Tilting characters for SL_3

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joint work with G. Williamson

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1)

G - semisimple alg. group $/ \bar{F}_p$, p = prime number

T - maximal torus

$X = Hom(T, \bar{F}_p^*)$, $Y = Hom(\bar{F}_p^*, T)$

$\check{\alpha}_i \in Y$ simple roots $i = 1, \dots, n$

$X^+ = \{ \lambda \in X ; \langle \check{\alpha}_i, \lambda \rangle \in \mathbb{N}, \forall i \}$

Rep G - category of f.d. rational representations of G $/ \bar{F}_p$

$M \in \text{Rep}\, G$ $\quad [M] = \sum_{\lambda \in X} \dim \underset{\substack{\uparrow \\ \lambda\text{-weight space}}}{M(\lambda)} \lambda \in \mathbb{Z}[X]$

Simple objects of Rep G $\leftrightarrow X^+$

$\mathcal{L}_\lambda \longleftrightarrow \lambda$

V_λ : Weyl module corr to $\lambda \in X^+$ ($\mathcal{L}_\lambda$ a quotient of V_λ)

V'_λ : co-Weyl module " " " ($\mathcal{L}_\lambda$: submod. of V'_λ)

$[V_\lambda] = [V'_\lambda]$ given by Weyl character formula

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2)

$\mathrm{Tilt}\, G$ - objects $M \in \mathrm{Rep}\, G$ which admit
- a finite filtration with subquotients Weyl modules
- a finite filtration with — " ——— co-Weyl modules

"Tilting modules".

Indecomposable objects of $\mathrm{Tilt}\, G \longleftrightarrow X^+$

$$T_\lambda \longleftrightarrow \lambda \in X^+$$

(Ringel 1991, Donkin 1993)

$$[T_\lambda] = \sum_{\mu \in X^+} m_{\mu\lambda} [V_\mu] \qquad \lambda \in X^+ \qquad m_{\mu\lambda} \in \mathbb{N}.$$

Problem : compute $m_{\mu,\lambda}$!

Actually $m_{\mu,\lambda}$ are in principle computable for any fixed μ, λ, p but the calculation is extremely complicated.

By work of Riche-Williamson, Achar-Makisumi-Riche-Williamson, Elias-Losev, Libedinsky-Williamson this computation can be given to a computer, but the calculation is still extremely long in each case.

So the problem can be restated as: [find a pattern / ~~closed~~ formula for $m_{\mu\lambda}$ uniform in p, for $p \gg 0$].

Example. $G = SL_2 \quad X = \mathbb{Z}, \quad \dim V_\lambda = \lambda + 1$

For $\lambda = 0, 1, \dots, p-1, \quad \mathcal{L}_\lambda = V_\lambda = V'_\lambda = T_\lambda$

For $\lambda = p, \quad T_\lambda = V_1 \otimes V_{p-1}, \quad [T_\lambda] = 2[\mathcal{L}_{\lambda-2}] + [\mathcal{L}_\lambda]$

$$= [V_\lambda] + [V_{\lambda-2}]$$

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3)

Replace Rep G by representations of a quantum group at $\sqrt[p]{1}$.

$\mathcal{L}_\lambda^q, V_\lambda^q, V_\lambda'^q, T_\lambda^q$: quantum analogues of $\mathcal{L}_\lambda, V_\lambda, V'_\lambda, T_\lambda$.

Then $[T_\lambda^q], [V_\lambda^q] \in \mathbb{Z}[X]$ defined and

$$[T_\lambda^q] = \sum_{\mu \in X^+} m_{\mu,\lambda}^q [V_\mu^q], \qquad m_{\mu\lambda}^q \in X^+$$

$m_{\mu,\lambda}^q$ can be expressed in terms of K-L polynomials of an affine Weyl group (Soergel 1997). Can write

$$[T_\lambda] = \sum_{\mu \in X^+} \tilde{m}_{\mu,\lambda} [T_\mu^q], \quad \lambda \in X^+, \ \tilde{m}_{\mu,\lambda} \in \mathbb{N}.$$

our problem reduced to

$*$ compute/understand $\tilde{m}_{\mu,\lambda}$

Now assume $G = SL_3$, $p \geq 3$.

In the affine space $X_{\mathbb{R}} = \mathbb{R} \otimes X$

consider the p-hyperplanes

$$H_k = \{x \in X_{\mathbb{R}} \mid \langle \check{\alpha}_1, x\rangle + 1 = kp\}, \quad k \in \mathbb{Z}$$

$$H'_k = \{x \in X_{\mathbb{R}} \mid \langle \check{\alpha}_2, x\rangle + 1 = kp\}, \quad -\text{"}-$$

$$H''_k = \{x \in X_{\mathbb{R}} \mid \langle \check{\alpha}_1 + \check{\alpha}_2, x\rangle + 2 = kp\}, \quad -\text{"}-$$

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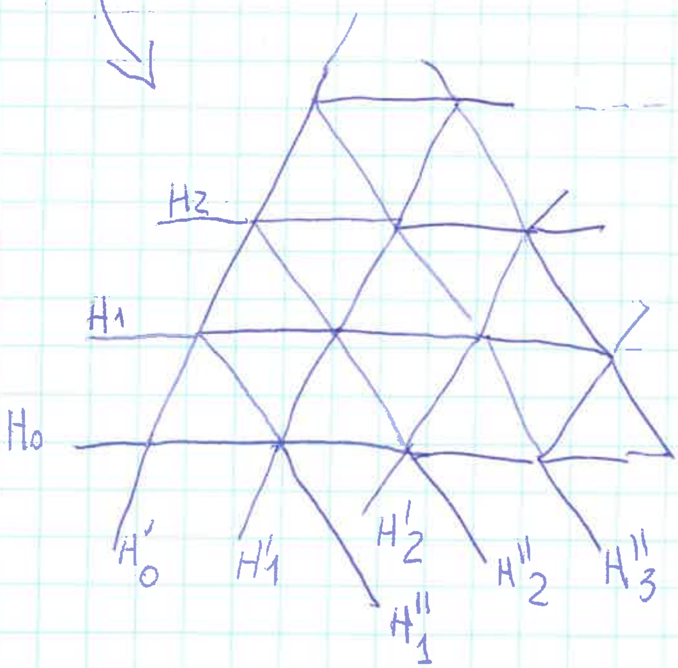
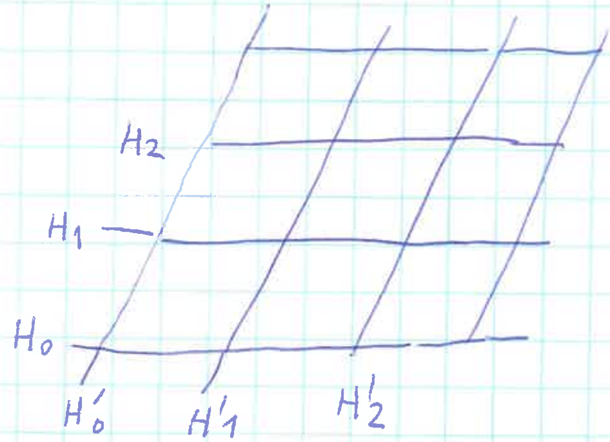
4)

p-alcoves: connected components of $X_{\mathbb{R}} - \bigcup$ p-hyperplanes.

dominant p-alcoves: those containing some $\lambda \in X^+$.



p-boxes: connected components of $X_{\mathbb{R}} - (\bigcup H_k) \cup H'_k$



$\exists$ matrix $\tilde{m}_{BA}$ indexed by dominant p-alcoves such that for $\mu, \lambda \notin$ any ~~Hap~~ p-hyperplane

$$\tilde{m}_{\mu,\lambda} = \begin{cases} \tilde{m}_{BA} & \text{if } \mu \in B, \lambda \in A \text{ are in some affine Weyl group orbit} \\ 0 & \text{if — '' — are not — '' —} \end{cases}$$

Our problem becomes:

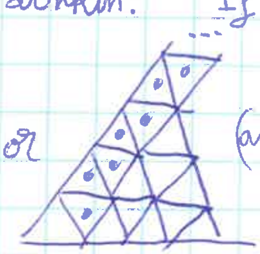
* compute/understand $\tilde{m}_{B,A}$ (B, A dominant p-alcoves)

Andersen: if B, A are not too far from 0 then $\tilde{m}_{B,A} = \delta_{BA}$

Donkin: If $\tilde{m}_{B,A}$ known for A of type

(any B)

or



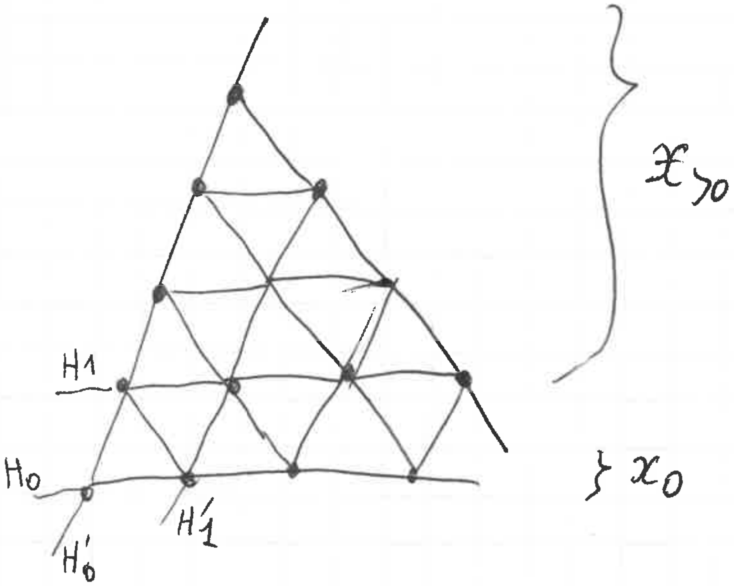
(any B) then $\tilde{m}_{BA}$ known for any B, A.

Denote:

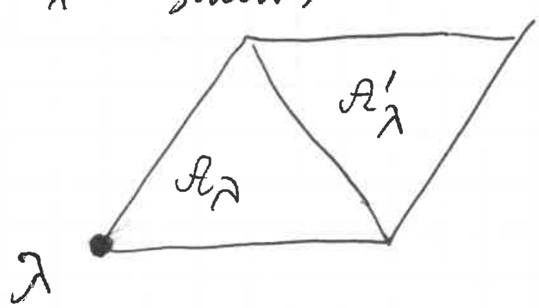
5) Set $\mathcal{X} = \{ \lambda \in X ; \langle \alpha_1^\vee, \lambda \rangle + 1 \in \{0, p, 2p, \dots\}$
$\langle \alpha_2^\vee, \lambda \rangle + 1 \in \{0, p, 2p, \dots\} \}$

$\mathcal{X}_0 = \{ \lambda \in \mathcal{X} ; \langle \alpha_1^\vee, \lambda \rangle + 1 = 0$

$\mathcal{X}_{>0} = \mathcal{X} - \mathcal{X}_0$



For each $\lambda \in \mathcal{X}_{>0}$ consider the two p-alcoves $\mathcal{A}_\lambda, \mathcal{A}'_\lambda$ as follows



6)

We will define explicitly a set Z and a function

$$Z \longrightarrow \mathcal{X}_{>0} \times \mathbb{N}_{>0} \times \{1,2\}$$

$$z \longrightarrow (\lambda_z, n_z, \delta_z)$$

so that, setting

$$\hat{m}_{\lambda, i} = \sum_{\substack{z \in Z \\ \lambda_z = \lambda,\ n_z = i}} \delta_z n_z \qquad \text{for } \lambda \in \mathcal{X}_{>0},\ \mathbb{N}_{>0}$$

we have conjecturally for $i \leq p^3$:

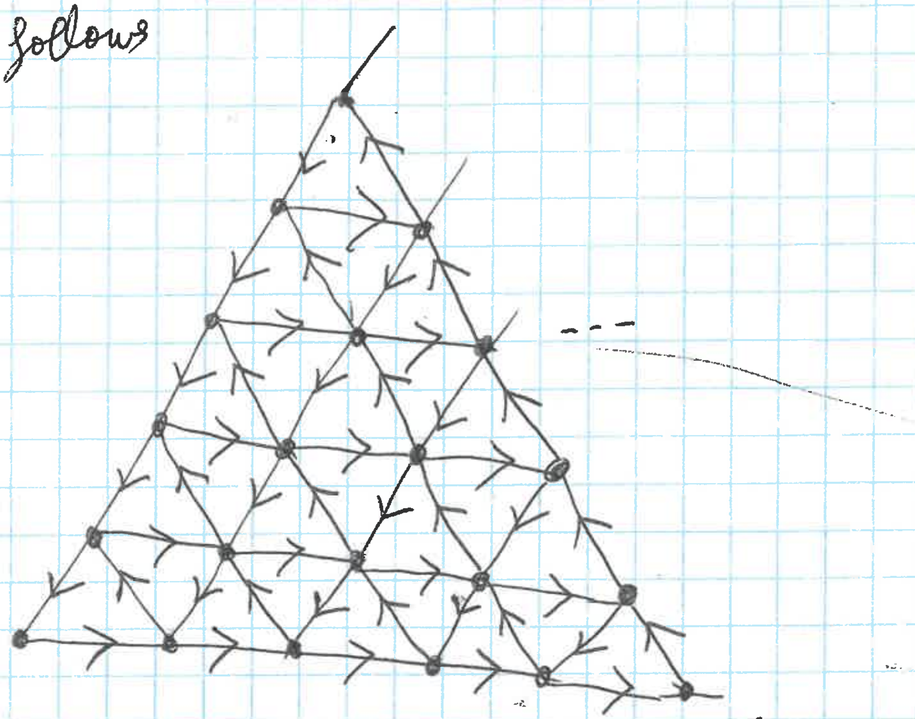
$$\sum_{\substack{B \\ \text{dom.} \\ p\text{-alcove}}} \left(\sum_{i \geq 1} \tilde{m}_{B A_i} [i] \right) B = \sum_{i \geq 1} [i] A_i +$$

$$+ \sum_{\lambda \in \mathcal{X}_{>0}} \left(\sum_{i \geq 1} \hat{m}_{\lambda i} \left([i] + [i+2] \right) \right) \mathcal{A}_\lambda + \sum_{i \geq 1} \left(\hat{m}_{\lambda i} [i+1] \right) \mathcal{A}'_\lambda$$

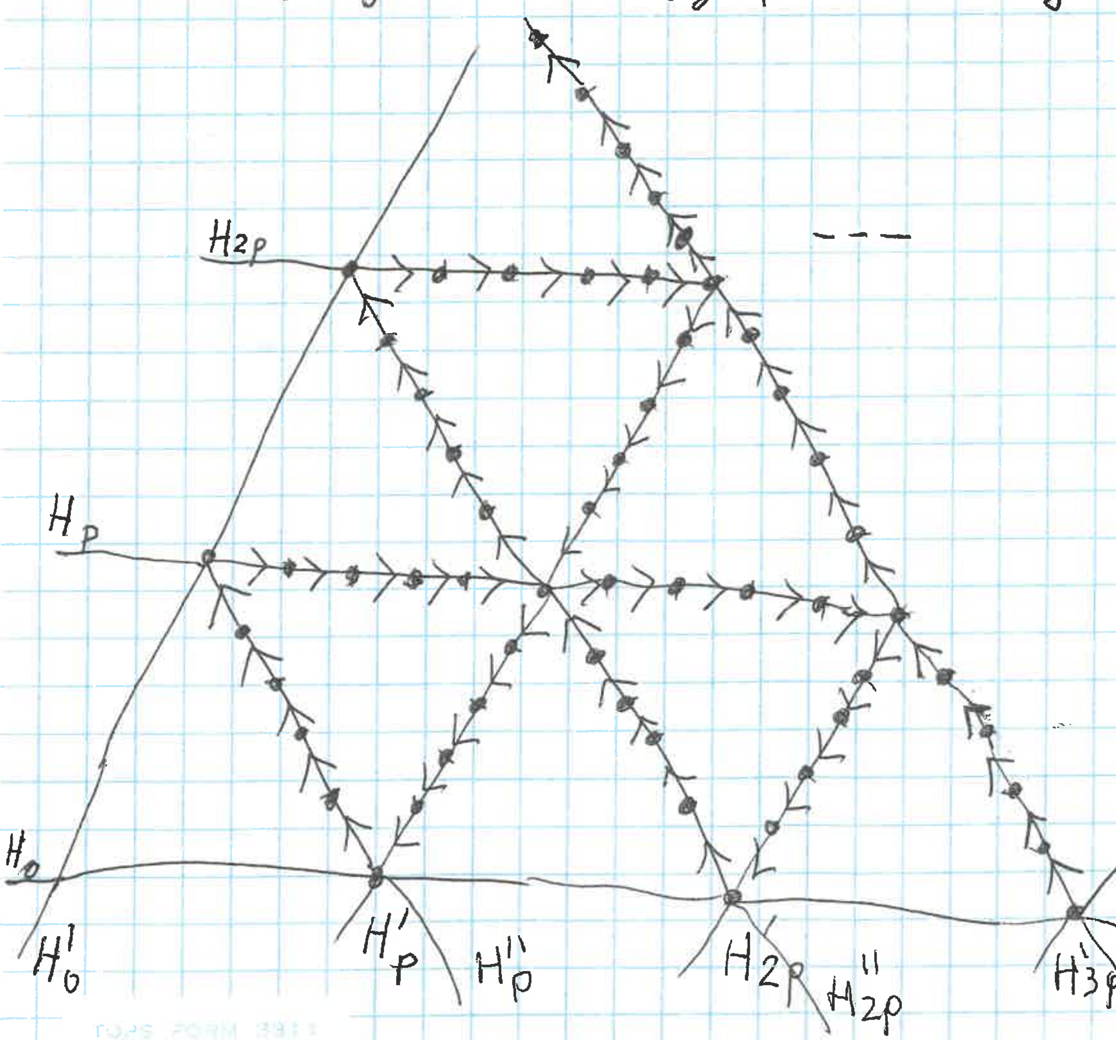
equality of formal sums.

~~explicit~~

7) We view $\mathfrak{X}$ as set of vertices of an oriented graph as follows



We now consider a subgraph $\mathfrak{X}^*$ of $\mathfrak{X}$ with induced orientation.



$\leftarrow \begin{cases} p^2 - \text{hyperplanes} \\ p^2 - \text{alcoves} \end{cases}$

$(p=5)$.

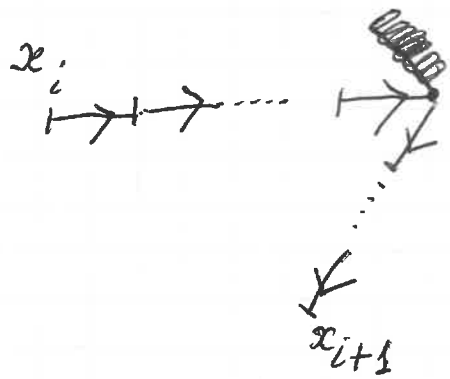
8)

Z is a certain set of sequences of vertices of $\mathcal{X}$ $(x_0, x_1, \dots x_n)$
such that
for $i = 0, \dots, n-1$ we have

$x_i = x_{i+1}$ (rest)

or $x_i \mapsto x_{i+1}$ (small step, distance 1)

or



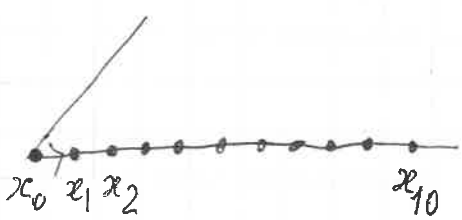
(big step - distance $p-1$ with a sharp turn all in the graph $\mathcal{X}^*$.)

A sequence in Z has three parts:

$$\underbrace{x_0, x_1, \dots, x_k}_{\text{Dull part}}, \underbrace{x_{k+1}, \dots, x_\ell}_{\text{"Brownian motion"}}, \underbrace{x_{\ell+1}, \dots x_n}_{\text{"Billiards"}}$$

this may be empty.

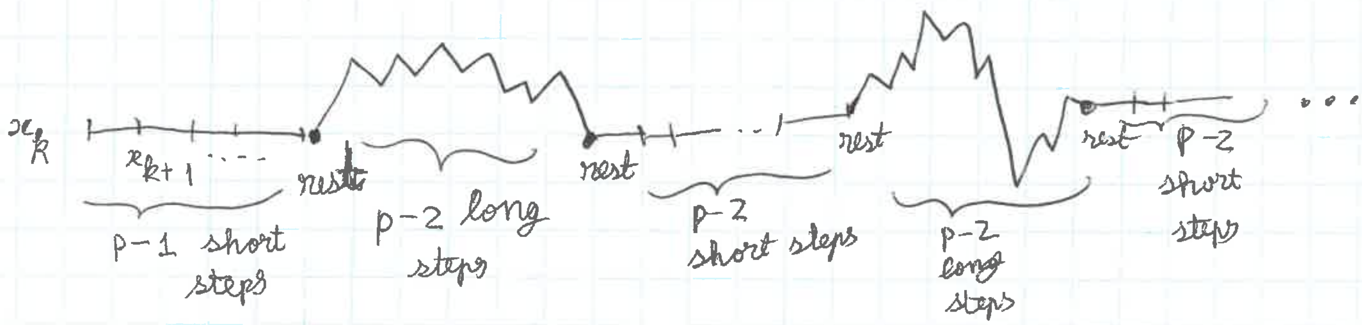
The dull part consists of $k = pr \geq p$ small steps starting with $x_0 = 0$ on the hyperplane $H_0 = \{x \mid \langle \check{\alpha}_1, x \rangle + 1 = 0\}$.

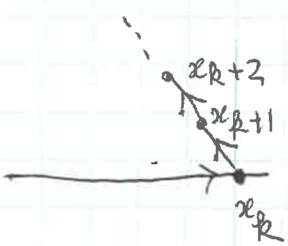


(example: $p = 5$, $r = 2$, $k = 10$)

The brownian motion part $x_{k+1}, \dots x_\ell$ takes place in $\mathcal{X}^*$; each x_i is on a p^2-hyperplane but not on a p^2-corner (intersection of two p^2-hyperplanes).

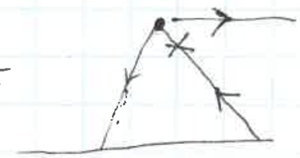
9) It is a succession of small steps followed by big steps, followed by small steps, followed by big steps, etc ~~set~~ with rest in between.



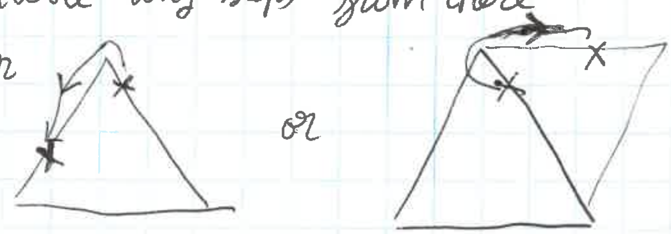


The first $p-1$ short steps are as shown.
We cannot have p steps without hitting a corner point.

After $p-1$ short steps ~~[illegible]~~ and a rest we are at at distance 1 from a corner point.



There are one or two available long steps from there and ~~[illegible]~~ we get either



which are distance 2 from a corner point.
From these points we can do one or two long steps. After $p-2$ long steps we cannot do long steps any more (we would hit a corner)
Instead we start doing small steps (after a rest). They are uniquely determined and we can do at most $p-2$ of them without hitting a corner.
If we do all $p-2$ small steps we rest and continue with long steps, etc. We can stop this process at any time But l must be $> k$.

In the case where x_ℓ could be a start of a big step we have the option of stopping completely or continuing the sequence by leaving the p^2-hyperplanes and getting inside (and remaining inside) a p^2-alcove. The sequence moves as a ball on a billiard table shaped as an equilateral triangle.

This defines the set Z.

The function $Z \to X_{>0}$ is $(x_0, \dots, x_n) \longrightarrow x_n$

The function $Z \to \mathbb{N}_{>0}$ is $|x_0, \dots, x_n) \to |x_n|$

where $|x_i|$ is defined by induction on i as follows

$|x_0| = 0$

if $x_i = x_{i+1}$ then $|x_{i+1}| = |x_i| + 3$

if $x_i \to x_{i+1}$ then $|x_{i+1}| = |x_i| + 2$

if x_i, x_{i+1} is a big step then $|x_{i+1}| = |x_i| + 2p+1$

Thus $|x_n|$ is approximately $2n$

The function $Z \to \{1,2\}$ is $(x_0, \dots x_n) \to \begin{cases} 1 & \text{if } x_0 \to x_1 \to \dots \to x_n \text{ are all small steps} \\ 2 & \text{otherwise.} \end{cases}$

<u>Remarks</u>. 1) The dimension of the tilting module T_λ $(\lambda \in A_i)$ grows at least exponentially in i (assuming the conjecture).

2) The conjecture implies a statement about the decomposition numbers of the symmetric group for partitions with ≤ 3 parts. (Erdmann)