NOTES ON CHARACTER SHEAVES

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To Pierre Deligne on the occasion of his 65-th birthday

Introduction

Let $\mathbf{k}$ be an algebraically closed field and let G be an affine algebraic group over $\mathbf{k}$ which is reductive (that is, its identity component G^0 is reductive). Let $\mathbf{N}_{\mathbf{k}}^*$ be the set of all integers ≥ 1 that are $\neq 0$ in $\mathbf{k}$. We fix a prime number $l \in \mathbf{N}_{\mathbf{k}}^*$. According to the theory of character sheaves (see [L1, I] for $G = G^0$ and [L4, VI] in the general case) one can define a natural class of simple perverse $\bar{\mathbf{Q}}_l$-sheaves on G (the "character sheaves" on G) whose properties mimic those of the irreducible characters of a reductive group over a finite field.

This note has two parts. In §1 we study the functor $\beta := \mathfrak{t}_* \mathfrak{s}^! \mathfrak{c}_! \mathfrak{d}^*$ (see below) introduced by Bezrukavnikov, Finkelberg and Ostrik [BFO] (who assumed that $G = G^0$ and that the characteristic of $\mathbf{k}$ is 0); they prove the remarkable result that the complex obtained by the application of β to a character sheaf is a perverse sheaf. Here the following notation is used:

D is a fixed connected component of G;

$\mathcal{B}$ is the variety of Borel subgroups of G^0; for any $B \in \mathcal{B}$, U_B is the unipotent radical of B;

$\tilde{Z} = \{(B, B', g) \in \mathcal{B} \times \mathcal{B} \times D; gBg^{-1} = B'\}$;

$Z = \{(B, B', gU_B); (B, B') \in \mathcal{B}^2, g \in D; gBg^{-1} = B'\}$;

$\mathfrak{Z} = \{(B_1, B, B', gU_B); (B_1, B, B') \in \mathcal{B}^3, g \in D, gBg^{-1} = B'; B_1, B \text{ opposed}\}$;

$Z' = \{(B_1, B', U_{B'} g U_{B_1}); (B_1, B') \in \mathcal{B}^2, g \in D; gB_1g^{-1}, B' \text{ opposed}\}$;

$\mathfrak{d} : \tilde{Z} \to D$ is $(B, B', g) \mapsto g$;

$\mathfrak{c} : \tilde{Z} \to Z$ is $(B, B', g) = (B, B', gU_B)$;

$\mathfrak{s} : \mathfrak{Z} \to Z$ is $(B_1, B, B', gU_B) \mapsto (B, B', gU_B)$;

$\mathfrak{t} : \mathfrak{Z} \to Z'$ is $(B_1, B, B', gU_B) \mapsto (B_1, B', U_{B'} g U_{B_1})$;

$\mathfrak{d}^* : \mathcal{D}(D) \to \mathcal{D}(\tilde{Z})$, $\mathfrak{c}_! : \mathcal{D}(\tilde{Z}) \to \mathcal{D}(Z)$, $\mathfrak{s}^! : \mathcal{D}(Z) \to \mathcal{D}(\tilde{Z})$, $\mathfrak{t}_* : \mathcal{D}(\tilde{Z}) \to \mathcal{D}(Z')$ are the corresponding Grothendieck functors.

(For an algebraic variety X over $\mathbf{k}$ we write $\mathcal{D}(X)$ for the bounded derived category of $\bar{\mathbf{Q}}_l$-sheaves on X.) Actually in [BFO] (where $G = G^0$), the varieties Z, Z' appear

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in a different (but equivalent) form as $T\backslash(G/U_B\times G/U_B)$, $T\backslash(G/U_B\times G/U_{B'})$, where B,B' are two opposed Borel subgroup and $T=B\cap B'$.

Note that (in 1987) I showed that the cohomology sheaves of $\mathfrak{c}_!\mathfrak{d}^*(A)$ for a character sheaf A on D (when $D=G^0$) have a particularly simple behaviour and that this behaviour characterizes character sheaves (see [MV], [Gi]). On the other hand, the functor $\mathfrak{t}_*\mathfrak{s}^!$ is essentially an intertwining operator.

In the remainder of this paper we assume that $\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$.

In §1 we restrict ourselves for simplicity to the case of unipotent character sheaves. (In some respects general character sheaves behave like unipotent character sheaves on a possibly smaller group.) The main result in §1 is an explicit computation of $\beta(A)$ in a Grothendieck group which takes weights into account (under a mild restriction on the characteristic of $\mathbf{k}$). See 1.2(b) for a precise statement.

We now describe the content of §2. We would like to understand how the tensor product of two irreducible representations ρ,ρ' (over $\bar{\mathbf{Q}}_l$) of a reductive group Γ over $\mathbf{F}_q$ decomposes into irreducibles. Take for example $\Gamma=PGL_2(\mathbf{F}_q)$. Let Θ be the (q^2-1) dimensional representation in which each irreducible representation of Γ (other than the unit representation) appears exactly once. If ρ,ρ' are two irreducible constituents of Θ then $\rho\otimes\rho'$ is equal (as a virtual representation) to Θ plus or minus the sum of at most three irreducible representations. (If $\rho=\rho'$ is the Steinberg representation then $\rho\otimes\rho'$ is Θ plus the unit representation; if ρ and ρ' are two principal series representations then, most of the time, $\rho\otimes\rho'$ is Θ plus a sum of two principal series representations; if ρ and ρ' are two discrete series representations then, most of the time, $\rho\otimes\rho'$ is Θ minus a sum of two discrete series representations; if ρ is a principal series representation and ρ' is a discrete series representation then $\rho\otimes\rho'=\Theta$.) We see that while the character of ρ,ρ' can be described in terms of character sheaves on $PGL_2(\mathbf{k})$, the character of $\rho\otimes\rho'$ cannot be described in terms of character sheaves (due to the presence of Θ). Note that the character of Θ is the function with value -1 at regular unipotent elements, value q^2-1 at 1 and value 0 elsewhere. This is a linear combination of two class functions on Γ which are characteristic functions of two simple perverse sheaves on $PGL_2(\mathbf{k})$ (one supported by the unipotent variety and one supported by the unit element). If we enlarge the class of character sheaves by including these two simple perverse sheaves the resulting class of simple perverse sheaves has the property that the tensor product of two members in the class is a suitable combination of members of the class, unlike the (unenlarged) class of character sheaves. In §2 we show how to enlarge (for a general G) the class of character sheaves to a larger class of simple perverse sheaves which behave under tensor product as in the case of $PGL_2(\mathbf{k})$.

Notation. We shall use extensively the notation and results of [BBD]. For $s\in\mathbf{Z}_{>0}$ let $\mathbf{F}_{q^s}$ be the subfield of $\mathbf{k}$ of cardinal q^s. Let $\mathcal{A}=\mathbf{Z}[v,v^{-1}]$ (v an indeterminate).

If X is an algebraic variety over $\mathbf{k}$ and $K \in \mathcal{D}(X)$, $n \in \mathbf{Z}$, we write $K[[n]]$ instead of $K[n](n/2)$. Let $\mathfrak{D}_X : \mathcal{D}(X) \longrightarrow \mathcal{D}(X)$ denote Verdier duality. For $K \in \mathcal{D}(X)$ let $\mathcal{H}^i(K)$ be the i-th cohomology sheaf of K and let $\mathcal{H}^i(K)_x$ be its stalk at $x \in X$; we write H^iK instead of ${}^pH^iK$. If X' is a closed subvariety of X, for any $K \in \mathcal{D}(X')$ we set $K^X = j_!K \in \mathcal{D}(X)$ where $j : X' \longrightarrow X$ is the inclusion.

If X has a given $\mathbf{F}_q$-structure we write $\mathcal{D}_m(X)$ for the corresponding mixed derived category of $\bar{\mathbf{Q}}_l$-sheaves. If $A \in \mathcal{D}_m(X)$ is perverse and $j \in \mathbf{Z}$, we denote by A_j the canonical subquotient of A wich is pure of weight j. We write X_s instead of $X(\mathbf{F}_{q^s})$ and we denote by $\bar{\mathbf{Q}}_l^{X_s}$ the $\bar{\mathbf{Q}}_l$-vector space consisting of all functions $X_s \longrightarrow \bar{\mathbf{Q}}_l$. If $K \in \mathcal{D}_m(X)$ and $s \in \mathbf{Z}_{>0}$, we define a function $\chi_{K,s} : X_s \longrightarrow \bar{\mathbf{Q}}_l$ by

$$\chi_{K,s}(\xi) = \sum_{i \in \mathbf{Z}} (-1)^i \mathrm{tr}(F^s, \mathcal{H}^i(K)_\xi)$$

where F is the Frobenius map relative to $\mathbf{F}_q$.

Let $\mathbf{W}$ be the set of G^0-orbits on $\mathcal{B} \times \mathcal{B}$ for the G^0-action given by conjugation on both factors. For $B, B' \in \mathcal{B}$ we write $\mathrm{pos}(B, B') = w$ if the G^0-orbit of (B, B') is w. We regard $\mathbf{W}$ as a finite Coxeter group with length function $l : \mathbf{W} \longrightarrow \mathbf{N}$ as in [L4, 26.1]; let $\mathbf{I} = \{w \in \mathbf{W}; l(w) = 1\}$. Let w_0 be the longest element of $\mathbf{W}$. Let $\leq$ be the standard partial order on $\mathbf{W}$.

We shall assume that on G we are given an $\mathbf{F}_q$-structure with Frobenius map F compatible with the group structure such that F acts as identity on G/G^0 and on $\mathbf{W}$.

For $g \in G$ let $Z_G(g_s)^0$ be the identity component of the centralizer in G of the semisimple part of g. Let $\mathcal{Z}_{G^0}^0$ be the identity component of the centre of G^0. For a connected component D of G let ${}^D\mathcal{Z}_{G^0}^0$ be the set of all $g \in \mathcal{Z}_{G^0}^0$ such that g commutes with some/any element of D. If G_1 is a subgroup of a group G_2 let $N_{G_2}(G_1)$ be the normalizer of G_1 in G_2.

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1. Study of the functor β

1.1. Let X be an algebraic variety over $\mathbf{k}$. Let $\mathcal{K}(X)$ be the Grothendieck group of the category of perverse sheaves on X; it has $\mathbf{Z}$-basis given by the isomorphism classes of simple perverse sheaves on X. Let $\mathcal{K}_\mathcal{A}(X) = \mathcal{A} \otimes \mathcal{K}(X)$. If A, K are perverse sheaves on X with A simple we denote by $(A : K)$ the multiplicity of A in a Jordan-Hölder series for K. Any perverse sheaf K on X gives rise to an element $\sum_A (A : K)A \in \mathcal{K}(X)$ (A runs over the isomorphism classes of simple perverse sheaves on X); this element is denoted again by K.

We define a symmetric bilinear inner product $(:) : \mathcal{K}_\mathcal{A}(X) \times \mathcal{K}_\mathcal{A}(X) \longrightarrow \mathcal{A}$ by $(A : A') = 1$ (resp. $(A : A') = 0$) if A, A' are isomorphic (resp. nonisomorphic) simple perverse sheaves on X. If A, K are perverse sheaves on X with A simple then the multiplicity $(A : K)$ and the inner product $(A : K)$ coincide.

If X has a given $\mathbf{F}_q$-structure and $K \in \mathcal{D}_m(X)$, we set

$$gr(K) = \sum_{i,j\in\mathbf{Z}} (-1)^i v^j H^i(K)_j \in \mathcal{K}_{\mathcal{A}}(X).$$

Let X, Y be algebraic varieties defined over $\mathbf{F}_q$ and let $f : X \to Y$ be a morphism defined over $\mathbf{F}_q$. We define a linear map $f_s^* : \bar{\mathbf{Q}}_l^{Y_s} \to \bar{\mathbf{Q}}_l^{X_s}$ by $f_s^*(\phi)(\xi) = \phi(f(\xi))$ for any $\phi \in \bar{\mathbf{Q}}_l^{Y_s}$, $\xi \in X_s$. We define a linear map $f_{!s} : \bar{\mathbf{Q}}_l^{X_s} \to \bar{\mathbf{Q}}_l^{Y_s}$ by

$$f_{!s}(\phi')(\xi') = \sum_{\xi\in X_s; f(x)=\xi'} \phi'(\xi)$$

for any $\phi' \in \bar{\mathbf{Q}}_l^{X_s}$, $\xi' \in Y_s$. Following Grothendieck we note that if $K \in \mathcal{D}_m(X)$ then $f_!K \in \mathcal{D}_m(Y)$ and we have $\chi_{f_!K,s} = f_{!s}(\chi_{K,s})$; if $K' \in \mathcal{D}_m(Y)$ then $f^*K' \in \mathcal{D}_m(X)$ and we have $\chi_{f^*K',s} = f_s^*(\chi_{K',s})$.

1.2. For $w \in \mathbf{W}$ let
$\bar{Z}^w = \{(B, B', xU_B) \in Z; \mathrm{pos}(B, B') \le w\}$,
$Z^w = \{(B, B', xU_B) \in Z; \mathrm{pos}(B, B') = w\}$
(an open dense smooth irreducible subvariety of $\bar{Z}^w$). Let
$\bar{Z}'^w = \{(B_1, B', U_{B'}xU_{B_1}) \in Z'; \mathrm{pos}(B_1, B') \le w\}$,
$Z'^w = \{(B_1, B', U_{B'}xU_{B_1}) \in Z'; \mathrm{pos}(B_1, B') = w\}$
(an open dense smooth irreducible subvariety of $\bar{Z}'^w$).

Let $L^w = IC(\bar{Z}^w, \bar{\mathbf{Q}}_l)^Z \in \mathcal{D}(Z)$, $\bar{\mathbf{Q}}_l^w = (\bar{\mathbf{Q}}_l)^Z$ where $\bar{\mathbf{Q}}_l$ is regarded as a local system on Z^w. Let $L'^w = IC(\bar{Z}'^w, \bar{\mathbf{Q}}_l)^{Z'} \in \mathcal{D}(Z')$ where $\bar{\mathbf{Q}}_l$ is regarded as a local system on Z'^w. We set
$\Delta = \dim G^0 = \dim D$, $M_w = \dim Z^w = \dim Z'^w = \Delta - l(w_0w)$
so that $L^w[[M_w]]$ is a simple perverse sheaf on Z and $L'^w[[M_w]]$ is a simple perverse sheaf on Z'. Let

$$K_D^w = \mathfrak{d}_!\mathfrak{c}^*(\bar{\mathbf{Q}}_l^w) \in \mathcal{D}(D), \bar{K}_D^w = \mathfrak{d}_!\mathfrak{c}^*(L^w) \in \mathcal{D}(D).$$

A simple perverse sheaf A on D is said to be a *unipotent character sheaf* if $(A : H^i(K_D^w)) \ne 0$ for some $w \in \mathbf{W}, i \in \mathbf{Z}$ or equivalently if $(A : H^i(\bar{K}_D^w)) \ne 0$ for some $w \in \mathbf{W}, i \in \mathbf{Z}$. Let $\hat{D}^{un}$ be the class of unipotent character sheaves on D. Let $\mathcal{K}_{\mathcal{A}}^{un}(D)$ be the $\mathcal{A}$-submodule of $\mathcal{K}_{\mathcal{A}}(D)$ spanned by the unipotent character sheaves on D. Let Ξ be a set of representatives for the isomorphism classes of objects in $\hat{D}^{un}$; note that Ξ is a finite set. For any $A \in \Xi$ we set $d_A = \dim \mathrm{supp}(A)$, $d'_A = \mathrm{codim}_D \mathrm{supp}(A)$. If $A \in \Xi$ then $\mathfrak{D}_D(A) \in \hat{D}^{un}$ and we denote by A^* the object of Ξ which is isomorphic to $\mathfrak{D}_D(A)$. Under the $\mathcal{A}$-linear involution $\mathbf{d} : \mathcal{K}_{\mathcal{A}}^{un}(D) \to \mathcal{K}_{\mathcal{A}}^{un}(D)$ (the "duality" in [L4, IX, §42]), for any $A \in \Xi$, we have $\mathbf{d}(A) = (-1)^{d'_A} A^\bullet$ where $A^\bullet$ is a well defined object of Ξ; moreover we have $d_A = d_{A^\bullet}$.

We shall assume that the given $\mathbf{F}_q$ structure on G is such that each $A \in \Xi$ satisfies $F^*A \cong A$. (Such an $\mathbf{F}_q$-structure exists since Ξ is finite.) For each $A \in \Xi$ we can find (and we fix) an object $\underline{A} \in \mathcal{D}_m(D)$ which gives rise to A and such that for any g in an open dense subset of $\mathrm{supp}(A)$ and any $s \in \mathbf{Z}_{>0}$ such that $F^s(g) = g$, the eigenvalues of F^s on the stalk $\mathcal{H}^{-d_A}(A)_g$ are roots of 1 times $q^{sd'_A/2}$. Note that $\underline{A}$ is pure of weight Δ. Now each of the varieties $\tilde{Z}, Z, \mathfrak{Z}, Z'$ has a natural $\mathbf{F}_q$-structure induced by that of G and the maps $\mathfrak{d}, \mathfrak{c}, \mathfrak{s}, \mathfrak{t}$ are defined over $\mathbf{F}_q$. Hence we have naturally $\beta(\underline{A}) \in \mathcal{D}_m(Z')$ (β as in §0) and $H^i(\beta(uA))_j$ is well defined for any $i, j \in \mathbf{Z}$.

In the remainder of §1 we shall make the following assumption:

(a) *either the characteristic of* $\mathbf{k}$ *is a good prime for* G^0 *or* G^0 *is of classical type.*

Proposition. *For any* $A \in \Xi$ *and any* $j \in \mathbf{Z}$ *we have*

$$\text{(b)} \qquad \sum_i (-1)^i H^i(\beta(\underline{A}))_j = \sum_{x \in \mathbf{W}} (A^\bullet : H^{j+l(w_0x)}(\bar{K}_D^{w_0x}))L'^x[[M_x]]$$

in $\mathcal{K}(Z')$.

The proof is given in 1.11.

Note that the right hand of (b) is the class in $\mathcal{K}(Z')$ of a perverse sheaf on Z'. This suggests that $\beta(\underline{A})$ is a perverse sheaf; by [BFO], this is actually the case if the characteristic of $\mathbf{k}$ is large enough.

1.3. Let H be the Iwahori-Hecke algebra attached to $\mathbf{W}$ that is, the free $\mathcal{A}$-module with basis $\{T_w; w \in \mathbf{W}\}$ and with $\mathcal{A}$-algebra structure given by $T_w T_{w'} = T_{ww'}$ if $l(ww') = l(w) + l(w')$, $(T_s + 1)(T_s - v^2) = 0$ if $s \in \mathbf{I}$. Note that T_w is invertible in H for any $w \in \mathbf{W}$. Define an $\mathcal{A}$-linear map $h \mapsto {}^th$, $H \to H$ by ${}^tT_w = T_{w^{-1}}$ for all w (an algebra antiautomorphism). Define an $\mathcal{A}$-linear map $h \mapsto h^\dagger$, $H \to H$ by $T_w^\dagger = (-v^2)^{l(w)} T_{w^{-1}}^{-1}$ for all w (an algebra involution commuting with $h \mapsto {}^th$). We have a ring involution $h \mapsto \bar{h}$, $H \to H$ such that $v^j T_w \mapsto v^{-j} T_{w^{-1}}^{-1}$ for all $w \in \mathbf{W}, j \in \mathbf{Z}$. For $w \in \mathbf{W}$ let

$$c_w = v^{-l(w)} \sum_{y \in \mathbf{W}} P_{y,w}(v^2) T_y \in H,$$

where $P_{y,w}$ are the polynomials in the indeterminate $\mathbf{q}$ defined in [KL1]. We have $P_{y,w} = 0$ unless $y \leq w$. Moreover, $\overline{c_w} = c_w$. According to [KL1], the matrix $(P_{y,w})$ indexed by $\mathbf{W} \times \mathbf{W}$ has an inverse $(Q_{y,w})$ where

(a) $Q_{y,w} = (-1)^{l(w)-l(y)} P_{w_0w, w_0y}$.

1.4. For $x, y \in \mathbf{W}$ we define $a_{x,y} \in \mathcal{A}$, $b_{x,y} \in \mathcal{A}$ by

$$\text{(a)} \qquad T_{w_0}^{-1} c_x = \sum_y a_{x,y} c_y, \quad (-v^2)^{-l(w_0)} T_{w_0} c_x = \sum_y b_{x,y} c_y.$$

In this subsection we prove for any $x, z \in \mathbf{W}$ that:

(b) $$a_{w_0x,w_0z} = a_{xw_0,zw_0} = (-1)^{l(x)-l(z)} b_{z,x}.$$

Let $\iota : H \longrightarrow H$ be the algebra involution of H such that $\iota(T_w) = T_{w_0ww_0}$ for all $w \in \mathbf{W}$. Note that $\iota(c_w) = c_{w_0ww_0}$ for all $w \in \mathbf{W}$. Applying ι to the first equation in (a) we obtain $T_{w_0}^{-1} c_{w_0xw_0} = \sum_y a_{x,y} c_{w_0yw_0}$. On the other hand we have $T_{w_0}^{-1} c_{w_0xw_0} = \sum_y a_{w_0xw_0,w_0yw_0} c_{w_0yw_0}$. It follows that $a_{w_0xw_0,w_0yw_0} = a_{x,y}$ for all x, y. This proves the first equality in (b).

To prepare for the proof of the second equality in (b), we set $H^* = \mathrm{Hom}_{\mathcal{A}}(H, \mathcal{A})$ and we define a basis $(\tilde{c}_w)_{w \in \mathbf{W}}$ of H^* by $\tilde{c}_x(c_y) = (-1)^{l(x)} \delta_{x,y}$. Define an H-module structure on H by the left multiplication. For $h \in H, \phi \in H^*$ define $h\phi \in H^*$ by $(h\phi)(h_1) = \phi(hh_1)$ for $h_1 \in H$; we define $h * \phi \in H^*$ by $h * \phi = ({}^t h^\dagger)\phi$. Then $(h, \phi) \mapsto h * \phi$ is an H-module structure on H^*. Define an $\mathcal{A}$-linear isomorphism $\Lambda : H^* \longrightarrow H$ by $\tilde{c}_w \mapsto c_{ww_0}$. We show that the second equality in (b) follows from (c) below:

(c) *Λ is H-linear.*

Let $h = T_{w_0}^{-1} \in H$. Then ${}^t h^\dagger = (-v^2)^{-l(w_0)} T_{w_0}$. By (c) we have $h * \tilde{c}_x = \sum_y a_{xw_0,yw_0} \tilde{c}_y$. Hence

$$(-1)^{l(z)} \alpha_{xw_0,zw_0} = \sum_y a_{xw_0,yw_0} \tilde{c}_y(c_z) = \tilde{c}_x({}^t h^\dagger c_z) = \tilde{c}_x(\sum_y b_{z,y} c_y) = (-1)^{l(x)} b_{z,x}$$

and the second equality in (b) follows.

In the remainder of this subsection we prove (c). (This is a $\mathbf{q}$-analogue of a result I proved in 1980 which is reproduced in [BV, 2.25].) It is enough to check that $\Lambda(T_s * \tilde{c}_w) = T_s \Lambda(\tilde{c}_w)$ for $w \in W, s \in \mathbf{I}$. Recall [KL1] that there exists a symmetric function $\mathbf{W} \times \mathbf{W} \longrightarrow \mathbf{N}$, $y, w \mapsto \mu(y, w)$ such that $\mu(y, w) = 0$ unless $(-1)^{l(y)-l(w)} = -1$ and such that

(d) $T_s c_w = -c_w + \sum_{y; sy<y} \mu(y, w) v c_y$ if $sw > w$; $T_s c_w = v^2 c_w$ if $sw < w$.

It follows that

$T_s \tilde{c}_w = -\tilde{c}_w$ if $sw > w$;

$T_s \tilde{c}_w = v^2 \tilde{c}_w + \sum_{y; sy>y} \mu(w, y)(-1)^{l(y)-l(w)} v \tilde{c}_y$ if $sw < w$.

The last equality can be written in the form $T_s \tilde{c}_w = v^2 \tilde{c}_w - \sum_{y; sy>y} \mu(w, y) v \tilde{c}_y$. Since $T_s^\dagger = -v^2 T_s^{-1}$, we have

$T_s * \tilde{c}_w = -\tilde{c}_w + \sum_{y; sy>y} \mu(w, y) v \tilde{c}_y$ if $sw < w$; $T_s * \tilde{c}_w = v^2 \tilde{c}_w$ if $sw > w$.

On the other hand, using (d) we have:

$T_s c_{ww_0} = -c_{ww_0} + \sum_{y; syw_0 < yw_0} \mu(w_0y, w_0w) v c_{yw_0}$ if $sww_0 > ww_0$,

$T_s c_{ww_0} = v^2 c_{ww_0}$ if $sww_0 < ww_0$.

Now the condition that $sww_0 > ww_0$ is equivalent to the condition that $sw < w$. Moreover, by [KL1], we have $\mu(y, w) = \mu(ww_0, yw_0)$ for any y, w. Hence $\Lambda(T_s * \tilde{c}_w) = T_s \Lambda(\tilde{c}_w)$. This proves (c) hence also (b).

1.5. Let $\mathcal{D}^{un}(Z)$ (resp. $\mathcal{D}^{un}(Z')$) be the subcategory of $\mathcal{D}(Z)$ (resp. $\mathcal{D}(Z')$) whose objects are those $L \in \mathcal{D}(Z)$ (resp. $L \in \mathcal{D}(Z')$) such that for any $i \in \mathbf{Z}$, any composition factor of $H^i(L)$ is isomorphic to $L^w[[M_w]]$ (resp. $L'^w[[M_w]]$) for some $w \in \mathbf{W}$. Let $\mathcal{K}^{un}_{\mathcal{A}}(Z)$ be the $\mathcal{A}$-submodule of $\mathcal{K}_{\mathcal{A}}(Z)$ spanned by the basis elements $L^w[[M_w]]$. Let $\mathcal{K}^{un}_{\mathcal{A}}(Z')$ be the $\mathcal{A}$-submodule of $\mathcal{K}_{\mathcal{A}}(Z')$ spanned by the basis elements $L'^w[[M_w]]$. We define an $\mathcal{A}$-linear isomorphism $\Psi : H \longrightarrow \mathcal{K}^{un}_{\mathcal{A}}(Z)$ by $(-1)^{-l(w)}c_w \mapsto L^w[[M_w]]$ for all $w \in \mathbf{W}$. We define an $\mathcal{A}$-linear isomorphism $\Psi' : H \longrightarrow \mathcal{K}^{un}_{\mathcal{A}}(Z')$ by $(-1)^{-l(w)}c_w \mapsto L'^w[[M_w]]$ for all $w \in \mathbf{W}$. Let $\tau = \mathfrak{t}_!\mathfrak{s}^* : \mathcal{D}(Z) \longrightarrow \mathcal{D}(Z')$, $\tilde{\tau} = \mathfrak{t}_*\mathfrak{s}^! : \mathcal{D}(Z) \longrightarrow \mathcal{D}(Z')$. From the definitions we see that τ restricts to a functor $\mathcal{D}^{un}(Z) \longrightarrow \mathcal{D}^{un}(Z')$ denoted again by τ. Also $\mathcal{D}_{un}(Z)$, $\mathcal{D}^{un}(Z')$ are stable under $\mathfrak{D}_Z, \mathfrak{D}_{Z'}$ hence $\tilde{\tau}$ restricts to a functor $\mathcal{D}^{un}(Z) \longrightarrow \mathcal{D}^{un}(Z')$ denoted again by $\tilde{\tau}$.

Now if $L \in \mathcal{D}^{un}(Z) \cap \mathcal{D}_m(Z)$ then $\tau(L), \tilde{\tau}(L)$ are naturally objects of $\mathcal{D}^{un}(Z') \cap \mathcal{D}_m(Z')$ and $gr(L) \in \mathcal{K}^{un}_{\mathcal{A}}(Z)$, $gr(\tau(L)) \in \mathcal{K}^{un}_{\mathcal{A}}(Z')$, $gr(\tilde{\tau}(L)) \in \mathcal{K}^{un}_{\mathcal{A}}(Z')$ are defined. Moreover, there are well defined $\mathcal{A}$-linear maps $gr(\tau) : \mathcal{K}^{un}_{\mathcal{A}}(Z) \longrightarrow \mathcal{K}^{un}_{\mathcal{A}}(Z')$, $gr(\tilde{\tau}) : \mathcal{K}^{un}_{\mathcal{A}}(Z) \longrightarrow \mathcal{K}^{un}_{\mathcal{A}}(Z')$ such that $gr(\tau)(gr(L)) = gr(\tau(L))$, $gr(\tilde{\tau})(gr(L)) = gr(\tilde{\tau}(L))$ for any L as above. We show:

(a) *$gr(\tau)(\Psi(h)) = \Psi'(T_{w_0}h)$ for any $h \in H$.*

Let $\bar{\mathfrak{Z}} = \{(B_1, B, B') \in \mathcal{B}^3; \mathrm{pos}(B_1, B) = w_0\}$. We have a diagram with cartesian squares

$$\begin{tikzcd} Z \arrow[d, "p_1"'] & \mathfrak{Z} \arrow[l, "\mathfrak{s}"'] \arrow[r, "\mathfrak{t}"] \arrow[d, "p_2"'] & Z' \arrow[d, "p_3"'] \\ \mathcal{B}^2 & \bar{\mathfrak{Z}} \arrow[l, "\bar{\mathfrak{s}}"'] \arrow[r, "\bar{\mathfrak{t}}"] & \mathcal{B}^2 \end{tikzcd}$$

where $\bar{\mathfrak{s}}(B_1, B, B') = (B, B')$, $\bar{\mathfrak{t}}(B_1, B, B') = (B_1, B')$ and p_1, p_2, p_3 are the obvious projections. Let $\mathcal{D}^{un}(\mathcal{B}^2)$ be the subcategory of $\mathcal{D}(\mathcal{B}^2)$ whose objects are those $L \in \mathcal{D}(\mathcal{B}^2)$ such that for any $i \in \mathbf{Z}$, any composition factor of $H^i(L)$ is equivariant for the diagonal G^0-action on $\mathcal{B}^2$. Let $\mathcal{K}^{un}_{\mathcal{A}}(\mathcal{B}^2)$ be the $\mathcal{A}$-submodule of $\mathcal{K}_{\mathcal{A}}(\mathcal{B}^2)$ spanned by the simple G^0-equivariant perverse sheaves on $\mathcal{B}^2$. Let $\bar{\tau} = \bar{\mathfrak{t}}_!\bar{\mathfrak{s}}^* : \mathcal{D}^{un}(\mathcal{B}^2) \longrightarrow \mathcal{D}^{un}(\mathcal{B}^2)$. We define $gr(\bar{\tau}) : \mathcal{K}^{un}_{\mathcal{A}}(\mathcal{B}^2) \longrightarrow \mathcal{K}^{un}_{\mathcal{A}}(\mathcal{B}^2)$ in terms of $\bar{\tau}$ in the same way as $gr(\tau)$ was defined in terms of τ. Note that p_1^*, p_3^* induce isomorphisms $\mathcal{K}^{un}_{\mathcal{A}}(\mathcal{B}^2) \xrightarrow{\sim} \mathcal{K}^{un}_{\mathcal{A}}(Z)$, $\mathcal{K}^{un}_{\mathcal{A}}(\mathcal{B}^2) \xrightarrow{\sim} \mathcal{K}^{un}_{\mathcal{A}}(Z')$. Moreover, from the cartesian diagram above we see that $p_3^*\bar{\tau} = \tau p_1^*$. We see that (a) is reduced to the well known description of $gr(\bar{\tau})$ in terms of left multiplication by T_{w_0} in H.

We show:

(b) *$gr(\tilde{\tau})(\Psi(h)) = \Psi'(T_{w_0}^{-1}h)$ for any $h \in H$.*

Note that $\mathfrak{D}_Z, \mathfrak{D}_{Z'}$ induce involutions of $\mathcal{K}^{un}_{\mathcal{A}}(Z)$, $\mathcal{K}^{un}_{\mathcal{A}}(Z')$ which are semilinear with respect to the ring involution $v^i \mapsto v^{-i}$ of $\mathcal{A}$ and are denoted again by $\mathfrak{D}_Z, \mathfrak{D}_{Z'}$. We have $\mathfrak{D}_Z\Psi(h) = \Psi(\bar{h})$, $\mathfrak{D}_{Z'}\Psi'(h) = \Psi'(\bar{h})$ for all $h \in H$. Moreover we have $\tilde{\tau} = \mathfrak{D}_{Z'}\tau\mathfrak{D}_Z$. Hence $gr(\tilde{\tau}) = \mathfrak{D}_{Z'}gr(\tau)\mathfrak{D}_Z$. Using (a) we see that

$$gr(\tilde{\tau})(\Psi(h)) = \mathfrak{D}_{Z'}gr(\tau)\mathfrak{D}_Z(\Psi(h)) = \mathfrak{D}_{Z'}gr(\tau)(\Psi(\bar{h}))$$
$$= \mathfrak{D}_{Z'}\Psi'(T_{w_0}\bar{h}) = \Psi'(\overline{T_{w_0}\bar{h}}) = \Psi'(T_{w_0}^{-1}h),$$

as required.

1.6. Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be algebraic numbers in $\bar{\mathbf{Q}}_l$ such that for $i \in [1,k]$ any complex conjugate of λ_i has absolute value $q^{t_i/2}$ where $t_i \in \mathbf{N}$ and let $e_1, e_2, \dots, e_k$ in $\{1, -1\}$ be such that $e_1\lambda_1^s + e_2\lambda_2^s + \dots + e_k\lambda_k^s = 0$ for all $s \in \mathbf{Z}_{>0}$. Then we have $e_1 v^{t_1} + e_2 v^{t_2} + \dots + e_k v^{t_k} = 0$ in $\mathcal{A}$. The proof is left to the reader.

1.7. Recall the assumption 1.2(a). Let $s \in \mathbf{Z}_{>0}$. By results in [L1] (when $D = G^0$) and [L4, X] (in the general case) for any $A, A' \in \Xi$ we have the orthogonality relation:

(a) $$\sum_{\xi \in D_s} \chi_{\underline{A},s}(\xi)\chi_{\underline{A'},s}(\xi) = \delta|G_s^0|\omega_A^s$$

where $\delta = 1$ if $A' = A^*$ and $\delta = 0$, otherwise; ω_A is a root of 1 in $\bar{\mathbf{Q}}_l$ not depending on s; moreover,

(b) *if $A \in \Xi$, $w \in \mathbf{W}$ and $j \in \mathbf{Z}$ are such that $(A : H^j(\bar{K}_D^w)) > 0$ then $j = d_A$ mod 2.*

By the relative hard Lefschetz theorem of Deligne [BBD, 5.4.10] applied to the projective morphism $\mathfrak{d}$ and to the simple perverse sheaf $\mathfrak{c}^*L^w[\Delta + l(w)]$ on $\tilde{Z}$, we see that for any $w \in \mathbf{W}$, $j \in \mathbf{Z}$ we have

(c) $H^j(\bar{K}_D^w) \cong H^{2\Delta+2\lambda(w)-j}(\bar{K}_D^w)$.

From the fact that $\mathfrak{d}_!$ commutes with Verdier duality we see that $\mathfrak{D}(\bar{K}_D^w) \cong \bar{K}_D^w[2\Delta + 2l(w)]$ hence $\mathfrak{D}(H^j(\bar{K}_D^w) \cong H^{2\Delta+2l(w)-j}(\bar{K}_D^w)$. Combining this with (c) we obtain $\mathfrak{D}(H^j(\bar{K}_D^w) \cong H^j(\bar{K}_D^w)$. Hence for any $A \in \Xi$ we have

(d) $(\mathfrak{D}_D(A) : H^j(\bar{K}_D^w)) = (A : H^j(\bar{K}_D^w))$.

1.8. Let $\mathcal{D}^{un}(D)$ be the subcategory of $\mathcal{D}(D)$ whose objects are those $K \in \mathcal{D}(D)$ such that for any $i \in \mathbf{Z}$, any composition factor of $H^i(K)$ is in $\hat{D}^{un}$. For $w \in \mathbf{W}$, $\bar{\mathbf{Q}}_l^w, L^w$ come naturally from objects $\underline{\bar{\mathbf{Q}}_l}^w, \underline{L}^w$ of $\mathcal{D}_m(Z)$ such that Frobenius acts trivially on $\mathcal{H}^0(\underline{\bar{\mathbf{Q}}_l}^w)_x$, $\mathcal{H}^0(\underline{L}^w)_x$ for any $\mathbf{F}_q$-rational point x of Z^w. Then we have naturally $\underline{L}^w[[M_w]] \in \mathcal{D}_m(Z)$ (it is perverse, pure of weight 0). We set $\underline{\bar{K}}_D^w = \mathfrak{d}_!\mathfrak{c}^*(\underline{L}^w) \in \mathcal{D}_m(D)$ (it is pure of weight zero) and $\underline{K}_D^w = \mathfrak{d}_!\mathfrak{c}^*(\underline{\bar{\mathbf{Q}}_l}^w) \in \mathcal{D}_m(D)$, so that $gr(\underline{\bar{K}}_D^w) \in \mathcal{K}_{\mathcal{A}}^{un}(D)$, $gr(\underline{K}_D^w) \in \mathcal{K}_{\mathcal{A}}^{un}(D)$ are defined.

Let $s \in \mathbf{Z}_{>0}$. From [KL2] we can deduce

(a) $$\chi_{\underline{L}^w,s} = \sum_{y \in \mathbf{W}} P_{y,w}(q^s)\chi_{\underline{\bar{\mathbf{Q}}_l}^y,s}.$$

Applying the linear map $\mathfrak{d}_{!s}\mathfrak{c}_s^*$ to both sides we obtain

$$\mathfrak{d}_{!s}\mathfrak{c}_s^*(\chi_{\underline{L}^w,s}) = \sum_{y\in\mathbf{W}} P_{y,w}(q^s)\mathfrak{d}_{!s}\mathfrak{c}_s^*(\chi_{\underline{\bar{\mathbf{Q}}_l}^y,s}).$$

We have

$$\chi_{\underline{K}_D^w,s} = \mathfrak{d}_{!s}\mathfrak{c}_s^*(\chi_{\underline{\bar{\mathbf{Q}}_l}^w,s}), \chi_{\underline{\bar{K}}_D^w,s} = \mathfrak{d}_{!s}\mathfrak{c}_s^*(\chi_{\underline{L}^w,s}).$$

It follows that

(b) $$\chi_{\underline{\bar{K}}_D^w,s} = \sum_{y\in\mathbf{W}} P_{y,w}(q^s)\chi_{\underline{K}_D^y,s}.$$

Let $A \in \Xi$. For any $j \in \mathbf{Z}$ the mixed perverse sheaf $H^j(\underline{\bar{K}}_D^w)$ (pure of weight j) is canonically of the form $\oplus_{A\in\Xi}V'_{j,A} \otimes \underline{A}$ where $V'_{j,A}$ are finite dimensional vector spaces on which the Frobenius map acts naturally with a (multi)set of eigenvalues $E'_{j,A}$ in $\bar{\mathbf{Q}}_l$ such that each $\lambda \in E'_{j,A}$ is an algebraic number all of whose complex conjugates have absolute value $q^{(j-\Delta)/2}$. For any $s \in \mathbf{Z}_{>0}$ and any $j \in \mathbf{Z}$ we have

$$\chi_{H^j(\underline{\bar{K}}_D^w),s} = \sum_{A\in\Xi}\sum_{\lambda\in E'_{j,A}} \lambda^s\chi_{\underline{A},s}$$

as functions on D_s. It follows that

$$\chi_{\underline{\bar{K}}_D^w,s} = \sum_{A\in\Xi} \mathcal{T}'_{w,A;s}\chi_{\underline{A},s}$$

where

$$\mathcal{T}'_{w,A;s} = \sum_{j\in\mathbf{Z}}(-1)^j \sum_{\lambda\in E'_{j,A}} \lambda^s.$$

We set

$$\mathcal{T}'_{w,A} = \sum_{j\in\mathbf{Z}}(-1)^j \dim V'_{j,A}v^j = \sum_{j\in\mathbf{Z}}(-1)^j(A : H^j(\bar{K}_D^w))v^{j-\Delta}.$$

Note that

$$\mathcal{T}'_{w,A} = v^{-\Delta}(A : gr(\underline{\bar{K}}_D^w)).$$

For any $i, j \in \mathbf{Z}$ the mixed perverse sheaf $H^i(\underline{K}_D^w)_j$ (pure of weight j) is canonically of the form $\oplus_{A\in\Xi}V''_{i,j,A} \otimes \underline{A}$ where $V''_{i,j,A}$ are finite dimensional vector spaces on which the Frobenius map acts naturally with a (multi)set of eigenvalues $E''_{i,j,A}$ in $\bar{\mathbf{Q}}_l$ such that each $\lambda \in E''_{i,j,A}$ is an algebraic number all of whose complex conjugates have absolute value $q^{(j-\Delta)/2}$. For any $s \in \mathbf{Z}_{>0}$ and any $i, j \in \mathbf{Z}$ we have

$$\chi_{H^i(\underline{K}^w_D)_j, s} = \sum_{A \in \Xi} \sum_{\lambda \in E''_{i,j,A}} \lambda^s \chi_{\underline{A},s}$$

as functions on D_s. It follows that

$$\chi_{\underline{K}^w_D, s} = \sum_{A \in \Xi} \mathcal{T}''_{w,A;s} \chi_{\underline{A},s}$$

where

$$\mathcal{T}''_{w,A;s} = \sum_{i,j \in \mathbf{Z}} (-1)^i \sum_{\lambda \in E''_{j,A}} \lambda^s.$$

We set

$$\mathcal{T}''_{w,A} = \sum_{i,j \in \mathbf{Z}} (-1)^i \dim V''_{j,A} v^j = \sum_{i,j \in \mathbf{Z}} (-1)^i (A : H^i(\underline{K}^w_D)_j) v^{j-\Delta}.$$

Note that

$$\mathcal{T}''_{w,A} = v^{-\Delta}(A : gr(\underline{K}^w_D)).$$

Using now (b) we see that

$$\sum_{A \in \Xi} \mathcal{T}'_{w,A;s} \chi_{\underline{A},s} = \sum_{y \in \mathbf{W}} P_{y,w}(q^s) \sum_{A \in \Xi} \mathcal{T}''_{w,A;s} \chi_{\underline{A},s}.$$

Since the functions $\chi_{\underline{A},s}$ (with $A \in \Xi$) are linearly independent (see 1.7(a)), it follows that

(c) $$\mathcal{T}'_{w,A;s} = \sum_{y \in \mathbf{W}} P_{y,w}(q^s) \mathcal{T}''_{y,A;s}$$

for any $w \in \mathbf{W}, A \in \Xi$. Applying 1.6 to (c) we obtain the equality

(d) $$\mathcal{T}'_{w,A} = \sum_{y \in \mathbf{W}} P_{y,w}(v^2) \mathcal{T}''_{y,A}$$

in $\mathcal{A}$. Equivalently,

(e) $$(A : gr(\underline{\bar{K}}^w_D)) = \sum_{y \in \mathbf{W}} P_{y,w}(v^2)(A : gr(\underline{K}^w_D)).$$

We define an $\mathcal{A}$-linear map $\Phi : H \to \mathcal{K}^{un}_{\mathcal{A}}(D)$ by $\Phi(T_w) = gr(\underline{K}^w_D)$ for all $w \in \mathbf{W}$. Now (e) shows that

(f) $$\Phi(v^{l(w)} c_w) = gr(\underline{\bar{K}}^w_D)$$

for all $w \in \mathbf{W}$.

1.9. We show that for x, y in W we have

(a)
$$\sum_{\zeta\in Z_s}\chi_{\underline{L}^x,s}(\zeta)\chi_{\underline{L}^y,s}(\zeta)=|G^0_s|\sum_{z\in\mathbf{W}}P_{z,x}(q^s)P_{z,y}(q^s)q^{-l(w_0z)s}.$$

Using 1.8(a) we see that the left hand side equals

$$\sum_{x',y'\in\mathbf{W}}P_{x',x}(q^s)P_{y',y}(q^s)\sum_{\zeta\in Z_s}\chi_{\underline{\bar{\mathbf{Q}}_l}^{x'},s}(\zeta)\chi_{\underline{\bar{\mathbf{Q}}_l}^{y'},s}(\zeta)$$
$$=\sum_{x',y'\in\mathbf{W}}P_{x',x}(q^s)P_{y',y}(q^s)|Z_s\cap Z^{x'}\cap Z^{y'}|=\sum_{z\in\mathbf{W}}P_{z,x}(q^s)P_{z,y}(q^s)|Z^z_s|$$

and it remains to use that $|Z^z_s|=|G^0_s|q^{-l(w_0z)s}$ for any $z\in\mathbf{W}$.

1.10. From [L3, 6.5] we see that $\mathfrak{e}:=\mathfrak{c}_!\mathfrak{d}_*:\mathcal{D}(D)\longrightarrow\mathcal{D}(Z)$ restricts to a functor $\mathcal{D}^{un}(D)\longrightarrow\mathcal{D}^{un}(Z)$ denoted again by $\mathfrak{e}$. Let $A\in\Xi$. We have $\mathfrak{e}(A)\in\mathcal{D}^{un}(Z)$ and $\mathfrak{e}(\underline{A})\in\mathcal{D}_m(Z)$ hence $gr(\mathfrak{e}(\underline{A}))\in\mathcal{K}^{un}_{\mathcal{A}}(Z)$ is defined. We have the following result:

(a)
$$gr(\mathfrak{e}(\underline{A}))=\Psi((-1)^{d_A}T_{w_0}\sum_{x\in\mathbf{W}}v^{-l(w_0x)}(A^\bullet:gr(\underline{\bar{K}}^{w_0x}_D)))(-1)^{l(x)}c_x).$$

For any $i,j\in\mathbf{Z}$ the mixed perverse sheaf $H^i(\mathfrak{e}(\underline{A}))_j$ (pure of weight j) is canonically of the form $\oplus_{x\in\mathbf{W}}V_{i,j,x}\otimes\underline{L}^x[[M_x]]$ where $V_{i,j,x}$ are finite dimensional vector spaces on which the Frobenius map acts naturally with a (multi)set of eigenvalues $E_{i,j,x}$ in $\bar{\mathbf{Q}}_l$ such that each $\lambda\in E_{i,j,x}$ is an algebraic number all of whose complex conjugates have absolute value $q^{j/2}$. For any $s\in\mathbf{Z}_{>0}$ and any $i,j\in\mathbf{Z}$ we have

$$\chi_{H^i(\mathfrak{e}(\underline{A}))_j,s}=\sum_{x\in\mathbf{W}}\sum_{\lambda\in E_{i,j,x}}\lambda^s\chi_{\underline{L}^x[[M_x]],s}$$

as functions on Z_s. It follows that

$$\chi_{\mathfrak{e}(\underline{A}),s}=\sum_{x\in\mathbf{W}}S_{x,A;s}\chi_{\underline{L}^x[[M_x]],s}$$

where

$$S_{x,A;s}=\sum_{i,j\in\mathbf{Z}}(-1)^i\sum_{\lambda\in E_{i,j,x}}\lambda^s.$$

We set

$$S_{x,A}=\sum_{i,j\in\mathbf{Z}}(-1)^i\dim V_{i,j,x}v^j=\sum_{i,j\in\mathbf{Z}}(-1)^i(L^x[[M_x]]:H^i(\mathfrak{e}(\underline{A}))_j)v^j.$$

Note that

$$S_{x,A}=(L^x[[M_x]]:gr(\mathfrak{e}(\underline{A}))).$$

For $w\in\mathbf{W}$ we have

$$\sum_{\xi\in D_s}\chi_{\underline{A},s}(\xi)\chi_{\underline{\bar K}^w,s}(\xi)=\sum_{\xi\in D_s}\chi_{\underline{A},s}(\xi)\sum_{\eta\in\tilde Z_s;\mathfrak{d}(\eta)=\xi}\chi_{\mathfrak{c}^*\underline{L}^w,s}(\eta)$$
$$=\sum_{\eta\in\tilde Z_s}\chi_{\underline{A},s}(\mathfrak{d}(\eta))\chi_{\underline{L}^w,s}(\mathfrak{c}(\eta)),$$

$$\sum_{\zeta\in Z_s}\chi_{\mathfrak{c}_!\mathfrak{d}^*\underline{A},s}(\zeta)\chi_{\underline{L}^w,s}(\zeta)$$
$$=\sum_{\zeta\in Z_s}\sum_{\eta\in\tilde Z_s;\mathfrak{c}(\eta)=\zeta}\chi_{\mathfrak{d}^*\underline{A},s}(\eta)\chi_{\underline{L}^w,s}(\zeta)=\sum_{\eta\in\tilde Z_s}\chi_{\mathfrak{d}^*\underline{A},s}(\eta)\chi_{\underline{L}^w,s}(\mathfrak{c}(\eta)).$$

It follows that

$$\sum_{\xi\in D_s}\chi_{\underline{A},s}(\xi)\chi_{\underline{\bar K}^w,s}(\xi)=\sum_{\zeta\in Z_s}\chi_{\mathfrak{c}_!\mathfrak{d}^*\underline{A},s}(\zeta)\chi_{\underline{L}^w,s}(\zeta).$$

We rewrite this as follows

$$\sum_{\xi\in D_s}\chi_{\underline{A},s}(\xi)\sum_{A'\in\Xi}\mathcal{T}'_{w,A';s}\chi_{\underline{A}',s}(\xi)=\sum_{\zeta\in Z_s}\sum_{x\in\mathbf{W}}S_{x,A;s}\chi_{\underline{L}^x[[M_x]],s}(\zeta)\chi_{\underline{L}^w,s}(\zeta)$$
$$=\sum_{x\in\mathbf{W}}S_{x,A;s}(-q^{s/2})^{-M_x}\sum_{z\in\mathbf{W}}P_{z,x}(q^s)P_{z,w}(q^s)|G_s^0|q^{-l(w_0z)s}$$

where the last equality follows from 1.9(a) and the equality

$$\chi_{\underline{L}^x[[M_x]],s}=(-q^{s/2})^{-M_x}\chi_{\underline{L}^x,s}.$$

Using now 1.7(a) we deduce

$$|G_s^0|\mathcal{T}'_{w,A^*;s}\omega_A^s=|G_s^0|\sum_{x,z\in\mathbf{W}}S_{x,A;s}(-q^{s/2})^{-M_x}P_{z,x}(q^s)P_{z,w}(q^s)q^{-l(w_0z)s}$$

or equivalently (see 1.8(c)):

$$\sum_{z\in\mathbf{W}}P_{z,w}(q^s)\mathcal{T}''_{z,A^*;s}\omega_A^s=\sum_{x,z\in\mathbf{W}}S_{x,A;s}(-q^{s/2})^{-M_x}P_{z,x}(q^s)P_{z,w}(q^s)q^{-l(w_0z)s}.$$

We multiply both sides by $Q_{w,u}(q^s)$ (entries of the inverse matrix of $(P_{y,w}(q^s))$ and sum over $w\in\mathbf{W}$. We obtain

$$\mathcal{T}''_{u,A^*;s}\omega_A^s=\sum_{x\in\mathbf{W}}S_{x,A;s}(-q^{s/2})^{-M_x}P_{u,x}(q^s)q^{-l(w_0u)s}.$$

We multiply both sides by $Q_{y,u}(q^s)q^{l(w_0u)s}$ and sum over $u\in\mathbf{W}$. We obtain

$$S_{y,A;s}(-q^{s/2})^{-M_y}=\sum_{u\in\mathbf{W}}Q_{y,u}(q^s)q^{l(w_0u)s}\mathcal{T}''_{u,A^*;s}\omega^s_A.$$

Applying 1.6 to the previous equality we obtain

$$S_{y,A}(-v)^{-M_y}=\sum_{u\in\mathbf{W}}Q_{y,u}(v^2)v^{2l(w_0u)}\mathcal{T}''_{u,A^*}.$$

Here we substitute $\mathcal{T}''_{u,A^*}=v^{-\Delta}(A^*:gr(\underline{\bar{K}}^u_D)))=v^{-\Delta}(A^*:\Phi(T_u))$ and $Q_{y,u}=(-1)^{l(y)-l(u)}P_{w_0u,w_0y}$. Note also that $(A^*:gr(\underline{\bar{K}}^u_D)))=(A:gr(\underline{\bar{K}}^u_D)))$, by 1.7(d). We obtain

$$\begin{aligned}S_{y,A}&=v^{-\Delta}(-v)^{M_y}(A:\Phi(\sum_{u\in\mathbf{W}}(-1)^{l(y)-l(u)}P_{w_0u,w_0y}(v^2)v^{2l(w_0u)}T_u))\\&=v^{-\Delta}(-v)^{M_y}(\mathbf{d}(A):\mathbf{d}(\Phi(\sum_{u\in\mathbf{W}}(-1)^{l(y)-l(u)}P_{w_0u,w_0y}(v^2)v^{2l(w_0u)}T_u))).\end{aligned}$$

(We use that $(\rho:\rho')=(\mathbf{d}(\rho):\mathbf{d}(\rho'))$ for any ρ,ρ' in $\mathcal{K}^{un}_{\mathcal{A}}(D)$.) From [L4, IX, 42.9] we have $\mathbf{d}(\Phi(h))=\Phi(h^\dagger)$ for any $h\in H$. Hence the previous formula for $S_{y,A}$ becomes

$$\begin{aligned}S_{y,A}&=v^{-\Delta}(-v)^{M_y}(-1)^{d'_A}(-1)^{l(y)}\\&\times(A^\bullet:\Phi(\sum_{u\in\mathbf{W}}(-1)^{l(u)}P_{w_0u,w_0y}(v^2)v^{2l(w_0u)}(-v^2)^{l(u)}T^{-1}_{u^{-1}})).\end{aligned}$$

We now replace
$T^{-1}_{u^{-1}}$ by $T^{-1}_{w_0}T_{w_0u}$, $(-1)^{l(u)}v^{2l(w_0u)}(-v^2)^{l(u)}$ by $v^{2l(w_0)}$,
$\sum_{u\in\mathbf{W}}P_{w_0u,w_0y}(v^2)T_{w_0u}$ by $v^{l(w_0y)}c_{w_0y}$,
$v^{-\Delta}(-v)^{M_y}(-1)^{d'_A}(-1)^{l(y)}$ by $(-1)^{l(w_0)}v^{-\lambda(w_0y)}(-1)^{d_A}$;
we obtain

$$S_{y,A}=(-v^2)^{l(w_0)}(-1)^{d_A}(A^\bullet:\Phi(T^{-1}_{w_0}c_{w_0y})).$$

We have

$$gr(\mathfrak{e}(\underline{A}))=\sum_{y\in\mathbf{W}}S_{y,A}L^y[[M_y]]$$

Hence

$$\begin{aligned}\Psi^{-1}(gr(\mathfrak{e}(\underline{A})))&=\sum_{y\in\mathbf{W}}S_{y,A}(-1)^{l(y)}c_y\\&=(-v^2)^{l(w_0)}(-1)^{d_A}\sum_{y\in\mathbf{W}}(A^\bullet:\Phi(T^{-1}_{w_0}c_{w_0y}))(-1)^{l(y)}c_y.\end{aligned}$$

Using now 1.4(a), 1.4(b) we obtain

$$\Psi^{-1}(gr(\mathfrak{e}(\underline{A}))) = (-v^2)^{l(w_0)}(-1)^{d_A} \sum_{x,y\in\mathbf{W}} (A^\bullet : \Phi(a_{w_0y,w_0x}c_{w_0x}))(-1)^{l(y)}c_y$$
$$= (-v^2)^{l(w_0)}(-1)^{d_A} \sum_{x,y\in\mathbf{W}} (A^\bullet : \Phi((-1)^{l(y)-l(x)}b_{x,y}c_{w_0x}))(-1)^{l(y)}c_y.$$

Here we substitute $(-v^2)^{l(w_0)}\sum_{y\in\mathbf{W}} b_{x,y}c_y = T_{w_0}c_x$ (see 1.4(a)); we obtain

$$\Psi^{-1}(gr(\mathfrak{e}(\underline{A}))) = (-1)^{d_A} \sum_{x\in\mathbf{W}} (A^\bullet : \Phi((-1)^{l(x)}c_{w_0x}))T_{w_0}c_x.$$

We now use 1.8(f) and apply Ψ to both sides; we obtain (a).

11. Proof of 1.2(b). Let $A \in \Xi$. From the definitions we have

$$gr(\beta(\underline{A})) = gr(\tilde{\tau}\mathfrak{e}(\underline{A})) = gr(\tilde{\tau})(gr(\mathfrak{e}(\underline{A}))).$$

We write the equation 1.5(b) for $h = \Psi^{-1}(gr(\mathfrak{e}(\underline{A}))) \in H$. We obtain

$$gr(\tilde{\tau})(gr(\mathfrak{e}(\underline{A}))) = \Psi'(T_{w_0}^{-1}\Psi^{-1}(gr(\mathfrak{e}(\underline{A})))).$$

Using now 1.10(a) we obtain

$$gr(\beta(uA)) = \Psi'((-1)^{d_A} \sum_{x\in\mathbf{W}} v^{-l(w_0x)}(A^\bullet : gr(\underline{\bar{K}_D^{w_0x}})))(-1)^{l(x)}c_x)$$
$$= \sum_{x\in\mathbf{W}}\sum_{j\in\mathbf{Z}} (-1)^{j+d_A}(A^\bullet : H^j(\bar{K}_D^{w_0x}))v^{j-l(w_0x)}L'^x[[M_x]].$$

By 1.7(b) we have $(A^\bullet : H^j(\bar{K}_D^{w_0x})) = 0$ unless $j + d_A = 0 \mod 2$ (note that $d_A = d_{A^\bullet}$). Hence we have

(a) $$gr(\beta(uA)) = \sum_{x\in\mathbf{W}}\sum_{j\in\mathbf{Z}} (A^\bullet : H^j(\bar{K}_D^{w_0x}))v^{j-l(w_0x)}L'^x[[M_x]].$$

Now 1.2(b) follows.

1.12. The functor $\tilde{\beta}$. Let $\tilde{\beta} = \mathfrak{t}_!\mathfrak{s}^*\mathfrak{c}_*\mathfrak{d}^! : \mathcal{D}(D) \to \mathcal{D}(Z')$. Let $A \in \Xi$. The following equality suggests that $\tilde{\beta}$ might be equal to β up to a twist:

(a) $$gr(\tilde{\beta}(\underline{A})) = v^{-2\Delta}gr(\beta(\underline{A})).$$

Note that $\tilde{\beta}(A) = \mathfrak{D}_{Z'}\beta(\mathfrak{D}_D A)$. It is enough to show that

$$gr(\mathfrak{D}_{Z'}\beta(\underline{A}^*)) = v^{-2\Delta}gr(\beta(\underline{A}))$$

or, by 1.11(a), that

$$\sum_{x\in\mathbf{W}}\sum_{j\in\mathbf{Z}}((\mathfrak{D}_D A)^\bullet : H^j(\bar K_D^{w_0x}))v^{-j+l(w_0x)}L'^x[[M_x]]$$
$$=\sum_{x\in\mathbf{W}}\sum_{j\in\mathbf{Z}}(A^\bullet : H^j(\bar K_D^{w_0x}))v^{-2\Delta+j-l(w_0x)}L'^x[[M_x]].$$

We have $(\mathfrak{D}_D A)^\bullet=\mathfrak{D}_D(A^\bullet)$. Hence by 1.7(c), 1.7(d) we have for any $x\in\mathbf{W}$:

$$((\mathfrak{D}_D A)^\bullet : H^j(\bar K_D^{w_0x}))=(A^\bullet : H^{2\Delta+2l(w_0x)-j}(\bar K_D^{w_0x})).$$

Hence it is enough to observe that for any $x\in\mathbf{W}$ we have

$$\sum_{j\in\mathbf{Z}}(A^\bullet : H^{2\Delta+2l(w_0x)-j}(\bar K_D^{w_0x}))v^{-j+l(w_0x)}=\sum_{j'\in\mathbf{Z}}(A^\bullet : H^{j'}(\bar K_D^{w_0x}))v^{-2\Delta+j'-l(w_0x)}. \blacksquare$$

(We use the substitution $j'=2\Delta+2l(w_0x)-j$.)

1.13. To simplify the notation, in the remainder of §1 we assume that $D=G^0$. (Similar results hold without this assumption.) Let $\mathcal{E}$ be a set of representatives for the isomorphism classes of simple $\mathbf{Q}[\mathbf{W}]$-modules; for each $E\in\mathcal{E}$ let E^v be the corresponding simple H^v-module where $H^v=\mathbf{Q}(v)\otimes_{\mathcal{A}}H$. Let $A\in\Xi$. By [L1, III, 14.11], for any $w\in\mathbf{W}$ we have

$$\sum_{j\in\mathbf{Z}}(A:H^j(\bar K_D^w))v^j=(-1)^{d'_A}\sum_{E\in\mathcal{E}}v^{\Delta+l(w)}\gamma_{A,E}\mathrm{tr}(c_w,E^v)$$

where $\gamma_{A,E}$ are certain rational numbers. Taking this into account we can rewrite 1.11(a) as follows:

$$\Psi'^{-1}(gr(\beta(uA)))=(-1)^{d'_A}v^\Delta\sum_{E\in\mathcal{E}}^{r}\gamma_{A^\bullet,E}\mathfrak{C}_E$$

(equality in H^v) where

(a) $$\mathfrak{C}_E=\sum_{x\in\mathbf{W}}(-1)^{l(x)}\mathrm{tr}(c_{w_0x},E^v)c_x\in H.$$

It is known that $(-1)^{d'_A}\gamma_{A^\bullet,E}=\gamma_{A,E^\dagger}$ where $E^\dagger\in\mathcal{E}$ is isomorphic to $E\otimes\mathrm{sgn}$ and sgn is the sign representation of $\mathbf{W}$. Hence we have

(b) $$\Psi'^{-1}(gr(\beta(uA)))=v^\Delta\sum_{E\in\mathcal{E}}^{r}\gamma_{A,E^\dagger}\mathfrak{C}_E.$$

Note that the sum over E in (b) can be restricted to the E which belong to a fixed two-sided cell (depending on A); this is a known property of the coefficients $\gamma_{A,E^\dagger}$. We show that for any $E \in \mathcal{E}$ we have

(c) $$T_{w_0}^{-1}\mathfrak{C}_E \in \text{centre}(H), T_{w_0}\mathfrak{C}_E \in \text{centre}(H).$$

(These two statements are equivalent since $T_{w_0}^2$ is in the centre of H.) It is enough to show that the image of $(T_{w_0}^{-1}\mathfrak{C}_E)^\dagger$ is in the centre of H. We have

$$T_{w_0}^{-1}\mathfrak{C}_E = \sum_{x,y,z\in\mathbf{W}} (-1)^{l(x)}v^{-l(w_0x)}v^{-l(x)}T_{w_0}^{-1}P_{w_0y,w_0x}P_{y,x}\text{tr}(T_{w_0y}, E^v)T_z.$$

Using 1.3(a) we obtain

$$T_{w_0}^{-1}\mathfrak{C}_E = \sum_{y,z\in\mathbf{W}} (-1)^{l(y)}v^{-l(w_0)}T_{w_0}^{-1}\delta_{y,z}\text{tr}(T_{w_0y}, E^v)T_z$$
$$= \sum_{y\in\mathbf{W}} (-1)^{l(y)}v^{-l(w_0)}\text{tr}(T_{w_0y}, E^v)T_{w_0}^{-1}T_y = \sum_{y\in\mathbf{W}} (-1)^{l(y)}v^{-l(w_0)}\text{tr}(T_{w_0y}, E^v)T_{y^{-1}w_0}^{-1}$$
$$= \sum_{u\in\mathbf{W}} (-1)^{l(w_0u)}v^{-l(w_0)}\text{tr}(T_u, E^v)T_{u^{-1}}^{-1}.$$

Hence

$$(T_{w_0}^{-1}\mathfrak{C}_E)^\dagger = (-v)^{-l(w_0)}\mathfrak{C}'_E$$

where

$$\mathfrak{C}'_E = \sum_{u\in\mathbf{W}} v^{-2l(u)}\text{tr}(T_{u^{-1}}, E^v)T_u.$$

It is well known that the elements $\mathfrak{C}'_E (E \in \mathcal{E})$ for a basis of the centre of H^v. This proves (c).

1.14. If X is an algebraic variety and $K \in \mathcal{D}(X)$ we set

$$gr_1(K) = \sum_{i\in\mathbf{Z}}(-1)^iH^i(K) \in \mathcal{K}(X).$$

Assume that $A \in \Xi$ is cuspidal. From 1.11(a) we have

(a) $$gr_1(\beta(A)) = (-1)^{d_A}\sum_{x\in\mathbf{W}}(A : gr_1(\bar{K}_D^{w_0x}))L'^x[[M_x]].$$

(We have $A^\bullet = A$ since A is cuspidal.) Let Γ_A be the set of all $x \in \mathbf{W}$ such that $L'^x[[M_x]]$ appears with $\neq 0$ coefficient in the sum (a) and x has maximum possible length with this property. Note that $\Gamma_A \neq \emptyset$. We show:

(b) $w_0\Gamma_A$ *is contained in a single conjugacy class in* $\mathbf{W}$.

Let Γ'_A be the set of all $w \in \mathbf{W}$ such that $(A : gr_1(\bar{K}_D^w) \neq 0\}$ and $l(w)$ is minimum possible with this property. We have $w_0\Gamma_A = \Gamma'_A$. Let Γ''_A be the set of all $w \in \mathbf{W}$ such that $(A : gr_1(K_D^w) \neq 0\}$ and $l(w)$ is minimum possible with this property. Since $gr_1(\bar{K}_D^w) = \sum_{y\in\mathbf{W};y\leq w}P_{y,w}(1)gr(K_D^y)$, we see that $\Gamma'_A = \Gamma''_A$. It is enough to show that Γ''_A is contained in a single conjugacy class in $\mathbf{W}$. This follows from the Corollary to Theorem 2.18 in [L2]. For a description of the conjugacy classes in $\mathbf{W}$ that arise in this manner, see [L2].

2. Tensor products of character sheaves

2.1. An element $g \in G$ is said to be *isolated* in G if there is no proper parabolic subgroup P of G^0 with Levi L such that $gPg^{-1} = P, gLg^{-1} = L$ and L and $Z_G(g_s)^0 \subset L$, see [L4, I, 2.2]. A subset C of G is said to be an *isolated stratum* of G if C is contained in a connected component D' of G, C is a single orbit of the action $(z,x) : y \mapsto xzyx^{-1}$ of ${}^{D'}\mathcal{Z}^0_{G^0} \times G^0$ on D' and if some/any element of C is isolated in G, see [L4, I, 3.3]. If C is an isolated stratum of G and $n \in \mathbf{N}^*_{\mathbf{k}}$, let $\mathcal{S}_n(C)$ be the category whose objects are the local systems on C that are equivariant for the transitive ${}^{D'}\mathcal{Z}^0_{G^0} \times G^0$-action $(z,x) : y \mapsto xz^nyx^{-1}$ on D' (D' is associated to C as above). Let $\mathcal{S}(C)$ be the category whose objects are the local systems on C that are in $\mathcal{S}_n(C)$ for some n as above.

Following [L4, I, 3.5], let $\mathbf{A}$ be the set of all pairs (L,S) where L is a Levi subgroup of some parabolic of G^0 and S is an isolated stratum of $N_G(L)$ with the following property: there exists a parabolic subgroup P of G^0 with Levi L such that $S \subset N_GP$. For $(L,S) \in \mathbf{A}$ let $\tilde{\mathcal{W}}_S = \{n \in N_{G^0}L; nSn^{-1} = S\}$ (a subgroup of $N_{G^0}L$) and $\mathcal{W}_S = \tilde{\mathcal{W}}_S/L$ (a subgroup of the finite group $N_{G^0}L/L$). Now $\tilde{\mathcal{W}}_S$ acts on S by conjugation. Hence if $\nu \in \tilde{\mathcal{W}}_S$ and $\mathcal{E} \in \mathcal{S}(S)$ then $\nu^*\mathcal{E}$ is a well defined local system (necessarily in $\mathcal{S}(S)$).

For $(L,S) \in \mathbf{A}$ let $S^* = \{g \in S; Z_G(g_s)^0 \subset L\}$ (an open dense subset of S, see [L4, I, 3.11]) and let $Y_{L,S} = \cup_{x \in G^0} xS^*x^{-1}$. By [L4, I, 3.16], $Y_{L,S}$ is a locally closed irreducible subvariety of G. Now G^0 acts on $\mathbf{A}$ by conjugation; moreover $(L,S) \in \mathbf{A}$ and $(L',S') \in \mathbf{A}$ are in the same G^0-orbit if and only if $Y_{L,S} = Y_{L',S'}$, see [L4, I, 3.12]. The subsets $Y_{L,S}$ with (L,S) running through a set of representatives for the G^0-orbits in $\mathbf{A}$ form a partition of G into finitely many subsets called the *strata* of G, see [L4, I, 3.12]. By [L4, I, 3.15], the closure of any stratum of G is a union of strata of G. For $(L,S) \in \mathbf{A}$ let $\tilde{Y}_{L,S} = \{(g,xL) \in G^0 \times G/L; x^{-1}gx \in S^*\}$. Define $\pi : \tilde{Y}_{L,S} \to Y_{L,S}$ by $(g,xL) \mapsto g$. Now $\mathcal{W}_S$ acts freely on $\tilde{Y}_{L,S}$ by $\nu : (g,xL) \mapsto (g,x\nu^{-1}L)$. This makes $\pi : \tilde{Y}_{L,S} \to Y_{L,S}$ into a principal $\mathcal{W}_S$-bundle, see [L4, I, 3.13]. By [L4, I, 3.17], $Y_{L,S}$ and $\tilde{Y}_{L,S}$ are smooth. Let $(L,S) \in \mathbf{A}$. We have a diagram $\tilde{Y}_{L,S} \xleftarrow{a} R \xrightarrow{b} S$ where $R = \{(g,x) \in G \times G^0; x^{-1}gx \in S^*\}$ and $a(g,x) = (g,xL)$, $b(g,x) = x^{-1}gx$. Let $\mathcal{E} \in \mathcal{S}(S)$. There is a well defined local system $\tilde{\mathcal{E}}$ on $\tilde{Y}_{L,S}$ such that $b^*\mathcal{E} = a^*\tilde{\mathcal{E}}$. Now $\tilde{\mathcal{W}}_S$ acts on $\tilde{Y}_{L,s}$ through its quotient $\mathcal{W}_S$, on Z by $\nu : (g,x) \mapsto (g,x\nu^{-1})$ and on S as above. These actions are compatible with a, b. Hence if $\nu \in \tilde{\mathcal{W}}_S$ represents $w \in \mathcal{W}_S$ we have $w^*\tilde{\mathcal{E}} \cong \widetilde{\nu^*\mathcal{E}}$ where $w^*\tilde{\mathcal{E}}$ is defined using the $\mathcal{W}_S$-action on $\tilde{Y}_{L,S}$. Note that $\pi_!\tilde{\mathcal{E}}$ is a local system on $Y_{L,S}$.

Let Y be a stratum of G. Let $\mathrm{Loc}(Y)$ be the category whose objects are the local systems on Y that are isomorphic to a direct summand of the local system $\pi_!\tilde{\mathcal{E}}$ for some $\mathcal{E} \in \mathcal{S}(S)$ (where $Y = Y_{L,S}, (L,S) \in \mathbf{A}$); this is independent of the choice of (L,S) such that $Y = Y_{L,S}$. Note that any object of $\mathrm{Loc}(Y)$ is semisimple. The local system $\bar{\mathbf{Q}}_l$ on Y belongs to $\mathrm{Loc}(Y)$. Clearly $\mathrm{Loc}(Y)$ is closed under direct sum. We show:

(a) $\mathrm{Loc}(Y)$ *is closed under* $\otimes$.
Let (L,S) be such that $Y=Y_{L,S}$. Let $\mathcal{L},\mathcal{L}'$ be objects of $\mathrm{Loc}(Y)$. By assumption we can find $\mathcal{E}\in\mathcal{S}(S)$, $\mathcal{E}'\in\mathcal{S}(S)$ such that $\mathcal{L}$ is a direct summand of $\pi_!\tilde{\mathcal{E}}$ and $\mathcal{L}'$ is a direct summand of $\pi_!\tilde{\mathcal{E}}'$. Then $\mathcal{L}\otimes\mathcal{L}'$ is a direct summand of the local system $\pi_!\tilde{\mathcal{E}}\otimes\pi_!\tilde{\mathcal{E}}'$. Hence it is enough to show that $\pi_!\tilde{\mathcal{E}}\otimes\pi_!\tilde{\mathcal{E}}'\in\mathrm{Loc}(Y)$. We have $\pi_!\tilde{\mathcal{E}}\otimes\pi_!\tilde{\mathcal{E}}'=\oplus_{w\in\mathcal{W}_S}\pi_!(\tilde{\mathcal{E}}\otimes w^*\tilde{\mathcal{E}}')$. Hence it is enough to show that for any $w\in\mathcal{W}_S$, $\pi_!(\tilde{\mathcal{E}}\otimes w^*\tilde{\mathcal{E}}')\in\mathrm{Loc}(Y)$. Thus we must show that $\pi_!(\tilde{\mathcal{E}}\otimes\widetilde{\nu^*\mathcal{E}'})=\pi_!(\widetilde{\mathcal{E}\otimes\nu^*\mathcal{E}'})\in\mathrm{Loc}(Y)$ (where $\nu\in\tilde{\mathcal{W}}_S$ represents w); this holds since $\mathcal{E}\otimes\nu^*\mathcal{E}'\in\mathcal{S}(S)$. This proves (a).

2.2. Let Y be a stratum of G. Let $\mathfrak{K}(Y)$ be the Grothendieck group of the category $\mathrm{Loc}(Y)$. Note that any object $\mathcal{L}$ of $\mathrm{Loc}(Y)$ can be viewed as an element of $\mathfrak{K}(Y)$ denoted by $\underline{\mathcal{L}}$. Let $\underline{\mathrm{Loc}}(Y)$ be a set of representatives for the isomorphism classes of irreducible local systems in $\mathrm{Loc}(Y)$.

From 2.1(a) we see that $\mathfrak{K}(Y)$ is naturally a commutative ring in which for any $\mathcal{L},\mathcal{L}'$ in $\mathrm{Loc}(Y)$ the product of $\underline{\mathcal{L}}$, $\underline{\mathcal{L}'}$ in $\mathfrak{K}(Y)$ is $\underline{\mathcal{L}\otimes\mathcal{L}'}$. This ring has a unit element: the class of the local system $\bar{\mathbf{Q}}_l$ on Y.

Let $\mathfrak{K}_{\mathcal{A}}(Y)=\mathcal{A}\otimes\mathfrak{K}(Y)$; this is naturally a commutative $\mathcal{A}$-algebra with 1. For any $\mathcal{L}\in\mathrm{Loc}(Y)$ we set $[\mathcal{L}]=v^{\mathrm{codim}Y}\underline{\mathcal{L}}\in\mathcal{K}_{\mathcal{A}}(Y)$. The elements $[\mathcal{L}])$ where $\mathcal{L}$ runs over $\underline{\mathrm{Loc}}(Y)$ form an $\mathcal{A}$-basis B_Y of $\mathfrak{K}_{\mathcal{A}}(Y)$.

2.3. Let $\mathcal{D}^*(G)$ be the subcategory of $\mathcal{D}(G)$ whose objects are those $K\in\mathcal{D}(G)$ such that for any stratum Y of G and any $i\in\mathbf{Z}$ we have $\mathcal{H}^i(K)|_Y\in\mathrm{Loc}(Y)$.

For any stratum Y of G and any irreducible local system $\mathcal{L}$ in $\mathrm{Loc}(Y)$ the simple perverse sheaf $A_{\mathcal{L}}:=IC(\bar{Y},\mathcal{L})[\dim Y]^G$ is in $\mathcal{D}^*(G)$, see [L4, V, 25.2]. Conversely, let A be a simple perverse sheaf in $\mathcal{D}^*(G)$. We can find an irreducible local system $\mathcal{L}'$ on an open dense smooth subvariety V of $\mathrm{supp}(A)$ such that $A=IC(\mathrm{supp}(A),\mathcal{L}')[\dim\mathrm{supp}(A)]^G$. The intersections of $\mathrm{supp}(A)$ with the various strata of G form a partition of $\mathrm{supp}(A)$ into finitely many locally closed subvarieties; hence we can find a stratum Y of G such that $Y\cap\mathrm{supp}(A)$ is open dense in $\mathrm{supp}(A)$. By assumption, $\mathcal{H}^{-\dim\mathrm{supp}(A)}A|_Y\in\mathrm{Loc}(Y)$. Hence $\mathcal{L}'|_{Y\cap V}=\mathcal{L}_{Y\cap V}$. Since $Y\cap V$ is open dense in $\mathrm{supp}(A)$ we have $A=IC(\bar{Y},\mathcal{L})[\dim Y]^G$. Moreover $\mathcal{L}$ is automatically irreducible. We see that the simple perverse sheaves in $\mathcal{D}^*(G)$ are precisely the complexes of the form $A_{\mathcal{L}}$ where $\mathcal{L}$ is an irreducible local system in $\mathrm{Loc}(Y)$ for some stratum Y of G.

2.4. Let $\mathfrak{K}_{\mathcal{A}}(G)=\oplus_Y\mathfrak{K}_{\mathcal{A}}(Y)$; here Y runs over the (finite) set of strata of G. We view $\mathfrak{K}_{\mathcal{A}}(G)$ with a commutative $\mathcal{A}$-algebra structure which is the direct sum of the algebras $\mathfrak{K}_{\mathcal{A}}(Y)$. Note that $B_G:=\sqcup_Y B_Y$ is an $\mathcal{A}$-basis of $\mathfrak{K}_{\mathcal{A}}(G)$.

Let $\mathcal{L}\in\mathrm{Loc}(Y_0)$ where Y_0 is a stratum of G. Let $A=A_{\mathcal{L}}$. If $s\in\mathbf{Z}_{>0}$ is sufficiently divisible then $F^{s*}A\cong A$ and we can choose an isomorphism $F^{s*}A\xrightarrow{\sim}A$ such that for any g in an open dense subset of $\bar{Y}_0$ and any $s'\in\mathbf{Z}_{>0}s$ such that $F^{s'}(g)=g$, the eigenvalues of $F^{s'}$ on the stalk $\mathcal{H}^{-dim\bar{Y}_0}(A)_g$ are roots of 1 times $q^{s'\mathrm{codim}Y_0/2}$. We can also assume that each stratum of G is F^s-stable. For any stratum Y of G and any $i \in \mathbf{Z}$, F^s induces an isomorphism $\phi_Y : F^{s*}\mathcal{H}^i(A)|_Y \xrightarrow{\sim} \mathcal{H}^i(A)|_Y$ and the local system $\mathcal{H}^i(A)|_Y$ has a canonical filtration compatible with ϕ_Y whose subquotients $(\mathcal{H}^i(A)|_Y)_j$ ($j \in \mathbf{Z}$) have the following property: for any $s' \in \mathbf{Z}_{>0}$ and any $g \in Y(F_{q^{s'}})$, any eigenvalue of ϕ_Y on the stalk $((\mathcal{H}^i(A)|_Y)_j)_g$ is an algebraic number all of whose complex conjugates have absolute value $q^{s'j/2}$. We set

$$\begin{aligned} Gr(A) &= (-1)^{\dim Y_0} \sum_Y \sum_{i,j \in \mathbf{Z}} (-1)^i \underline{(\mathcal{H}^i(A)|_Y)_j} v^j \\ &= (-1)^{\dim Y_0} \sum_Y \sum_{i,j \in \mathbf{Z}} (-1)^i [(\mathcal{H}^i(A)|_Y)_j] v^{j-\mathrm{codim} Y} \in \mathfrak{K}_{\mathcal{A}}(G) \end{aligned}$$

where Y runs over the strata of G. Note that $Gr(A)$ is independent of the choice of s, ϕ_Y. Using Gabber's purity theorem [BBD] and the fact that $\bar{Y}_0$ is a union of strata of G (see [L4, I, 3.15]) we see that in the second sum defininig $Gr(A)$, j can be assumed to satisfy $j - \mathrm{codim} Y < 0$ if $Y \subset \bar{Y}_0, Y \neq Y_0$ and $j - \mathrm{codim} Y = 0$ if $Y = Y_0$. Thus $Gr(A_{\mathcal{L}})$ is equal to $[\mathcal{L}]$ plus a $v^{-1}\mathbf{Z}[v^{-1}]$-linear combination of elements in $\sqcup_{Y_1} B_{Y_1}$ where Y_1 runs over the strata of G such that $Y_1 \subset \bar{Y}_0$, $Y_1 \neq Y_0$. We see that the elements $Gr(A_{\mathcal{L}})$ with $[\mathcal{L}]$ running through B_G form an $\mathcal{A}$-basis $\tilde{B}_G$ of $\mathfrak{K}_{\mathcal{A}}(G)$.

Hence if $[\mathcal{L}], [\mathcal{L}'] \in B_G$ we can write the product $Gr(A_{\mathcal{L}})Gr(A_{\mathcal{L}'})$ in $\mathfrak{K}_{\mathcal{A}}(G)$ uniquely in the form

$$Gr(A_{\mathcal{L}})Gr(A_{\mathcal{L}'}) = \sum_{[\mathcal{L}''] \in B_G} f_{\mathcal{L},\mathcal{L}',\mathcal{L}''} Gr(A_{\mathcal{L}''})$$

where $f_{\mathcal{L},\mathcal{L}',\mathcal{L}''} \in \mathcal{A}$ is 0 for all but finitely many $[\mathcal{L}''] \in B_G$. We see that the family of elements $(Gr(A_{\mathcal{L}}))_{\mathcal{L} \in B_G}$ span a module that is closed under multiplication. Hence the class of simple perverse sheaves $(A_{\mathcal{L}})_{\mathcal{L} \in B_G}$ on G is of the kind described in the Introduction. Note that this class contains the character sheaves on G; moreover it contains only few non-character sheaves (compared to character sheaves).

2.5. Example. In this subsection we assume that $G = PGL_2(\mathbf{k})$. There are exactly three strata of G: the set Y_{rs} of regular semisimple elements; the set Y_{ru} of regular unipotent elements; the set $\{1\}$.

We have $Y_{rs} = Y_{T,T}$ where T is a maximal torus of G. Let $\mathcal{S}^1(T)$ be the subcategory of $\mathcal{S}(T)$ whose objects are the $\mathcal{E} \in \mathcal{S}(T)$ which have rank 1. For each $\mathcal{E} \in \mathcal{S}^1(T)$ we set $\mathcal{L}_{\mathcal{E}} = \pi_!(\tilde{\mathcal{E}})$. If $\mathcal{E}^{\otimes 2} \ncong \bar{\mathbf{Q}}_l$ then $\mathcal{L}_{\mathcal{E}} = \mathcal{L}_{\mathcal{E}^*}$ is irreducible (of rank 2). If $\mathcal{E}^{\otimes 2} \cong \bar{\mathbf{Q}}_l$ then $\mathcal{L}_{\mathcal{E}} \cong \mathcal{L}'_{\mathcal{E}} \oplus \mathcal{L}''_{\mathcal{E}}$ where $\mathcal{L}'_{\mathcal{E}}, \mathcal{L}''_{\mathcal{E}}$ are local systems of rank 1 on Y_{rs} such that $\mathcal{L}'_{\mathcal{E}}$ extends to a local system on G and $\mathcal{L}''_{\mathcal{E}}$ does not. Now $B_{Y_{rs}}$ consists of $[\mathcal{L}_{\mathcal{E}}] = [\mathcal{L}_{\mathcal{E}^*}]$ (with $\mathcal{E} \in \mathcal{S}^1(T)$, $\mathcal{E}^{\otimes 2} \ncong \bar{\mathbf{Q}}_l$) and of $\mathcal{L}'_{\mathcal{E}}, \mathcal{L}''_{\mathcal{E}}$ (with $\mathcal{E} \in \mathcal{S}^1(T)$, $\mathcal{E}^{\otimes 2} \cong \bar{\mathbf{Q}}_l$). If $\mathcal{E}, \mathcal{E}_1 \in \mathcal{S}^1(T)$ we have

$[\mathcal{L}_{\mathcal{E}}][\mathcal{L}_{\mathcal{E}_1}] = [\mathcal{L}_{\mathcal{E}\otimes\mathcal{E}_1}] + [\mathcal{L}_{\mathcal{E}\otimes\mathcal{E}_1^*}]$.
If in addition we have $\mathcal{E}^{\otimes 2} \cong \bar{\mathbf{Q}}_l$ then
$[\mathcal{L}'_{\mathcal{E}}][\mathcal{L}_{\mathcal{E}_1}] = [\mathcal{L}''_{\mathcal{E}}][\mathcal{L}_{\mathcal{E}_1}] = [\mathcal{L}_{\mathcal{E}\otimes\mathcal{E}_1}]$.
If in addition we have $\mathcal{E}^{\otimes 2} \cong \bar{\mathbf{Q}}_l$, $\mathcal{E}_1^{\otimes 2} \cong \bar{\mathbf{Q}}_l$ then
$[\mathcal{L}'_{\mathcal{E}}][\mathcal{L}'_{\mathcal{E}_1}] = [\mathcal{L}''_{\mathcal{E}}][\mathcal{L}''_{\mathcal{E}_1}] = [\mathcal{L}'_{\mathcal{E}\otimes\mathcal{E}_1}]$,
$[\mathcal{L}'_{\mathcal{E}}][\mathcal{L}''_{\mathcal{E}_1}] = [\mathcal{L}''_{\mathcal{E}\otimes\mathcal{E}_1}]$.
We have $Y_{ru} = Y_{G,S}$ where $S = Y_{ru}$ and $B_{Y_{ru}}$ consists of $[\bar{\mathbf{Q}}_l^{ru}]$ where $\bar{\mathbf{Q}}_l^{ru}$ is the local system $\bar{\mathbf{Q}}_l$ on Y_{ru}. We have $[\bar{\mathbf{Q}}_l^{ru}][\bar{\mathbf{Q}}_l^{ru}] = v[\bar{\mathbf{Q}}_l^{ru}]$.

We have $\{1\} = Y_{G,S}$ where $S = \{1\}$ and $B_{\{1\}}$ consists of $[\bar{\mathbf{Q}}_l^1]$ where $\bar{\mathbf{Q}}_l^1$ is the local system $\bar{\mathbf{Q}}_l$ on $\{1\}$. We have $[\bar{\mathbf{Q}}_l^1][\bar{\mathbf{Q}}_l^1] = v^3[\bar{\mathbf{Q}}_l^1]$.

Let $\mathcal{E} \in \mathcal{S}^1(T)$, $\mathcal{E}^{\otimes 2} \not\cong \bar{\mathbf{Q}}_l$. We write $A_{\mathcal{E}}$ instead of $A_{\mathcal{L}_{\mathcal{E}}}$. Let $\mathcal{E} \in \mathcal{S}^1(T)$, $\mathcal{E}^{\otimes 2} \cong \bar{\mathbf{Q}}_l$. We write $A'_{\mathcal{E}}$ instead of $A_{\mathcal{L}'_{\mathcal{E}}}$ and $A''_{\mathcal{E}}$ instead of $A_{\mathcal{L}''_{\mathcal{E}}}$. We have
$Gr(A_{\mathcal{E}}) = Gr(A_{\mathcal{E}^*}) = [\mathcal{L}_{\mathcal{E}}] + v^{-1}[\bar{\mathbf{Q}}_l^{ru}] + (v^{-1} + v^{-3})[\bar{\mathbf{Q}}_l^1]$ if $\mathcal{E}^{\otimes 2} \not\cong \bar{\mathbf{Q}}_l$,
$Gr(A'_{\mathcal{E}}) = [\mathcal{L}'_{\mathcal{E}}] + v^{-1}[\bar{\mathbf{Q}}_l^{ru}] + v^{-3}[\bar{\mathbf{Q}}_l^1]$, $Gr(A''_{\mathcal{E}}) = [\mathcal{L}''_{\mathcal{E}}] + v^{-1}[\bar{\mathbf{Q}}_l^1]$ if $\mathcal{E}^{\otimes 2} \cong \bar{\mathbf{Q}}_l$,
$Gr(A_{\bar{\mathbf{Q}}_l^{ru}}) = [\bar{\mathbf{Q}}_l^{ru}] + v^{-2}[\bar{\mathbf{Q}}_l^1]$,
$Gr(A_{\bar{\mathbf{Q}}_l^1}) = [\bar{\mathbf{Q}}_l^1]$.

From these formulas and from the multiplication table with respect to the basis B_G we can easily compute the product of any two elements in the basis $\tilde{B}_G$ as an $\mathcal{A}$-linear combination of elements in $\tilde{B}_G$. For example, if $\mathcal{E}, \mathcal{E}_1 \in \mathcal{S}^1(T)$, $\mathcal{E}^{\otimes 2} \not\cong \bar{\mathbf{Q}}_l$, $\mathcal{E}_1^{\otimes 2} \not\cong \bar{\mathbf{Q}}_l$, $(\mathcal{E}\otimes\mathcal{E}_1)^{\otimes 2} \not\cong \bar{\mathbf{Q}}_l$, $(\mathcal{E}\otimes\mathcal{E}_1^*)^{\otimes 2} \not\cong \bar{\mathbf{Q}}_l$, we have
$Gr(A_{\mathcal{E}})Gr(A_{\mathcal{E}_1}) = Gr(A_{\mathcal{E}\otimes\mathcal{E}_1}) + Gr(A_{\mathcal{E}\otimes\mathcal{E}_1^*}) - v^{-1}Gr(A_{\bar{\mathbf{Q}}_l^{ru}}) + vGr(A_{\bar{\mathbf{Q}}_l^1})$.

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