_{\mathbf{u}}](l_{\mathbf{u}}/2) \cong p_{\mathbf{w}'!}\tilde{\mathcal{L}}_{\mathbf{w}'}[l_{\mathbf{w}'}](l_{\mathbf{w}'}/2).$$

Replacing $\mathbf{w}$ by $\mathbf{w}'$ we see that we may assume in addition that $w_j = w_{j+1}$ for some $j \in [1, r-1]$. We have a partition $Z_{\mathbf{w}} = Z'_{\mathbf{w}} \cup Z''_{\mathbf{w}}$ where $Z'_{\mathbf{w}}$ (resp.

$Z''_{\mathbf{w}}$) is defined by the condition $B_{j-1} = B_{j+1}$ (resp. $B_{j-1} \neq B_{j+1}$). Let $\mathbf{w}' = (w_1, w, \dots, w_{j-1}, w_{j+2}, \dots, w_r)$, $\mathbf{w}'' = (w_1, w, \dots, w_{j-1}, w_{j+1}, \dots, w_r)$. Define $c : Z'_{\mathbf{w}} \to Z_{\mathbf{w}'}$ by

$$(B_0, B_1, \dots, B_r, g) \mapsto (B_0, B_1, \dots, B_{j-1}, B_{j+2}, \dots, B_r, g).$$

This is an affine line bundle and $\tilde{\mathcal{L}}_{\mathbf{w}}|_{Z'_{\mathbf{w}}} = c^* \tilde{\mathcal{L}}_{\mathbf{w}'}$. Let $p'_{\mathbf{w}}$ be the restriction of $p_{\mathbf{w}}$ to $Z'_{\mathbf{w}}$. We have $p'_{\mathbf{w}} = p_{\mathbf{w}'} c$. Since the induction hypothesis applies to $\mathbf{w}'$ we have

$$\begin{aligned} & p'_{\mathbf{w}!}(\tilde{\mathcal{L}}_{\mathbf{w}}|_{Z'_{\mathbf{w}}})[l_{\mathbf{w}}](l_{\mathbf{w}}/2) = p_{\mathbf{w}'!} c_! c^* \tilde{\mathcal{L}}_{\mathbf{w}'}[l_{\mathbf{w}}](l_{\mathbf{w}}/2) \\ \text{(b)} \qquad & = p_{\mathbf{w}'!} \tilde{\mathcal{L}}_{\mathbf{w}'}[-2](-1)[l_{\mathbf{w}}](l_{\mathbf{w}}/2) = p_{\mathbf{w}'!} \tilde{\mathcal{L}}_{\mathbf{w}'}[l_{\mathbf{w}'}](l_{\mathbf{w}'}/2) = p_{\emptyset!} \tilde{\mathcal{L}}_{\emptyset}. \end{aligned}$$

Define $e : Z''_{\mathbf{w}} \to Z_{\mathbf{w}''}$ by

$$(B_0, B_1, \dots, B_r, g) \mapsto (B_0, B_1, \dots, B_{j-1}, B_{j+1}, \dots, B_r, g).$$

Let $p''_{\mathbf{w}}$ be the restriction of $p_{\mathbf{w}}$ to $Z''_{\mathbf{w}}$. We have $p''_{\mathbf{w}} = p_{\mathbf{w}''} e$. We show that $p''_{\mathbf{w}!}(\tilde{\mathcal{L}}_{\mathbf{w}}|_{Z''_{\mathbf{w}}}) = 0$. It is enough to show that

$$\text{(c)} \qquad p_{\mathbf{w}''!} e_!(\tilde{\mathcal{L}}_{\mathbf{w}}|_{Z''_{\mathbf{w}}}) = 0.$$

Hence it is enough to show that $e_!(\tilde{\mathcal{L}}_{\mathbf{w}}|_{Z''_{\mathbf{w}}}) = 0$. It is also enough to show that, if E is a fibre of e, then $H^i_c(E, \tilde{\mathcal{L}}_{\mathbf{w}}|_E) = 0$ for any i. As in the proof of [L3, VI, 28.10] we may identify $E = \mathbf{k}^*$ in such a way that $\tilde{\mathcal{L}}_{\mathbf{w}}|_E$ becomes $\check{\alpha}^*(\mathcal{L})$ for some coroot $\check{\alpha} : \mathbf{k}^* \to T$. We then use that $H^i_c(\mathbf{k}^*, \check{\alpha}^* \mathcal{L}) = 0$ which follows from $\check{\alpha}^* \mathcal{L} \not\cong \bar{\mathbf{Q}}_l$.

Using (c) and the exact triangle

$$(p_{\mathbf{w}''!} e_!(\tilde{\mathcal{L}}_{\mathbf{w}}|_{Z''_{\mathbf{w}}}), p_{\mathbf{w}!} \tilde{\mathcal{L}}_{\mathbf{w}}, p'_{\mathbf{w}!}(\tilde{\mathcal{L}}_{\mathbf{w}}|_{Z'_{\mathbf{w}}}))$$

we see that

$$p_{\mathbf{w}!} \tilde{\mathcal{L}}_{\mathbf{w}}[l_{\mathbf{w}}](l_{\mathbf{w}}/2) = p'_{\mathbf{w}!}(\tilde{\mathcal{L}}_{\mathbf{w}}|_{Z'_{\mathbf{w}}})[l_{\mathbf{w}}])(l_{\mathbf{w}}/2) = p_{\emptyset!} \tilde{\mathcal{L}}_{\emptyset}$$

(the last equality follows from (b)). The lemma is proved.

Lemma 1.5. *Assume that $\mathcal{L}$ (as in 1.3) satisfies 1.1(iii). Then $\mathcal{L}$ satisfies 1.4(i).*

Let $R_{\mathcal{L}}$ be the set of roots $\alpha : T \to \mathbf{k}^*$ such that the corresponding coroot $\check{\alpha}$ satisfies $\check{\alpha}^* \mathcal{L} \cong \bar{\mathbf{Q}}_l$. Let $W_{\mathcal{L}}$ be the subgroup of W generated by the reflections with respect to the various $\alpha \in R_{\mathcal{L}}$. Since $\underline{D}^* \mathcal{L} \cong \mathcal{L}$ we have $\underline{D}(W_{\mathcal{L}}) = W_{\mathcal{L}}$. Assume that 1.4(i) does not hold. Then $R_{\mathcal{L}} \neq \emptyset$ and $W_{\mathcal{L}} \neq \{1\}$. By [DL, 5.17] the fixed point set of $\underline{D} : W_{\mathcal{L}} \to W_{\mathcal{L}}$ is $\neq \{1\}$. Let $w \in W_{\mathcal{L}} - \{1\}$ be such that $\underline{D}(d) w = w$. Since $w \in W_{\mathcal{L}}$ we have $\underline{w}^* \mathcal{L} \cong \mathcal{L}$ (see [L3, VI, 28.3(b)]). Thus 1.1(iii) does not hold. The lemma is proved.

2. Constructing representations of G^F

2.1. In this section we construct some representations of G^F using the method of [DL]. See [M],[DM] for other results in this direction.

Let $\mathcal{L}$ be a local system of rank 1 on T such that 1.1(i) holds. For any $t \in T$ let $\mathcal{L}_t$ be the stalk of $\mathcal{L}$ at t. Assume that we are given $w \in W$ and $x \in [w]$ such that

(i) $F_x^*\mathcal{L} \cong \mathcal{L}$;

($F_x : T \to T$ as in 1.1). Let $\phi : F_x^*\mathcal{L} \to \mathcal{L}$ be the unique isomorphism of local systems on T which induces the identity map on $\mathcal{L}_1$. For $t \in T$, ϕ induces an isomorphism $\mathcal{L}_{F_x(t)} \xrightarrow{\sim} \mathcal{L}_t$. When $t \in T^{F_x}$ this is an automorphism of the 1-dimensional vector space $\mathcal{L}_t$ given by multiplication by $\theta(t) \in \bar{\mathbf{Q}}_l^*$. It is well known that $t \mapsto \theta(t)$ is a group homomorphism $T^{F_x} \to \bar{\mathbf{Q}}_l^*$.

Following [DL] we define

$$Y = \{hU^* \in G^0/U^*; h^{-1}F(h) \in U^*xU^*\}.$$

For $(g,t) \in G^{0F} \times T^{F_x}$ we define $e_{g,t} : Y \to Y$ by $hU^* \mapsto ght^{-1}U^*$. Note that $(g,t) \mapsto e_{g,t}$ is an action of $G^{0F} \times T^{F_x}$ on Y. Hence $G^{0F} \times T^{F_x}$ acts on $H_c^i(Y) := H_c^i(Y, \bar{\mathbf{Q}}_l)$ by $(g,\tau) \mapsto e^*_{g^{-1},\tau^{-1}}$. We set

$$H_c^i(Y)_\theta = \{\xi \in H_c^i(Y); e^*_{1,t^{-1}}\xi = \theta(t)^{-1}\xi \text{ for all } t \in T^{F_x}\};$$

this is a $G^{0F} \times T^{F_x}$-stable subspace of $H_c^i(Y)$.

For $g \in G^{0F}$ we define $\epsilon_g : H_c^i(Y)_\theta \to H_c^i(Y)_\theta$ by $\epsilon_g(\xi) = e^*_{g^{-1},1}$. This makes $H_c^i(Y)_\theta$ into a G^{0F}-module.

We can find an integer $r \geq 1$ such that

$$F^r(x) = x, \quad xF(x)\dots F^{r-1}(x) = 1.$$

Indeed we first find an integer $r_1 \geq 1$ such that $F^{r_1}(x) = x$ and then we find an integer $r_2 \geq 1$ such that $(xF(x)\dots F^{r_1-1}(x))^{r_2} = 1$. Then $r = r_1r_2$ has the required properties. Then $hU^* \mapsto F^r(h)U^*$ is a well defined map $Y \to Y$ denoted again by F^r. Also,

$$F^r = F_x^r : G \to G.$$

(We have $F_x^r(g) = (xF(x)\dots F^{r-1}(x))F^r(g)(xF(x)\dots F^{r-1}(x))^{-1} = F^r(g)$.) Hence F^r acts trivially on T^{F_x}. We see that $F^r : Y \to Y$ commutes with $e_{g,t} : Y \to Y$ for any $(g,t) \in G^{0F} \times T^{F_x}$. Hence $(F^r)^* : H_c^i(Y) \to H_c^i(Y)$ leaves stable the subspace $H_c^i(Y)_\theta$. Note that:

for any i, all eigenvalues of $(F^r)^* : H_c^i(Y) \to H_c^i(Y)$ are of the form root of 1 times $q^{nr/2}$ where $n \in \mathbf{Z}$.

(See [L1, 6.1(e)] and the references there.)

Replacing r by an integer multiple we may therefore assume that r satisfies in addition the following condition:

(a) for any i, all eigenvalues of $(F^r)^* : H_c^i(Y) \to H_c^i(Y)$ are of the form $q^{nr/2}$ where $n \in \mathbf{Z}$.

2.2. We preserve the setup of 2.1 and assume in addition that $\mathcal{L}$ satisfies 1.4(i). Let $i_0 = 2\dim U^* - l(w)$. Note that

(a) *$H^i_c(Y)_\theta = 0$ for $i \ne i_0$; if $i = i_0$ then all eigenvalues of $(F^r)^* : H^i_c(Y)_\theta \longrightarrow H^i_c(Y)_\theta$ are of the form $q^{ir/2}$.*

For the first statement in (a) see [DL, 9.9] and the remarks in the proof of [L1, 8.15]. The second statement in (a) is deduced from 2.1(a) as in the proof of [L1, 6.6(c)].

2.3. We preserve the setup of 2.1 and assume in addition that $\mathcal{L}$ satisfies 1.1(ii) and that $w \in W$ satisfies $\underline{D}(w) = w$. From the definitions we see that $\underline{D} : T \longrightarrow T$ commutes with $F_x : T \longrightarrow T$ hence $\underline{D}$ restricts to an automorphism of T^{F_x} and that

(a) $\theta(\underline{D}(t)) = \theta(t)$ for any $t \in T^{F_x}$.

We show:

(b) *there exists a homomorphism $\tilde\theta : \tilde T^{F_x} \longrightarrow \bar{\mathbf{Q}}_l^*$ such that $\tilde\theta|_{T^{F_x}} = \theta$.*

Let $d \in \tilde T^{F_x}_D$. Let $n = |G/G^0| = |\tilde T^{F_x}/T^{F_x}|$. Then $t_0 := d^n \in T^{F_x}$. Let $c \in \bar{\mathbf{Q}}_l^*$ be such that $c^n = \theta(t_0)$. For any $t \in T^{F_x}$ and $j \in \mathbf{Z}$ we set $\tilde\theta(d^j t) = c^j\theta(t)$. This is well defined: if $d^j t = d^{j'} t'$ with $j, j' \in \mathbf{Z}$, $t, t' \in T^{F_x}$ then $j' = j + nj_0$, $j_0 \in \mathbf{Z}$ and $t' = t_0^{j_0} t$ so that $\theta(t') = c^{nj_0}\theta(t)$ and $c^j\theta(t) = c^{j'}\theta(t')$. We show that if $j, j' \in \mathbf{Z}$, $t, t' \in T^{F_x}$ then $\tilde\theta(d^j t d^{j'} t') = \tilde\theta(d^j t)\tilde\theta(d^{j'} t')$ that is $c^{j+j'}\theta(\underline{D}^{-j'}(t)t') = c^j\theta(t)c^{j'}\theta(t')$; this follows from (a). This proves (b).

Let $\Gamma = \{(g, \tau) \in G^F \times \tilde T^{F_x}; g\tau^{-1} \in G^0\}$, a subgroup of $G^F \times \tilde T^{F_x}$. For $(g, \tau) \in \Gamma$ we define $e_{g,\tau} : Y \longrightarrow Y$ by $hU^* \mapsto gh\tau^{-1}U^*$. To see that this is well defined we assume that $h \in G^0$ satisfies $h^{-1}F(h) \in U^*xU^*$ and $(g, \tau) \in \Gamma$; we compute

$$\begin{aligned}&(gh\tau^{-1})^{-1}F(gh\tau^{-1}) = \tau h^{-1}g^{-1}gF(h)F(\tau^{-1})\\ &= \tau h^{-1}F(h)F(\tau^{-1}) \in \tau U^*xU^*F(\tau^{-1}) = U^*\tau xF(\tau^{-1})U^* = U^*xU^*,\end{aligned}$$

since $\tau xF(\tau^{-1}) = x$ (that is $F_x(\tau) = \tau$). Note that $(g, \tau) \mapsto e_{g,\tau}$ is an action of Γ on Y (extending the action of $G^{0F} \times T^{F_x}$). Hence Γ acts on $H^i_c(Y)$ by $(g, \tau) \mapsto e^*_{g^{-1},\tau^{-1}}$. Note that $H^i_c(Y)_\theta$ is a Γ-stable subspace of $H^i_c(Y)$. This follows from the identity

$$e_{g^{-1},\tau^{-1}}e_{1,t^{-1}} = e_{1,\tau^{-1}t^{-1}\tau}e_{g^{-1},\tau^{-1}}$$

for $g \in G^F$, $\tau \in \tilde T^{F_x}$, $t \in T^{F_x}$ together with the identity $\theta(t) = \theta(\tau^{-1}t\tau)$ which is a consequence of (a).

For $g \in G^F$ we define $\epsilon_g : H^i_c(Y)_\theta \longrightarrow H^i_c(Y)_\theta$ by

$$\epsilon_g(\xi) = \tilde\theta(\tau)e^*_{g^{-1},\tau^{-1}}\xi$$

for any $\xi \in H^i_c(Y)_\theta$ and any $\tau \in \tilde T^{F_x}$ such that $g\tau^{-1} \in G^0$. Assume that $\tau' \in \tilde T^{F_x}$ is another element such that $g\tau'^{-1} \in G^0$. Then $\tau' = \tau t$ with $t \in T^{F_x}$ and

$$\tilde\theta(\tau')e^*_{g^{-1},\tau'^{-1}}\xi = \tilde\theta(\tau)\theta(t)e^*_{g^{-1},\tau^{-1}}e^*_{1,t^{-1}}\xi = \tilde\theta(\tau)e^*_{g^{-1},\tau^{-1}}\xi$$

so that ϵ_g is well defined. For g, g' in G^F we choose τ, τ' in $\tilde{T}^{F_x}$ such that $g\tau^{-1} \in G^0, g'\tau'^{-1} \in G^0$; we have

$$\epsilon_g \epsilon_{g'} \xi = \tilde{\theta}(\tau')\tilde{\theta}(\tau) e^*_{g^{-1},\tau^{-1}} e^*_{g'^{-1},\tau'^{-1}} \xi = \tilde{\theta}(\tau\tau') e^*_{(gg')^{-1},(\tau\tau')^{-1}} \xi = \epsilon_{gg'} \xi.$$

We see that

$g \mapsto \epsilon_g$ *defines a* G^F*-module structure on* $H^i_c(Y)_\theta$ *extending the* G^{0F}*-module structure in 2.1.*

(Note that this extension depends on the choice of $\tilde{\theta}$.) We show:

(c) *If* $(g, \tau) \in \Gamma$ *then* $F^r e_{g,\tau} : Y \longrightarrow Y$ *is the Frobenius map of an* $\mathbf{F}_q$*-rational structure on* Y*.*

Since $e_{g,t}$ is a part of a Γ-action, it has finite order. Since $F^r = F^r_x : G \longrightarrow G$ (see 2.1), we see that $F^r : Y \longrightarrow Y$ commutes with $e_{g,\tau} : Y \longrightarrow Y$. Hence (c) holds.

2.4. We preserve the setup of 2.3 and assume in addition that $\mathcal{L}$ satisfies 1.3(i). Let $i_0 = 2 \dim U^* - l(w)$. Using 2.2(a), 2.3(c) and Grothendieck's trace formula we see that for $(g, d) \in \Gamma$ we have

$$\begin{aligned}
&(-1)^{l(w)} \tilde{\theta}(d) q^{i_0 r/2} \mathrm{tr}(\epsilon_g, H^{i_0}_c(Y)_\theta) \\
&= \tilde{\theta}(d) \sum_i (-1)^i \mathrm{tr}((F^r)^* \epsilon_g, H^i_c(Y)_\theta) = \sum_i (-1)^i \mathrm{tr}((F^r)^* e^*_{g^{-1},d^{-1}}, H^i_c(Y)_\theta) \\
&= \sum_i (-1)^i |T^{F_x}|^{-1} \sum_{t \in T^{F_x}} \mathrm{tr}((F^r)^* e^*_{g^{-1},d^{-1}} e^*_{1,t^{-1}}, H^i_c(Y)) \theta(t) \\
&= |T^{F_x}|^{-1} \sum_{t \in T^{F_x}} \sum_i (-1)^i \mathrm{tr}((F^r)^* e^*_{g^{-1},(dt)^{-1}}, H^i_c(Y)) \theta(t) \\
&= |T^{F_x}|^{-1} \sum_{t \in T^{F_x}} |Y^{F^r e_{g^{-1},(dt)^{-1}}}| \theta(t) \\
&= |T^{F_x}|^{-1} \sum_{t \in T^{F_x}} |\{hU^* \in (G^0/U^*); h^{-1}F(h) \in U^*xU^*, h^{-1}g^{-1}F^r(h)dt \in U^*\}| \theta(t).
\end{aligned}$$

∎

3. Proof of Theorem 1.2

3.1. Let A, ψ, χ_ψ be as in 1.2. Let $\mathcal{L}, w$ be as in the end of 1.2. Let $x \in [w]$. From 1.2(a) we see that 2.1(i) holds. Let $r \geq 1$ be as in 2.1. Let

$$\mathbf{w} = (w, F(w), \dots, F^{r-1}(w)).$$

By the choice of r we have $wF(w) \dots F^{r-1}(w) = 1$. Define a morphism $\tilde{F} : Z_\mathbf{w} \longrightarrow Z_\mathbf{w}$ by

$$\tilde{F}(B_0, B_1, \dots, B_r, g) = (F(g^{-1}B_{r-1}g), F(B_0), F(B_1), \dots, F(B_{r-1}), F(g)).$$

We show:

(a) *Let* $g \in D^F$ *and let* $\tilde{F}_g : p_{\mathbf{w}}^{-1}(g) \longrightarrow p_{\mathbf{w}}^{-1}(g)$ *be the restriction of* $\tilde{F} : Z_{\mathbf{w}} \longrightarrow Z_{\mathbf{w}}$*. Then* $\tilde{F}_g$ *is the Frobenius map of an* $\mathbf{F}_q$*-rational structure on* $p_{\mathbf{w}}^{-1}(g)$*.*
It is enough to note that the map $\mathcal{B}^{r+1} \longrightarrow \mathcal{B}^{r+1}$ given by

$$(B_0, B_1, \dots, B_r) \mapsto (F(g^{-1}B_{r-1}g), F(B_0), F(B_1), \dots, F(B_{r-1}))$$

is the composition of the map

$$F' : (B_0, B_1, \dots, B_r) \mapsto (F(B_0), F(B_1), \dots, F(B_r))$$

(the Frobenius map of an $\mathbf{F}_q$-rational structure on $\mathcal{B}^{r+1}$) with the automorphism

$$(B_0, B_1, \dots, B_r) \mapsto (g^{-1}B_{r-1}g, B_0, B_1, \dots, B_{r-1})$$

of $\mathcal{B}^{r+1}$ which commutes with F' and has finite order (since g has finite order in G).

Let $d \in \tilde{T}_D^{F_x}$. Define a morphism $\tilde{F}' : \dot{Z}_{\mathbf{w},d} \longrightarrow \dot{Z}_{\mathbf{w},d}$ by

$$\tilde{F}'(h_0U^*, h_1B^*, \dots, h_{r-1}B^*, h_rU^*, g) = (h_0'U^*, h_1'B^*, \dots, h_{r-1}'B^*, h_r'U^*, F(g))$$

where

$$h_0' = F(g^{-1}h_{r-1}k(h_{r-1}^{-1}h_r))x^{-1}d, \quad h_r' = F(h_{r-1}k(h_{r-1}^{-1}h_r)x^{-1},$$

$$h_i' = F(h_{i-1}) \text{ for } i \in [1, r-1].$$

This is well defined since

$$(F(h_{r-1}k(h_{r-1}^{-1}h_r)x^{-1})^{-1}F(g)F(g^{-1}h_{r-1}k(h_{r-1}^{-1}h_r))x^{-1})dd^{-1} = 1.$$

We show that the T-action on $\dot{Z}_{\mathbf{w},d}$ (see 1.3) satisfies $\tilde{F}'(t_0\tilde{x}) = F_x(t_0)\tilde{F}'(\tilde{x})$ for $t_0 \in T, \tilde{x} \in \dot{Z}_{\mathbf{w},d}$. Let (h_i) be as above. We must show:

$$F(g^{-1}h_{r-1}k(h_{r-1}^{-1}h_rdt_0^{-1}d^{-1}))x^{-1}d = F(g^{-1}h_{r-1}k(h_{r-1}^{-1}h_r))x^{-1}dxF(t_0^{-1})x^{-1},$$

$$F(h_{r-1}k(h_{r-1}^{-1}h_rdt_0^{-1}d^{-1})x^{-1} = F(h_{r-1}k(h_{r-1}^{-1}h_r)x^{-1}dxF(t_0)^{-1}x^{-1}d^{-1},$$

which follow from $F(d) = x^{-1}dx$. Note that

(b) $a_{\mathbf{w}}\tilde{F}' = \tilde{F}a_{\mathbf{w}} : \dot{Z}_{\mathbf{w},d} \longrightarrow Z_{\mathbf{w}}$.

We show:

(c) $|a_{\mathbf{w}}^{-1}(y)^{\tilde{F}'}| = |T^{F_x}|$ *for any* $y \in Z_{\mathbf{w}}^{\tilde{F}}$*.*

Since $a_{\mathbf{w}}^{-1}(y)$ is a homogeneous T-space this follows from Lang's theorem applied to (T, F_x).

We have

(d) $p_{\mathbf{w}}\tilde{F} = Fp_{\mathbf{w}} : Z_{\mathbf{w}} \longrightarrow D$.

3.2. We show:

(a) $b_{\mathbf{w}}\tilde{F}' = F_x b_{\mathbf{w}} : \dot{Z}_{\mathbf{w},d} \to T$.

Let $(h_0, h_1, \dots, h_r, g) \in (G^0)^{r+1} \times D$ be such that

$$(h_0U^*, h_1B^*, \dots, h_{r-1}B^*, h_rU^*, g) \in \dot{Z}_{\mathbf{w},d}.$$

Let $(h'_1, h'_2, \dots, h'_r)$ be as in 3.1. We set

$$\mu = k(h_0^{-1}h_1)k(h_1^{-1}h_2)\dots k(h_{r-1}^{-1}h_r) \in T,$$

$$\mu' = k(h_0^{-1}h_1)k(h_1^{-1}h_2)\dots k(h_{r-2}^{-1}h_{r-1}) \in B^*F^{r-1}(x)^{-1}B^*$$

$$\tilde{\mu} = k({h'_0}^{-1}h'_1)k({h'_1}^{-1}h'_2)\dots k({h'_{r-1}}^{-1}h'_r) \in T$$

so that $\mu = \mu' k(h_{r-1}^{-1}h_r)$ and

$$\begin{aligned}
&\tilde{\mu} = k(d^{-1}xF(k(h_{r-1}^{-1}h_r)^{-1}h_{r-1}^{-1}gh_0))\\
&\times k(F(h_0^{-1}h_1))\dots k(F(h_{r-3}^{-1}h_{r-2}))k(F(h_{r-2}^{-1}h_{r-1}k(h_{r-1}^{-1}h_r))x^{-1})\\
&= d^{-1}xF(k(h_{r-1}^{-1}h_r)^{-1})F(d)k(F(d^{-1})F(h_{r-1}^{-1}gh_0))F(\mu')F(k(h_{r-1}^{-1}h_r))x^{-1}\\
&= d^{-1}xF(d)F(\mu)x^{-1} = xF(\mu)x^{-1} = F_x(\mu),
\end{aligned}$$

as required.

3.3. Let $\phi : F_x^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$, $\theta : T^{F_x} \to \bar{\mathbf{Q}}_l^*$ be as in 2.1. We shall denote by ? the various isomorphisms induced by ϕ such as:

(a) $\tilde{F}'^*b_{\mathbf{w}}^*\mathcal{L} = b_{\mathbf{w}}^*F_x^*\mathcal{L} \xrightarrow{\sim} b_{\mathbf{w}}^*\mathcal{L}$ (see 3.2(a)),

(b) $\tilde{F}'^*a_{\mathbf{w}}^*\tilde{\mathcal{L}}_{\mathbf{w}} \xrightarrow{\sim} a_{\mathbf{w}}^*\tilde{\mathcal{L}}_{\mathbf{w}}$ (coming from (a)),

(c) $a_{\mathbf{w}}^*\tilde{F}^*\tilde{\mathcal{L}}_{\mathbf{w}} \xrightarrow{\sim} a_{\mathbf{w}}^*\tilde{\mathcal{L}}_{\mathbf{w}}$ (see (b) and 3.1(b)),

(d) $\tilde{F}^*\tilde{\mathcal{L}}_{\mathbf{w}} \xrightarrow{\sim} \tilde{\mathcal{L}}_{\mathbf{w}}$ (coming from (c)),

(e) $p_{\mathbf{w}!}\tilde{F}^*\tilde{\mathcal{L}}_{\mathbf{w}} \xrightarrow{\sim} p_{\mathbf{w}!}\tilde{\mathcal{L}}_{\mathbf{w}}$ (coming from (d)),

(f) $F^*p_{\mathbf{w}!}\tilde{\mathcal{L}}_{\mathbf{w}} \xrightarrow{\sim} p_{\mathbf{w}!}\tilde{\mathcal{L}}_{\mathbf{w}}$ (coming from (e) and 3.1(d)).

(g) $F^*(p_{\mathbf{w}!}\tilde{\mathcal{L}}_{\mathbf{w}}[l_{\mathbf{w}}]) \xrightarrow{\sim} p_{\mathbf{w}!}\tilde{\mathcal{L}}_{\mathbf{w}}[l_{\mathbf{w}}]$ (coming from (f)).

3.4. For any $g \in D^F$ we compute

$$\sum_i(-1)^i\mathrm{tr}(?, \mathcal{H}_g^i(p_{\mathbf{w}!}\tilde{\mathcal{L}}_{\mathbf{w}})) = \sum_i(-1)^i\mathrm{tr}(?, H_c^i(p_{\mathbf{w}}^{-1}(g), \tilde{\mathcal{L}}_{\mathbf{w}}))$$
$$= \sum_{y\in p_{\mathbf{w}}^{-1}(g);\tilde{F}(y)=y}\mathrm{tr}(?, (\tilde{\mathcal{L}}_{\mathbf{w}})_y)$$

where $\mathcal{H}^i$ is the i-th cohomology sheaf. (The last two sums are equal by the Grothendieck trace formula applied in the context of 3.1(a).) Using 3.1(c) we see that the last sum equals

$$|T^{F_x}|^{-1}\sum_{\tilde{y}\in a_{\mathbf{w}}^{-1}(p_{\mathbf{w}}^{-1}(g))^{\tilde{F}'}}\mathrm{tr}(?, (a_{\mathbf{w}}^*\tilde{\mathcal{L}}_{\mathbf{w}})_{\tilde{y}}) = |T^{F_x}|^{-1}\sum_{\tilde{y}\in a_{\mathbf{w}}^{-1}(p_{\mathbf{w}}^{-1}(g))^{\tilde{F}'}}\mathrm{tr}(?, (b_{\mathbf{w}}^*\mathcal{L}_{\mathbf{w}})_{\tilde{y}})$$
$$= |T^{F_x}|^{-1}\sum_{\tilde{y}\in a_{\mathbf{w}}^{-1}(p_{\mathbf{w}}^{-1}(g))^{\tilde{F}'}}\mathrm{tr}(?, (\mathcal{L}_{\mathbf{w}})_{b_{\mathbf{w}}(\tilde{y})}).$$

Now $a_{\mathbf{w}}^{-1}(p_{\mathbf{w}}^{-1}(g))^{\tilde{F}'}$ can be identified with the set of all

$(h_0U^*, h_1B^*, \dots, h_{r-1}B^*, h_rU^*) \in (G^0/U^*) \times (G^0/B^*) \times \dots \times (G^0/B^*) \times (G^0/U^*)$

such that
(a) $k(h_{i-1}^{-1}h_i) \in F^{i-1}(x)T$ for $i \in [1, r]$,
(b) $h_r^{-1}gh_0d^{-1} \in U^*$,
(c) $h_0U^* = F(g^{-1}h_{r-1}k(h_{r-1}^{-1}h_r))x^{-1}dU^*$,
(d) $h_iB^* = F(h_{i-1})B^*$ for $i \in [1, r-1]$.
(We then have automatically $h_rU^* = F(h_{r-1}k(h_{r-1}^{-1}h_r)x^{-1}U^*.)$ If h_0U^* is given, then (d) determines successively $h_2B^*, \dots h_{r-1}B^*$ in a unique way and (b) determines h_rU^* in a unique way. We see that the equations (a)-(d) are equivalent to the following equations for h_0U^*:

$$h_0^{-1}F(h_0) \in B^*xB^*, \quad F^{r-1}(h_0)^{-1}gh_0d^{-1} \in B^*F^{r-1}(x)B^*,$$

$$F^r(h_0)^{-1}gh_0d^{-1}U^* = k(F^r(h_0)^{-1}gF(h_0)F(d^{-1}))x^{-1}U^*$$

(if $r \geq 2$) and

$$h_0^{-1}gh_0d^{-1} \in B^*xB^*, \quad F(h_0)^{-1}gh_0d^{-1}U^* = k(F(h_0)^{-1}gF(h_0)F(d^{-1}))x^{-1}U^*$$

(if $r = 1$). In both cases these equations are equivalent to

(e) $$h_0^{-1}F(h_0) \in U^*txF(t)^{-1}U^*, \quad F^r(h_0)^{-1}gh_0d^{-1} \in F^r(t)U^*$$

for some $t \in T$. We then have $F^{r-1}(h_0)^{-1}gh_0d^{-1} \in U^*F^{r-1}(t)F^{r-1}(x)U^*$. For h_0U^*, t as in (e) we compute

$$\begin{aligned}
&k(h_0^{-1}F(h_0))k(F(h_0)^{-1}F^2(h_0))\dots k(F^{r-2}(h_0)^{-1}F^{r-1}(h_0))k(F^{r-1}(h_0)^{-1}gh_0d^{-1})\\
&= (txF(t)^{-1})(F(t)F(x)F^2(t^{-1}))\dots(F^{r-2}(t)F^{r-2}(x)F^{r-1}(t)^{-1})(F^{r-1}(t)F^{r-1}(x))\\
&= txF(x)\dots F^{r-1}(x) = t.
\end{aligned}$$

∎

By 3.2(a) the result of the last computation is necessarily in T^{F_x}. Thus $F_x(t) = t$. Hence $F^r(t) = t$ and the equations (e) become

(f) $$h_0^{-1}F(h_0) \in U^*xU^*, \quad F^r(h_0)^{-1}gh_0d^{-1} \in T^{F_x}U^*.$$

We see that

$$\sum_i (-1)^i \mathrm{tr}(?, \mathcal{H}_g^i(p_{\mathbf{w}!}\tilde{\mathcal{L}}_{\mathbf{w}})) = |T^{F_x}|^{-1} \sum_{t \in T^{F_x}} a_t = |T^{F_x}|^{-1} \sum_{t' \in T^{F_x}} a'_{t'}$$

where

$$a_t = |\{hU^* \in (G^0/U^*); h^{-1}F(h) \in U^*xU^*, dh^{-1}g^{-1}F^r(h)t \in U^*\}|\theta(t),$$

$$a'_{t'} = |\{hU^* \in (G^0/U^*); h^{-1}F(h) \in U^*xU^*, h^{-1}g^{-1}F^r(h)dt' \in U^*\}|\theta(dt'd^{-1}).$$

Comparing with the last formula in 2.4 and using $\theta(dt'd^{-1}) = \theta(t')$ for $t' \in T^{F_x}$ we obtain (with i_0 as in 2.4):

$$\sum_i (-1)^i \mathrm{tr}(?, \mathcal{H}_g^i(p_{\mathbf{w}!}\tilde{\mathcal{L}}_{\mathbf{w}})) = (-1)^{l(w)}\tilde{\theta}(d)q^{i_0 r/2}\mathrm{tr}(\epsilon_g, H_c^{i_0}(Y)_\theta).$$

Let us choose an isomorphism $p_{\mathbf{w}!}\tilde{\mathcal{L}}_{\mathbf{w}}[l_{\mathbf{w}}] \cong p_{\emptyset!}\tilde{\mathcal{L}}_\emptyset$. (This exists by 1.4; note that 1.4(i) holds by 1.5.) Via this isomorphism, the isomorphism 3.3(g) corresponds to an isomorphism $F^*(p_{\emptyset!}\tilde{\mathcal{L}}_\emptyset) \longrightarrow p_{\emptyset!}\tilde{\mathcal{L}}_\emptyset$ that is to an isomorphism $\psi' : F^*A \xrightarrow{\sim} A$ so that

$$\sum_i (-1)^i \mathrm{tr}(?, \mathcal{H}_g^i(p_{\mathbf{w}!}\tilde{\mathcal{L}}_{\mathbf{w}})) = \sum_i (-1)^i \mathrm{tr}(\psi', \mathcal{H}_g^i(A))$$

for any $g \in D^F$. (We use that $l_{\mathbf{w}}$ is even.) Since A is irreducible, we must have $\psi = \lambda'\psi'$ for some $\lambda' \in \bar{\mathbf{Q}}_l^*$. It follows that

$$\sum_{i \in \mathbf{Z}} (-1)^i \mathrm{tr}(\psi, \mathcal{H}_g^i(A)) = \lambda'(-1)^{l(w)}\tilde{\theta}(d)q^{i_0 r/2}\mathrm{tr}(\epsilon_g, H_c^{i_0}(Y)_\theta)$$

for any $g \in D^F$. Thus Theorem 1.2 holds with V being the G^F-module $H_c^{i_0}(Y)_\theta$, which is irreducible (even as a G^{0F}-module) if G^0 has connected centre, but is not necessarily irreducible in general.

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