Singularities, character formulas, and a q-analog of weight multiplicities

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1. The purpose of this paper is to discuss examples in which the intersection cohomology theory of Deligne-Goresky-MacPherson [4] enters in an essential way in the character formula for some irreducible representation of a semisimple group or Lie algebra. Thus, sections 3-5 are an exposition of the connection between singularities of Schubert varieties and multiplicities in Verma modules. In sections 6-11 we give an interpretation in terms of intersection cohomology for the multiplicities of weights in a finite dimensional representation of a simple Lie algebra. I wish to thank J. Bernstein for allowing me to use his unpublished results on the center of a Hecke algebra. (I learned about his results from D. Kazhdan.) These are used in the proof of Theorem 6.1 ; the original proof of that Theorem was based on [10] and on Macdonald's formulas for spherical functions.

2. Notations. For an irreducible complex algebraic variety X, we denote by $\mathcal{H}^i(X)$ the i-th cohomology sheaf of the intersection cohomology complex of X.

Let $\underline{g}$ be a simple complex Lie algebra, $\underline{b} \subset \underline{g}$ a Borel subalgebra, $\underline{h} \subset \underline{b}$ a Cartan subalgebra, $\underline{h}^*$ its dual space. Let $W \subset \mathrm{Aut}(\underline{h}^*)$ be the Weyl group, and let $S \subset W$ be the set of simple reflections (with respect to $\underline{b}$). $Q \subset \underline{h}^*$ is the subgroup generated by the roots.

$P \subset h^*$ is the subgroup consisting of those elements of $\underline{h}^*$ which take integral values on any coroot. Then Q has finite index in P.

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$\widetilde{W}_a \subset$ {affine transformations of $\underline{h}^*$} is the semidirect of W and of P (acting by translations). We shall regard $\widetilde{W}_a$ as acting on the <u>right</u> on $\underline{h}^*$. The transform of $\lambda \in \underline{h}^*$ under $w \in \widetilde{W}_a$ will be denoted $(\lambda)w$.

W_a is the subgroup of $\widetilde{W}_a$ generated by W and Q. This is the affine Weyl group. It is a Coxeter group whose set S_a of simple reflections is S together with the reflection in W_a whose fixed point set is $\{x \in \underline{h}^* \mid \langle x, \check{\alpha}_o \rangle = 1\}$; here $\check{\alpha}_o \in \underline{h}$ is the highest coroot. Let Ω be the normalizer of S_a in $\widetilde{W}_a$. Then $\widetilde{W}_a$ is a semi-direct product $\Omega \cdot W_a$.

For $\lambda \in P$, we denote by p_λ the same element, regarded in $\widetilde{W}_a$. Since the group law in $\widetilde{W}_a$ is written multiplicatively, we have $p_{\lambda+\lambda'} = p_\lambda \cdot p_{\lambda'}$ for $\lambda, \lambda' \in P$. ℓ is the length function on the Coxeter group W_a. We extend it to $\widetilde{W}_a$ by $\ell(\gamma w) = \ell(w\gamma) = \ell(w)$, $w \in W_a$, $\gamma \in \Omega$. For $s \in S$, let $\alpha_s \in Q$ be the corresponding simple root and let $\check{\alpha}_s \in \underline{h}$ be the corresponding simple coroot.

Let $P^{++} = \{p \in P \mid \langle p, \check{\alpha}_s \rangle \geq 0, \forall s \in S\}$. Then P^{++} parametrizes the double cosets $W \backslash \widetilde{W}_a / W : \lambda \leftrightarrow W p_\lambda W$. For $\lambda \in P^{++}$, W_λ denotes the stabilizer of λ in W, m_λ is the element of minimal length of $W p_\lambda W$, n_λ is the element of maximal length of $W p_\lambda W$, ν_λ is the number of reflections in W_λ, $P_\lambda = \sum_{w \in W_\lambda} q^{\ell(w)}$ (q is an indeterminate). For $\lambda = 0$, we set $\nu_o = \nu$, $P_o = P$; $\rho \in P$ denotes half the sum of all positive roots ; $\check{\rho} \in \underline{h}$ denotes half the sum of all positive coroots.

The fundamental alcove A_o is the open simplex in $P \otimes \mathbb{R}$ (embedded in $\underline{h}^*$) bounded by the fixed hyperplanes of the various reflections in S_a. An alcove is an open simplex in $P \otimes \mathbb{R}$ of the form $(A_o)w$, $w \in W_a$ (which is unique). Define a new (left) action of W_a on the set of alcoves (denotes $A \to yA$) by the rule $y((A_o)w) = (A_o)yw$. For each $\lambda \in P$, we denote $A_\lambda^+ = (A_o)p_\lambda$, $A_\lambda^- = (-A_o)p_\lambda$. Let $\leq$ be the standard partial order on the Coxeter group W_a. It is generated by the relations $s_1 s_2 \ldots \hat{s}_i \ldots s_n \leq s_1 s_2 \ldots s_n$ for any reduced expression $s_1 \ldots s_n$ ($s_i \in S_a$), $1 \leq i \leq n$. We extend it to a partial order $\leq$ on $\widetilde{W}_a$ by $\gamma w \leq \gamma' w' \Leftrightarrow \gamma = \gamma'$ and $w \leq w'$ ($\gamma, \gamma' \in \Omega$, $w, w' \in W_a$). Let $\leq$ be the partial order on P defined by

$\lambda \leq \lambda' \Leftrightarrow \lambda' - \lambda$ is a linear combination of positive roots, with ≥ 0 integral coefficients. If $\lambda, \lambda' \in P^{++}$, we have $\lambda \leq \lambda'$ if and only if $n_\lambda \leq n_{\lambda'}$, (in $\widetilde{W}_a$). For $\lambda \in \underline{h}^*$, M_λ denotes the Verma module for $\underline{g}$ with highest weight λ (with respect to $\underline{b}$) and L_λ denotes the unique irreducible quotient $\underline{g}$-module of M_λ.

3. We will restrict our attention to the Verma modules $M_{-\rho w-\rho}$ $(w \in W)$. In the Grothendieck group of $\underline{g}$-modules, $L_{-\rho w-\rho}$ is a linear combination with integral coefficients of the $\underline{g}$-modules $M_{-\rho y-\rho}$ $(y \leq w)$. The $\underline{g}$-module $M_{-\rho w-\rho}$ appears with coefficient 1, but the other coefficients were rather mysterious. A study of representations of Hecke algebras has led Kazhdan and the author [7] to give a (conjectural) algorithm for these coefficients and to interpret them in terms of singularities of Schubert varieties. Let us define the Schubert varieties. Consider the adjoint group G of $\underline{g}$, and let B be the Borel subgroup corresponding to $\underline{b}$, G_w the B-B double coset of G containing a representative of $w \in W$, $\mathcal{O}_w = G_w/B \subset G/B$. The Zariski closure $\overline{\mathcal{O}}_w$ of $\mathcal{O}_w$ in G/B is said to be a Schubert variety. It is the union of the various $\mathcal{O}_y$ for $y \leq w$.

The following result was conjectured by D. Kazhdan and the author [7],[8] and was proved by J.L. Brylinski and M. Kashiwara [3] and independently by A.A. Beilinson and J.N. Bernstein [1], using the theory of holonomic systems.

<u>Theorem 3.1</u>. <u>In the Grothendieck group of $\underline{g}$-modules, we have, for any</u> $w \in W$:

$$L_{-\rho w-\rho} = \sum_{y \leq w} (-1)^{\ell(w)-\ell(y)} \Big(\sum_i (-1)^i \dim \mathcal{H}^i_{\mathcal{O}_y}(\overline{\mathcal{O}}_w)\Big) M_{-\rho y-\rho} \tag{3.2}$$

<u>where</u> $\dim \mathcal{H}^i_{\mathcal{O}_y}(\overline{\mathcal{O}}_w)$ <u>is the dimension of the stalk of</u> $\mathcal{H}^i(\overline{\mathcal{O}}_w)$ <u>at a point in</u> $\mathcal{O}_y$.

4. We shall now describe the integers $\dim \mathcal{H}^i_{\mathcal{O}_y}(\overline{\mathcal{O}}_w)$ following [7],[8]. Let us recall the definition of the Hecke algebra H associated to (W,S). It consists of all formal linear combinations $\sum_{w \in W} a_w T_w$ with $a_w \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$ with multiplication defined by the rules $T_w T_{w'} = T_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w')$ and $(T_s+1)(T_s-q) = 0$ if $s \in S$; here $q^{1/2}$ is an indeterminate. There is a unique ring involution $h \to \overline{h}$ of H which takes $q^{1/2}$ to $q^{-1/2}$ and T_w to $T^{-1}_{w^{-1}}$ $(w \in W)$.

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It is semilinear with respect to the ring involution $q^{1/2} \to q^{-1/2}$ of $\mathbb{Z}[q^{1/2},q^{-1/2}]$. According to [7,1.1], for each $w \in W$, there is a unique element $C'_w \in H$ of the form $C'_w = q^{-\ell(w)/2} \sum_{y \leq w} P_{y,w} T_y$, where $P_{y,w}$ are polynomials in q satisfying $P_{w,w} = 1$ and $\deg P_{y,w} \leq 1/2(\ell(w)-\ell(y)-1)$ for $y < w$, and such that $\overline{C}'_w = C'_w$. The uniqueness of C'_w holds also if $P_{y,w}$ for $y < w$ is only assumed to be a polynomial in q and q^{-1} in which only powers q^i with $i \leq 1/2(\ell(w)-\ell(y)-1)$ are allowed to occur. It follows automatically that the $P_{y,w}$ are polynomials in q. The proof in [7] applies without change. (The discussion so far in this section, applies to an arbitrary Coxeter group and in particular to (W_a,S_a). It also applies word by word to $(\widetilde{W}_a,S_a)$ which although is not a Coxeter group, possesses the length function and the partial order $\leq$ which give a sense to the previous definitions and results.)

We can now state

<u>Theorem 4.1.</u> <u>Let</u> $y \leq w$ <u>be two elements in the Weyl group</u> W. <u>Then</u>

$$(4.2) \qquad \dim \mathcal{H}^i_{\mathcal{O}_y}(\overline{\mathcal{O}}_w) = 0 \quad \underline{\text{if}} \quad i \quad \underline{\text{is odd}}$$

$$(4.3) \qquad \sum_i \dim \mathcal{H}^{2i}_{\mathcal{O}_y}(\overline{\mathcal{O}}_w) q^i = P_{y,w} \quad .$$

Besides the original proof in [8], there is another proof in [12] which has the advantage that it also applies in the case where $\overline{\mathcal{O}}_w$ is replaced by the closure of a K-orbit on G/B, where K is the centralizer of an involution in G. (This plays a role in a character formula for real semisimple Lie groups.) Both proofs make use of reduction to characteristic > 1 and of a form of Weil's conjectures. Combining Theorems 3.1, 4.1, we can rewrite (3.2) in the form

$$(4.4) \qquad L_{-\rho w-\rho} = \sum_{y \leq w} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1) M_{-\rho y-\rho}$$

where $P_{y,w}(1)$ is the value of $P_{y,w}$ at $q = 1$. Using the inversion formula [7 3.1] for the matrix $(P_{y,w})$, this can be also written as

$$(4.5) \qquad M_{\rho w-\rho} = \sum_{w \leq y} P_{w,y}(1) L_{\rho y-\rho} \quad .$$

5. Remarks. (a) In the case where $y,w \in W_a$. the polynomials $P_{y,w}$ have been interpreted in [7] in terms analogous to (4.3), as intersection cohomology of certain generalized Schubert varieties. (In particular, they have ≥ 0 coefficients).

(b) There is a (conjectural) formula analogous to (3.2) for the characters of irreducible rational representations of a semisimple group over an algebraically closed field of characteristic > 1 . It involves the polynomials $P_{y,w}$ for y,w in an affine Weyl group. (See [9] for a precise statement).

6. If $\lambda \in P^{++}$, the $\underline{g}$-module L_λ is finite dimensional. With respect to the action of h , it decomposes into direct sum of weight spaces parametrized by elements $\mu \in P$. For $\mu \in P^{++}$, we denote $d_\mu(L_\lambda)$ the dimension of the μ-weight space in L_λ . It is well known that $d_\mu(L_\lambda) = 0$ unless $\mu \leq \lambda$. The remainder of this paper is mainly concerned with the proof of the following result.

Theorem 6.1. If $\mu,\lambda \in P^{++}$, $\mu \leq \lambda$, then $d_\mu(L_\lambda) = P_{n_\mu,n_\lambda}(1)$.

Here, P_{n_μ,n_λ} is defined in terms of the Hecke algebra of $\widetilde{W}_a$, see section 4 . (This Hecke algebra will be denoted $\widetilde{H}$; from now on, we shall reserve the letter H to denote the Hecke algebra of W_a . It is a subalgebra of $\widetilde{H}$.) Note that $P_{\gamma y,\gamma w} = P_{y,w}$ ($\gamma \in \Omega$, $y,w \in W_a$) so that the polynomials $P_{y',w'}$, for $y',w' \in \widetilde{W}_a$, have ≥ 0 coefficients. For type A, Theorem 6.1 follows from the results of [11], where P_{n_μ,n_λ} are interpreted as Green-Foulkes polynomials. In general, 6.1 would be a consequence of the conjecture 5(b) together with the Steinberg tensor product theorem. The integers $d_\mu(L_\lambda)$ are given by Weyl's character formula. To state the formula, we consider the elements

(6.2) $$k_\lambda = \frac{1}{|W|}\sum_{w\in Wp_\lambda W} w,\ (\lambda \in P^{++}),\ j_\lambda = \Big(\sum_{w\in W}(-1)^{\ell(w)}w^{-1}\Big)p_\lambda\Big(\sum_{w\in W} w\Big),\ (\lambda \in P^{++}+\rho)$$

of the group algebra $\mathbb{Q}[\widetilde{W}_a]$. Then k_λ $(\lambda \in P^{++})$ form a $\mathbb{Z}$-basis for the subgroup $K^1 = \{x \in \frac{1}{|W|}\mathbb{Z}[\widetilde{W}_a] : (\sum_{w\in W} w)x = x(\sum_{w\in W} w) = |W|\cdot x\} \subset \mathbb{Q}[\widetilde{W}_a]$ and j_λ $(\lambda \in P^{++}+\rho)$ form a $\mathbb{Z}$-basis for the subgroup

$$J^1 = \{y \in \mathbb{Z}[\widetilde{W}_a] : (\sum_{w\in W}(-1)^{\ell(w)}w^{-1})y = y(\sum_{w\in W} w) = |W|\cdot y\} .$$

It follows that K^1 is a subring of $\mathbb{Q}[\widetilde{W}_a]$ with unit element $\frac{1}{|W|} \sum_{w \in W} w$ and that, with respect to the product in $\mathbb{Q}[\widetilde{W}_a]$, we have $J^1 \cdot K^1 \subset J^1$, i.e. J^1 is a right K^1-module. Moreover, the map $K^1 \longrightarrow J^1$ given by $k \to j_\rho \cdot k$ is an isomorphism of right K^1-modules. (This is a reformulation of [2, Ch. VI, 3.3, Prop. 2(iii)] . We can now stat Weyl's character formula as follows

(6.3) __For__ $\lambda \in P^{++}$, __let__ $C'^{1}_{\lambda} = \sum_{\mu \in P^{++}} d_\mu(L_\lambda) k_\mu \in K^1$. __Then__ C'^{1}_{λ} __is the unique element in__ K^1 __such that__ $j_\rho C'^{1}_{\lambda} = j_{\lambda+\rho}$.

(This is equivalent to the usual formulation in which the character of L_λ appears as a quotient of two alternating expressions.)

We wish to consider a q-analog of the multiplicity $d_\mu(L_\lambda)$. The q-analogs of the elements (6.2) are the following elements of the Hecke algebra $\widetilde{H}$:

(6.4) $$K_\lambda = \frac{1}{P} \sum_{w \in W p_\lambda W} T_w = \frac{q^{-\nu+\nu_\lambda}}{P \cdot P_\lambda} \Big(\sum_{w \in W} T_w\Big) T_{p_\lambda} \Big(\sum_{w \in W} T_w\Big), \quad (\lambda \in P^{++})$$

(6.5) $$J_\lambda = \Big(\sum_{w \in W} (-q)^{\ell(w)} T_w^{-1}\Big) q^{-\ell(m_\lambda)/2} T_{m_\lambda} \Big(\sum_{w \in W} T_w\Big)$$

and therefore

$$J_\lambda = q^{-\nu/2} \Big(\sum_{w \in W} (-q)^{\ell(w)} T_w^{-1}\Big) q^{-\ell(p_\lambda)/2} T_{p_\lambda} \Big(\sum_{w \in W} T_w\Big) \quad \text{for } \lambda \in P^{++}+\rho .$$

Then $K_\lambda (\lambda \in P^{++})$ form a $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-basis for

$$K = \{x \in \tfrac{1}{P} \cdot \widetilde{H} : \Big(\sum_{w \in W} T_w\Big) x = x \Big(\sum_{w \in W} T_w\Big) = P \cdot x\} \subset \widetilde{H} \otimes \mathbb{Q}(q^{1/2})$$

and $J_\lambda (\lambda \in P^{++}+\rho)$ form a $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-basis for

$$J = \{y \in \widetilde{H} : \Big(\sum_{w \in W} (-q)^{\ell(w)} T_w^{-1}\Big) y = y \Big(\sum_{w \in W} T_w\Big) = P \cdot y\} .$$

Note that K is a subring of $\widetilde{H} \otimes \mathbb{Q}(q^{1/2})$ with unit element $\frac{1}{P} \sum_{w \in W} T_w$ and that, with respect to the product in $\widetilde{H} \otimes \mathbb{Q}(q^{1/2})$, we have $J \cdot K \subset J$, i.e. J is a right K-module.

In the statement of the following theorem, we shall give a meaning to $J_\lambda \in J$ for arbitrary $\lambda \in P$: if $(\lambda)w \neq \lambda$ for all $w \in W$, $w \neq e$, we set $J_\lambda = (-1)^{\ell(w)} J_{(\lambda)w}$ where w is the unique element of W such that $(\lambda)w \in P^{++}+\rho$.

For the remaining $\lambda \in P$, we set $J_\lambda = 0$.

Theorem 6.6. For any $\lambda \in P^{++}$, we have

$$J_\rho \cdot (q^{-\ell(p_\lambda)/2} K_\lambda) = \frac{1}{\overline{P}_\lambda} \sum_I (-q)^{-|I|} J_{\lambda+\rho-\alpha_I} \tag{6.7}$$

(sum over all subsets I of the set of positive roots); here α_I denotes the sum of the roots in I.

The proof will be given in Section 7.

If I is as in the previous sum and if $w \in W$ is such that $\lambda+\rho-\alpha_I = (\lambda'+\rho)w$, $\lambda' \in P^{++}$, then $\lambda-\lambda' = \lambda-(\lambda)w^{-1}-(\rho)w^{-1}+\rho+(\alpha_I)w^{-1} = \lambda-(\lambda)w^{-1}+\alpha_J$ where J is the set of positive roots β such that $(\beta)w \in I$ or such that $-(\beta)w$ is positive, $\notin I$. Since $\lambda \geq (\lambda)w^{-1}$ $(\lambda \in P^{++})$ and $\alpha_J \geq 0$, it follows that $\lambda \geq \lambda'$. Thus, the right hand side of (6.7) is a linear combination of elements $J_{\lambda'+\rho}$ $(\lambda' \leq \lambda)$ with formal power series in q^{-1} without terms of form q^i $(i > 0)$ as coefficients; moreover for $\lambda' < \lambda$, the coefficient doesn't have a constant term. On the other hand, since the left hand side of (6.7) is in J, these coefficients must be polynomials in $q^{1/2}$, $q^{-1/2}$. It follows that they are polynomials in q^{-1} (without constant term if $\lambda' < \lambda$). The coefficient of $J_{\lambda+\rho}$ is equal to 1; this follows from the identity $\frac{1}{\overline{P}_\lambda} \sum_{\substack{I \\ w\in W_\lambda \\ \alpha_I = \rho-(\rho)w}} (-q)^{-|I|} = 1$.

Since a triangular matrix with 1's on diagonal has an inverse of the same form, we see that for any $\lambda \in P^{++}$, the element $J_{\lambda+\rho}$ is a linear combination of elements $J_\rho(q^{-\ell(p_{\lambda'})/2}K_{\lambda'})$, $\lambda' \leq \lambda$, with coefficients polynomials in q^{-1} (without constant term, if $\lambda' < \lambda$ and $\equiv 1$, if $\lambda' = \lambda$). Hence we have

Corollary 6.8. For any $\lambda \in P^{++}$, there is a unique element $C'_\lambda \in K$ such that

$$J_\rho \cdot C'_\lambda = J_{\lambda+\rho} \tag{6.9}$$

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$$C'_\lambda = q^{-\ell(p_\lambda)/2} \sum_{\substack{\mu \in P^{++} \\ \mu \leq \lambda}} d_\mu(L_\lambda;q) K_\mu \tag{6.10}$$

where $d_\mu(L_\lambda;q)$ are polynomials in q and q^{-1} with integer coefficients ; moreover, the powers q^i appearing in $d_\mu(L_\lambda;q)$ satisfy $i < \frac{1}{2}(\ell(p_\lambda)-\ell(p_\mu))$ if $\mu < \lambda$ and $d_\lambda(L_\lambda;q) \equiv 1$. In particular, the map $h \to J_\rho h$ defines an isomorphism of right K-modules of K onto J .

Note that, if $\mu \leq \lambda$, then $\frac{1}{2}(\ell(p_\lambda)-\ell(p_\mu))$ is an integer. Indeed, it is known [5] that, for $\lambda \in P^{++}$,

$$\ell(p_\lambda) = \langle\lambda, 2\check{\rho}\rangle \quad . \tag{6.11}$$

Hence $\frac{1}{2}(\ell(p_\lambda)-\ell(p_\mu)) = \frac{1}{2}(\langle\lambda,2\check{\rho}\rangle - \langle\mu,2\check{\rho}\rangle) = \langle\lambda-\mu,\check{\rho}\rangle$ and this is an integer since $\lambda - \mu \in Q$.

We shall now show that $d_\mu(L_\lambda;q)$ are actually polynomials in q with ≥ 0 coefficients.

We have

Theorem 6.12. $C'_\lambda = q^{\nu/2} P^{-1} C'_{n_\lambda}$ $(\lambda \in P^{++})$. In particular, for $\mu \leq \lambda$ in P^{++} , we have

$$d_\mu(L_\lambda;q) = P_{n_\mu, n_\lambda} \tag{6.13}$$

hence $d_\mu(L_\lambda;q)$ is a polynomial in q with ≥ 0 coefficients.

For the proof of 6.12, we need the following result.

Lemma 6.14. If $\lambda \in P^{++}$, then $\overline{J}_{\lambda+\rho} = J_{\lambda+\rho}$.

In the case where $\lambda \in Q \cap P^{++}$, this is just Lemma 11.7 of [10] . The general case is proved in the same way.

The definition of K shows that K is stable under $h \to \overline{h}$ (which is extended to a ring involution of $\mathcal{H} \otimes \mathbb{Q}(q^{1/2})$. (Note that $\overline{P^{-1} \sum_{w\in W} T_w} = P^{-1} \sum_{w \in W} T_w$.) From (6.9) it then follows that $J_\rho \overline{C}'_\lambda = J_{\rho+\lambda}$. Thus $J_\rho(C'_\lambda - \overline{C}'_\lambda) = 0$ and, since $C'_\lambda - \overline{C}'_\lambda \in K$, we have $C'_\lambda = \overline{C}'_\lambda$, by the last sentence in Corollary 6.8.

The element $q^{-\nu/2} P C'_\lambda$ is also fixed by $h \mapsto \overline{h}$, since $\overline{q^{-\nu/2} P} = q^{-\nu/2} P$. This element is equal to

$$q^{-\ell(n_\lambda)/2} \sum_{y \leq n_\lambda} d_{\mu(y)}(L_\lambda;q) T_y$$

where $\mu(y) \in P^{++}$ is defined by $y \in W\, p_{\mu(y)} W$.

We now use the bounds on the powers of q appearing in $d_\mu(L_\lambda;q)$ given in Corollary 6.8. If follows that $q^{-\nu/2} P C'_\lambda$ satisfies the defining property of C'_{n_λ}, hence is equal to it. Thus Theorem 6.12 follows from Theorem 6.6. On the other hand, it implies Theorem 6.1. Indeed, under the specialization $\mathbb{Z}[q^{1/2},q^{-1/2}] \to \mathbb{Z}$, given by $q^{1/2} \to 1$, $\tilde{H}$ becomes the group ring $\mathbb{Z}[\tilde{W}_a]$, K_λ becomes k_λ ($\lambda \in P^{++}$), J_λ becomes j_λ ($\lambda \in P^{++}+\rho$) and (6.9) becomes (6.3). It follows that for $\mu,\lambda \in P^{++}$, $\mu \leq \lambda$, $d_\mu(L_\lambda)$ is the value of $d_\mu(L_\lambda;q)$ at $q=1$ and theorem 6.1 follows.

7. For the proof of Theorem 6.6 we shall need several preliminary steps. We shall begin with a definition (due to J. Bernstein) of a large commutative subalgebra of $\tilde{H}$, which is a q-analogue of the subring $\mathbb{Z}[P]$ of $\mathbb{Z}[\tilde{W}_a]$. To each $\lambda \in P$, Bernstein associates an element $\tilde{T}_{p_\lambda} \in \tilde{H}$ defined by $\tilde{T}_{p_\lambda} = (q^{-\ell(p_{\lambda_1})/2} T_{p_{\lambda_1}}) \cdot (q^{-\ell(p_{\lambda_2})/2} T_{p_{\lambda_2}})^{-1}$ where λ_1,λ_2 are elements of P^{++} such that $\lambda = \lambda_1-\lambda_2$. This is independent of the choice of λ_1,λ_2, since for $\lambda',\lambda'' \in P^{++}$ we have the identity $T_{p_{\lambda'}} T_{p_{\lambda''}} = T_{p_{\lambda''}} T_{p_{\lambda'}} = T_{p_{\lambda'} p_{\lambda''}} = T_{p_{\lambda'+\lambda''}}$. (Indeed, we have $\ell(p_{\lambda'})+\ell(p_{\lambda''}) = \ell(p_{\lambda'} \cdot p_{\lambda''})$, by (6.11).) It follows also that if $\lambda',\lambda'' \in P$, we have $\tilde{T}_{p_{\lambda'}} \tilde{T}_{p_{\lambda''}} = \tilde{T}_{p_{\lambda''}} \tilde{T}_{p_{\lambda'}} = \tilde{T}_{p_{\lambda'+\lambda''}}$ and $\tilde{T}^{-1}_{p_{\lambda'}} = \tilde{T}_{p_{-\lambda'}}$. We shall prove the following

<u>Lemma 7.1</u>. (J. Bernstein) <u>Let</u> $\lambda \in P$ <u>and let</u> $s \in S$. <u>We have</u>

$$T_s(\tilde{T}_{p_\lambda}+\tilde{T}_{p_{(\lambda)s}}) = (\tilde{T}_{p_\lambda}+\tilde{T}_{p_{(\lambda)s}})T_s \quad .$$

<u>Proof</u> : We may clearly assume that $<\lambda,\check{\alpha}_s> \geq 0$. Assume first that $<\lambda,\check{\alpha}_s> = 0$. We can write $\lambda = \lambda_1-\lambda_2$ with $\lambda_1,\lambda_2 \in P^{++}$, $<\lambda_1,\alpha_s> = <\lambda_2,\check{\alpha}_s> = 0$. To prove the identity $T_s\tilde{T}_{p_\lambda} = \tilde{T}_{p_\lambda} \cdot T_s$, we are thus reduced to the case where $\lambda \in P^{++}$, $<\lambda,\check{\alpha}_s> = 0$. But then $\ell(sp_\lambda) = \ell(p_\lambda s) = \ell(p_\lambda)+1$ hence $T_s T_{p_\lambda} = T_{sp_\lambda} = T_{p_\lambda s} = T_{p_\lambda} T_s$ as required.

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Next, we consider the case where $<\lambda, \check{\alpha}_s> = 1$, i.e. $(\lambda)s = \lambda - \alpha_s$. In this case, the result follows from Lemma 4.4.(b) in (G. Lusztig, Some examples of square integrable representations of semisimple p-adic groups, preprint IHES, 1982).

Next, we assume that $<\lambda, \check{\alpha}_s> = d \geq 2$ and that the result is already known when d is replaced by d' , $0 \leq d' < d$. We can write $\lambda = \lambda_1 + \lambda_2$ where $<\lambda_1, \check{\alpha}_s> = d-1$, $<\lambda_2, \check{\alpha}_s> = 1$. Then $<\lambda_1 + (\lambda_2)s, \check{\alpha}_s> = d-2$. The induction hypothesis is applicable to λ_1, λ_2 and to $\lambda_1 + (\lambda_2)s$. Hence T_s commutes with $A = \tilde{T}_{p_{\lambda_1}} + \tilde{T}_{p_{(\lambda_1)s}}$, $B = \tilde{T}_{p_{\lambda_2}} + \tilde{T}_{p_{(\lambda_2)s}}$, $C = \tilde{T}_{p_{\lambda_1 + (\lambda_2)s}} + \tilde{T}_{p_{(\lambda_1)s + \lambda_2}}$. But $\tilde{T}_{p_{\lambda_1+\lambda_2}} + \tilde{T}_{p_{(\lambda_1+\lambda_2)s}} = A \cdot B - C$ hence T_s commutes with $\tilde{T}_p + \tilde{T}_{p_{(\lambda)s}}$. The lemma is proved.

We now define, for any $\lambda \in P$ an element $\tilde{J}_\lambda \in J$ by the formula

$$\tilde{J}_\lambda = q^{-\nu/2} \theta' \, \tilde{T}_{p_\lambda} \theta$$

where $\theta = \sum_{w \in W} T_w$, $\theta' = \sum_{w \in W} (-q)^{\ell(w)} T_w^{-1}$. When $\lambda \in P^{++} + \rho$, we have clearly $\tilde{J}_\lambda = J_\lambda$. In general, we have

<u>Lemma 7.3.</u> $\tilde{J}_{(\lambda)w} = (-1)^{\ell(w)} \tilde{J}_\lambda$ <u>for any</u> $\lambda \in P$, $w \in W$; <u>hence,</u> $\tilde{J}_\lambda = J_\lambda$ <u>for all</u> $\lambda \in P$.

<u>Proof</u> : We may assume that $w = s \in S$. Note that $T_s \theta = q\theta$, $\theta' T_s^{-1} = -\theta'$, hence

$$\begin{aligned} \tilde{J}_\lambda + \tilde{J}_{(\lambda)s} &= q^{-\nu/2} \theta' (\tilde{T}_\lambda + \tilde{T}_{(\lambda)s}) \theta \\ &= q^{-\nu/2} \theta' T_s^{-1} (\tilde{T}_\lambda + \tilde{T}_{(\lambda)s}) T_s \theta \\ &= -q \cdot q^{-\nu/2} \theta' (\tilde{T}_\lambda + \tilde{T}_{(\lambda)s}) \theta \qquad \text{by lemma (7.1)} \\ &= -q (\tilde{J}_\lambda + \tilde{J}_{(\lambda)s}) \, . \end{aligned}$$

Thus, $\tilde{J}_\lambda + \tilde{J}_{(\lambda)s} = 0$, as required.

<u>Lemma 7.4.</u> <u>There is a unique function</u> $f : Q + \rho \to \mathbb{Z}[q, q^{-1}]$ <u>with finite support</u> <u>satisfying properties</u> (i), (ii), (iii) <u>below</u> :

(i) $f(\rho) = q^\nu$

(ii) $f(\lambda) \neq 0 \Rightarrow \lambda \leq \rho$

(iii) *Let* $X \subset Q+\rho$ *be an* α_s*-string* : $X = \{x+n\alpha_s, n \in \mathbb{Z}\}$, *where* x *is any fixed element of* $Q+\rho$ *and* α_s *is any fixed simple root. Let* $a \geq 0$ *be an integer such that* $\langle \lambda, \check{\alpha}_s \rangle \equiv a \pmod 2$ *for all* $\lambda \in X$. *Then*

$$\sum_{\substack{\lambda \in X \\ \langle \lambda, \check{\alpha}_s \rangle \geq a}} f(\lambda) = -q^{-(a-1)} \sum_{\substack{\lambda \in X \\ \langle \lambda, \check{\alpha}_s \rangle \leq -a}} f(\lambda) .$$

This function is given by the formula

$$(7.5) \qquad f(\lambda) = (-1)^{\nu} \sum_{\substack{I \\ \alpha_I = \lambda+\rho}} (-q)^{|I|} q^{-\langle \lambda-\rho, \check{\rho} \rangle} ,$$

where I *runs through the subsets of the set of positive roots, and* α_I *is defined as in 6.6.*

Proof : The function f defined by (7.5) clearly satisfies (i) and (ii). We now verify that it satisfies (iii). We shall set $\alpha_s = \alpha, \check{\alpha}_s = \check{\alpha}$. We have, with the notations of (iii) :

$$\sum_{\substack{\lambda \in X \\ \langle \lambda, \check{\alpha} \rangle \geq 0}} f(\lambda) = (-1)^{\nu} \sum_{\substack{\lambda \in X \\ I \\ \alpha_I = \lambda+\rho \\ \langle \lambda, \check{\alpha} \rangle \geq a}} (-q)^{|I|} q^{-\langle \lambda-\rho, \check{\rho} \rangle}$$

$$= (-1)^{\nu} \sum_{\substack{\lambda \in X \\ I \not\ni \alpha \\ \alpha_I = \lambda+\rho \\ \langle \lambda, \check{\alpha} \rangle \geq a}} (-q)^{|I|} q^{-\langle \lambda-\rho, \check{\rho} \rangle} + \Sigma'$$

where

$$\Sigma' = (-1)^{\nu} \sum_{\substack{\lambda \in X \\ I \ni \alpha \\ \alpha_I = \lambda+\rho \\ \langle \lambda, \check{\alpha} \rangle \geq a}} (-q)^{|I|} q^{-\langle \lambda-\rho, \check{\rho} \rangle} = (-1)^{\nu} \sum_{\substack{\lambda \in X \\ I' \not\ni \alpha \\ \alpha_{I'} = \lambda-\alpha+\rho \\ \langle \lambda, \check{\alpha} \rangle \geq a}} (-q)^{|I|+1} q^{-\langle \lambda-\rho, \check{\rho} \rangle} =$$

$$= (-1)^{\nu} \sum_{\substack{\lambda' \in X \\ I' \not\ni \alpha \\ \alpha_{I'} = \lambda'+\rho \\ \langle \lambda'+\alpha, \check{\alpha} \rangle \geq a}} (-q)^{|I'|+1} q^{-\langle \lambda'+\alpha-\rho, \check{\rho} \rangle} = (-1)^{\nu} \sum_{\substack{\lambda \in X \\ I \not\ni \alpha \\ \alpha_I = \lambda+\rho \\ \langle \lambda, \check{\alpha} \rangle \geq a-2}} (-q)^{|I|} q^{-\langle \lambda-\rho, \check{\rho} \rangle}$$

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$$(7.6)\qquad \sum_{\substack{\lambda\in X\\ \langle\lambda,\check{\alpha}\rangle\geq a}} f(\lambda) = -(-1)^{\nu} \sum_{\substack{\lambda\in X\\ I\not\ni\alpha\\ \alpha_I=\lambda+\rho\\ \langle\lambda,\check{\alpha}\rangle=a-2}} (-q)^{|I|}q^{-\langle\lambda-\rho,\check{\rho}\rangle} \quad .$$

A similar computation shows that

$$\sum_{\substack{\lambda\in X\\ \langle\lambda,\alpha\rangle\leq -a}} f(\lambda) = (-1)^{\nu} \sum_{\substack{\lambda\in X\\ I\not\ni\alpha\\ \alpha_I=\lambda+\rho\\ \langle\lambda,\check{\alpha}\rangle=-a}} (-q)^{|I|}q^{-\langle\lambda-\rho,\check{\rho}\rangle} \quad .$$

Now the simple reflection s maps the set of positive roots $\neq \alpha$ onto itself. Hence the last sum is equal to

$$(-1)^{\nu} \sum_{\substack{\lambda\in X\\ I\not\ni\alpha\\ \alpha_{(I)s}=(\lambda+\rho)s\\ \langle\lambda,\check{\alpha}\rangle=-a}} (-q)^{|I|}q^{-\langle\lambda-\rho,\check{\rho}\rangle} = (-1)^{\nu} \sum_{\substack{\lambda\in X\\ I'\not\ni\alpha\\ \alpha_{I'}=\lambda+(a-1)\alpha+\rho\\ \langle\lambda,\check{\alpha}\rangle=-a}} (-q)^{|I'|}q^{-\langle\lambda-\rho,\check{\rho}\rangle}$$

$$= (-1)^{\nu} \sum_{\substack{\lambda'\in X\\ I'\not\ni\alpha\\ \alpha_{I'}=\lambda'+\rho\\ \langle\lambda'-(a-1)\alpha,\check{\alpha}\rangle=-a}} (-q)^{|I'|}q^{-\langle\lambda'-(a-1)\alpha-\rho,\check{\rho}\rangle}$$

$$= (-1)^{\nu}q^{a-1} \sum_{\substack{\lambda'\in X\\ I'\not\ni\alpha\\ \alpha_{I'}=\lambda'+\rho\\ \langle\lambda',\check{\alpha}\rangle=a-2}} (-q)^{|I'|}q^{\langle\lambda'-\rho,\check{\rho}\rangle}$$

Comparing with the right hand side of (7.6), we conclude that f satisfies (iii).

To prove the converse it is enough to show that if a function $g : Q+\rho\to \mathbb{Z}[q,q^{-1}]$ with finite support satisfies $g(\rho) = 0$, $g(\lambda) \neq 0 \Rightarrow \lambda \leq \rho$ and the identity (iii) with f replaced by g , then $g \equiv 0$. Assume that $g \not\equiv 0$, and let $x \in Q+\rho$ be an element of maximal possible length (with respect to some positive definite, W-invariant scalar product on $P\otimes\mathbb{R}$) such that $g(x) \neq 0$. Let X be the string through x corresponding to the simple root α_s . Then $x' = (x)s$ is also in X . Let a be the absolute value of $\langle x,\check{\alpha}_s\rangle = -\langle x',\check{\alpha}_s\rangle$. If $y \in X$ satisfies

$|\langle y,\check{\alpha}_s\rangle| > a$ then clearly the length of y is strictly bigger than that of x hence $g(y) = 0$. Hence the identity (iii) for g, and X, a, as above, reduces to $g(x) = -q^{\pm a-1}g(x')$. It follows that $g(x') \neq 0$. Note also that x,x' have the same length. Iterating this, we see that $g((x)w) \neq 0$ for all $w \in W$; moreover, $(x)w$ has the same length as x. For suitable $w \in W$, we have $\langle (x)w,\check{\alpha}_s\rangle \geq 0$ for all simple roots α_s. Replacing x by $(x)w$, we may thus assume that $\langle x,\check{\alpha}_s\rangle \geq 0$ for all simple roots α_s. If we had $\langle x-\rho,\check{\alpha}_s\rangle \geq 0$ for all simple roots α_s then it would follow that $\langle x-\rho,\check{\rho}\rangle \geq 0$; since $g(x) \neq 0$, we would have $\rho - x \geq 0$, hence $\rho - x = \Sigma\, n_s\alpha_s$ (α_s simple, $n_s \geq 0$ integers), hence $\langle -\Sigma\, n_s\alpha_s,\check{\rho}\rangle \geq 0$. Thus $-\Sigma\, n_s = 0$, hence $n_s = 0$ for all simple roots α_s, hence $x = \rho$. But $g(\rho) = 0$ and this is a contradiction with $g(x) \neq 0$. Thus, there exists a simple root α_s such that $\langle x-\rho,\check{\alpha}_s\rangle < 0$; since $\langle x,\check{\alpha}_s\rangle \geq 0$, it follows that $\langle x,\check{\alpha}_s\rangle = 0$. Consider the string X through x corresponding to the simple root α_s. The equality $\langle x,\check{\alpha}_s\rangle = 0$ shows that among the elements of X, the element x has minimal length. It follows that $g(y) = 0$ for all $y \in X$, $y \neq x$. Let us now write the identity (iii) for g, this X, and $a = 0$. We get $g(x) = -q^{-1}g(x)$ hence $g(x) = 0$. This contradiction shows that $g \equiv 0$ and the Lemma is proved.

We shall now introduce as in [10] an H-module $\mathcal{M}$ as follows. $\mathcal{M}$ is the free $\mathbb{Z}[q^{1/2},q^{-1/2}]$ module with basis (A) where A are the various alcoves in $P \otimes \mathbb{R}$. For each $s \in S_a$, we define an endomorphism T_s of this $\mathbb{Z}[q^{1/2},q^{-1/2}]$-module by

$$T_s(A) = \begin{cases} sA, & \text{if } \exists \text{ positive coroot } \check{\alpha} \text{ with } \langle x,\check{\alpha}\rangle > n \text{ for } x \in sA,\ \langle x,\check{\alpha}\rangle < n \text{ for } x \in A \\ q\cdot sA+(q-1)A, & \text{otherwise.} \end{cases}$$

These endomorphisms make $\mathcal{M}$ into an H-module.

Let W' be the subgroup of W_a generated by those $s \in S_a$ for which $s(A_\rho^+)$ contains ρ in its closure. (This is a parabolic subgroup of W_a conjugate to W under an element in Ω.)

Lemma 7.7. Let $y \in W_a$. We define a function $f : Q+\rho \longrightarrow \mathbb{Z}[q^{1/2}, q^{-1/2}]$ as follows: $f(\lambda)$ is the coefficient with which A^-_λ appears in

$$(\sum_{w \in W'} (-q)^{\ell(w)} T_w^{-1}) T_y (\sum_{w \in W} T_w) A^-_o \in \mathcal{M} \quad .$$

Then

(i) If $y(A^+_o) = A^-_\lambda$, $\lambda \in P^{++}+$, $\lambda \in Q+\rho$, then $f(\lambda) = q^\nu$; moreover $\lambda' \in Q+\rho$, $f(\lambda') \neq 0$ implies $\lambda' \leq \lambda$.

(ii) In general, let $X \subset Q+\rho$ be an α_s-string (α_s a simple root) and let $a \geq 0$ be an integer such that $\langle \lambda, \check{\alpha}_s \rangle \equiv a (\mathrm{mod}\ 2)$ for all $\lambda \in X$. Then

$$\sum_{\substack{\lambda \in X \\ \langle \lambda, \check{\alpha}_s \rangle \geq a}} f(\lambda) = -q^{-(a-1)} \sum_{\substack{\lambda \in X \\ \langle \lambda, \check{\alpha}_s \rangle \leq -a}} f(\lambda) \quad .$$

Proof : (i) Follows from [10, 4.2 (a)] and (ii) is a consequence of [10, 9.2] applied to the element $T_y (\sum_{w \in W} T_w) A^-_o$.

Corollary 7.8. If y in the previous lemma is such that $y(A^+_o) = A^-_\rho$, then

$$(7.9) \qquad (\sum_{w \in W'} (-q)^{\ell(w)} T_w^{-1}) T_y (\sum_{w \in W} T_w) A^-_o = q^{-\nu} (\sum_{w \in W'} (-q)^{\ell(w)} T_w^{-1}) \sum_{\lambda \in Q+\rho} f(\lambda) h_\lambda A^-_o \ ,$$

where, for $\lambda \in Q+\rho$, $f(\lambda)$ is given by (7.5), and h_λ is an element of H such that $h_\lambda A^-_o = A^-_\lambda$.

Proof : In our case, the function f of Lemma 7.7 satisfies the conditions (i), (ii), (iii) of Lemma 7.4, hence is given by (7.5). It follows that for any $\lambda \in Q+\rho$, A^-_λ appears with the same coefficient in the two sides of (7.9) and the corollary follows.

Since the H-module $\mathcal{M}$ is faithful, we can erase A^-_o from the two sides of (7.9) and we obtain an identity in H . We can rewrite this identity as follows. Let $\gamma \in \Omega$ be such that $\gamma W' \gamma^{-1} = W$. We multiply both sides of our identity on the left by T_γ . Note that $T_\gamma T_y = T_{\gamma y} = T_{m_\rho}$. Moreover $T_\gamma h_\lambda = q^{\ell(p_\lambda)/2} \tilde{T}_{p_\lambda}$. Thus, we have

$$\theta' T_{m_\rho} \theta = \Big(\sum_{w\in W} (-q)^{\ell(w)} T_w^{-1}\Big) T_{m_\rho} \sum_{w\in W} T_w =$$

$$= q^{-\nu}\Big(\sum_{w\in W} (-q)^{\ell(w)} T_w^{-1}\Big) \sum_{\lambda\in Q+\rho} f(\lambda) q^{\ell(p_\lambda)/2} \widetilde{T}_{p_\lambda} .$$

We can now compute for $\lambda \in P^{++}$:

$$J_\rho(q^{-\ell(p_\lambda)/2} K_\lambda) = q^{-\ell(m_\lambda)/2} \theta' T_{m_\rho} \theta \cdot \frac{1}{P\cdot P_\lambda} \theta \cdot q^{-\ell(p_\lambda)/2} q^{-\nu+\nu_\lambda} T_{p_\lambda} \theta$$

$$= \frac{1}{P_\lambda} q^{-\ell(m_\rho)/2} \cdot q^{-2\nu+\nu_\lambda} \cdot \sum_{\mu\in Q+\rho} f(\mu) q^{\langle\mu,\check{\rho}\rangle} \theta' \widetilde{T}_{p_\lambda} \cdot \widetilde{T}_{p_\mu} \theta$$

$$= \frac{1}{\overline{P}_\lambda} q^{-\langle\rho,\check{\rho}\rangle+\nu/2} \cdot q^{-2\nu} (-1)^\nu \sum_I (-q)^{|I|} \cdot q^{\langle\rho,\check{\rho}\rangle} q^{\nu/2} J_{\lambda+\alpha_I-\rho}$$

$$= \frac{1}{\overline{P}_\lambda} \sum_I (-q)^{|I|-\nu} J_{\lambda+\alpha_I-\rho} .$$

Here I runs through the subsetsof the set of positive roots. We make a change of variable $I \longrightarrow I'$ = complement of I . Then $\alpha_I+\alpha_{I'} = 2\rho$, $|I|+|I'| = \nu$ hence

$$\frac{1}{\overline{P}_\lambda} \sum_I (-q)^{|I|-\nu} J_{\lambda+\alpha_I-\rho} = \frac{1}{\overline{P}_\lambda} \sum_{I'} (-q)^{-|I'|} J_{\lambda+\rho-\alpha_{I'}}$$

and Theorem 6.6 is proved.

8. The following result describes the centre Z of $\widetilde{H}$.

<u>Theorem 8.1</u>. (J. Bernstein). <u>Let</u> $\lambda \in P^{++}$ <u>and let</u> $(\lambda)W$ <u>be its</u> W-<u>orbit in</u> P . <u>Then</u> $z_\lambda = \sum_{\lambda'\in(\lambda)W} \widetilde{T}_{p_{\lambda'}}$ <u>is in</u> Z . <u>Moreover</u>, Z <u>is the free</u> $\mathbb{Z}[q^{1/2},q^{-1/2}]$-<u>module with basis</u> z_λ $(\lambda \in P^{++})$.

<u>Proof</u> : Let $s \in S$. Then $T_s z_\lambda = z_\lambda T_s$ by 7.1. It follows that $T_w z_\lambda = z_\lambda T_w$ for all $w \in W$. It is obvious that, for any $\mu \in P^{++}$, T_{p_μ} commutes with z_λ . But the elements T_w $(w \in W)$ and T_{p_μ} $(\mu \in P^{++})$ generate H as an algebra. Hence $z_\lambda \in Z$.

Let z_λ^1 be the specializations of z_λ under the homomorphism $H \longrightarrow \mathbb{Z}[\widetilde{W}_a]$ given by $q^{1/2} \longrightarrow 1$. Then clearly z_λ^1 form a set of $\mathbb{Z}$-generators for the centre of $\mathbb{Z}[\widetilde{W}_a]$: the elements of P are the only elements of $\widetilde{W}_a$ whose conjugacy class is finite. Using a version of Nakayama's lemma it follows that any element z of Z

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is a linear combination of the elements z_λ with coefficients being allowed to be in the localization of $\mathbb{Z}[q^{1/2},q^{-1/2}]$ at the ideal generated by $q^{1/2}-1$. Since $z \in H$, these coefficients must automatically be in $\mathbb{Z}[q^{1/2},q^{-1/2}]$. The fact that the elements z_λ are linearly independent is obvious. The Theorem is proved.

Let us now define, for $\lambda \in P^{++}$, an element

(8.2)
$$S_\lambda = \sum_{\substack{\mu \in P^{++} \\ \mu \leq \lambda}} d_\mu(L_\lambda) z_\mu \in Z \quad .$$

It is clear that for $\lambda, \lambda' \in P^{++}$, we have

(8.3)
$$S_\lambda S_{\lambda'} = \sum_{\lambda'' \in P^{++}} m(\lambda,\lambda';\lambda'') S_{\lambda''}$$

where the ≥ 0 integers $m(\lambda,\lambda';\lambda'')$ are the multiplicities in the tensor product of $\underline{g}$-modules :

(8.4)
$$L_\lambda \otimes L_{\lambda'} = \sum_{\lambda'' \in P^{++}} m(\lambda,\lambda';\lambda'') L_{\lambda''} \quad .$$

By Weyl's character formula (6.3) we have

$$\Big(\sum_{w \in W} (-1)^{\ell(w)} \widetilde{T}_{(\rho)w}\Big) S_\lambda = \sum_{w \in W} (-1)^{\ell(w)} \widetilde{T}_{(\lambda+\rho)w} \quad .$$

It follows that

$$\begin{aligned}
J_\rho S_\lambda &= |W|^{-1} \sum_{w \in W} (-1)^{\ell(w)} J_{(\rho)(w)} S_\lambda \\
&= |W|^{-1} \sum_{w \in W} q^{-\nu/2} (-1)^{\ell(w)} \theta' \widetilde{T}_{(\rho)w} \theta S_\lambda \qquad \text{by lemma (7.3)} \\
&= |W|^{-1} \theta' \sum_{w \in W} q^{-\nu/2} (-1)^{\ell(w)} \widetilde{T}_{(\rho)w} S_\lambda \theta \\
&= |W|^{-1} \theta' \sum_{w \in W} q^{-\nu/2} (-1)^{\ell(w)} \widetilde{T}_{(\lambda+\rho)w} \theta \\
&= |W|^{-1} \sum_{w \in W} (-1)^{\ell(w)} J_{(\lambda+\rho)w} \\
&= J_{\lambda+\rho} \quad .
\end{aligned}$$

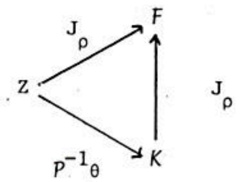
The identity

(8.5)
$$J_\rho \cdot S_\lambda = J_{\lambda+\rho} \qquad (\lambda \in P^{++})$$

shows that the map $Z \longrightarrow J$ given by $x \longrightarrow J_\rho z$ is an isomorphism of $\mathbb{Z}[q^{1/2},q^{-1/2}]$-modules. From this we shall deduce

<u>Proposition 8.6</u> : <u>The map</u> $Z \longrightarrow K$ <u>given by</u> $z \longrightarrow (\frac{1}{P} \sum_{w\in W} T_w)z = P^{-1}\theta z$ <u>is an isomorphism of</u> $\mathbb{Z}[q^{1/2},q^{-1/2}]$<u>-algebras preserving the unit element</u>. <u>Under this isomorphism</u> $S_\lambda \in Z$ <u>correspond to</u> $C'_\lambda \in K$, <u>i.e.</u> $C'_\lambda = P^{-1}\theta S_\lambda$.

Indeed, we have a commutative diagram



(since $P^{-1}J_\rho\theta = J_\rho$) and the maps $Z \longrightarrow J$, $K \longrightarrow J$ given by multiplication by J_ρ are known to be isomorphisms (see 6.8). Our map $Z \longrightarrow K$ preserves multiplication : $P^{-1}\theta z\cdot P^{-1}\theta z' = P^{-2}\theta^2 zz' = P^{-1}\theta zz'$. Finally $S_\lambda \in Z$ corresponds to $C'_\lambda \in K$, since both correspond to $J_{\lambda+\rho} \in J$ (see (6.9), (8.5)). The isomorphsim $Z \longrightarrow K$ is a version of the Satake isomorphism. It shows in particular that K is a commutative algebra.

<u>Corollary 8.7</u>. <u>If</u> $\lambda,\lambda' \in P^{++}$, <u>we have</u>

$$C'_\lambda \cdot C'_{\lambda'} = \sum_{\lambda''\in P^{++}} m(\lambda,\lambda';\lambda'')C'_{\lambda''}$$

<u>where</u> $m(\lambda,\lambda';\lambda'')$ <u>are defined by</u> (8.4).

(The remarkable fact in (8.7) is that the coefficients with which $C'_{\lambda''}$ appears in the decomposition of $C'_\lambda \cdot C'_{\lambda'}$ are independent of q .)

<u>Corollary 8.8</u>. <u>For any</u> $\lambda \in P^{++}$, <u>we have</u> $\overline{z}_\lambda = z_\lambda$.

Indeed, the isormophism given in 8.6 is compatible with $h \longrightarrow \overline{h}$ (since $\overline{P^{-1}\theta} = P^{-1}\theta$). Since $\overline{C'_\lambda} = C'_\lambda$, it follows that $\overline{S}_\lambda = S_\lambda$. But z_λ is a $\mathbb{Z}$-linear combination of element $S_{\lambda'}$, $(\lambda' \le \lambda)$ hence $\overline{z}_\lambda = z_\lambda$.

<u>Corollary 8.9</u>. <u>If</u> $\lambda \in P^{++}$, <u>we have</u>

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Proposition 9.

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$$(8.10)\qquad \sum_{\substack{\mu\in P^{++}\\ \mu\leq\lambda}} \frac{P}{P_\mu}\, q^{\langle\mu,2\check{\rho}\rangle-\nu+\nu_\mu} d_\mu(L_\lambda;q) = \frac{\prod_{\alpha>0}\left(q^{\langle\lambda+\rho,\check{\alpha}\rangle}-1\right)}{\prod_{\alpha>0}\left(q^{\langle\rho,\check{\alpha}\rangle}-1\right)}$$

(product over all positive roots $\check{\alpha}$)

Proof : The left hand side of (8.10) is $\chi(q^{\ell(p_\lambda)/2}C'_\lambda)$ (see 6.10) where $\chi : \widetilde{H} \longrightarrow \mathbb{Z}[q^{1/2},q^{-1/2}]$ is the algebra homomorphism defined by $\chi(T_w) = q^{\ell(w)}$, $\forall w \in \widetilde{W}_a$. Note that $\chi(\widetilde{T}_{p_\mu}) = q^{\langle\mu,\check{\rho}\rangle}$ for any $\mu \in P^{++}$, (see (6.11)). We have

$$\begin{aligned}\chi(q^{\ell(p_\lambda)/2}C'_\lambda) &= \chi(q^{\ell(p_\lambda)/2}P^{-1}{}_\theta S_\lambda)\\ &= q^{\ell(p_\lambda)/2}\chi(S_\lambda)\\ &= q^{\langle\lambda,\check{\rho}\rangle} \sum_{\substack{\mu\in P^{++}\\ \mu\leq\lambda}} d_\mu(L_\lambda) \sum_{\mu'\in(\mu)W} q^{\langle\mu',\check{\rho}\rangle}\end{aligned}$$

and this is known to be equal to the right hand side of (8.10). (See the proof of Weyl's character formula in [6]) .

9. Let $\mu \leq \lambda$ be two elements of P . According to [10] if $\tau \in P$ is such that $\langle\tau,\check{\alpha}_s\rangle \gg 0$ for all $s \in S$ (so that, in particular, $\mu+\tau \in P^{++}$, $\lambda+\tau \in P^{++}$) , the polynomial $P_{n_{\mu+\tau},n_{\lambda+\tau}}$ is independent of the choice of τ . In particular, it only depends on the difference $\lambda-\mu$. Using now (6.13), we see that there exists a well defined function

$$\widehat{P} : \{\kappa\in Q \mid \kappa \geq 0\} \longrightarrow \mathbb{Z}[q^{-1}]$$

such that for any $\mu \leq \lambda$ in P , with $\lambda-\mu=\kappa$, we have

$$(9.1)\qquad q^{-\langle\kappa,\rho\rangle} d_{\mu+\tau}(L_{\lambda+\tau};q) = \widehat{P}(\kappa)$$

for any $\tau \in P$ such that $\langle\tau,\alpha_s\rangle \gg 0$, for all $s \in S$.

Proposition 9.2.

$$(9.3)\qquad \widehat{P}(\kappa) = \sum_{\substack{n_1,\ldots,n_\nu\geq 0\\ n_1\alpha_1+\ldots+n_\nu\alpha_\nu=\kappa}} q^{-(n_1+\ldots+n_\nu)} .$$

Here $\alpha_1,\dots,\alpha_\nu$ is the list of all positive roots and $n_1,\dots,n_\nu$ are required to be integers. In particular for $q = 1$, $P(\kappa)$ reduces to the Kostant partition function.

Proof : The formulas (6.7), (6.9), (6.10) show that $\hat{P}(\kappa)$ satisfies the recurrence relation

$$\sum_I (-q)^{-|I|}\hat{P}(\kappa-\alpha_I) = \begin{cases} 1 & \text{if } \kappa = 0 \\ 0 & \text{if } \kappa > 0 \end{cases}$$

(sum over all subsets I of the set of positive roots), with the convention that $\hat{P}(\kappa) = 0$ if $\kappa \not\geq 0$. From this, the required formula for $\hat{P}(\kappa)$ follows immediately.

It may be conjectured that, for any $\mu \leq \lambda$ in P^{++}, we have

$$\text{(9.4)} \qquad q^{-\langle\lambda-\mu,\check{\rho}\rangle} d_\mu(L_\lambda;q) = \sum_{w\in W} (-1)^{\ell(w)}\hat{P}((\lambda+\rho)w-(\mu+\rho)) \quad .$$

For $q = 1$ this reduces to a well knwon formula of Kostant.

(Note added May 1982 : Conjecture (9.4) has been recently proved by S. Kato, to appear in Inventiones Math.)

For type A, formula (9.4) follows from a statement in [13, p. 131]; indeed, in that case, the left hand side of (9.4) is a Green-Foulkes polynomial (cf. [11]).

The right hand side of (9.4), in the special case $\mu = 0$, appears also in the work of D. Peterson, in connection with the $\underline{g}$-module structure of the (graded) coordinate ring of the nilpotent variety of $\underline{g}$.

10. If λ is the highest root, we have $d_\mu(L_\lambda;q) = 1$ for any $\mu \in P^{++}$, $0 < \mu \leq \lambda$. Indeed, the multiplicity $d_\mu(L_\lambda)$ is 1 in this case (it is a dimension of a root space in the adjoint representation of $\underline{g}$). Since $d_\mu(L_\lambda;q)$ has ≥ 0 coefficients and constant term 1, it must be identically 1. If we write the formula (8.10) for λ, the only unknown term is, therefore, $d_0(L_\lambda;q)$. We can compute it from (8.10) and we find $d_0(L_\lambda;q) = \sum q^{e_i-1}$ where e_i $(i = 1,\dots,rk(\underline{g}))$ are the exponents of $\underline{g}$.

11. We shall now describe the (generalized) Schubert varieties $\overline{\mathcal{O}}_\lambda$ $(\lambda \in P^{++})$ with the following properties :

a) $\overline{\mathcal{O}}_\lambda$ is an irreducible, projective complex variety of dimension $\langle\lambda, 2\overset{\vee}{\rho}\rangle$.

b) If $\mu,\lambda \in P^{++}$, are such that $\mu \leq \lambda$ then $\overline{\mathcal{O}}_\mu \subset \overline{\mathcal{O}}_\lambda$.

c) Let $x \in \overline{\mathcal{O}}_\lambda$ be such that $x \in \overline{\mathcal{O}}_\mu$ $(\mu \leq \lambda)$ but $x \notin \overline{\mathcal{O}}_{\mu'}$, for any $\mu' < \mu$. Then the stalks $H^i_x(\overline{\mathcal{O}}_\lambda)$ are zero if i is odd and $\sum_i \dim H^{2i}_x(\overline{\mathcal{O}}_\lambda)q^i = d_\mu(L_\lambda;q) = P_{n_\mu,n_\lambda}$.

Let $\underline{g}'$ be a simple complex Lie algebra which is dual to $\underline{g}$ in the following sense. There is a Cartan subalgebra $\underline{h}' \subset \underline{g}'$ with a given isomorphism onto $\underline{h}^*$ which carries the set of coroots of $\underline{g}'$ with respect to $\underline{h}'$ onto the set of roots of $\underline{g}'$ with respect to $\underline{h}$. Let $\hat{\underline{g}}' = \underline{g}' \otimes \mathbb{C}((t))$. For each coroot $\overset{\vee}{\alpha} \in \underline{h}$ of $\underline{g}$ we denote by X_α a non-zero vector in the corresponding root space of $\underline{g}'$. For each $\lambda \in P^{++}$, we denote by L_λ the $\mathbb{C}[[t]]$-submodule of $\hat{\underline{g}}'$ generated by the vectors $t^{\langle\lambda,\overset{\vee}{\alpha}\rangle}X_\alpha$ and by $\underline{h} \otimes \mathbb{C}[[t]]$. This is a lattice in $\hat{\underline{g}}'$ (i.e. a $\mathbb{C}[[t]]$-submodule of maximal rank.) It is moreover an order in $\hat{\underline{g}}'$ (i.e. a lattice closed under the Lie bracket). Let $(\, , \,)$ be the Killing form on $\underline{g}'$; we extend it to a symmetric bilinear form on $\hat{\underline{g}}'$ with values in $\mathbb{C}((t))$. Then $L_\lambda = L_\lambda^\#$ where for any lattice L we denote by $L^\#$ the dual lattice $\{x \in \hat{\underline{g}}' \mid (x,y) \in \mathbb{C}[[t]]$ for all $y \in L\}$. It is easy to check that if L is any order in $\hat{\underline{g}}'$, then $L \subset L^\#$. It follows that any self dual order is a maximal order, hence, by a theorem of Bruhat-Tits, it is a "maximal parahoric" order. It moreover, must correspond to a special vertex of the extended diagram of $\underline{g}'$. Indeed, if L is a maximal parahoric order corresponding to a non-special vertex v , then $\dim(L^\#/L)$ is equal to the number of roots of $\underline{g}'$ minus the number of roots in a proper semisimple subalgebra of $\underline{g}'$ (whose Coxeter diagram is obtained by removing v from the extended diagram of $\underline{g}'$); hence L is not self-dual. It follows that the group G' of automorphisms of the Lie algebra $\hat{\underline{g}}'$ inducing identity on the Weyl group, acts transitively on the set X of all self dual orders in $\hat{\underline{g}}'$. Let G'_o be the stabilizer of L_o in G' . It is known that the sets $\mathcal{O}_\lambda$ ($\mathcal{O}_\lambda = G'_o$ - orbit of L_λ in X) $(\lambda \in P^{++})$ are disjoint and cover the whole of X . For any integer $n \geq 0$, we consider the subset

$X_n \subset X$ defined by $X_n = \{L \in X \mid t^n L_o \subset L \subset t^{-n} L_o\}$. Then $X_o \subset X_1 \subset X_2 \subset \dots$ and their union is X : indeed for any lattice L we can find $n \geq 0$ such that $t^n L_o \subset L$ and we then have by duality $L \subset t^{-n} L_o$.

We will show that X_n is in a natural way a projective algebraic variety. To give a self-dual lattice L , $t^n L_o \subset L \subset t^{-n} L_o$, is the same as to give a subspace $\overline{L}$ of $t^{-n}L_o/t^nL_o$ which is t-stable and is maximal isotropic for the symmetric $\mathbb{C}$-bilinear form on $t^{-n}L_o/t^nL_o$ defined by $\mathrm{Res}(x,y)$. Moreover, L gives rise to a subspace $\widetilde{L} \subset t^{-n}L_o/t^{2n}L_o$ of codimension $= \dim L_o/t^nL_o$. Now $t^{-n}L_o/t^{2n}L_o$ carries a canonical alternating 3-form with values in $\mathbb{C}$, defined by $\mathrm{Res}([x,y],z)$. The condition that L is an order (if we assume that L is already known to be a self-dual lattice) is that this 3-form is identically zero on $\widetilde{L}$.

Thus, we have a 1-1 correspondence $L \leftrightarrow \overline{L}$ between X_n and the set of maximal isotropic subspaces of $t^{-n}L_o/t^nL_o$, stable under the nilpotent endomorphism t , and whose inverse image in $t^{-n}L_o/t^{2n}L_o$ is such that the canonical alternating 3-form vanishes identically on it.

This is a subset of a Grassmannian, defined by algebraic equations, hence is a projective algebraic variety. Thus X can be regarded as an increasing union of projective varieties. If $\lambda \in P^{++}$ satisfies $\langle \lambda, \check{\alpha} \rangle \leq n$ for all roots then $\mathcal{O}_\lambda \subset X_n$. It is then a locally closed subset of X_n , since it can be regarded as an orbit of the algebraic group $G_o'/\{g' \in G_o' \mid g' \equiv 1 \text{ on } L_o/t^nL_o\}$ acting on X_n .

We then define $\overline{\mathcal{O}}_\lambda$ to be the Zariski closure of $\overline{\mathcal{O}}_\lambda$ in X_n . One could define similarly the varieties $\overline{\mathcal{O}}_\lambda$ over a finite field F_{p^s} (instead of over $\mathbb{C}$). The number of rational points (over F_{p^s}) of $\overline{\mathcal{O}}_\lambda$ (in the sense of intersection cohomology) i.e., with each rational point x counted with a multiplicity equal to the trace of the Frobenius map on $\Sigma(-1)^i \mathcal{H}^i_x(\overline{\mathcal{O}}_\lambda)$ is the left hand side of (8.10), hence it is given by the right hand side of (8.10), with q replaced by p^s .

In particular, the Euler characteristic of $\overline{\mathcal{O}}_\lambda$ (in the sense of intersection cohomology) is equal to $\dim(L_\lambda)$.

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