THE INVERSE OF A CARTAN MATRIX

George Lusztig and Jacques Tits

M. I. T., Cambridge, MA 02139, USA and
Collège de France, 75231 Paris Cedex 05, France

1. The purpose of this note is to give an explicit formula for the entries of the inverse of a Cartan matrix. More generally, we shall treat the case of a matrix modelled on the incidence matrix of a tree.

2. Let Δ be a finite graph, that is, a finite set I together with a subset E of $I \times I$ such that $(i,i) \notin E$ for all $i \in I$ and such that $(i,j) \in E$ implies $(j,i) \in E$. Let R be a commutative ring with 1. Let $C = (c_{ij})_{i,j \in I}$ be a matrix indexed by $I \times I$ with entries $c_{ij} \in R$ such that $c_{ij} = 0$ whenever $(i,j) \notin E$ and $i \neq j$.

More generally, if J is a subset of I, let $C(J) = (c'_{ij})_{i,j \in I-J}$ be the matrix given by $c'_{ij} = c_{ij}$ for all $i,j \in I - J$.

Let $C^* = (c^*_{ij})_{i,j \in I}$ be the matrix of cofactors of C; thus,

$$c^*_{ij} = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{k : k \neq j} c_{k,\sigma(k)} \tag{2.1}$$

where σ runs through all bijections $I \to I$ such that $\sigma(j) = i$. We have $CC^* = (\det C)\mathbf{1}$; this equation can be taken as definition of C^*, in the case where $\det C$ is invertible in R.

3. From now on, we assume that Δ is a forest, that is, a disjoint union of trees. Given $i,j \in I$ in the same connected component of Δ, we define a subset $[i,j]$ of I and an element $f_{ij} \in R$ as follows.

If $i = j$, we set $[i,j] = \{i\}$ and $f_{ij} = 1$. If $i \neq j$, there is a unique path joining i,j, by which we mean a sequence $i = i_0, i_1, i_2, \ldots, i_m = j$ such that $(i_\alpha, i_{\alpha+1}) \in E$ and $i_\alpha \neq i_{\alpha+2}$ for all relevant α. Let $[i,j] = \{i_0, i_1, \ldots, i_m\}$ and let $f_{ij} = (-c_{i_0 i_1})(-c_{i_1 i_2}) \ldots (-c_{i_{m-1} i_m})$.

The matrix $C_{[i,j]}$ (notation of §2) is a matrix of the same kind as C, with respect to the full subgraph of Δ whose set of vertices is $I - [i,j]$.

If $I - [i,j]$ is empty, we set $\det C_{[i,j]} = 1$.

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Proposition. *Assume that I is non-empty. The entries of C^* are given by*

$$c^*_{ij} = f_{ij} \det C_{[i,j]} \tag{3.1}$$

if i, j are in the same connected component of Δ and

$$c^*_{ij} = 0 \tag{3.2}$$

if i, j are in different connected components of Δ.

Formula (3.2) is obvious.

Now assume that i, j are in the same connected component. For any permutation σ of I such that $\sigma(j) = i$, let $J(\sigma)$ denote the smallest σ-stable subset of I containing i; it consists of $m+1$ elements $i, \sigma(i), \sigma^2(i), \dots, \sigma^m(i) = \sigma^{-1}(i) = j$ for some $m \geq 0$. If the contribution of σ to the sum (2.1) is non-zero, we have $c_{i,\sigma(i)} c_{\sigma(i),\sigma^2(i)} \dots c_{\sigma^{m-1}(i),\sigma^m(i)} \neq 0$. It follows that $\sigma^r(i) = i_r$ for $r = 0, \dots, m$, where $(i_0, i_1, i_2, \dots, i_m)$ is the sequence defined by i, j as in §2 (we take $i_0 = i, m = 0$ if $i = j$) and $J(\sigma) = [i, j]$. Hence the contribution of σ to the sum (2.1) is

$$\text{sgn } (\sigma)(-1)^m f_{ij} \prod_{k \in I - [i,j]} c_{k,\sigma(k)}$$

or, equivalently

$$\text{sgn } (\tau) f_{ij} \prod_{k \in I - [i,j]} c_{k,\tau(k)}$$

where $\tau : I - [i, j] \to I - [i, j]$ is the bijection defined by the restriction of σ. Thus, we have

$$c^*_{ij} = f_{ij} \sum_{\tau} \text{sgn } (\tau) \prod_{k \in I - [i,j]} c_{k,\tau(k)}$$

where τ runs over all bijections $I - [i, j] \to I - [i, j]$ and (3.1) follows.

4. Let us choose a vertex of Δ which we denote by ∞, and let $P(\infty)$ be the set of neighbours of ∞ in Δ. From (3.1), we deduce

$$\det C = \sum_{j \in I} c_{\infty j} c^*_{j \infty} = c_{\infty\infty} \det C_{[\infty,\infty]} - \sum_{j \in P(\infty)} c_{\infty j} c_{j \infty} \det C_{[j,\infty]}. \tag{4.1}$$

For $i \in I$, we set $c_{\{i\}} = c_{ii}$. For an undirected edge $\{i, j\}$ of Δ, we set $c_{\{i,j\}} = -c_{ij} c_{ji}$.

Let $\mathcal{P}$ be the set of all partitions of I into subsets which are either of cardinality 1 or are undirected edges. For $\pi = (\pi_\alpha)$ in $\mathcal{P}$, let $c_\pi = \prod_\alpha c_{\pi_\alpha}$. By a straightforward induction, (4.1) implies:

Corollary. *The determinant of C is given by*

$$\det C = \sum_{\pi \in \mathcal{P}} c_\pi. \tag{4.2}$$

5. From the proposition, we see that, if C is invertible, the i,j-entry of C^{-1} is zero if i,j are not in the same connected component and is equal to $f_{ij} \det C_{[i,j]} / \det C$ if i,j are in the same connected component; here $\det C_{[i,j]}$ and $\det C$ are given explicitly by (4.2).

Assume now that $R = \mathbf{Q}$, that C is a Cartan matrix and that Δ is the associated Coxeter graph. Then $C_{[i,j]}$ is again a Cartan matrix and the determinants $\det C$ and $\det C_{[i,j]}$ are known to be the orders of the centres of the simply connected compact groups corresponding to C and $C_{[i,j]}$; hence they are integers > 0. (We recall that this order is $n+1$ for a simple Lie group of type A_n, 2 for types B_n, C_n, E_7, 3 for type E_6, 4 for type $D_n (n \geq 4)$, and 1 for types G_2, F_4, E_8.)

Hence, in this case, our proposition explains why the entries of the inverse matrix C^{-1} are strictly positive rational numbers.

Appendix

Another, more naive approach to the computation of C^{-1} or, as the case may be, to the recognition of the noninvertibility of C, consists in just solving the system of linear equations provided by the relation $CC^{-1} = \mathbf{1}$. We want to show that, due to the special form of C, the computation can be organized in a very efficient way. For simplicity, we assume that Δ is a tree, that R is a field and that $c_{ij} \neq 0$ whenever $(i,j) \in E$. Let C'_i denote the column of C^{-1} indexed by i. If C is not invertible, C'_i is not defined and this is then revealed by the outcome of the algorithm we are going to describe.

Computation of C'_∞ for a given element $\infty \in I$. We set $C'_\infty = {}^t(c'_i)_{i \in I}$. Let us order I partially by setting $j < i$ (for $i,j \in I, \quad i \neq j$) when $i \in [j, \infty]$. Each element i of $I - \{\infty\}$ has a unique succesor which we denote by $\sigma(i)$. For any $j \in I$, we set $P(j) = \{i \in I - \{\infty\} | \sigma(i) = j\}$. For $i \in I - \{\infty\}$, we write "formally"

$$c'_i = d_i c'_{\sigma(i)}. \tag{$A.1$}$$

Expressing that the (i, ∞)-entry of CC^{-1} vanishes when $i \neq \infty$, we get

$$\sum_{j \in P(i)} c_{ij} c'_j + c_{ii} c'_i + c_{i,\sigma(i)} c'_{\sigma(i)} = 0,$$

hence

$$-\Big(\sum_{j \in P(i)} c_{ij} d_j + c_{ii}\Big) d_i = c_{i,\sigma(i)} \quad (i \neq \infty). \tag{$A.2_i$}$$

Similarly, equating to 1 the (∞,∞)-entry, we find

$$(A.2_\infty) \qquad (\sum_{j\in P(\infty)} c_{\infty j}d_j + c_{\infty\infty})c'_\infty = 1.$$

Now, our algorithm runs as follows: the relation $(A.2_i)$ $(i\neq\infty)$ provides the values of all d_i $(i\neq\infty)$ by increasing induction, then $(A.2_\infty)$ gives c'_∞ and, finally, $(A.1)$ enables us to compute all c'_i by descending induction. If the coefficient on the left of $(A.2_j)$ turns out to be 0 for some $j\in I$, one replaces c_{jj} by $c_{jj}+\epsilon_j$ and adjoins the indeterminate ϵ_j to the ground field. The values obtained for c'_j at the end are rational functions in the auxiliary indeterminates introduced in this computation and one must specialize them by setting all indeterminates equal to 0. If this leads to undefined expressions with 0 in denominators, it means that C is not invertible.

Remarks. (a) Almost always, the algorithm goes through without any ϵ_i: indeed, for a given Δ, the need for those indeterminates only concerns a proper subvariety of the affine space of all C. Of course, the cases where several indeterminates are needed are even more exceptional.

(b) In applications, it does happen that one is content with the knowledge of one column of C^{-1}.

Example. We compute the column C'_5 of the inverse of the matrix

$$C=\begin{pmatrix}1&1&0&0&0\\1&2&1&0&0\\0&3&3&1&0\\0&0&1&2&2\\0&0&0&1&1\end{pmatrix}$$

with rows and columns indexed in the usual way by $\{1,2,3,4,5\}$. We have

$d_1=-1,\quad d_2=-1\cdot(-1+2)^{-1}=-1,\quad d_3=-1\cdot(-3+3)^{-1}$, impossible.

Replacing c_{33} by $c_{33}+\epsilon$, we obtain $d_3=-1\cdot(-3+3+\epsilon)^{-1}=-\epsilon^{-1}$. We have $d_4=-2(-\epsilon^{-1}+2)^{-1}=\frac{2\epsilon}{1-2\epsilon},\quad c'_5=(\frac{2\epsilon}{1-2\epsilon}+1)^{-1}=1-2\epsilon(=1)$,

$c'_4=(1-2\epsilon)\frac{2\epsilon}{1-2\epsilon}=2\epsilon(=0),\quad c'_3=2\epsilon(-\epsilon^{-1})=-2,\quad c'_2=(-1)(-2)=2,\quad c'_1=-2$.

Thus, $C'_5={}^t(-2,2,-2,0,1)$.

From the Proposition we see that $c'_1=2(\det C)^{-1}$. Since $c'_1=-2$, we deduce that $\det C=-1$.

Computation of the full matrix C^{-1}. One can of course compute C^{-1} by determining all C'_i, but this is not the best procedure. Here we shall assume that no diagonal entry of C vanishes (i. e. $c_{ii}\neq 0$ for all i). For $i\in I$, let $V(i)$ be the set of all neighbours of i in Δ and let $\mathbf{1}_i$ be i-th column of the identity matrix (a column with δ_{ij} in the j-th position). We have, for $i\in I$,

$$(A.3) \qquad c_{ii}C'_i + \sum_{j\in V(i)} c_{ji}C'_j = \mathbf{1}_i.$$

This relation can be used in two ways: if C'_j is known for all $j \in V(i)$, it provides C'_i as a linear combination of those C'_j and $\mathbf{1}_i$; and if C'_i is known for all but one of the $C'_j \quad (j \in V(i))$, it enables one similarly to compute that last one. For $J \subset I$, let EJ denote the set of all $x \in I$ such that either $x \in J$, or $V(x) \subset J$, or $V(j) \subset J \cup \{x\}$ for some $j \in J$. The subsets J of I such that $J = EJ$ are the closed subsets of a topology on I; the closure $\bar{J}$ of any set J is the limit of the increasing sequence $J, EJ, E^2J, \ldots$. The above remark on the use of $(A.3)$ can be restated as follows: if $J \subset I$, then for any $k \in EJ$, $(A.3)$ yields an *explicit* expression of C'_k as a linear combination of all $C'_j \quad (j \in J)$ and some $\mathbf{1}_s$. By successive applications of $(A.3)$, one gets a similar expression of C'_k for all $k \in \bar{J}$, where of course several $\mathbf{1}_s$ are involved in general. If J is *dense* in I, one gets in this way the full matrix C^{-1}.

Here are two examples of dense subsets of I.

(a) Any set consisting of all ends of Δ but one.

(b) The set $V(r)$, if Δ has a single ramification point r.

Example. Let $\Delta = (I, E)$ be a graph with exactly one ramification, of any order r, let $n_1, n_2, \ldots, n_r$ be the lengths of the branches; thus, I consists of $1 + (n_1 + n_2 + \cdots + n_r)$ vertices. Let $v \in I$ be the ramification vertex. Let C be the matrix given by $c_{ii} = 2$ for all i, $c_{ij} = -1$ for all $(i, j) \in E$ and $c_{ij} = 0$ for all (i, j) such that $i \neq j$, $(i, j) \notin E$.

The algorithm for computing the column C'_v quickly yields the following value for the (v, v)-entry of C^{-1}:

$$(2 - \sum_1^r \frac{n_k}{n_k + 1})^{-1}.$$

From this and our proposition, one deduces

$$\det C = (2 - \sum_1^r \frac{n_k}{n_k + 1}) \prod_1^r (n_k + 1) = (2 - r) \prod_1^r (n_k + 1) + \sum_{l=1}^r \prod_{k \neq l} (n_k + 1).$$

With this formula, our proposition readily gives the inverse of any matrix C of the above type.

Here we assume of course that the inverses we are seeking, exist. This will be automatically true if we replace each 2 on the diagonal of C by $T + T^{-1}$ where T is an indeterminate which we adjoin to R; the other entries of C are unchanged. Let $\tilde{C}$ be the resulting matrix. Using again the algorithm for computing the column $\tilde{C}'_v$, we obtain the following value for the (v, v)-entry of $\tilde{C}^{-1}$:

$$(T + T^{-1} - \sum_1^r \frac{T^{n_k} - T^{-n_k}}{T^{n_k+1} - T^{-n_k-1}})^{-1}.$$

From this and our proposition, we deduce

$$\det \tilde{C} = (T + T^{-1} - \sum_1^r \frac{T^{n_k} - T^{-n_k}}{T^{n_k+1} - T^{-n_k-1}}) \prod_1^r \frac{T^{n_k+1} - T^{-n_k-1}}{T - T^{-1}}.$$