CANONICAL BASES ARISING FROM QUANTIZED ENVELOPING ALGEBRAS, II

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Introduction

This paper is a continuation of [L2]; in that paper we have defined some canonical bases for the finite dimensional simple modules of the Drinfeld-Jimbo quantized enveloping algebra corresponding to a symmetric positive definite Cartan matrix and also for the + part of that algebra.

In this paper we shall compare our definition of the canonical bases with a very recent definition given by Kashiwara at the Yukawa International Seminar (Kyoto, May 1990); we will show that the two definitions agree.

As a consequence, we are able to study a braid group action on finite dimensional simple modules and show that the action of the element T_{w_0} corresponding to the longest element in the Weyl group is given in the canonical basis by a monomial matrix. (See Corollary 5.9.)

We also construct a canonical basis in the space of coinvariants in a triple tensor product (§4) and we give a purely combinatorial description of its dimension.

Kashiwara defined certain operators $\tilde{E}_i$ and $\tilde{F}_i$ which map the canonical basis (at $v = \infty$) of a simple module into itself union with $\{0\}$. We will give a purely combinatorial description of these operators. (See 6.5(e).)

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We also give a (conjectural) definition, in terms of intersection cohomology, of a canonical basis of the + part of the Drinfeld-Jimbo algebra corresponding to the Cartan matrix given by a more or less arbitrary graph.

The elements of that basis are parametrized by the irreducible components of certain algebraic varieties, which seem to be of independent interest.

One of the ingredients in the definition of these algebraic varieties (see 8.2) is the idea to consider representations of a graph with all orientations simultaneously.

(I have learned this idea from Michael Atiyah, in a discussion, at the Kyoto Seminar, in which he described to me some recent work of Kronheimer [Kr] on Kleinian singularities and quivers.)

1. Recollections from [L2]

1.1 Let v be an indeterminate, let $A = \mathbf{Z}[v, v^{-1}]$ and let $A' = \mathbf{Q}(v)$ be its quotient field. Let $^- : A' \to A'$ be the field automorphism (of order two) which takes v to v^{-1}; it restricts to an automorphism of the ring A. Given an integer $N \geq 0$, we define

$$[N]! = \prod_{h=1}^{N} \frac{v^h - v^{-h}}{v - v^{-1}} \in A.$$

1.2 Let $(a_{ij})_{1 \leq i,j \leq n}$ be a symmetric, positive definite Cartan matrix. Let Q be the free abelian group with basis $\{\alpha_1, \dots, \alpha_n\}$. Define an inner product $(,)$ on Q by $(\alpha_i, \alpha_j) = a_{ij}$.

For any i, we define a homomorphism $s_i : Q \to Q$ by $\alpha_j \to \alpha_j - a_{ij}\alpha_i$ for all j. Let W be the subgroup of $\mathrm{Aut}(Q)$ generated by the reflections $s_i (1 \leq i \leq n)$ (a Coxeter group). Let $l : W \to \mathbf{N}$ be the usual length function; let w_0 be the unique element of W of maximal length and let $l(w_0) = \nu$. Let $i \to \hat{i}$ be the permutation of $[1, n]$ defined by $w_0 s_i w_0^{-1} = s_{\hat{i}}$. If $\alpha \in Q - \{0\}$, we say that $\alpha > 0$ (resp. $\alpha < 0$) if all its coordinates with respect to the basis α_i are ≥ 0 (resp. ≤ 0).

Let $\mathbf{U}$ be the Drinfeld-Jimbo quantized enveloping algebra corresponding to (a_{ij}); it is the A'-algebra defined by the generators $E_i, F_i, K_i, K_i^{-1} (1 \leq i \leq n)$ and the $q-$ analogues of the Serre relations (see [L2, 1.2(a1-a5]). Let $\mathbf{U}^+$ (resp. $\mathbf{U}^-$) be the A'-subalgebra of $\mathbf{U}$ generated by the elements $E_i (1 \leq i \leq n)$ (resp. $F_i (1 \leq i \leq n)$). Let U be the A-subalgebra of $\mathbf{U}$ generated by the elements

$$E_i^{(N)} = E_i^N/[N]!, \quad F_i^{(N)} = F_i^N/[N]!, \quad K_i, \quad K_i^{-1}.$$

Let U^+ be the A-subalgebra of $\mathbf{U}^+$ generated by the elements $E_i^{(N)}$ for $1 \leq i \leq n, N \geq 0$. Let $^- : \mathbf{U} \to \mathbf{U}$ be the $\mathbf{Q}$-algebra involution defined by

$$E_j \to E_j, \quad F_j \to F_j, \quad K_j \to K_j^{-1}$$

for all j and $v \to v^{-1}$. This maps $\mathbf{U}^+, \mathbf{U}^-, U, U^+$ into themselves.

1.3 For $i \in [1, n]$, let $\tilde{T}_i : \mathbf{U} \to \mathbf{U}$ be the A'-algebra automorphism defined in [L2, 1.3]. If $w \in W$ and $w = s_{i_1} \dots s_{i_k}$ with $k = l(w)$, then $\tilde{T}_{i_1} \dots \tilde{T}_{i_k} : \mathbf{U} \to \mathbf{U}$ depends only on w; we denote it $\tilde{T}_w$.

There is a unique A'-algebra isomorphism $\Psi : \mathbf{U} \cong \mathbf{U}^{\mathrm{opp}}$ such that $E_i \to E_i, F_i \to F_i, K_i \to K_i^{-1}$. We have $\Psi\tilde{T}_i = \tilde{T}_i^{-1}\Psi$ for all i.

There is a unique $\mathbf{Q}$-algebra isomorphism $\Lambda' : \mathbf{U} \cong \mathbf{U}$ such that $E_i \to F_i, F_i \to E_i, K_i \to K_i, v \to v^{-1}$.

There is a unique A'-algebra isomorphism $\Lambda : \mathbf{U} \cong \mathbf{U}$ such that $E_i \to F_{\hat{i}}, F_i \to E_{\hat{i}}, K_i \to K_{\hat{i}}^{-1}$.

Let $\mathcal{X}$ be the set of all sequences $\mathbf{i} = (i_1, \ldots, i_\nu)$ in $[1, n]$ such that $s_{i_1} \ldots s_{i_\nu} = w_0$.

Given $\mathbf{i} = (i_1, \ldots, i_\nu) \in \mathcal{X}$ and $\mathbf{c} = (c_1, \ldots, c_\nu) \in \mathbf{N}^\nu$ we define

$$E_{\mathbf{i}}^{\mathbf{c}} = E_{i_1}^{(c_1)}\tilde{T}_{i_1}(E_{i_2}^{(c_2)})\tilde{T}_{i_1}\tilde{T}_{i_2}(E_{i_3}^{(c_3)}) \ldots \tilde{T}_{i_1}\tilde{T}_{i_2} \ldots \tilde{T}_{i_{\nu-1}}(E_{i_\nu}^{(c_\nu)}).$$

These elements are contained in U^+. According to [**L2**, 2.3], the $\mathbf{Z}[v^{-1}]$-submodule of $\mathbf{U}^+$ generated by the elements

(a) $$E_{\mathbf{i}}^{\mathbf{c}} \quad (\mathbf{c} \in \mathbf{N}^\nu)$$

(for fixed $\mathbf{i}$) is independent of $\mathbf{i}$; we denote it $\mathcal{L}$. Moreover, the elements (a) form a $\mathbf{Z}[v^{-1}]$ basis of $\mathcal{L}$ (denoted $B_{\mathbf{i}}$) and their images under the canonical projection $\pi : \mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L}$ form a $\mathbf{Z}$-basis of $\mathcal{L}/v^{-1}\mathcal{L}$, independent of $\mathbf{i} \in \mathcal{X}$; we denote it B. The following result has been proved in [L2, §3].

Theorem 1.4. *(a) The restriction of $\pi : \mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L}$ defines an isomorphism of $\mathbf{Z}$-modules $\pi' : \mathcal{L} \cap \bar{\mathcal{L}} \cong \mathcal{L}/v^{-1}\mathcal{L}$. Hence, if we set $\mathbf{B} = (\pi')^{-1}(B)$ then $\mathbf{B}$ is a $\mathbf{Z}$-basis of $\mathcal{L} \cap \bar{\mathcal{L}}$.*

(b) $\mathbf{B}$ is a $\mathbf{Z}[v^{-1}]$-basis of $\mathcal{L}$, an A-basis of U^+ and an A'-basis of $\mathbf{U}^+$.

(c) Each element of $\mathbf{B}$ is fixed by $^- : U^+ \to U^+$.

$\mathbf{B}$ is called the canonical basis of $\mathbf{U}^+$. We have $\Psi(\mathbf{B}) = \mathbf{B}$.

1.5 Let $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbf{N}^n$, $\mathbf{d}' = (d'_1, \ldots, d'_n) \in \mathbf{N}^n$ and let $I_{\mathbf{d}}$ (resp. $J_{\mathbf{d}'}$) be the left (resp. right) ideal in $\mathbf{U}^+$ generated by the elements $E_1^{d_1+1}, \ldots, E_n^{d_n+1}$ (resp. $E_1^{d'_1+1}, \ldots, E_n^{d'_n+1}$). We have $\Psi(I_{\mathbf{d}}) = J_{\mathbf{d}}$.

The following result is proved in [L2, §8].

Theorem 1.6. *(a) The intersection $\mathbf{B} \cap I_{\mathbf{d}}$ is an A'-basis of $I_{\mathbf{d}}$.*

(b) The intersection $\mathbf{B} \cap J_{\mathbf{d}'}$ is an A'-basis of $J_{\mathbf{d}'}$.

1.7 It is known that there is a simple $\mathbf{U}$-module $L_{\mathbf{d}}$ of finite dimension over A' with a non-zero vector x_0 such that $F_i x_0 = 0, K_i x_0 = v^{-d_i} x_0$ for all i; moreover $(L_{\mathbf{d}}, x_0)$ is unique up to a unique isomorphism. It is known that the map $\mathbf{U}^+/I_{\mathbf{d}} \to L_{\mathbf{d}}$, $u \to ux_0$ is an isomorphism. From 1.6(a) it follows that the image of $\mathbf{B}$ under the map $\mathbf{U}^+ \to L_{\mathbf{d}}$, $u \to ux_0$ is the union of an A'-basis $\mathbf{B}[\mathbf{d}]$ of $L_{\mathbf{d}}$ with the set $\{0\}$. We call $\mathbf{B}[\mathbf{d}]$ the canonical basis of $L_{\mathbf{d}}$.

From 1.6(a) we see that

(a) the map $u \to ux_0$ defines a bijection

$$\mathbf{B} - \mathbf{B} \cap I_{\mathbf{d}} \cong \mathbf{B}[\mathbf{d}].$$

Let $\mathcal{L}_{\mathbf{d},A}$ be the A-submodule Ux_0 of $L_{\mathbf{d}}$. Let $\mathcal{L}_{\mathbf{d}}$ be the $\mathbf{Z}[v^{-1}]$-submodule $\mathcal{L}x_0$ of $L_{\mathbf{d}}$. The image of B under the map $\mathcal{L}/v^{-1}\mathcal{L} \to \mathcal{L}_{\mathbf{d}}/v^{-1}\mathcal{L}_{\mathbf{d}}$ induced by $u \to ux_0$ is the union of a $\mathbf{Z}$ basis $B[\mathbf{d}]$ of $\mathcal{L}_{\mathbf{d}}/v^{-1}\mathcal{L}_{\mathbf{d}}$ with the set $\{0\}$. (See [L2, 8.9].)

By [L2, 8.8], there is a unique isomorphism of $\mathbf{Q}$-vector spaces $^- : L_{\mathbf{d}} \cong L_{\mathbf{d}}$ such that $\overline{ux} = \bar{u}\bar{x}$ for all $u \in \mathbf{U}, x \in L_{\mathbf{d}}$ and $\bar{x_0} = x_0$. Its square is equal to 1.

Let $\tau : \mathcal{L}_{\mathbf{d}} \to \mathcal{L}_{\mathbf{d}}/v^{-1}\mathcal{L}_{\mathbf{d}}$ be the canonical projection. The following result is contained in [L2, 8.10].

Theorem 1.8. *(a) For any $b \in B[\mathbf{d}]$ there is a unique element $\tilde{b} \in \mathcal{L}_{\mathbf{d}}$ such that $\tau(\tilde{b}) = b$ and $\bar{\tilde{b}} = \tilde{b}$.*

(b) The map $b \to \tilde{b}$ is a bijection $B[\mathbf{d}] \cong \mathbf{B}[\mathbf{d}]$; its inverse is given by the restriction of τ.

(c) $\mathbf{B}[\mathbf{d}]$ is a $\mathbf{Z}[v^{-1}]$-basis of $\mathcal{L}_{\mathbf{d}}$ and an $A-$basis of $\mathcal{L}_{\mathbf{d},A}$.

2. Comparison with Kashiwara's results

2.1 In his lecture at the International Yukawa Seminar, Kashiwara announced a different construction of canonical bases for $\mathbf{U}^+$ and $L_{\mathbf{d}}$. (Earlier, he constructed canonical bases at $v = 0$ for classical types [Ka].) His definition at $v = 0$ is quite different from ours, and he extends it to a global basis by the method of [L2] but with a different proof. We will show that his basis is the same as ours.

2.2 Kashiwara defines a basis of $\mathbf{U}^-$ at $v = 0$; using the isomorphism Λ' this can be reformulated as a basis of $\mathbf{U}^+$ at $v = \infty$. (The same applies to other results of Kashiwara quoted below.) Consider some $i \in [1, n]$. Kashiwara defines an endomorphism $\tilde{E_i} : \mathbf{U}^+ \to \mathbf{U}^+$ as follows. He notes that

(a) any $\xi \in \mathbf{U}^+$ can be written uniquely in the form $\xi = \sum_{k \geq 0} E_i^{(k)} \xi_k$ where $\xi_k \in \mathbf{U}^+$ satisfy $F_i\xi_k - \xi_k F_i \in K_i\mathbf{U}^+$ and $\xi_k = 0$ for large k.

(The existence of such an expression will be re-proved in the course of the proof of 2.4; the uniqueness is easy.) Kashiwara then defines $\tilde{E_i}(\xi) = \sum_{k \geq 0} E_i^{(k+1)} \xi_k$. Let A'_∞ be the set of all $f \in A'$ which are regular at $v = \infty$. Let $\mathcal{L}'$ be the A'_∞-submodule of $\mathbf{U}^+$ generated by the elements

(b) $$\tilde{E_{i_1}} \dots \tilde{E_{i_s}} 1$$

for various sequences $i_1, \dots, i_s$ in $[1, n]$. Let B' be the subset of $\mathcal{L}'/v^{-1}\mathcal{L}'$ consisting of the images of the elements (b) under the canonical projection $\mathcal{L}' \to \mathcal{L}'/v^{-1}\mathcal{L}'$, except for 0. Kashiwara shows that

(c) B' is a $\mathbf{Q}$-basis of $\mathcal{L}'/v^{-1}\mathcal{L}'$ and that for each $b \in B'$ there is a unique $\tilde{b} \in U^+$ such that $\tilde{b} \in \mathcal{L}' \cap \bar{\mathcal{L}}'$ and such that the image of $\tilde{b}$ under the canonical map $\mathcal{L}' \to \mathcal{L}'/v^{-1}\mathcal{L}'$ is b.

Moreover,

(d) the set $\mathbf{B}' = \{\tilde{b} | b \in B'\}$ is an A-basis of U^+.

Now let $L_{\mathbf{d}}, x_0$ be as in 1.7.

Following Kashiwara, we define for fixed $i \in [1, n]$ some linear maps $\tilde{F_i}, \tilde{E_i} : L_{\mathbf{d}} \to L_{\mathbf{d}}$ as follows. Note that any vector $x \in \mathrm{L}_{\mathbf{d}}$ can be written uniquely in the form $x=\sum_{0\le k\le k'} F_i^{(k)}x_{k,k'}$ where $x_{k,k'}$ satisfy $E_ix_{k,k'}=0$ and $K_ix_{k,k'}=v^{k'}x_{k,k'}$; we then set $\tilde F_i(x)=\sum_{0\le k\le k'} F_i^{(k+1)}x_{k,k'}$, $\tilde E_i(x)=\sum_{1\le k\le k'} F_i^{(k-1)}x_{k,k'}$. An equivalent definition is as follows. Note that any vector $x\in L_{\mathbf{d}}$ can be written uniquely in the form $x=\sum_{0\le k\le k'} E_i^{(k)}x'_{k,k'}$ where $x'_{k,k'}$ satisfy $F_ix'_{k,k'}=0$ and $K_ix'_{k,k'}=v^{-k'}x'_{k,k'}$; we then set $\tilde E_i(x)=\sum_{0\le k\le k'} E_i^{(k+1)}x'_{k,k'}$, $\tilde F_i(x)=\sum_{1\le k\le k'} E_i^{(k-1)}x'_{k,k'}$. Kashiwara defines $\mathcal{L}'_{\mathbf{d}}$ to be the A'_∞-submodule of $L_{\mathbf{d}}$ generated by the elements

$$\tilde E_{i_1}\dots\tilde E_{i_s}x_0$$

for various sequences $i_1,\dots,i_s$ in $[1,n]$. The images of these elements under the canonical projection $\mathcal{L}'_{\mathbf{d}}\to\mathcal{L}'_{\mathbf{d}}/v^{-1}\mathcal{L}'_{\mathbf{d}}$ (except for 0) form a subset $B'[\mathbf{d}]$ of $\mathcal{L}'_{\mathbf{d}}/v^{-1}\mathcal{L}'_{\mathbf{d}}$.

Kashiwara shows that $B'[\mathbf{d}]$ is a $\mathbf{Q}$-basis of $\mathcal{L}'_{\mathbf{d}}/v^{-1}\mathcal{L}'_{\mathbf{d}}$. Moreover he shows the statements (e)-(h) below.

(e) $u\to ux_0$ defines a surjective map $\mathcal{L}'\to\mathcal{L}'_{\mathbf{d}}$. The image of B' under the canonical map $\mathcal{L}'/v^{-1}\mathcal{L}'\to\mathcal{L}'_{\mathbf{d}}/v^{-1}\mathcal{L}'_{\mathbf{d}}$ defined by $u\to ux_0$ is equal to $B'[\mathbf{d}]\cup\{0\}$.

(f) $\mathcal{L}'_{\mathbf{d}}$ is stable under $\tilde E_i,\tilde F_i,\quad(1\le i\le n)$. Moreover $\tilde E_i$ and $\tilde F_i$ map $B'[\mathbf{d}]$ into $B'[\mathbf{d}]\cup\{0\}$.

(g) If $b,b'\in B'[\mathbf{d}]$ then the conditions $b'=\tilde E_i(b)$ and $b=\tilde F_i(b')$ are equivalent.

(h) For any $b\in B'[\mathbf{d}]$ there is a unique element $\tilde b\in\mathcal{L}'_{\mathbf{d}}$ such that $\tilde b\to b$ under the canonical map $\mathcal{L}'_{\mathbf{d}}\to\mathcal{L}'_{\mathbf{d}}/v^{-1}\mathcal{L}'_{\mathbf{d}}$ and $\bar{\tilde b}=\tilde b$; the map $b\to\tilde b$ is a bijection of $B'[\mathbf{d}]$ onto a basis $\mathbf{B}'[\mathbf{d}]$ of $\mathcal{L}'_{\mathbf{d}}$ as an A'_∞-module and of $L_{\mathbf{d}}$ as an $A'-$ vector space.

We shall prove the following result.

Theorem 2.3. *(a)* $\mathcal{L}'=A'_\infty\mathcal{L}$.

(b) The canonical imbedding $\mathcal{L}/v^{-1}\mathcal{L}\to\mathcal{L}'/v^{-1}\mathcal{L}'$ *defines a bijection* $B\cong B'$.

(c) We have $\mathbf{B}=\mathbf{B}'$.

The proof will be given in 2.7.

Lemma 2.4. *We fix* $i\in[1,n]$. *Let* $R_i=\{\xi\in\mathbf{U}^+|F_i\xi-\xi F_i\in K_i\mathbf{U}^+\}$. *Let* R'_i *be the* A'*-subalgebra of* $\mathbf{U}^+$ *generated by the elements* $\tilde T_i(E_j)$ *for* j *such that* $a_{ij}=-1$ *and by* E_j *for* j *such that* $j\ne i$. *Choose* $\mathbf{i}=(i_1,\dots,i_\nu)\in\mathcal{X}$ *be such that* $i_1=i$ *and such that* $\mathbf{i}$ *is adapted to some orientation of the Dynkin graph (see [***L2***, 4.7, 4.12(b)]). Let* R''_i *be the* A'*-subspace of* $\mathbf{U}^+$ *spanned by the elements* $E_{\mathbf{i}}^{\mathbf{c}}$ *for various* $\mathbf{c}=(c_1,\dots,c_\nu)\in\mathbf{N}^\nu$ *such that* $c_1=0$. *We have* $R_i=R'_i=R''_i$.

It is clear that R_i is an A'-subalgebra of $\mathbf{U}^+$. It is easy to check that the generators of R'_i are contained in R_i. It follows that $R'_i\subset R_i$. From [**L2**, 5.4(b)] we see that

(a) $$R''_i=R'_i.$$

It remains to show that $R_i\subset R''_i$. Let $\xi\in\mathbf{U}^+$. We can write $\xi=\sum_{\mathbf{c}\in\mathbf{N}^\nu} f(\mathbf{c})E_{\mathbf{i}}^{\mathbf{c}}$ with $f(\mathbf{c})\in A'$ (finite sum). Let $\xi_k=\sum_{\mathbf{c}\in\mathbf{N}^\nu:c_1=0} f(\mathbf{c}+(k,0,\dots,0))E_{\mathbf{i}}^{\mathbf{c}}$. We have $\xi=\sum_{k\ge0} E_i^{(k)}\xi_k$ and $\xi_k\in R''_i$. Using (a) we see that $\xi_k\in R_i$. Hence the existence part of 2.2(a) is established. Using the uniqueness in 2.2(a) we see that, if we assume $\xi \in R_i$, then $\xi_k = 0$ for all $k > 0$; hence, $\xi = \xi_0 \in R''_i$. Thus we have $R_i \subset R''_i$ and the lemma is proved.

In a similar way we see that the three spaces in the lemma are equal to $\mathbf{U}^+ \cap \tilde{T}_i(\mathbf{U}^+)$.

Corollary 2.5. *Let $i, \mathbf{i}$ be as in 2.4; let $\mathbf{c} \in \mathbf{N}^\nu$, $r \geq 0$ and let $\mathbf{c}^+ = \mathbf{c} + (r, 0, \dots, 0)$. We have*

$$\tilde{E_i}^r (E_{\mathbf{i}}^{\mathbf{c}}) = E_{\mathbf{i}}^{\mathbf{c}^+}.$$

Lemma 2.6. *We have $\mathcal{L}' \subset A'_\infty \mathcal{L}$.*

We first show that, for any $i \in [1, n]$, we have

(a) $$\tilde{E}_i(\mathcal{L}) \subset \mathcal{L}.$$

Choose $\mathbf{i}$ as in 2.4. The elements $E_{\mathbf{i}}^{\mathbf{c}}$ for various $\mathbf{c} \in \mathbf{N}^\nu$ span $\mathcal{L}$ as a $\mathbf{Z}[v^{-1}]$ module; hence (a) follows from 2.5. Now the lemma follows by repeated applications of (a).

Lemma 2.7. *Given $b \in \mathbf{B} - \{1\}$, there exists some $\mathbf{i} \in \mathcal{X}$ as in 2.4 and $\mathbf{c} \in \mathbf{N}^\nu$ such that $\pi(b) = \pi(E_{\mathbf{i}}^{\mathbf{c}})$ (π as in 1.3) and such that $c_1 \geq 1$.*

Assume that this is not so. Let $\mathbf{i}, \mathbf{j} \in \mathcal{X}$ and t be as in [**L2**, 8.3]. Let $\mathbf{c}, \mathbf{c}' \in \mathbf{N}^\nu$ be such that $\pi(b) = \pi(E_{\mathbf{i}}^{\mathbf{c}}) = \pi(E_{\mathbf{j}}^{\mathbf{c}'})$. By our assumption we have $c_i = 0$ for all $i \in [1, t]$ and $c'_i = 0$ for all $i \in [1, n-t]$. Then, by [**L2**, 8.13], $\Psi(b)$ applied to the lowest weight vector of $L_{0,\dots,0}$ gives a vector in that module which is not a multiple of the lowest weight vector; this is absurd, since that module is 1 dimensional. This contradiction proves the lemma.

2.8 We now prove Theorem 2.3. Let $b \in \mathbf{B} - \{1\}$ and let $\mathbf{i}, \mathbf{c}$ be as in 2.7. Let $i = i_1$. We have $c_1 \geq 1$.

By the argument in [**L2**, 8.5], we see that b is a $\mathbf{Z}[v^{-1}]$-linear combination of elements $E_{\mathbf{i}}^{\mathbf{a}}$ with $\mathbf{a} \in \mathbf{N}^\nu$ such that $a_1 \geq c_1 \geq 1$. Now each such $E_{\mathbf{i}}^{\mathbf{a}}$ is in $\tilde{E}_i(\mathcal{L})$ (by 2.5). Hence we have $b \in \tilde{E}_i(\mathcal{L})$ so that $b = \sum_{b' \in \mathbf{B}} \gamma_{b'} \tilde{E}_i(b')$ where $\gamma_{b'} \in \mathbf{Z}[v^{-1}]$. We now apply the same argument for each b' in the sum except for $b' = 1$ and we iterate this. This process must terminate by homogeneity reasons and we see that any $b \in \mathbf{B}$ is a $\mathbf{Z}[v^{-1}]$-linear combination of elements of form $\tilde{E_{k_1}} \dots \tilde{E_{k_s}} 1$ for various sequences $k_1, \dots, k_s$ in $[1, n]$. Hence $\mathcal{L} \subset \mathcal{L}'$. This, combined with 2.6, proves 2.3(a).

From 2.6(a) we see that, for any $i \in [1, n]$, $\tilde{E}_i$ defines an endomorphism of $\mathcal{L}/v^{-1}\mathcal{L}$ which, by 2.5, maps B into itself. The argument in the earlier part of this proof shows that any element of B other than 1 is in the image of $\tilde{E}_i$ for some i; these remarks imply the truth of 2.3(b).

Now, using the definitions, we see that 2.3(a),(b) imply the truth of 2.3(c). The theorem is proved.

2.9 The previous proof shows that 2.3 remains true if A'_∞ is replaced throughout by $\mathbf{Z}[v^{-1}]$.

We now state the following result.

Theorem 2.10. *Let $L_{\mathbf{d}}, x_0$ be as in 1.7.*

(a) $\mathcal{L}'_{\mathbf{d}} = A'_\infty \mathcal{L}_{\mathbf{d}}$.

(b) The canonical imbedding $\mathcal{L}_{\mathbf{d}}/v^{-1}\mathcal{L}_{\mathbf{d}} \to \mathcal{L}'_{\mathbf{d}}/v^{-1}\mathcal{L}'_{\mathbf{d}}$ defines a bijection $B[\mathbf{d}] \cong B'[\mathbf{d}]$.

(c) We have $\mathbf{B}[\mathbf{d}] = \mathbf{B}'[\mathbf{d}]$.

(a),(b) follow from 2.3(a),(b) and 2.2(e); (c) follows from (a),(b) and the definitions.

3. An involution of $L_{\mathbf{d}}$

3.1 In this section, we fix $L_{\mathbf{d}}, x_0$ as in 1.7. Let $x_1 \in \mathbf{B}[\mathbf{d}] = \mathbf{B}'[\mathbf{d}]$ be the vector defined by the conditions $E_i x_1 = 0, \quad K_i x_1 = v^{d_{\hat{i}}} x_1$ for all i. (Thus, x_0 is a lowest weight vector and x_1 is a highest weight vector.)

Lemma 3.2. *(a) There is a unique A'-linear map $\Lambda : L_{\mathbf{d}} \to L_{\mathbf{d}}$ such that $\Lambda(ux) = \Lambda(u)\Lambda(x)$ for all $u \in \mathbf{U}$ and all $x \in L_{\mathbf{d}}$ ($\Lambda : \mathbf{U} \to \mathbf{U}$ is defined in 1.3) and such that $\Lambda(x_0) = x_1$.*

(b) Λ leaves stable $\mathcal{L}_{\mathbf{d},A}$.

The proof of (a) is almost the same as that of [L2, 8.8]. We can consider a new $\mathbf{U}$-module structure on $L_{\mathbf{d}}$: u acts in the new module structure as $\Lambda(u)$ in the original module structure. The new module is simple and in it we have $F_i x_1 = 0, K_i x_1 = v^{-d_i} x_1$ for all i. It remains to use the uniqueness of $L_{\mathbf{d}}, x_0$ up to unique isomorphism. Now (b) follows from the fact that $\mathcal{L}_{\mathbf{d},A}$ is generated by x_0 as a U-module and it contains x_1. (Clearly, $\Lambda : \mathbf{U} \to \mathbf{U}$ leaves U stable.)

Theorem 3.3. *(a) We have $\Lambda(\mathbf{B}[\mathbf{d}]) = \mathbf{B}[\mathbf{d}]$.*

(b) The square of Λ is the identity map on $L_{\mathbf{d}}$.

After a number of lemmas, the proof will be given in 3.8.

Lemma 3.4. *For any i, we have $\Lambda \circ \tilde{E}_i = \tilde{F}_{\hat{i}} \circ \Lambda$ and $\Lambda \circ \tilde{F}_i = \tilde{E}_{\hat{i}} \circ \Lambda$ as endomorphisms of $L_{\mathbf{d}}$.*

Let $x = \sum_{0 \le k \le k'} F_i^{(k)} x_{k,k'} \in L_{\mathbf{d}}$ where $x_{k,k'}$ satisfy $E_i x_{k,k'} = 0$ and $K_i x_{k,k'} = v^{k'} x_{k,k'}$ Then $\Lambda(x) = \sum_{0 \le k \le k'} E_{\hat{i}}^{(k)} \Lambda(x_{k,k'})$ and $\Lambda(x_{k,k'})$ satisfy $F_{\hat{i}} \Lambda(x_{k,k'}) = 0$ and $K_{\hat{i}} \Lambda(x_{k,k'}) = v^{-k'} \Lambda(x_{k,k'})$. By the definitions in 2.2, we have

$$\tilde{F}_i(x) = \sum_{0 \le k \le k'} F_i^{(k+1)} x_{k,k'}$$

$$\tilde{E}_{\hat{i}}(\Lambda(x)) = \sum_{0 \le k \le k'} E_{\hat{i}}^{(k+1)} \Lambda(x_{k,k'})$$

hence

$$\Lambda(\tilde{F}_i(x)) = \sum_{0 \le k \le k'} E_{\hat{i}}^{(k+1)} \Lambda(x_{k,k'})$$

and the second identity in the lemma is proved; the first identity is proved in a similar way.

Lemma 3.5. *(a) We have $\Lambda(\mathcal{L}'_{\mathbf{d}}) \subset \mathcal{L}'_{\mathbf{d}}$. Hence Λ induces an endomorphism of $\mathcal{L}'_{\mathbf{d}}/v^{-1}\mathcal{L}'_{\mathbf{d}}$.*

(b) The last endomorphism maps $B'[\mathbf{d}]$ into $B'[\mathbf{d}] \cup \{0\}$.

By 3.4, we have $\Lambda(\tilde{E_{i_1}} \dots \tilde{E_{i_k}} x_0) = \tilde{F_{\hat{i}_1}} \dots \tilde{F_{\hat{i}_k}} x_1$ and it remains to use the fact that $x_1 \in \mathcal{L}'_{\mathbf{d}}$ and 2.2(f).

Lemma 3.6. *(a) We have $\Lambda(x_1) = \zeta x_0$ where $\zeta \in 1 + v^{-1}\mathbf{Z}[v^{-1}]$.*

(b) We have $\Lambda(B'[\mathbf{d}]) = B'[\mathbf{d}]$.

It is clear that the square of Λ is the identity map of $L_{\mathbf{d}}$ times a non zero scalar $\zeta \in A'$. In particular, $\Lambda(x_1) = \zeta x_0$. Using 3.2(b) we see that $\zeta \in A$. Using 3.5(a) and 2.2(h) we see that $\zeta \in A'_\infty$. Hence we have $\zeta \in \mathbf{Z}[v^{-1}]$.

We shall denote the image of a vector $x \in \mathcal{L}'_{\mathbf{d}}$ under $\mathcal{L}'_{\mathbf{d}} \to \mathcal{L}'_{\mathbf{d}}/v^{-1}\mathcal{L}'_{\mathbf{d}}$ by $\underline{x}$. Since $x_1 \in \mathbf{B}'[\mathbf{d}]$, we must have $\underline{x}_1 \in B'[\mathbf{d}]$. Hence $\underline{x}_1 = \tilde{E_{i_1}} \dots \tilde{E_{i_k}} \underline{x}_0$ for some $i_1, \dots, i_k$. Using 2.2(g) it follows that $\underline{x}_0 = \tilde{F_{i_k}} \dots \tilde{F_{i_1}} \underline{x}_1$; applying Λ we obtain $\underline{x}_1 = \zeta|_{v=\infty} \tilde{E_{\hat{i}_k}} \dots \tilde{E_{\hat{i}_1}} \underline{x}_0$; this shows that $\zeta|_{v=\infty} \neq 0$. We have $\Lambda(\underline{x}_1) = \zeta|_{v=\infty} \underline{x}_0$; this belongs to $B'[\mathbf{d}] \cup \{0\}$ by 3.5(b). Since $\underline{x}_0 \in B'[\mathbf{d}]$ and $\zeta|_{v=\infty} \neq 0$ it follows that $\zeta|_{v=\infty} = 1$ and (a) is proved. Let $b \in B'[\mathbf{d}]$. Then $b = \tilde{E_{j_1}} \dots \tilde{E_{j_k}} \underline{x}_0$ for some $j_1, \dots, j_k$. We then have $\Lambda(\tilde{F_{\hat{j}_1}} \dots \tilde{F_{\hat{j}_k}} \underline{x}_1) = b$ by (a) and 3.4. In particular, $\tilde{F_{\hat{j}_1}} \dots \tilde{F_{\hat{j}_k}} \underline{x}_1 \neq 0$. This is contained in $B'[\mathbf{d}] \cup \{0\}$ by 2.2(f) hence it is contained in $B'[\mathbf{d}]$. Thus we have $b = \Lambda(b')$ for some $b' \in B'[\mathbf{d}]$. Thus, $\Lambda(B'[\mathbf{d}])$ contains $B'[\mathbf{d}]$. But $\Lambda(B'[\mathbf{d}])$ has at most as many elements as $B'[\mathbf{d}]$. This forces $\Lambda(B'[\mathbf{d}]) = B'[\mathbf{d}]$.

Lemma 3.7. *(a) The involution $^- : \mathbf{U} \to \mathbf{U}$ commutes with $\Lambda : \mathbf{U} \to \mathbf{U}$.*

(b) The involution $^- : L_{\mathbf{d}} \to L_{\mathbf{d}}$ commutes with $\Lambda : L_{\mathbf{d}} \to L_{\mathbf{d}}$.

It is enough to check (a) on generators, where it is obvious. Let $x \in L_{\mathbf{d}}$. We can write $x = ux_0$ with $u \in \mathbf{U}$. We have, using (a): $\overline{\Lambda(ux_0)} = \overline{\Lambda(u)x_1} = \Lambda(\bar{u})\overline{x_1}$. On the other hand, we have $\Lambda(\overline{ux_0}) = \Lambda(\bar{u}x_0) = \Lambda(\bar{u})x_1$ since $\overline{x_0} = x_0$. It remains to use that $\overline{x_1} = x_1$.

3.8 We now prove Theorem 3.3. Applying $^-$ to the equality $\Lambda(x_1) = \zeta x_0$ we obtain (using 3.7): $\Lambda(x_1) = \bar{\zeta} x_0$. Thus, $\bar{\zeta} x_0 = \zeta x_0$. Hence $\bar{\zeta} = \zeta$. This together with 3.6(a) shows that $\zeta = 1$. Thus 3.8(b) is proved. Let $b \in B'[\mathbf{d}]$ and let $\tilde{b} \in \mathbf{B}'[\mathbf{d}]$ be the corresponding element (see 2.2(h)). Thus $\bar{\tilde{b}} = \tilde{b}$; moreover, $\tilde{b} \in \mathcal{L}'_{\mathbf{d}}$ and its image under the canonical map $\mathcal{L}'_{\mathbf{d}} \to \mathcal{L}'_{\mathbf{d}}/v^{-1}\mathcal{L}'_{\mathbf{d}}$ is b. Consider the element $\Lambda(\tilde{b})$. This element is contained in $\mathcal{L}'_{\mathbf{d},A}$ by 3.2(b), in $\mathcal{L}'_{\mathbf{d}}$ by 3.5(a), and is fixed by $^-$, by 3.7(b); its image in $\mathcal{L}'_{\mathbf{d}} \to \mathcal{L}'_{\mathbf{d}}/v^{-1}\mathcal{L}'_{\mathbf{d}}$ is $\Lambda(b)$ which, by 3.6(b), is in $B'[\mathbf{d}]$. By 2.2(h), we have $\Lambda(\tilde{b}) \in \mathbf{B}'[\mathbf{d}]$. Since any element of $\mathbf{B}'[\mathbf{d}]$ is of the form $\tilde{b}$ we see that $\Lambda(\mathbf{B}'[\mathbf{d}]) \subset \mathbf{B}'[\mathbf{d}]$. Since $\Lambda^2 = 1$, we have $\Lambda(\mathbf{B}'[\mathbf{d}]) = \mathbf{B}'[\mathbf{d}]$. It remains to use 2.10(c).

4. Coinvariants in triple tensor products

4.1 Recall that $\mathbf{U}$ is a Hopf algebra with comultiplication Δ, antipode S and counit ϵ defined by

$$\Delta E_i = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta F_i = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta K_i = K_i \otimes K_i$$

$$SE_i = -K_i^{-1}E_i, \quad SF_i = -F_iK_i, \quad SK_i = K_i^{-1}$$

$$\epsilon E_i = \epsilon F_i = 0, \quad \epsilon K_i = 1.$$

Then $S' = S^{-1}$ is given by

$$S'E_i = -E_iK_i^{-1}, \quad S'F_i = -K_iF_i, \quad SK_i = K_i^{-1}.$$

These maps induce a Hopf algebra structure on U.

Let $\mathbf{d} = (d_1, \dots, d_n) \in \mathbf{N}^n$, $\mathbf{d}' = (d'_1, \dots, d'_n) \in \mathbf{N}^n$ be as in 1.5. Let $L_{\mathbf{d}}, x_0$ be as in 1.7.

Proposition 4.2. *(a) $\mathbf{B} \cap I_{\mathbf{d}} \cap J_{\mathbf{d}'}$ is a basis of $I_{\mathbf{d}} \cap J_{\mathbf{d}'}$.*

(b) The map $u \to ux_0$ defines a bijection of $(\mathbf{B} \cap J_{\mathbf{d}'}) - (\mathbf{B} \cap I_{\mathbf{d}} \cap J_{\mathbf{d}'})$ with an $A'-$basis of $J_{\mathbf{d}'}L_{\mathbf{d}} = \sum_i E_i^{(d'_i+1)}L_{\mathbf{d}}$.

(a) follows immediately from 1.6(a),(b). The isomorphism $\mathbf{U}^+/I_{\mathbf{d}} \cong L_{\mathbf{d}}$ given by $u \to ux_0$ restricts to an isomorphism $J_{\mathbf{d}'}/(I_{\mathbf{d}} \cap J_{\mathbf{d}'}) \cong J_{\mathbf{d}'}L_{\mathbf{d}}$. Hence (b) follows from (a) and 1.6(b).

4.3 We shall regard the dual space $L_{\mathbf{d}}^*$ of $L_{\mathbf{d}}$ as an $\mathbf{U}-$module with $(u\xi)(x) = \xi(S'(u)x)$ for $x \in L_{\mathbf{d}}, \xi \in L_{\mathbf{d}}^*, u \in \mathbf{U}$. We define

$$R_{\mathbf{d}'}L_{\mathbf{d}}^* = \{\xi \in L_{\mathbf{d}}^* | E_i^{(d'_i+1)}\xi = 0 \quad \forall i\}.$$

$$R'_{\mathbf{d}'}L_{\mathbf{d}}^* = \{\xi \in L_{\mathbf{d}}^* | F_i^{(d'_i+1)}\xi = 0 \quad \forall i\}.$$

It is clear that

(a) $R_{\mathbf{d}'}L_{\mathbf{d}}^*$ is precisely the annihilator of $J_{\mathbf{d}'}L_{\mathbf{d}}$ in $L_{\mathbf{d}}^*$.

Let $\mathbf{B}[\mathbf{d}]^*$ be the basis of $L_{\mathbf{d}}^*$ dual to the basis $\mathbf{B}[\mathbf{d}]$ of $L_{\mathbf{d}}$.

A basis of a vector space is said to be compatible with a collection of subspaces of that vector space if its intersection with any subspace in the collection is a basis for that subspace.

Theorem 4.4. *(a) The basis $\mathbf{B}[\mathbf{d}]$ of $L_{\mathbf{d}}$ is compatible with the collection of subspaces $\sum_i E_i^{(d'_i+1)}L_{\mathbf{d}} \quad (\mathbf{d}' \in \mathbf{N}^n)$.*

(b) The basis $\mathbf{B}[\mathbf{d}]$ of $L_{\mathbf{d}}$ is compatible with the collection of subspaces $\sum_i F_i^{(d'_i+1)}L_{\mathbf{d}}$ $(\mathbf{d}' \in \mathbf{N}^n)$.

(c) The basis $\mathbf{B}[\mathbf{d}]^*$ of $L_{\mathbf{d}}^*$ is compatible with the collection of subspaces $R_{\mathbf{d}'}L_{\mathbf{d}}^* \quad (\mathbf{d}' \in \mathbf{N}^n)$.

(d) The basis $\mathbf{B}[\mathbf{d}]^*$ of $L_{\mathbf{d}}^*$ is compatible with the collection of subspaces $R'_{\mathbf{d}'}L_{\mathbf{d}}^* \quad (\mathbf{d}' \in \mathbf{N}^n)$.

(The existence of a basis as in (c) has been conjectured for $v = 1$ by Gelfand, Zelevinskii and Baclawski and recently proved by Mathieu [M].) Now (a) follows from 4.2(b); (c) follows from (a) and 4.3(a); (b) and (d) follow from (a) and (c) respectively using the involution Λ and 3.3(a).

4.5 A tensor product (over A') of two or more $\mathbf{U}-$modules will be regarded as a $\mathbf{U}-$ module in the usual way, using the comultiplication in $\mathbf{U}$. For a $\mathbf{U}-$ module M we denote

$$M^{\mathbf{U}} = \{x \in M | ux = \epsilon(u)x \quad \forall u \in \mathbf{U}\}.$$

Similarly $M^{\mathbf{U}^+} = \{x \in M | ux = \epsilon(u)x \quad \forall u \in \mathbf{U}^+\}$.

We shall denote by $M_{\mathbf{U}}$ the quotient of M by the subspace generated by the elements $ux - \epsilon(u)x$ for all $u \in \mathbf{U}$ and all $x \in M$.

The following result and its proof is a $q-$analog of [B, Ex. 14,§9,Ch.VIII].

Proposition 4.6. *Let $\mathbf{d}, \mathbf{d}', \mathbf{d}'' \in \mathbf{N}^n$. Let $(x_0, x_1, L_{\mathbf{d}})$ be as in 3.1, and let $(x'_0, x'_1, L_{\mathbf{d}'})$, $(x''_0, x''_1, L_{\mathbf{d}''})$ be defined in the same way. The rule $\xi'' \otimes \xi' \otimes \xi \to \xi''(x''_1)\xi'(x'_0)\xi$ defines an isomorphism*

(a) $$(L^*_{\mathbf{d}''} \otimes L^*_{\mathbf{d}'} \otimes L^*_{\mathbf{d}})^{\mathbf{U}} \cong R_{\mathbf{d}'}(L^*_{\mathbf{d}})_{(\hat{\mathbf{d}''} - \mathbf{d}')}.$$

(A subscript $(\mathbf{d})$ means taking the subspace on which K_i acts as multiplication by v^{d_i} for all i. We set $\hat{\mathbf{d}}'' = (d''_{\hat{1}}, \dots, d''_{\hat{n}})$.)

The proof will be given in 4.10, after a number of lemmas.

Lemma 4.7. *The rule $\xi' \otimes \xi \to [x \to \xi'(x)\xi]$ defines an isomorphism*

$$(L^*_{\mathbf{d}'} \otimes L^*_{\mathbf{d}})^{\mathbf{U}^+} \cong Hom_{\mathbf{U}^+}(L_{\mathbf{d}'}, L^*_{\mathbf{d}}).$$

The proof is standard.

Lemma 4.8. *The rule $f \to f(x'_0)$ defines an isomorphism*

$$Hom_{\mathbf{U}^+}(L_{\mathbf{d}'}, L^*_{\mathbf{d}}) \cong R_{\mathbf{d}'}(L^*_{\mathbf{d}}).$$

This follows from the fact that $L_{\mathbf{d}'} \cong \mathbf{U}^+/I_{\mathbf{d}'}$ as an $\mathbf{U}^+$-module and the obvious fact that

$$\mathrm{Hom}_{\mathbf{U}^+}(\mathbf{U}^+/I_{\mathbf{d}'}, L^*_{\mathbf{d}}) \cong R_{\mathbf{d}'}(L^*_{\mathbf{d}}).$$

Lemma 4.9. *Let $M = L^*_{\mathbf{d}'} \otimes L^*_{\mathbf{d}}$.*

(a) The rule $f \to f(x''_1)$ defines an isomorphism

$$Hom_{\mathbf{U}}(L_{\mathbf{d}''}, M) \cong M^{\mathbf{U}^+}_{(\hat{\mathbf{d}''})}.$$

(b) The rule $\xi'' \otimes m \to [x \to \xi''(x)m]$ defines an isomorphism

$$(L^*_{\mathbf{d}''} \otimes M)^{\mathbf{U}} \cong Hom_{\mathbf{U}}(L_{\mathbf{d}''}, M).$$

The proof is standard.

4.10 We now prove Proposition 4.6. The left hand side of 4.6(a) is isomorphic to $M^{\mathbf{U}^+}_{(\hat{\mathbf{d}''})}$ (by 4.9(b) and (a)); this is isomorphic to $\mathrm{Hom}_{\mathbf{U}^+}(L_{\mathbf{d}'}, L^*_{\mathbf{d}})_{(\hat{\mathbf{d}''})}$ (by 4.7); this is isomorphic to the right hand side of 4.6(a) (by 4.8). The proposition follows by composing these isomorphisms.

Corollary 4.11. *The rule*

$$y \to \text{ coset of } x_1'' \otimes x_0' \otimes y$$

defines an isomorphism

$$(L_{\mathbf{d}}/J_{\mathbf{d}'}L_{\mathbf{d}})_{(-\hat{\mathbf{d}''}+\mathbf{d}')} \cong (L_{\mathbf{d}''} \otimes L_{\mathbf{d}'} \otimes L_{\mathbf{d}})_{\mathbf{U}}.$$

This follows from 4.6 by duality.

4.12 The statement of 4.11 remains true in the following integral form: the same rule as in 4.11 defines an isomorphism

$$(\mathcal{L}_{\mathbf{d},A}/\sum_j E_j^{(d_j'+1)}\mathcal{L}_{\mathbf{d},A})_{(-\hat{\mathbf{d}''}+\mathbf{d}')} \cong (\mathcal{L}_{\mathbf{d}'',A} \otimes L_{\mathbf{d}'} \otimes L_{\mathbf{d}})_U.$$

(Here the tensor products are taken over A and the subscript U denotes $U-$coinvariants, i.e. the largest quotient on which U acts via ϵ.) The proof is the same except that one should replace $I_{\mathbf{d}}$ by the left ideal $\mathcal{I}_{\mathbf{d}}$ in U^+ generated by the elements $E_i^{(N)}$ for all i and all $N \geq d_i+1$; similarly for $J_{\mathbf{d}}$. (We have $U^+/\mathcal{I}_{\mathbf{d}} \cong \mathcal{L}_{\mathbf{d},A}$.)

4.13 For any subset $\hat{\mathbf{B}}$ of $\mathbf{B}$ and any $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{Z}^n$, we shall denote

$$\hat{\mathbf{B}}_{(\mathbf{a})} = \{b \in \hat{\mathbf{B}} | K_i b K_i^{-1} = v^{a_i} b \quad \forall i\}.$$

The following result answers a question raised in [BZ] (see also RZ).

Proposition 4.14. *The map*

(a) $$u \to \text{ coset of } x_1'' \otimes x_0' \otimes u x_0$$

defines a bijection of

$$(\mathbf{B} - (\mathbf{B} \cap I_{\mathbf{d}}) - (\mathbf{B} \cap J_{\mathbf{d}'}))_{(-\hat{\mathbf{d}''}+\mathbf{d}'+\mathbf{d})}$$

onto an $A'-$basis of $(L_{\mathbf{d}''} \otimes L_{\mathbf{d}'} \otimes L_{\mathbf{d}})_{\mathbf{U}}$ and onto an $A-$basis of $(\mathcal{L}_{\mathbf{d}'',A} \otimes \mathcal{L}_{\mathbf{d}',A} \otimes \mathcal{L}_{\mathbf{d},A})_U$.

This follows from 4.11, 4.2(b), 1.7(a) and 4.12.

5. Braid group action on $L_{\mathbf{d}}$

5.1 The automorphisms $\tilde{T}_i : \mathbf{U} \to \mathbf{U}$ define an action of the braid group corresponding to W on the algebra $\mathbf{U}$. Let $L_{\mathbf{d}}, x_0, x_1$ be as in 3.1. For each $i \in [1,n]$, there exists a vector space isomorphism $\tilde{T}_i : L_{\mathbf{d}} \to L_{\mathbf{d}}$ such that $\tilde{T}_i(ux) = \tilde{T}_i(u)\tilde{T}_i(x)$ for all $u \in \mathbf{U}$ and all $x \in L_{\mathbf{d}}$. (See [L1,1.3].) Moreover, this isomorphism is uniquely determined up to a non-zero scalar factor. We will normalize it as follows. Note that $\tilde{T}_i^{-1}(x_0)$ is a non-zero scalar times $E_i^{(d_i)}x_0$. Indeed, it is clear that $E_i^{(d_i)}x_0 \neq 0, \tilde{T}_i^{-1}(x_0) \neq 0$, that these two vectors have the same weight and that there is, up to a scalar, exactly one vector of that weight in $L_{\mathbf{d}}$; our assertion follows. We can therefore normalize uniquely $\tilde{T}_i : L_{\mathbf{d}} \to L_{\mathbf{d}}$ so that $\tilde{T}_i^{-1}(x_0) = E_i^{(d_i)}x_0$. It is clear that the isomorphisms $\tilde{T}_i^{-1} : L_{\mathbf{d}} \to L_{\mathbf{d}}$ define a projective representation of the braid group on $L_{\mathbf{d}}$; in other words, we have $\tilde{T}_i^{-1}\tilde{T}_j^{-1}\tilde{T}_i^{-1} = c_{ij}\tilde{T}_j^{-1}\tilde{T}_i^{-1}\tilde{T}_j^{-1}$ if $a_{ij} = -1$ and $\tilde{T}_i^{-1}\tilde{T}_j^{-1} = c_{ij}\tilde{T}_j^{-1}\tilde{T}_i^{-1}$ if $a_{ij} = 0$ where $c_{ij} \in A' - \{0\}$.

Proposition 5.2. *The isomorphisms $\tilde T_i^{-1}: L_{\mathbf d}\to L_{\mathbf d}$ $(1\le i\le n)$ just defined satisfy the braid group relations.*

Assume first that $a_{ij}=0$. We have

$$\tilde T_i^{-1}\tilde T_j^{-1}(x_0)=\tilde T_i^{-1}(E_j^{(d_j)}x_0)=E_j^{(d_j)}\tilde T_i^{-1}(x_0)=E_j^{(d_j)}E_i^{(d_i)}x_0.$$

Similarly,

$$\tilde T_j^{-1}\tilde T_i^{-1}(x_0)=E_i^{(d_i)}E_j^{(d_j)}x_0=E_j^{(d_j)}E_i^{(d_i)}x_0.$$

Thus $\tilde T_i^{-1}\tilde T_j^{-1}(x_0)=\tilde T_j^{-1}\tilde T_i^{-1}(x_0)$ hence $c_{ij}=1$.

Assume next that $a_{ij}=-1$. We have

$$\begin{aligned}\tilde T_i^{-1}\tilde T_j^{-1}\tilde T_i^{-1}(x_0)&=\tilde T_i^{-1}\tilde T_j^{-1}(E_i^{(d_i)}x_0)=\tilde T_i^{-1}(\tilde T_j^{-1}(E_i^{(d_i)})E_j^{(d_j)}x_0)\\&=\tilde T_i^{-1}\tilde T_j^{-1}(E_i^{(d_i)})\tilde T_i^{-1}(E_j^{(d_j)})E_i^{(d_i)}x_0.\end{aligned}$$

Similarly,

$$\tilde T_j^{-1}\tilde T_i^{-1}\tilde T_j^{-1}(x_0)=\tilde T_j^{-1}\tilde T_i^{-1}(E_j^{(d_j)})\tilde T_j^{-1}(E_i^{(d_i)})E_j^{(d_j)}x_0.$$

If we can show that $\tilde T_i^{-1}\tilde T_j^{-1}\tilde T_i^{-1}(x_0)=\tilde T_j^{-1}\tilde T_i^{-1}\tilde T_j^{-1}(x_0)$ then it will follow that $c_{ij}=1$ and we would be done. Hence it is enough to verify the identity

(a) $$\tilde T_i^{-1}\tilde T_j^{-1}(E_i^{(d_i)})\tilde T_i^{-1}(E_j^{(d_j)})E_i^{(d_i)}x_0=\tilde T_j^{-1}\tilde T_i^{-1}(E_j^{(d_j)})\tilde T_j^{-1}(E_i^{(d_i)})E_j^{(d_j)}x_0.$$

By an identity in the proof of [L2, 2.3] we have

$$E_i^{(d_j)}E_j^{(d_i+d_j)}E_i^{(d_i)}=\sum_{k=0}^{d_i}a_k\tilde T_j^{-1}\tilde T_i^{-1}(E_j^{(d_i+d_j-k)})\tilde T_j^{-1}(E_i^{(k)})E_j^{(d_i+d_j-k)}$$

where $a_k\in A$ and $a_{d_i}=1$. The terms of the last sum corresponding to $k<d_i$ are contained in $I_{\mathbf d}$ hence they annihilate x_0; it follows that

$$E_i^{(d_j)}E_j^{(d_i+d_j)}E_i^{(d_i)}x_0=\tilde T_j^{-1}\tilde T_i^{-1}(E_j^{(d_j)})\tilde T_j^{-1}(E_i^{(d_i)})E_j^{(d_j)}x_0.$$

Interchanging i,j we have

$$E_j^{(d_i)}E_i^{(d_i+d_j)}E_j^{(d_j)}x_0=\tilde T_i^{-1}\tilde T_j^{-1}(E_i^{(d_i)})\tilde T_i^{-1}(E_j^{(d_j)})E_i^{(d_i)}x_0.$$

Hence (a) is equivalent to the identity

$$E_i^{(d_j)}E_j^{(d_i+d_j)}E_i^{(d_i)}x_0=E_j^{(d_i)}E_i^{(d_i+d_j)}E_j^{(d_j)}x_0$$

which follows from the following known identity

$$E_i^{(d_j)}E_j^{(d_i+d_j)}E_i^{(d_i)}=E_j^{(d_i)}E_i^{(d_i+d_j)}E_j^{(d_j)}$$

(a q-analog of Verma's identity). The proposition is proved.

Corollary 5.3. *For any $w \in W$ there is a unique isomorphism $\tilde{T}_w : L_{\mathbf{d}} \to L_{\mathbf{d}}$ such that $\tilde{T}_w = \tilde{T}_{i_1} \dots \tilde{T}_{i_k} : \mathbf{U} \to \mathbf{U}$ whenever $w = s_{i_1} \dots s_{i_k}$ with $k = l(w)$; we have $\tilde{T}_w(ux) = \tilde{T}_w(u)\tilde{T}_w(x)$ for all $u \in \mathbf{U}$ and all $x \in L_{\mathbf{d}}$.*

5.4 Let $P : \mathbf{U} \to \mathbf{U}$ be the $A'-$algebra automorphism defined by $P(E_i) = -E_iK_i$, $P(F_i) = -K_i^{-1}F_i$, $\quad P(K_i) = K_i$. The inverse $P' : \mathbf{U} \to \mathbf{U}$ of P is given by $P'(E_i) = -E_iK_i^{-1}$, $P'(F_i) = -K_iF_i$, $\quad P'(K_i) = K_i$. Clearly, there is a vector space isomorphism (unique up to a non-zero scalar) $P : L_{\mathbf{d}} \cong L_{\mathbf{d}}$ such that $P(ux) = P(u)P(x)$ for all $u \in \mathbf{U}$ and all $x \in L_{\mathbf{d}}$. It takes x_0 to a non-zero mulptiple of x_0; we normalize it so that $P(x_0) = x_0$. Its inverse is denoted $P' : L_{\mathbf{d}} \cong L_{\mathbf{d}}$.

Consider the $\mathbf{z}-$weight space of $L_{\mathbf{d}}$ where $\mathbf{z} = (z_1, \dots, z_n)$ is given by $z_i = -d_i + \sum_j a_{ij}t_j$ for some $(t_1, \dots, t_n) \in \mathbf{Z}^n$. It is easy to see that P acts on this weight space as multiplication by the scalar

$$(-1)^s v^{s'} \text{ where } s = \sum_i t_i, s' = \sum_{i,j} a_{ij}t_it_j/2 - \sum_i (d_i - t_i). \tag{a}$$

Proposition 5.5. *(a) We have $\tilde{T}_{w_0} = P \circ \Lambda$ hence $\tilde{T}_{w_0}^{-1} = \Lambda \circ P'$ as automorphisms of $\mathbf{U}$.*

(b) We have $\tilde{T}_{w_0}^{-1} = \Lambda \circ P'$ hence $\tilde{T}_{w_0} = P \circ \Lambda$ as automorphisms of $L_{\mathbf{d}}$.

Before giving the proof we need a lemma.

Lemma 5.6. *Let $w \in W$ and let $i_1, \dots, i_k$ be a sequence in $[1, n]$ such that $w = s_{i_1} \dots s_{i_k}$ and $k = l(w)$. We have*

$$\tilde{T}_{w^{-1}}^{-1}(x_0) = \tilde{T}_{i_1}^{-1} ... \tilde{T}_{i_{k-1}}^{-1}(E_{i_k}^{(d_{i_k})}) \dots \tilde{T}_{i_1}^{-1}\tilde{T}_{i_2}^{-1}(E_{i_3}^{(d_{i_3})})\tilde{T}_{i_1}^{-1}(E_{i_2}^{(d_{i_2})})E_{i_1}^{(d_{i_1})}x_0. \tag{a}$$

We have

$$\tilde{T}_{w^{-1}}^{-1}(x_0) = E_{i_1}^{(N)}\tilde{T}_{w^{-1}s_{i_1}}^{-1}(x_0). \tag{b}$$

where $N = -\sum_h d_h(w^{-1}(\alpha_{i_1}))_h \in \mathbf{N}$ and $(w^{-1}(\alpha_{i_1}))_h$ is the $h-th$ coordinate of $w^{-1}(\alpha_{i_1})$.

We use induction on k; for $k = 0, 1$ both (a) and (b) are known, hence we may assume that $k \geq 2$. Using the identity $\tilde{T}_{i_1}^{-1}(x_0) = E_{i_1}^{(d_{i_1})}x_0$, the induction hypothesis and the fact that $\tilde{T}_{i_1}^{-1}$ is a homomorphism we see that the right hand side of (a) is equal to $\tilde{T}_{i_1}^{-1}(\tilde{T}_{w^{-1}s_{i_1}}(x_0))$ which, by 5.3(a), is equal to $\tilde{T}_{w^{-1}}^{-1}(x_0)$. Thus (a) is proved. We set $i_1 = i$. We can find $j \in [1, n]$ such that $l(s_js_iw) < l(s_iw)$; we have $i \neq j$. Let $< s_i, s_j >$ be the dyhedral subgroup of W generated by s_i, s_j and let z be the element of minimal length in the coset $< s_i, s_j > w$. We must be in one of the following three cases:

$$a_{ij} = 0, w = s_is_jz, l(z) = k - 2 \tag{c}$$

(d) $$a_{ij}=-1, w=s_is_js_iz, l(z)=k-3$$

(e.) $$a_{ij}=-1, w=s_is_jz, l(z)=k-2$$

Assume first that we are in case (c). Using the induction hypothesis we have

$$\tilde T^{-1}_{w^{-1}}(x_0)=\tilde T_i^{-1}\tilde T_j^{-1}\tilde T^{-1}_{z^{-1}}(x_0)=\tilde T_j^{-1}\tilde T_i^{-1}\tilde T^{-1}_{z^{-1}}(x_0)=\tilde T_j^{-1}(E_i^{(N)}\tilde T^{-1}_{z^{-1}}(x_0))$$

hence

$$\tilde T^{-1}_{w^{-1}}(x_0)=E_i^{(N)}\tilde T_j^{-1}\tilde T^{-1}_{z^{-1}}(x_0)$$

as required. Assume now that $a_{ij}=-1$. Let

$$N'=\sum_h d_h(z^{-1}(\alpha(i))_h\in\mathbf{N},\quad N''=\sum_h d_h(z^{-1}(\alpha(j))_h\in\mathbf{N}$$

where the subscript h denotes the $h-$th coordinate. In case (d) we have $N=N''$, while in case (e) we have $N=N'+N''$. Assume that we are in case (d). Using the induction hypothesis we have

$$\begin{aligned}\tilde T^{-1}_{w^{-1}}(x_0)&=\tilde T_i^{-1}\tilde T_j^{-1}\tilde T_i^{-1}\tilde T^{-1}_{z^{-1}}(x_0)=\tilde T_j^{-1}\tilde T_i^{-1}\tilde T_j^{-1}\tilde T^{-1}_{z^{-1}}(x_0)=\tilde T_j^{-1}\tilde T_i^{-1}(E_j^{(N'')}\tilde T^{-1}_{z^{-1}}(x_0))\\&=E_i^{(N'')}\tilde T_j^{-1}\tilde T_i^{-1}\tilde T^{-1}_{z^{-1}}(x_0)\end{aligned}$$ ∎

as required. Finally, assume that we are in case (e). Using the induction hypothesis we have

$$\begin{aligned}\tilde T^{-1}_{w^{-1}}(x_0)&=\tilde T_i^{-1}\tilde T_j^{-1}\tilde T^{-1}_{z^{-1}}(x_0)=\tilde T_i^{-1}(E_j^{(N'')}\tilde T^{-1}_{z^{-1}}(x_0))=\tilde T_i^{-1}(E_j^{(N'')})\tilde T_i^{-1}\tilde T^{-1}_{z^{-1}}(x_0))\\&=\tilde T_i^{-1}(E_j^{(N'')})E_i^{(N')}\tilde T^{-1}_{z^{-1}}(x_0).\end{aligned}$$

On the other hand,

$$E_i^{(N)}\tilde T_{w^{-1}s_i}(x_0)=E_i^{(N)}\tilde T_j^{-1}\tilde T^{-1}_{z^{-1}}(x_0)=E_i^{(N)}E_j^{(N'')}\tilde T^{-1}_{z^{-1}}(x_0).$$

It remains to prove the identity

(f) $$\tilde T_i^{-1}(E_j^{(N'')})E_i^{(N')}\tilde T^{-1}_{z^{-1}}(x_0)=E_i^{(N)}E_j^{(N'')}\tilde T^{-1}_{z^{-1}}(x_0).$$

Note that

$$E_i^{(N'+1)}\tilde T^{-1}_{z^{-1}}(x_0)=0.$$

(From the definition of N' we see that the left hand side of this equation has weight such that the corresponding weight space of $L_{\mathbf{d}}$ is zero.)

Hence to prove (f) it is enough to show that

(g) $$\tilde T_i^{-1}(E_j^{(N'')})E_i^{(N')}-E_i^{(N'+N'')}E_j^{(N'')}\in\mathbf{U}^+E_i^{(N'+1)}.$$

By an identity in the proof of [**L2**, 2.3] we have

$$E_i^{(N'+N'')}E_j^{(N'')}=\sum_{h=0}^{N''}a_hE_j^{(N''-h)}\tilde T_j^{-1}(E_i^{(h)})E_i^{(N'+N''-h)}$$

where $a_h\in A$ and $a_{N''}=1$. The terms corresponding to $h<N''$ are contained in $\mathbf{U}^+E_i^{(N'+1)}$ and (g) follows. The lemma is proved.

5.7 We now prove Proposition 5.5. Let $i\in[1,n]$ and let $\mathbf{i}=(i_1,\dots,i_\nu)\in\mathcal{X}$ be such that $i_\nu=i$. We have

(a) $$\tilde T_{w_0}(F_i)=\tilde T_{i_1}\dots\tilde T_{i_{\nu-1}}\tilde T_i(F_i)=-\tilde T_{i_1}\dots\tilde T_{i_{\nu-1}}(E_iK_i).$$

Since $s_{i_1}\dots s_{i_{\nu-1}}(\alpha_i)=\alpha_{\hat i}$ we deduce from (a) and [L2, 1.3(c)] that

$$\tilde T_{w_0}(F_i)=-E_{\hat i}K_{\hat i}.$$

Similarly,

$$\tilde T_{w_0}(E_i)=-K_{\hat i}^{-1}F_{\hat i}$$

$$\tilde T_{w_0}(K_i)=K_{\hat i}^{-1}$$

and 5.5(a) follows. From 5.5(a) it follows immediately that for some non-zero $c\in A'$ we have $\tilde T_{w_0}^{-1}=c\Lambda\circ P'$ as automorphisms of $L_{\mathbf{d}}$. We have $\Lambda(x_0)=x_1$ and $P'(x_0)=x_0$. Hence $\tilde T_{w_0}^{-1}(x_0)=cx_1$ and we must only show that $c=1$. From 5.6(b) we see by induction on $l(w)$ that $\tilde T_{w^{-1}}^{-1}(x_0)$ is fixed by $^-:L_{\mathbf{d}}\to L_{\mathbf{d}}$. (To start the induction note that $\bar{x_0}=x_0$.) In particular, $\tilde T_{w_0}^{-1}(x_0)$ is fixed by $^-$ hence cx_1 is fixed by $^-$. Since x_1 is part of the canonical basis of $L_{\mathbf{d}}$, it is fixed by $^-$. It follows that $\bar c=c$. Let $\mathbf{i}=(i_1,\dots,i_\nu)\in\mathcal{X}$ and let $\mathbf{i}'=(i'_1,\dots,i'_\nu)\in\mathcal{X}$ be defined by $s_{i'_j}=w_0s_{i_{\nu-j+1}}w_0^{-1}$. By 5.6(a) we have

$$\tilde T_{w_0}^{-1}(x_0)=\tilde T_{i_1}^{-1}...\tilde T_{i_{\nu-1}}^{-1}(E_{i_\nu}^{(d_{i_\nu})})\dots\tilde T_{i_1}^{-1}\tilde T_{i_2}^{-1}(E_{i_3}^{(d_{i_3})})\tilde T_{i_1}^{-1}(E_{i_2}^{(d_{i_2})})E_{i_1}^{(d_{i_1})}x_0.$$

As in [L2, 2.11]) we see that the right hand side of this equality is equal to

$$E_{i'_1}^{(d_{i'_1})}\tilde T_{i'_1}(E_{i'_2}^{(d_{i_2})})\dots\tilde T_{i'_1}...\tilde T_{i'_{\nu-1}}(E_{i'_\nu}^{(d_{i'_\nu})})x_0.$$

Thus we have $\tilde T_{w_0}^{-1}(x_0)=bx_0$ where $b\in\mathcal{L}$ is such that its image in $\mathcal{L}/v^{-1}\mathcal{L}$ is in B. (See 1.3.) By the results in 1.7, we see that $bx_0\in\mathcal{L}_{\mathbf{d}}$ and its image in $\mathcal{L}_{\mathbf{d}}/v^{-1}\mathcal{L}_{\mathbf{d}}$ is in $B[\mathbf{d}]$. But $bx_0=cx_1$. Thus $cx_1\in\mathcal{L}_{\mathbf{d}}$; since x_1 is part of a $\mathbf{Z}[v^{-1}]$ basis of $\mathcal{L}_{\mathbf{d}}$ it follows that $c\in\mathbf{Z}[v^{-1}]$. As we have seen, the image of cx_1 in $\mathcal{L}_{\mathbf{d}}/v^{-1}\mathcal{L}_{\mathbf{d}}$ is in $B[\mathbf{d}]$. But so is the image of x_1; it follows that $c\in1+v^{-1}\mathbf{Z}[v^{-1}]$. As we have seen earlier, we have $\bar c=c$. This now forces $c=1$. The proposition is proved.

Corollary 5.8. *$\tilde{T}_w : L_{\mathbf{d}} \to L_{\mathbf{d}}$ defines an isomorphism $\mathcal{L}_{\mathbf{d},A} \cong \mathcal{L}_{\mathbf{d},A}$, for any $w \in W$.*

Assume first that $w = s_i$ so that $\tilde{T}_w = \tilde{T}_i$. If $x \in \mathcal{L}_{\mathbf{d},A}$, we can write $x = ux_0$ for some $u \in U$. We have $\tilde{T}_i^{-1}(x) = \tilde{T}_i^{-1}(u)\tilde{T}_i^{-1}(x_0) = \tilde{T}_i^{-1}(u)E_i^{(d_i)}x_0$. We have $\tilde{T}_i^{-1}(U) \subset U$. It follows that $\tilde{T}_i^{-1}$ maps $\mathcal{L}_{\mathbf{d},A}$ into itself. Now for any $w \in W$ we have that $\tilde{T}_w^{-1} : L_{\mathbf{d}} \to L_{\mathbf{d}}$ is a composition of automorphisms of the form $\tilde{T}_i^{-1}$; by the first part of the proof we have that $\tilde{T}_w^{-1}$ maps $\mathcal{L}_{\mathbf{d},A}$ into itself. Since $\tilde{T}_{w_0} = P \circ \Lambda$ and P, Λ map $\mathcal{L}_{\mathbf{d},A}$ into itself, we see that $\tilde{T}_{w_0}$ maps $\mathcal{L}_{\mathbf{d},A}$ into itself. For any $w \in W$ we have $\tilde{T}_w = \tilde{T}_{w_0}\tilde{T}_{w^{-1}w_0}^{-1}$ hence $\tilde{T}_w$ maps $\mathcal{L}_{\mathbf{d},A}$ into itself. The corollary follows.

Corollary 5.9. *The action of $\tilde{T}_{w_0}$ with respect to the canonical basis $\mathbf{B}[\mathbf{d}]$ is given by a monomial matrix: for each $b \in \mathbf{B}[\mathbf{d}]$ there exists $b' \in \mathbf{B}[\mathbf{d}]$ and a scalar λ_b such that $\tilde{T}_{w_0}(b) = \lambda_b b'$. Moreover, the scalar λ_b is $\pm$ a power of v (see 5.4(a)) and it is independent of b for b running through a fixed weight space.*

6. Combinatorics

6.1 Assume that the Dynkin graph corresponding to (a_{ij}) is connected. There is a unique partition $[1, n] = C' \sqcup C''$ with the following property: whenever $i \neq j$ are both in C' or both in C'' we have $a_{ij} = 0$. We may assume that the numbering is such that $C' = [1, t], \quad C'' = [t+1, n]$ for some $t \in [1, n]$. We define two particular elements $\mathbf{i}' = (i'_1, \dots, i'_\nu)$ and $\mathbf{i}'' = (i''_1, \dots, i''_\nu)$ of $\mathcal{X}$ by $\mathbf{i}' = C', C'', C', \dots$ and $\mathbf{i}'' = C'', C', C'', \dots$; in both cases there are h groups (h is the Coxeter number) and the indices in each group (C' or C'') are written in the natural order. Following [L2, 2.6] we define a bijection $R : \mathbf{N}^\nu \to \mathbf{N}^\nu$ as follows. It is known that we can choose a sequence $\mathbf{i}^1, \dots, \mathbf{i}^N$ of elements of $\mathcal{X}$ with the following properties. $\mathbf{i}^1 = \mathbf{i}', \quad \mathbf{i}^N = \mathbf{i}''$ and for each $k \in [1, N-1]$, $\mathbf{i}^{k+1}$ is obtained from $\mathbf{i}^k$ by

(a) replacing three consecutive indices i, j, i (with $a_{ij} = -1$) by j, i, j or

(b) replacing two consecutive indices i, j (with $a_{ij} = 0$) by j, i.

We define $R_k : \mathbf{N}^\nu \to \mathbf{N}^\nu$ by $R_k(\mathbf{c}) = \mathbf{c}'$ where $\mathbf{c}'$ is obtained by replacing the three (resp. two) consecutive coordinates (a, b, c) (resp. (a, b)) corresponding to the positions on which i, j, i (resp. i, j) appear in $\mathbf{i}$ in case (a) (resp. (b)) by $(b+c-\min(a, c), \min(a, c), a+c-\min(a, c))$ (resp. (b, a)); the other coordinates are unchanged. We then define $R : \mathbf{N}^\nu \to \mathbf{N}^\nu$ as the composition $R = R_{N-1} \dots R_2 R_1$. It is known [L2, 2.6] that R is independent of the choices; it is a piecewise linear bijection.

It turns out that the various canonical bases we considered can be naturally parame-
trized by some combinatorial objects defined in terms of R.

6.2 Let Z be the subset of $\mathbf{N}^\nu \times \mathbf{N}^\nu$ consisting of all pairs $\mathbf{c}' = (c'_1, \dots, c'_\nu)$, $\mathbf{c}'' = (c''_1, \dots, c''_\nu)$ such that $R(\mathbf{c}') = \mathbf{c}''$. For any $i \in [1, n]$ we define maps $\lambda_i, \mu_i : Z \to \mathbf{N}$ as follows. Assume first that $i \in C'$. We set $\lambda_i(\mathbf{c}', \mathbf{c}'') = c'_i$. If the last group in $\mathbf{i}'$ (resp. in $\mathbf{i}''$) is C' we set $\mu_i(\mathbf{c}', \mathbf{c}'') = c'_{\nu-t+i}$ (resp. $\mu_i(\mathbf{c}', \mathbf{c}'') = c''_{\nu-t+i}$).

Assume next that $i \in C''$. We set $\lambda_i(\mathbf{c}', \mathbf{c}'') = c''_{i-t}$. If the last group in $\mathbf{i}'$ (resp. in $\mathbf{i}''$) is C'' we set $\mu_i(\mathbf{c}', \mathbf{c}'') = c'_{\nu-n+t+i}$ (resp. $\mu_i(\mathbf{c}', \mathbf{c}'') = c''_{\nu-n+t+i}$).

6.3 Define a map $\chi : Z \to \mathbf{N}^n$ as follows. Write $s_{i'_1} \dots s_{i'_{k-1}}(\alpha_{i'_k}) = \sum_{j=1}^n p_j^k \alpha_j$ for $1 \le k \le \nu$ and set $\chi(\mathbf{c}', \mathbf{c}'') = \sum_{k=1}^{\nu} p_j^k c'_k$. Alternatively we can write $s_{i''_1} \dots s_{i''_{k-1}}(\alpha_{i''_k}) = \sum_{j=1}^n {p'}_j^k \alpha_j$ for $1 \le k \le \nu$ and set $\chi(\mathbf{c}', \mathbf{c}'') = \sum_{k=1}^{\nu} {p'}_j^k c''_k$. (The two definitions coincide by [**L2**, 2.9].)

6.4 For any $i \in [1, n]$ we define an (injective) map $\phi_i : Z \to Z$ by

$$\phi_i(\mathbf{c}', \mathbf{c}'') = (\tilde{\mathbf{c}'}, \tilde{\mathbf{c}''})$$

as follows. If $1 \le i \le t$, $\tilde{\mathbf{c}'}$ is obtained from $\mathbf{c}'$ by replacing c'_i by $c'_i + 1$ and leaving the other coordinates unchanged; we set $\tilde{\mathbf{c}''} = R(\tilde{\mathbf{c}'})$.

If $t+1 \le i \le n$, $\tilde{\mathbf{c}''}$ is obtained from $\mathbf{c}''$ by replacing c''_{i-t} by $c'_{i-t}+1$ and leaving the other coordinates unchanged; we set $\tilde{\mathbf{c}'} = R^{-1}(\tilde{\mathbf{c}''})$.

In the following result we preserve the notations of 4.6.

Theorem 6.5. *(a) We have a natural bijection $B \cong Z$ defined by $b \to (\mathbf{c}', \mathbf{c}'')$ where*

$$b = \pi(E_{\mathbf{i}'}^{\mathbf{c}'}) = \pi(E_{\mathbf{i}''}^{\mathbf{c}''}) \in \mathcal{L}/v^{-1}\mathcal{L}.$$

(b) Under the bijection (a), the map $\phi_i : Z \to Z$ (see 6.4) corresponds to the map $B \to B$ defined by $\tilde{E}_i$.

(c) The composition of $\pi : \mathbf{B} \cong B$ with the bijection (a) restricts to a bijection

$$\mathbf{B} \cap J_{\mathbf{d}'} \cong \{(\mathbf{c}', \mathbf{c}'') \in Z | \lambda_i(\mathbf{c}', \mathbf{c}'') \ge d'_i + 1 \quad \forall i\},$$

to a bijection

$$\mathbf{B} \cap I_{\mathbf{d}} \cong \{(\mathbf{c}', \mathbf{c}'') \in Z | \mu_i(\mathbf{c}', \mathbf{c}'') \ge d_i + 1 \quad \forall i\},$$

and to a bijection

$$\mathbf{B} \cap I_{\mathbf{d}} \cap J_{\mathbf{d}'} \cong \{(\mathbf{c}', \mathbf{c}'') \in Z | \lambda_i(\mathbf{c}', \mathbf{c}'') \ge d'_i + 1, \quad \mu_i(\mathbf{c}', \mathbf{c}'') \ge d_i + 1 \quad \forall i\}.$$

(d) Let

$$Z_{\mathbf{d}} = \{(\mathbf{c}', \mathbf{c}'') \in Z | \mu_i(\mathbf{c}', \mathbf{c}'') \le d_i \quad \forall i\}.$$

We have a natural bijection

$$B[\mathbf{d}] \cong Z_{\mathbf{d}}$$

defined by $b \to (\mathbf{c}', \mathbf{c}'')$ where

$$b = \tau(E_{\mathbf{i}'}^{\mathbf{c}'} x_0) = \tau(E_{\mathbf{i}''}^{\mathbf{c}''} x_0) \in \mathcal{L}_{\mathbf{d}}/v^{-1}\mathcal{L}_{\mathbf{d}}.$$

(e) Let $\phi_i[\mathbf{d}] : Z_{\mathbf{d}} \to Z_{\mathbf{d}} \cup \{0\}$ be the map defined by $\phi_i[\mathbf{d}](z) = \phi_i(z)$ if $\phi_i(z) \in Z_{\mathbf{d}}$ and $\phi_i[\mathbf{d}](z) = 0$ if $\phi_i(z) \notin Z_{\mathbf{d}}$. Under the bijection (d), the map $\phi_i[\mathbf{d}] : Z_{\mathbf{d}} \to Z_{\mathbf{d}} \cup \{0\}$ corresponds to the map $B[\mathbf{d}] \to B[\mathbf{d}] \cup \{0\}$ defined by $\tilde{E}_i$.

(f) Let $Z(\mathbf{d}'', \mathbf{d}', \mathbf{d})$ be the set

$$\{(\mathbf{c}', \mathbf{c}'') \in Z | \mu_i(\mathbf{c}', \mathbf{c}'') \le d_i, \quad \lambda_i(\mathbf{c}', \mathbf{c}'') \le d'_i \quad \forall i, \quad \chi(\mathbf{c}', \mathbf{c}'') = -\hat{\mathbf{d}}'' + \mathbf{d}' + \mathbf{d}\}.$$

Consider the composition

$$Z(\mathbf{d}'', \mathbf{d}', \mathbf{d}) \to Z \cong B \cong \mathbf{B} \to (L_{\mathbf{d}''} \otimes L_{\mathbf{d}'} \otimes L_{\mathbf{d}})_{\mathbf{U}}$$

(the first map is the obvious imbedding, the second map is as in (a), the third map is defined by π, the fourth map is given by 4.14(a)). This composition defines a bijection of $Z(\mathbf{d}'', \mathbf{d}', \mathbf{d})$ onto an A'−basis of $(L_{\mathbf{d}''} \otimes L_{\mathbf{d}'} \otimes L_{\mathbf{d}})_{\mathbf{U}}$.

(a) follows from [L2, §2,3]; (b) follows from 2.5; (c) and (d) follow from [**L2**, §8]; (e) follows from (d) and (b); (f) follows from (c) and 4.14.

7. Multiplication by $E_i^{(r)}$

7.1 In this section we shall fix $i \in [1, n]$. We shall choose, as we may, an element $\mathbf{i} = (i_1, \dots, i_\nu) \in \mathcal{X}$ such that $i_1 = i$ and an orientation of the Dynkin graph to which $\mathbf{i}$ is adapted. (See [L2, 4.7, 4.12(b)].) We write $h \to k$ when an edge of the Dynkin graph is oriented from h to k. Note that i is a *sink* for the orientation, that is, $i \to j$ never holds. For any $\mathbf{d} \in \mathbf{N}^n$, we consider the vector space $\mathbf{E}_{\mathbf{d}} = \oplus_{h \to k} \mathrm{Hom}(\mathbf{C}^{d_h}, \mathbf{C}^{d_k})$ with the action of the algebraic group $G_{\mathbf{d}} = \prod_h Aut(\mathbf{C}^{d_h})$ given by $(g_h) : (f_k^h) \to (g_k f_k^h g_h^{-1})$. The orbits of $G_{\mathbf{d}}$ on $\mathbf{E}_{\mathbf{d}}$ are indexed, as in [L2, 4.16], by the elements $\mathbf{c} \in \mathbf{N}^\nu$ of $\mathbf{i}$−homogeneity $\mathbf{d}$. (See [L2, 2.8].) We shall write $\mathcal{O}_{\mathbf{c}}$ for the $G_{\mathbf{d}}$−orbit on $\mathbf{E}_{\mathbf{d}}$ corresponding to $\mathbf{c}$.

A point of $\mathbf{E}_{\mathbf{d}}$ may be regarded as a module for the quiver defined by the given orientation.

For any $r' \ge 0$ we define $\mathbf{E}_{\mathbf{d},r'}$ to be the set of all $f \in \mathbf{E}_{\mathbf{d}}$ such that the sum (over all j such that $j \to i$) of the images of $f_i^j : \mathbf{C}^{d_j} \to \mathbf{C}^{d_i}$ has codimension r' in $\mathbf{C}^{d_i}$. The $\mathbf{E}_{\mathbf{d},r'}$ form a partition of $\mathbf{E}_{\mathbf{d}}$ so that, for any $r \ge 0$, the union $\cup_{r' \ge r} \mathbf{E}_{\mathbf{d},r'}$ is closed in $\mathbf{E}_{\mathbf{d}}$.

We have

(a) $\mathcal{O}_{\mathbf{c}} \subset \mathbf{E}_{\mathbf{d},r'}$ if and only if $c_1 = r'$.

Proposition 7.2. *Let $b \in \mathbf{B}[\mathbf{d}]$, $N \ge 0$. Then:*

(a) $E_i^{(N)} b = \sum_{b' \in \mathbf{B}[\mathbf{d}]} c_{b'} b'$ where the $c_{b'} \in A$ have ≥ 0 coefficients.

(b) $F_i^{(N)} b = \sum_{b' \in \mathbf{B}[\mathbf{d}]} c'_{b'} b'$ where the $c'_{b'} \in A$ have ≥ 0 coefficients.

(a) is just [L2, 10.11]; (b) follows from (a) by applying Λ and using 3.3.

Proposition 7.3. *Let $s \ge 0$. The intersection $\mathbf{B} \cap E_i^{(s)} \mathbf{U}^+$ is an A'−basis of $E_i^{(s)} \mathbf{U}^+$.*

The proof is almost the same as that of [L2, 8.5]. Let $B_{\mathbf{i},s}$ be the set of all elements $E_{\mathbf{i}}^{\mathbf{c}}$ with $\mathbf{c} = (c_1, \dots, c_\nu) \in \mathbf{N}^\nu$ such that $c_1 \ge s$. Clearly, $B_{\mathbf{i},s}$ is an A'−basis of $E_i^{(s)} \mathbf{U}^+$. We have

$$\overline{E_{\mathbf{i}}^{\mathbf{c}}} = \sum_{\mathbf{c}' \in \mathbf{N}^\nu} \omega_{\mathbf{c}'}^{\mathbf{c}} E_{\mathbf{i}}^{\mathbf{c}'}$$

with $\omega^{\mathbf{c}}_{\mathbf{c}'} \in A$. Now $E_i^{(s)}\mathbf{U}^+$ is stable under $^- : \mathbf{U}^+ \to \mathbf{U}^+$. Hence

(a) if $\mathbf{c} = (c_1, \dots, c_\nu)$ satisfies $c_1 \ge s$ and $\omega^{\mathbf{c}}_{\mathbf{c}'} \ne 0$, then $\mathbf{c}' = (c'_1, \dots, c'_\nu)$ satisfies $c'_1 \ge s$.

Let $\mathcal{E}^{\mathbf{c}}$ be the unique element of $\mathbf{B}$ such that $\pi(\mathcal{E}^{\mathbf{c}}) = \pi(E^{\mathbf{c}}_{\mathbf{i}})$. We have

$$\mathcal{E}^{\mathbf{c}} = \sum_{\mathbf{c}' \in \mathbf{N}^\nu} \zeta^{\mathbf{c}}_{\mathbf{c}'} E^{\mathbf{c}'}_{\mathbf{i}}$$

where $\zeta^{\mathbf{c}}_{\mathbf{c}'} \in \mathbf{Z}[v^{-1}]$. From the system of equations [L2, 7.11(a)] we see, using (a), that

(b) if $\mathbf{c} = (c_1, \dots, c_\nu)$ satisfies $c_1 \ge s$ and $\zeta^{\mathbf{c}}_{\mathbf{c}'} \ne 0$, then $\mathbf{c}' = (c'_1, \dots, c'_\nu)$ satisfies $c'_1 \ge s$.

From (b) it follows that, for any $\mathbf{c} \in \mathbf{N}^\nu$ such that $c_1 \ge s$, the element $\mathcal{E}^{\mathbf{c}}$ is a linear combination of elements in $B_{\mathbf{i},s}$; in particular, it is contained in $E_i^{(s)}\mathbf{U}^+$. This linear combination is given by a "unipotent" matrix (with rows and columns indexed by the elements $\mathbf{c}$ as above). It follows that any element of $B_{\mathbf{i},s}$ is a linear combination of elements $\mathcal{E}^{\mathbf{c}}$ with $\mathbf{c} \in \mathbf{N}^\nu$ such that $c_1 \ge s$. Since $B_{\mathbf{i},s}$ is an $A'-$basis of $E_i^{(s)}\mathbf{U}^+$, it follows that

(c) $\{\mathcal{E}^{\mathbf{c}} | \mathbf{c} \in \mathbf{N}^\nu$ such that $c_1 \ge s\}$ is an $A'-$basis of $E_i^{(s)}\mathbf{U}^+$.

The Proposition follows.

Corollary 7.4. . *Let $b \in \mathbf{B}$. Let $\rho_i(b)$ be the largest integer $r \ge 0$ such that $b \in E_i^{(r)}\mathbf{U}^+$.*

(a) If $b = \mathcal{E}^{\mathbf{c}}$ is as in 7.3, then $\rho_i(b) = c_1$.

(b) Moreover, we have $b = \sum_{\mathbf{c}' \in \mathbf{N}^\nu} \zeta^{\mathbf{c}}_{\mathbf{c}'} E^{\mathbf{c}'}_{\mathbf{i}}$ where $\zeta^{\mathbf{c}}_{\mathbf{c}'} \in \mathbf{Z}[v^{-1}]$ is zero unless $\mathbf{c}'$ satisfies $c'_1 \ge \rho_i(b)$.

(c) $\rho_i(b)$ is the largest integer $r \ge 0$ such that $\pi(b) \in \tilde{E}_i^r(B)$.

(a) and (b) follow from 7.3(b) together with the fact that $\zeta^{\mathbf{c}}_{\mathbf{c}} = 1$ (in the proof of 7.3); (c) follows from (a) and 2.5.

The following result singles out a leading term in the product of $E_i^{(r)}$ with an element of $\mathbf{B}$.

Theorem 7.5. *Let $b \in \mathbf{B}$ and let $r \ge 0$. We set $s = \rho_i(b)$. Let $b' \in \mathbf{B}$ be defined by the condition $\pi(b') = \tilde{E}_i{}^r \pi(b)$. We have $\rho_i(b') = s + r$ and $E_i^{(r)} b = [s + r]!([s]!)^{-1}([r]!)^{-1}b' +$ a linear combination of elements $b'' \in \mathbf{B}$ satisfying $\rho_i(b'') > s + r$.*

The special case where $\rho_i(b) = 0$ will be considered in 7.6. Here we shall deduce the general case from this special case.

By 7.4(c) we can find $b_1 \in \mathbf{B}$ such that $\pi(b) = \tilde{E}_i^s \pi(b_1)$. By our assumption we have $E_i^{(s)} b_1 = b+$ an element in $E_i^{(s+1)}\mathbf{U}^+$. Multiplying by $E_i^{(r)}$ we obtain

$$[s + r]!([s]!)^{-1}([r]!)^{-1} E_i^{(s+r)} b_1 = E_i^{(r)} b + \text{ an element in } E_i^{(s+r+1)}\mathbf{U}^+.$$

We have $\pi(b') = \tilde{E}_i^{s+r}\pi(b_1)$. Using again our assumption we have

$$E_i^{(r+s)}b_1 = b' + \text{ an element in } E_i^{(s+r+1)}\mathbf{U}^+.$$

It follows that

$$E_i^{(r)}b = [s+r]!([s]!)^{-1}([r]!)^{-1}b' + \text{ an element in } E_i^{(s+r+1)}\mathbf{U}^+;$$

we then use 7.3 and our assertion follows.

7.6 We shall now prove Theorem 7.5 under the assumption that $\rho_i(b) = 0$. As in 7.4, we have $b = \mathcal{E}^{\mathbf{c}}$ with $c_1 = 0$. By 2.5 we have $b' = \mathcal{E}^{\mathbf{c}^+}$ where $\mathbf{c}^+ = \mathbf{c} + (r, 0, \dots, 0)$. Assume that $\mathbf{c}$ has $\mathbf{i}-$homogeneity $\mathbf{d} \in \mathbf{N}^n$ and that $\mathbf{c}^+$ has $\mathbf{i}-$homogeneity $\mathbf{d}' \in \mathbf{N}^n$. Then $d'_i = d_i + r$ and $d'_j = d_j$ for $j \neq i$.

Consider the closures $\bar{\mathcal{O}}_{\mathbf{c}}$, $\bar{\mathcal{O}}_{\mathbf{c}^+}$ of the orbits $\mathcal{O}_{\mathbf{c}}$, $\mathcal{O}_{\mathbf{c}^+}$ in $\mathbf{E}_{\mathbf{d}}$, $\mathbf{E}_{\mathbf{d}'}$.

Let Y be the set of pairs (f', V) where $f' \in \bar{\mathcal{O}}_{\mathbf{c}^+}$ and V is a submodule of the module for the quiver, defined by f', such that V is isomorphic to the module for the quiver, defined by some point of $\bar{\mathcal{O}}_{\mathbf{c}}$.

We define a (proper) morphism $p : Y \to \bar{\mathcal{O}}_{\mathbf{c}^+}$ by $p(f', V) = f'$. Let K be the intersection cohomology complex of Y. The direct image $p_!K$ is a direct sum of intersection cohomology complexes (with shifts) corresponding to various $G_{\mathbf{d}'}-$orbits contained in $\bar{\mathcal{O}}_{\mathbf{c}^+}$. By the geometric description of multiplication by $E_i^{(r)}$ (see [L2, §9]) we see that it is enough to prove the following statement.

If the intersection cohomology complex corresponding to an orbit $\mathcal{O}_{\mathbf{c}'} \subset \bar{\mathcal{O}}_{\mathbf{c}^+}$ appears (with shift σ) in $p_!K$ then either $\mathcal{O}_{\mathbf{c}'} = \mathcal{O}_{\mathbf{c}^+}$ and $\sigma = 0$, or else $c'_1 > r$ (i.e. we have $\mathcal{O}_{\mathbf{c}'} \subset \sqcup_{r'>r}\mathbf{E}_{\mathbf{d}',r'}$, see 7.1(a)).

It is then enough to show that the restriction of p defines an isomorphism $p^{-1}(\mathcal{U}) \cong \mathcal{U}$, where $\mathcal{U} = \bar{\mathcal{O}}_{\mathbf{c}^+} \cap \mathbf{E}_{\mathbf{d}',r}$.

(Note that $\mathcal{U}$ is an open subset of $\bar{\mathcal{O}}_{\mathbf{c}^+}$ containing $\mathcal{O}_{\mathbf{c}^+}$.)

We define a map $\phi \to (\phi, V)$ from $\mathcal{U}$ to $p^{-1}(\mathcal{U})$ as follows: the $i-$component of V is the sum (over all j such that $j \to i$) of the images of $\phi_i^j : \mathbf{C}^{d'_j} \to \mathbf{C}^{d'_i}$; for $h \neq i$, the $h-$component of V is $\mathbf{C}^{d'_h}$. This map provides an inverse for $p : p^{-1}(\mathcal{U}) \to \mathcal{U}$. This completes the proof.

8. Equidimensionality

8.1 Let I be a finite non-empty set and let $H \subset I \times I$ be a subset such that H is stable under the involution $(i,j) \to (j,i)$ and H does not meet the diagonal. Then I can be regarded as the set of vertices of a graph in which i, j are joined precisely when $(i,j) \in H$ or, equivalently, $(j,i) \in H$. We assume that this graph is connected.

8.2 A module (for our graph) is, by definition, a collection of $\mathbf{C}-$vector spaces $V_i \quad (i \in I)$ and of linear maps $f_j^i : V_i \to V_j$ defined for any $(i,j) \in H$. (This is a module for the quiver in which all orientations of the graph are simultaneously considered.) The dimension of (V_i, f_j^i) is, by definition, the collection $\{\dim V_i | i \in I\}$. A submodule of the module (V_i, f_j^i) is, by definition, a collection of subspaces

$V_i' \subset V_i (i \in I)$ such that f_j^i maps V_i' into V_j' for any $(i,j) \in H$; then the V_i' with restrictions of the f_j^i form a module. An isomorphism between two modules $(\bar{V}_i, \bar{f}_j^i), (V_i, f_j^i)$ is, by definition, a collection of isomorphisms $h_i : \bar{V}_i \cong V_i \quad (i \in I)$ such that $h_j \bar{f}_j^i = f_j^i h_i$ for all $(i,j) \in H$.

We fix $d \in \mathbf{N}^I$; thus d is a function $i \to d_i \in \mathbf{N}$. Let

$$\tilde{\mathbf{E}}_d = \oplus_{(i,j)\in H} \mathrm{Hom}(\mathbf{C}^{d_i}, \mathbf{C}^{d_j}).$$

Clearly, any point of $\tilde{\mathbf{E}}_d$ may be regarded as a module with $V_i = \mathbf{C}^{d_i}$.

An element $(f_j^i) \in \tilde{\mathbf{E}}_d$ is said to be *nilpotent* if there exists a number $N \geq 2$ such that the composition $f_{i_N}^{i_{N-1}} \dots f_{i_3}^{i_2} f_{i_2}^{i_1}$ is equal to zero for any sequence of vertices $i_1, \dots, i_N$ such that $(i_1, i_2) \in H, (i_2, i_3) \in H, \dots, (i_{N-1}, i_N) \in H$.

Define a map

$$\psi : \tilde{\mathbf{E}}_d \to \oplus_{i \in I} \mathrm{Hom}(\mathbf{C}^{d_i}, \mathbf{C}^{d_i})$$

by $\psi(f_j^i) = (\xi_i)$ where $\xi_i = \sum_{j:(i,j)\in H} f_i^j f_j^i$. We have

$$\psi^{-1}(0) = \{(f_j^i) \in \tilde{\mathbf{E}}_d | \sum_{j:(i,j)\in H} f_i^j f_j^i = 0 \quad \forall i\}.$$

Let $\tilde{\mathbf{E}}_d^0$ be the set of all nilpotent elements in $\psi^{-1}(0)$.

The group $G_d = \prod_i Aut(\mathbf{C}^{d_i})$ acts on $\tilde{\mathbf{E}}_d$ by $(g_i) : (f_j^i) \to (g_j f_j^i g_i^{-1})$. This action leaves stable the subvarieties $\psi^{-1}(0)$ and $\tilde{\mathbf{E}}_d^0$ of $\tilde{\mathbf{E}}_d$.

8.3 Given d as in 8.2 and $k \in I$, we define a partition

$$\tilde{\mathbf{E}}_d^0 = \sqcup_{s \geq 0} \tilde{\mathbf{E}}_d^0(k,s)$$

by letting $\tilde{\mathbf{E}}_d^0(k,s)$ be the set of all $(f_j^i) \in \tilde{\mathbf{E}}_d^0$ such that the linear map $\oplus_{j:(j,k)\in H} \mathbf{C}^{d_j} \to \mathbf{C}^{d_k}$ with components f_k^j has cokernel of dimension s.

It is clear that for fixed $s_0 \geq 0$, the union $\cup_{s: s \leq s_0} \tilde{\mathbf{E}}_d^0(k,s)$ is an open subvariety of $\tilde{\mathbf{E}}_d^0$; hence, $\tilde{\mathbf{E}}_d^0(k,s)$ is a locally closed subvariety of $\tilde{\mathbf{E}}_d^0$. It is clearly G_d-stable.

8.4 Given d as in 8.2, $k \in I$ and an integer $s \geq 0$, we define $d' \in \mathbf{N}^I$ by $d'_j = d_j$ if $j \neq k$ and $d'_k = d_k + s$. We want to compare the varieties $P = \tilde{\mathbf{E}}_d^0(k,0)$ and $P' = \tilde{\mathbf{E}}_{d'}^0(k,s)$.

For this purpose we introduce several auxiliary varieties.

Let P_1 be the set of all triples $(f, \bar{f}, h)$ where $f = (f_j^i) \in P$; $\bar{f} = (\bar{f}_j^i) \in P'$; and $h = (h_i)$ is an isomorphism from the module defined by f onto a submodule of the module defined by $\bar{f}$.

Let P_2 be the set of all collections $h = (h_i)$ of injective linear maps $h_j : \mathbf{C}^{d_j} \to \mathbf{C}^{d'_j} \quad (j \in I)$.

The group G_d acts on P_1 by

$$(g_i) : ((f_j^i), (\bar{f}_j^i), (h_i)) \to ((g_j f_j^i g_i^{-1}), (\bar{f}_j^i), (h_i g_i^{-1}))$$

and on P_2 by $(g_i) : (h_i) \to (h_i g_i^{-1})$; hence it acts (diagonally) on $P \times P_2$.

Proposition 8.5. *(a) The map $\alpha : P_1 \to P \times P_2$ given by $(f, \bar f, h) \to (f, h)$ is a G_d-equivariant affine-space bundle with fibres of dimension $m = \sum_{j:(j,k)\in H} sd_j - sd_k$.*

(b) The second projection $p'' : P_1 \to P'$ is a principal G_d-bundle.

(c) If R is an irreducible component of P, then $R' = p''(\alpha^{-1}(R \times P_2))$ is an irreducible component of P'.

(d) We have

$$\dim R' = \dim R + (\dim \tilde{\mathbf{E}}_{d'} - \dim \tilde{\mathbf{E}}_d)/2.$$

(e) $R \to R'$ defines a 1-1 correspondence between the set of irreducible components of P and the set of irreducible components of P'.

(b) is obvious. We prove (a).

Assume that $(f, h) \in P \times P_2$ has been fixed; let $\mathcal{F}$ be the set of all $\bar f \in P'$ such that $(f, \bar f, h) \in P_1$. We must show that $\mathcal{F} \cong \mathbf{C}^m$. We denote by V the image of h_k and by V' a complement of V in $\mathbf{C}^{d'_k}$. If $\bar f \in \mathcal{F}$, we have $\bar f^i_j = h_j f^i_j h_i^{-1}$ for $i \neq k$ and $f^k_j|_V = h_j f^i_j h'_k$ for $(j, k) \in H$ where $h'_k : V \cong \mathbf{C}^{d_k}$ is such that $h_k h'_k$ is the identity map of V. Hence $\bar f$ is completely determined by the linear maps $\phi_j = h_j^{-1} \bar f^k_j|_{V'} : V' \to \mathbf{C}^{d_j}$ for $(j, k) \in H$.

Conversely, if linear maps $\phi_j : V' \to \mathbf{C}^{d_j}$ are given for each $j \in I$ with $(k, j) \in H$, then they define by the formulas above an element $\bar f \in \tilde{\mathbf{E}}_{d'}$.

This element is automatically nilpotent (this follows from the corresponding condition for f) and satisfies the rank condition in the definition of P'. However, this point will be in $\psi^{-1}(0)$ (with ψ defined as in 8.2, in terms of d') precisely when

$$\sum_{j:(k,j)\in H} f^j_k \phi_j = 0$$

as a linear map from V' to $\mathbf{C}^{d_k}$. Thus, $\bar f \to (\phi_j)$ identifies $\mathcal{F}$ with the kernel of the linear map $\oplus_{j:(j,k)\in H} \mathrm{Hom}(V', \mathbf{C}^{d_j}) \to \mathrm{Hom}(V', \mathbf{C}^{d_k})$ given by $(\phi_j) \to \sum_{j:(k,j)\in H} f^j_k \phi_j$.

Since $f \in P$, this linear map is surjective, hence its kernel has dimension $\sum_{j:(j,k)\in H} sd_j - sd_k = m$. This proves (a). Now (c) and (e) follow immediately from (a),(b). From these we see also that in (d), we have $\dim R' + \dim G_d = \dim R + \dim P_2 + m$. It is clear that $\dim P_2 + m - \dim G_d = sd_k + m = \sum_{j:(j,k)\in H} sd_j$. Thus, $\dim R' - \dim R = \sum_{j:(j,k)\in H} sd_j$. On the other hand, we have $\dim \tilde{\mathbf{E}}_d = \sum_{(i,j)\in H} d_i d_j$ and, similarly, $\dim \tilde{\mathbf{E}}_{d'} = \sum_{(i,j)\in H} d'_i d'_j$. Substracting, we get $\dim \tilde{\mathbf{E}}_{d'} - \dim \tilde{\mathbf{E}}_d = 2 \sum_{j:(j,k)\in H} sd_j$, and (d) follows.

Proposition 8.6. *Assume that d as in 8.2 is not identically zero. Then*

$$\tilde{\mathbf{E}}^0_d = \cup_{k\in I, s>0} \tilde{\mathbf{E}}^0_d(k, s).$$

Assume that this is not so. This means that we can find $f = (f^i_j) \in \tilde{\mathbf{E}}^0_d$ such that the following condition is satisfied:

(a) for any $k \in I$, any $x \in \mathbf{C}^{d_k}$ is of the form $x = \sum_{j:(j,k)\in H} f_k^j(x_j)$, for suitable $x_j \in \mathbf{C}^{d_j}$. $\blacksquare$

We apply this to each x_j in the sum instead of x. We obtain $x_j = \sum_{j':(j',j)\in H} f_j^{j'}(x_{j,,'})$, for suitable $x_{j,j'} \in \mathbf{C}^{d_{j'}}$. Substituting this into into the previous expression for x, we obtain

$$x = \sum_{j,j':(j,k)\in H,(j',j)\in H} f_k^j(f_j^{j'}(x_{j,j'}).$$

We iterate this procedure and after sufficiently many steps we obtain, using the nilpotency of f, that $x = 0$. Since this holds for any k, we see that $\oplus_i \mathbf{C}^{d_i} = 0$ contradicting the assumption that $d_i \neq 0$ for some i. The proposition is proved.

Theorem 8.7. *(a) $\tilde{\mathbf{E}}_d^0$ has pure dimension equal to* $\dim \tilde{\mathbf{E}}_d/2$.
(b) $\tilde{\mathbf{E}}_d^0(k,s)$ has pure dimension equal to $\dim \tilde{\mathbf{E}}_d/2$.

This is trivial when d is identically zero; in that case, both varieties are reduced to a point. Assume now that d is not identically zero. We may assume that the result is known when d is replaced by a d' with $\sum_i d'_i < \sum_i d_i$. Now the statement of (b), in the case where $s > 0$, follows from 8.5(c),(d),(e) and the induction hypothesis. From this special case of (b) we deduce that (a) holds, using the (finite) decomposition provided by 8.6. Finally, we note that (b) in the case where $s = 0$ follows from (a) since $\tilde{\mathbf{E}}_d^0(k,0)$ is open in $\tilde{\mathbf{E}}_d^0$. The theorem is proved.

8.8 Let $\mathcal{Z}_d$ be the set of irreducible components of the variety $\tilde{\mathbf{E}}_d^0$ and let $\mathcal{Z} = \sqcup_{d\in\mathbf{N}^I}\mathcal{Z}_d$. Given an irreducible component X of $\mathcal{Z}_d$ and some $k \in I$, we see from 8.3 that there is a unique integer $s \geq 0$ such that the intersection $X \cap \tilde{\mathbf{E}}_d^0(k,s)$ is open dense in X; thus X determines a map $k \to s$ from I to $\mathbf{N}$; associating this map to X gives a map $\mathcal{Z}_d \to \mathbf{N}^I$. The fibre of this map at $d' \in \mathbf{N}^I$ is denoted $\mathcal{Z}_d^{d'}$. Thus we have a partition $\mathcal{Z}_d = \sqcup_{d'\in\mathbf{N}^I}\mathcal{Z}_d^{d'}$.

8.9 For each $k \in I$ we will define a map $\tilde{E}_k : \mathcal{Z} \to \mathcal{Z}$ as follows. Let $X \in \mathcal{Z}$. We have $X \in \mathcal{Z}_d^{d'}$ for well defined d, d'. By definition, the intersection $X \cap \tilde{\mathbf{E}}_d^0(k,s)$ is an irreducible variety of dimension $\dim X = \dim \tilde{\mathbf{E}}_d/2$ (see 8.7(a)) hence it is contained in a unique irreducible component R' of $\tilde{\mathbf{E}}_d(k,s)$ (see 8.7(b)).

Let $\delta, \delta' \in \mathbf{N}^I$ be defined by $\delta_i = \delta'_i = d_i$ for $i \neq k$, and $\delta_k = d_k - s$, $\delta'_k = d_k + 1$. Under the correspondence in 8.5(e), R' corresponds to an irreducible component R of $\tilde{\mathbf{E}}_\delta^0(k,0)$. Under the analogous correspondence (for $s+1$ instead of s), R corresponds to an irreducible component R'' of $\tilde{\mathbf{E}}_{\delta'}^0(k,s+1)$. By 8.7, R'' is contained in a unique irreducible component X'' of $\tilde{\mathbf{E}}_{\delta'}^0$. By definition, $\tilde{E}_k(X) = X''$.

The following property of $\tilde{E}_k$ is clear from the definition.

For any $k \in I$ and $s \geq 0$, $\tilde{E}_k$ restricts to a bijection $\sqcup_{d,d';d'_k=s}\mathcal{Z}_d^{d'} \cong \sqcup_{d,d';d'_k=s+1}\mathcal{Z}_d^{d'}$. $\blacksquare$

From 8.6, we see that

(a) $$\cup_k \tilde{E}_k(\mathcal{Z}) = \sqcup_{d\neq 0}\mathcal{Z}_d.$$

From (a) it follows that

(b) any element of $\mathcal{Z}$ is of the form $\tilde{E}_{i_1} \dots \tilde{E}_{i_N} 1$ for some sequence $i_1, \dots, i_N$ in I, where 1 denotes the unique element of $\mathcal{Z}_0$.

8.10 We define an involution $\Psi : \tilde{\mathbf{E}}_d \cong \tilde{\mathbf{E}}_d$ by $\Psi : (f_j^i) \to (\phi_j^i)$ where $\phi_j^i : \mathbf{C}^{d_i} \to \mathbf{C}^{d_j}$ is the transpose of $f_i^j : \mathbf{C}^{d_j} \to \mathbf{C}^{d_i}$. It is clear that Ψ leaves $\tilde{\mathbf{E}}_d^0$ stable. Hence it induces a involution of the set $\mathcal{Z}_d$ of irreducible components of $\tilde{\mathbf{E}}_d^0$.

9. Symplectic structure

9.1 We shall make the following assumption (parity condition):

(a) There exists a function $\gamma : I \to \{\pm 1\}$ such that $\gamma(i) + \gamma(j) = 0$ whenever $(i, j) \in H$.

Clearly, if such a function exists then it is unique (up to replacing γ by $-\gamma$). We define a non-degenerate symplectic form $<,>$ on $\tilde{\mathbf{E}}_d$ with values in $\mathbf{C}$ by

$$< (f_j^i), (\phi_j^i) >= \sum_{(i,j) \in H} \gamma(j) \ \mathrm{tr}(f_j^i \phi_i^j).$$

This form is clearly G_d-invariant. (Note that the map ψ in 8.2 is the moment map for the (symplectic) action of G_d on $\tilde{\mathbf{E}}_d$.)

9.2 Assume given a subset H' of H such that, for any $(i, j) \in H$ we have $(i, j) \in H'$ if and only if $(j, i) \notin H'$. (Thus H' is an orientation of our graph.) We consider the $\mathbf{C}$-vector space

$$\mathbf{E}_d = \oplus_{(i,j) \in H'} \mathrm{Hom}(\mathbf{C}^{d_i}, \mathbf{C}^{d_j})$$

with the action of G_d given by $(g_i) : (f_j^i) \to (g_j f_j^i g_i^{-1})$. Let

$$\mathbf{E}'_d = \oplus_{(i,j) \in H'} \mathrm{Hom}(\mathbf{C}^{d_j}, \mathbf{C}^{d_i}).$$

We shall identify $\mathbf{E}'_d$ with the dual space $\mathbf{E}_d^*$ by the pairing

$$((\phi_i^j), (f_j^i)) = \sum_{(i,j) \in H'} \gamma(j) \ \mathrm{tr}(f_j^i \phi_i^j)$$

where $\phi_i^j \in \mathrm{Hom}(\mathbf{C}^{d_j}, \mathbf{C}^{d_i})$ and $f_j^i \in \mathrm{Hom}(\mathbf{C}^{d_i}, \mathbf{C}^{d_j})$ for $(i, j) \in H'$. Then the obvious isomorphism

(b) $$\mathbf{E}_d \times \mathbf{E}'_d \cong \tilde{\mathbf{E}}_d$$

identifies $\tilde{\mathbf{E}}_d$ with the cotangent bundle of $\mathbf{E}_d$ as symplectic vector spaces.

For any $f = (f_j^i) \in \mathbf{E}_d$ we denote by T_f the tangent space at f to the G_d-orbit of f, translated to 0.

It is easy to see that T_f is the subspace of $\mathbf{E}_d$ consisting of all vectors (e_j^i) such that there exists $(\xi_i) \in \oplus_i \mathrm{Hom}(\mathbf{C}^{d_i}, \mathbf{C}^{d_i})$ with $e_j^i = \xi_j f_j^i - f_j^i \xi_i$ for all $(i, j) \in H'$.

Lemma 9.3. *With the identifications in 9.2, the subvariety $\psi^{-1}(0)$ is the set of all pairs $(f,\phi)\in\mathbf{E}_d\times\mathbf{E}'_d$ such that ϕ vanishes on T_f.*

Consider the linear map

$$\psi^{-1}(0)\to\mathbf{E}_d \tag{a}$$

defined by forgetting the coordinates f^i_j with $(j,i)\in H'$. Let $\psi^{-1}(0)_f$ be the inverse image under (a) of $f=(f^i_j)\in\mathbf{E}_d$.

The annihilator of T_f in $\mathbf{E}^*_d$ (identified as above with $\mathbf{E}'_d$) is the subspace of $\mathbf{E}'_d$ consisting of all vectors $(\phi^j_i)\in\mathbf{E}'_d$ such that for all $(\xi_i)\in\oplus_i\mathrm{Hom}(\mathbf{C}^{d_i},\mathbf{C}^{d_i})$ we have

$$\sum_{(i,j)\in H'}\gamma(j)\ \mathrm{tr}((\xi_j f^i_j-f^i_j\xi_i)\phi^j_i)=0$$

or equivalently

$$\sum_j\gamma(j)\ \mathrm{tr}(\xi_jA_j)=0$$

where $A_j=\sum_{i:(i,j)\in H'}f^i_j\phi^j_i+\sum_{i:(j,i)\in H'}\phi^i_jf^j_i$, or equivalently $A_j=0$ for all j. But this is exactly $\psi^{-1}(0)_f$. The Lemma is proved.

Proposition 9.4. *Assume that the graph defined by I,H is a Dynkin graph (of type A,D or E). Then any element in the subset $\psi^{-1}(0)$ of $\mathbf{E}_d$ is automatically nilpotent. Hence, we have $\tilde{\mathbf{E}}_d=\psi^{-1}(0)$.*

The proof will be given elsewhere.

10. Conjectures

10.1 In this section we assume that the graph defined by I,H in 8.1 satisfies the parity condition 9.1(a). We choose an orientation H' as in 9.2. One can show that $\tilde{\mathbf{E}}_d^{\,0}$ is a lagrangian subvariety of $\tilde{\mathbf{E}}_d$.

Let $\mathcal{P}_d$ be the set of isomorphism classes of irreducible G_d-equivariant perverse sheaves on $\mathbf{E}_d$ whose characteristic variety is contained in $\tilde{\mathbf{E}}_d^{\,0}$. Let $\mathcal{P}=\sqcup_d\mathcal{P}_d$.

It should be true that

(a) There is a bijection $\mathcal{Z}_d\cong\mathcal{P}_d$ which to any irreducible component X of $\tilde{\mathbf{E}}_d^{\,0}$ attaches $\gamma_X\in\mathcal{P}_d$ such that X is contained in the characteristic variety of γ_X.

10.2 If we are also given an irreducible component X' of $\tilde{\mathbf{E}}^0_{d'}$ we can define the convolution $\gamma_X\star\gamma_{X'}$ in derived category (exactly as in [L2, 9.5]). By the decomposition theorem, this is a direct sum of shifts of intersection cohomology complexes on $\mathbf{E}_{d+d'}$ which are in $\mathcal{P}_{d+d'}$ hence by 10.1(a) are of the form $\gamma_{X''}$ for various irreducible components X'' of $\tilde{\mathbf{E}}^0_{d+d'}$. This then defines an associative $A-$algebra structure on the free $A-$ module with basis (γ_X) indexed by the elements $X\in\mathcal{Z}$. (Multiplication by v corresponds to shift.) We denote this algebra by $\mathbf{K}$. In this algebra we have $\gamma_{X'}\gamma_{X''}=\sum_X G_{X',X'',X}\gamma_X$ where $X'\in\mathcal{Z}_{d'}$, $X''\in\mathcal{Z}_{d''}$ and X runs over $\mathcal{Z}_d$ with $d=d'+d''$; we have $G_{X',X'',X}\in A$.

We now set

$$\tilde{G}_{X',X'',X} = v^{m-D(X')-D(X'')+D(X)} G_{X',X'',X}$$

where

$$m = -\sum_i d_i' d_i'' - \sum_{(i,j)\in H'} d_i' d_j'',$$

$D(X)$ is the dimension of the support of γ_X and $D(X'), D(X'')$ are defined similarly.

It is clear that we can define a new associative $A-$algebra structure on the $A-$module $\mathbf{K}$ by the rule $\gamma_{X'}\gamma_{X''} = \sum_X \tilde{G}_{X',X'',X}\gamma_X$. We denote this algebra by $\tilde{\mathbf{K}}$.

For $i \in I$, let $X(i)$ be the unique irreducible component (a point) of $\tilde{\mathbf{E}_d}^0$ where $d_i = 1$ and $d_j = 0$ for $j \neq i$.

10.3 Let $\mathbf{U}^+$ be the $+$ part of the quantized enveloping algebra (over A') attached by Drinfeld and Jimbo to the Kac-Moody Lie algebra corresponding to the generalized Cartan matrix (a_{ij}) with indices in I, given by $a_{ij} = -1$ if $(i,j) \in H$, $a_{ii} = 2$ and $a_{ij} = 0$ for all other (i,j). Let E_i $(i \in I)$ be the standard generators of $\mathbf{U}^+$.

Let $\Psi : \mathbf{U}^+ \cong (\mathbf{U}^+)^{opp}$ be the algebra isomorphism such that $\Psi(E_i) = E_i$ for all i.

It should be true that

(a) there is a unique isomorphism of algebras with unit $\tilde{\mathbf{K}} \otimes_A A' \cong \mathbf{U}^+$ which takes $\gamma_{X(i)}$ to E_i for all $i \in I$.

This isomorphism carries the element $\gamma_X \otimes 1 \in \tilde{\mathbf{K}} \otimes_A A'$ to an element $\mathcal{E}(X) \in \mathbf{U}^+$ for any $X \in \mathcal{Z}$.

The map $\mathcal{Z} \to \mathbf{U}^+$ given by $X \to \mathcal{E}(X)$ has been defined in terms of the orientation H'; it should be true that

(b) this map is independent of the choice of H'.

Thus the elements $\mathcal{E}(X)$ form a canonical basis of $\mathbf{U}^+$ parametrized by $\mathcal{Z}$.

10.4 In his lecture at the Yukawa International Seminar, Kashiwara gave an algebraic definition of a canonical basis $B_{I,H}$ of $\mathbf{U}^+$ (like the one in §2). Kashiwara shows that $B_{I,H}$ is equipped with injective maps $\tilde{E}_i : B_{I,H} \to B_{I,H}$ $(i \in I)$ such that $1 \in B_{I,H}$ and any element of $B_{I,H}$ is of the form $\tilde{E}_{i_1} \dots \tilde{E}_{i_N} 1$ for some sequence $i_1, \dots, i_N$ in I.

It should be true that

(a) $B_{I,H} = \{\mathcal{E}(X) | X \in \mathcal{Z}\}$ and that the map $\tilde{E}_i : B_{I,H} \to B_{I,H}$ coincides with the map $\mathcal{E}(X) \to \mathcal{E}(\tilde{E}_i(X)$ where $\tilde{E}_i : \mathcal{Z} \to \mathcal{Z}$ is as in 8.9. Moreover, $\Psi(\mathcal{E}(X)) = \mathcal{E}(\Psi(X))$ (notation of 8.10, 10.3).

10.5 We consider an example. Assume that

$$m \geq 2, \quad I = \{1, 2, \dots, 2m\}, \quad H' = \{(i, i+1) | 1 \leq i < 2m\} \cup \{(2m, 1)\}.$$

This corresponds to an (oriented) affine Dynkin graph of type A_{2m-1}. Let d be given by $d_i = 1$ for all i. We may identify $\mathbf{E}_d = \mathbf{C} \oplus \cdots \oplus \mathbf{C}$, ($2m$ copies). According to 10.3(a), we should expect that $\tilde{\mathbf{E}}_d{}^0$ has $2^{2m}-2$ irreducible components. They are the closures of the conormal bundles of $2^{2m}-2$ smooth G_d-invariant submanifolds of $\mathbf{E}_d$. These are the $2^{2m}-2$ orbits of G_d on the subset of $\mathbf{E}_d$ consisting of sequences of $2m$ numbers such that at least one is non-zero but not all are zero. (The conormal bundle of the orbit $\{0\}$ contains non-nilpotent elements.)

10.6 We consider a second example. Assume that

$$I = \{0,1,2,3,4\}, \quad H' = \{(i,0)|1 \leq i \leq 4\}.$$

This corresponds to an (oriented) affine Dynkin graph of type D_4. Let d be given by $d_0 = 2$ and $d_i = 1$ for $i \neq 0$. We may identify $\mathbf{E}_d = \mathbf{C}^2 \oplus \mathbf{C}^2 \oplus \mathbf{C}^2 \oplus \mathbf{C}^2$. According to 10.3(a), we should expect that $\tilde{\mathbf{E}}_d{}^0$ has 49 irreducible components. They are the closures of the conormal bundles of 49 smooth G_d-invariant submanifolds of $\mathbf{E}_d$. We want to describe this collection $\mathcal{F}$ of smooth submanifolds in terms of orbits of G_d on $\mathbf{E}_d$. Let O_1 be the open subset of $\mathbf{E}_0$ consisting of all quadruples of vectors in $\mathbf{C}^2$ such that any two of them form a basis of $\mathbf{C}^2$. This is a union of uncountably many orbits (all of dimension 7); we have $O_1 \in \mathcal{F}$. The complement of $\mathbf{E}_1 - O_1$ is the union of 51 orbits of which 48 are in $\mathcal{F}$. The union of the three excluded orbits is the set of all quadruples of non-zero vectors of $\mathbf{C}^2$ such that two of them are proportional, the other two are proportional, but the four vectors generate $\mathbf{C}^2$. (These three orbits are 6-dimensional; their conormal bundles contain some non-nilpotent elements.) This completes the description of the 49 submanifolds in $\mathcal{F}$.

11. Comments on type A

11.1 For type A certain interesting bases for the finite dimensional simple modules where constructed by De Concini and Kazhdan in [DK] (in the classical case, i.e. for $v = 1$). We will discuss here very briefly the connection of their approach with the canonical bases of [L2]. Details will be appear in [GL].

It is known that any irreducible representation of the symmetric group S_N can be realized (Springer) in the top homology of the variety of Borel subgroups containing a unipotent element in $GL_N(\mathbf{C})$. The irreducible components of that variety provide a canonical basis for that representation. The main idea of [DK] was to use this canonical basis for an irreducible representation of S_N and the well known relation (Schur, Weyl) between GL_m and S_N provided by the tensor space $(\mathbf{C}^m)^{\otimes N}$ to obtain certain bases of irreducible representations of $GL_n(\mathbf{C})$. These bases were defined only up to scalar factors and it was not clear whether they were unique. In addition, these bases were defined only for $v = 1$. One can however use a $q-$ analog of the construction of [DK]; instead of the Springer's model of representations of S_N one can use the model of [KL] for the representations of the corresponding Iwahori-Hecke algebra H; instead of the tensor space one can use Jimbo's $q-$analog of the tensor space. In this way one can obtain certain bases for the representations $L_{\mathbf{d}}$ (in type A) by a $q-$analog of the method of [DK]. In [GL] it is shown that these bases coincide with the bases constructed in [L2]. Our method uses in an essential way the point of view of [BLM]. In [BLM] the algebra $\mathbf{U}$ has been realized in a geometric setting; namely certain big quotients of it were realized in terms of the geometry of relative positions of pairs of $m-$step filtrations of some large $\mathbf{C}^N$. This language is quite appropriate for the discussion of the canonical bases of [L2]. On the other hand the Iwahori-Hecke algebra of S_N is realized in terms of the geometry of relative positions of pairs of complete flags in $\mathbf{C}^N$ and this language is quite appropriate for the discussion of the bases of [KL]. We want to get a bridge between the two points of view. The main idea is to realize the tensor space (and the action of GL_m and S_N on it) in a similar geometric setting namely in terms of the geometry of relative positions of pairs consisting of an $m-$step filtration of $\mathbf{C}^N$ and a complete flag in $\mathbf{C}^N$. (Note that the number of such relative positions is $m^N = \dim(\mathbf{C}^m)^{\otimes N}$.) In this setting, the two points of view ([KL] and [L2]) coexist and can be successfully compared.

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