HISTORY OF THE CANONICAL BASIS AND CRYSTAL BASIS

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1. Introduction

This document clarifies the historical development of the canonical basis theory (see §3). Such a clarification is necessary in view of the persistent false and misleading claims in the literature with regard to the history of the canonical basis and crystal basis (see §2). Over the years I thought that the evidence in the literature about this history is clear and that most people accept this evidence. However Kashiwara repeatedly made misleading claims about priority and this has influenced some people who do not know the history. Since such false and misleading information has now spread out of the mathematical community to the public, I feel that I must now publish this document. I present this document to restore the correct historical record and to defend academic integrity in the mathematical community.

2. False claims

2.1. An evolving narrative.

In 1995, Kashiwara [K95] claimed that his basis and mine were "constructed independently" despite clear evidence to the contrary, see §3.

In 2006 at my birthday conference he gave a talk where he again made the independence claim. After his talk I pointed out this error to him and then he **acknowleged in** [EK] **that I defined the canonical basis first for type ADE.**

Yet, in 2018, Kashiwara, [K18] (version 1 on arXiv), repeated the independence claim. After I protested, he **acknowleged in version 2 on arXiv that I defined the canonical basis first for type ADE.**

Kashiwara's repeated attempts to rewrite history have also influenced other people who do not know the history. See 2.2 and 2.3 for examples of this.

2.2. The Carter incident. In the original version of [C], Carter states: "In 1990 Lusztig made another discovery of fundamental importance by proving the existence of a remarkable basis of a quantized enveloping algebra called the canonical basis. This basis was also subsequently proved to exist by Kashiwara..." However, in the published version, **somehow the word "subsequently" was changed to**

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"independently", without Carter's consent, falsely implying parallel discovery. After I pointed out this distorsion to the Editorial Committee, **they issued an Erratum** [EC]**, restoring Carter's original wording.**

2.3. Misleading awards citations. Throughout the past eight years, Kashiwara has received various awards based in part on inaccurate information provided by nominators.

-In the Chern Medal initial citation it is stated that "Kashiwara's discovery of crystal basis is another landmark in representation theory".

-In the Kyoto Prize initial citation it is stated that "...Kashiwara...for construction of the crystal basis theory".

-In the Abel Prize initial citation it is stated that "Kashiwara introduced the notion of crystal bases and proved the existence of crystal bases for integrable highest weight modules for quantum groups...Kashiwara also generalized crystal bases to global bases which were independently discovered by George Lusztig under the name canonical bases".

Each of these statements is false, see 3.5, 3.6. In each of these cases the citations were modified after I explained the facts.

2.4. Public misinformed. In the New York Times (March 26, 2025) it is stated that Kashiwara invented the crystal basis.

In Nature (March 26, 2025) it is stated: "In particular, Kashiwara's notion of a crystal base has enabled mathematicians to interpret any representation ..."

In Scientific American (March 26, 2025) it is stated:"...Kashiwara introduced the concept of crystal bases."

In the Wikipedia page on Abel Prize it is stated: "(Kashiwara) ... and the discovery of crystal bases".

These statements reflect the statements in the original Abel Prize citation and that the public was misinformed.

3. Overview of the canonical basis

3.1. In my 1990 paper [L90] I observed that for the irreducible representations of a quantized enveloping algebra of simply laced type as well as for the positive part of that algebra there is a new, extremely rigid structure in which the objects of the theory are provided with canonical bases with rather remarkable properties; in particular, all the structure constants with respect to the canonical basis are in $\mathbf{N}[v, v^{-1}]$ (v is an indeterminate).

My construction employed two methods: an algebraic approach utilizing braid group actions and PBW (Poincaré-Birkhoff-Witt) bases and a topological approach using intersection cohomology.

Key consequences include:

Specializing the parameter v to $v = 1$ yields canonical bases for the irreducible representations of the corresponding simple Lie algebra in which all the structure constants are in $\mathbf{N}$, a property that was not previously known. When specializing

v to 0 or to ∞ the canonical basis produces a shadow structure ("crystal basis") which Kashiwara also considered (for classical types) in his 1990 paper [K90].

The graph structure on the crystal basis considered by Kashiwara is also a shadow of the canonical basis, see [L90b, Theorem 7.5], in the sense that the graph structure can be deduced from the canonical basis. A property similar to [L90b, Theorem 7.5] is stated in Kashiwara's later paper [K95], Lemma 12.1, but he fails to give a reference to my paper.

3.2. In [K90], written at the same time as [L90], Kashiwara gives a conjectural definition of the crystal basis ("basis at $v = 0$") for the irreducible representations of a quantized enveloping algebra. He proves that the definition is correct for classical types. He was motivated by mathematical physics and his focus on looking at $v = 0$ is related to the absolute temperature being 0. The underlying philosophy, as indicated in the paper [K90], is that at the absolute temperature being 0, modules should have some special properties. This philosophy would not lead to canonical bases.

3.3. His subsequent announcement [K90a] -which explicitly cites [L90] and states that his basis "appears to agree" with mine-proves that he encountered the concept in my work first. He developed his approach after reading [L90] though without crediting this foundation.

The main ingredients in lifting the crystal basis to the canonical bases in [L90] are

-the $\mathbf{Z}[v, v^{-1}]$-form of quantized enveloping algebras [L88, L90a],

-the bar involution [L90].

In [K90a] **Kashiwara copies (see (1.5), page 278 and §5, page 279) the definitions of these ingredients from** [L88]**,** [L90]**,** [L90a] **without reference, thus giving the impression that he is the first to define these concepts.**

He also copies (see the line after Theorem 5, page 279) the method of lifting the crystal basis to the canonical basis from [L90] **without reference.**

3.4. The concept of canonical basis was subsequently extended to the broader setting of Kac-Moody Lie algebras. I provided such an extension using a topological approach in [L91] (see also [L93]) while Kashiwara extended it through an algebraic approach in [K91].

3.5. It is **not** correct to say that Kashiwara defined the canonical basis independently of me. The reason is as follows.

My 1990 paper [L90] contains the first construction of canonical bases for ADE types.

Kashiwara's 1990 paper [K90] written concurrently with [L90] contains nothing about canonical bases. Both the motivation and the technical tools of his paper [K91] are inherited from [L90], although Kashiwara does not acknowledge that.

Therefore it is clear that without my discovery of canonical bases [L90], Kashiwara could not have written [K91].

3.6. It is **not** correct to say that Kashiwara is the sole discoverer of the crystal basis. The reason is as follows.

My 1990 paper [L90] defines the crystal basis for ADE types, including the most challenging E_8 type.

Kashiwara's 1990 paper [K90] defines the crystal basis for classical types but in type E his definition is only conjectural.

In [K91], [L91] (see also [L93]) the crystal basis is defined for all types. But to do so, [K91] needed the idea that the crystal basis exists for the + part of the quantum group (not only for irreducible representations, which was the only case considered in [K90]). This idea came from my paper [L90], without acknowledgment.

5. Conclusion

By publishing this document I aim to rectify the historical narrative for the benefit of the mathematical community and of the general public and to ensure that proper attribution and academic integrity is upheld by all.

I trust that all readers -including Kashiwara- will recognize these established facts:

(a) The canonical basis was first defined in my work [L90] and Kashiwara's subsequent contribution built directly on this foundation.

(b) The crystal basis is not solely Kashiwara's discovery.

And everyone who knows the history would suggest Kashiwara to publicly acknowledge (a) and (b), to correct all false and misleading information once and for all.

Appendix. Technical details and comparisons

A.1. In this appendix we give more details of the papers [L90], [K90], [K90a].

We introduce some notation. Let $\mathbf{U}$ be the quantum group over $\mathbf{Q}(v)$ attached by Drinfeld, Jimbo to a generalized Cartan matrix $C = (a_{ij})_{i,j \in I}$. Let E_i, F_i, K_i $(i \in I)$ be the standard generators of $\mathbf{U}$; recall that $\mathbf{U}^+$ is the subalgebra of $\mathbf{U}$ generated by $E_i, i \in I$. Now let $\lambda : I \to \mathbf{N}$ and let $\mathbf{L}_\lambda = \mathbf{U}^+ / \sum_{i \in I} \mathbf{U}^+ E_i^{\lambda(i)+1}$. Let x_0 be the image of $1 \in \mathbf{U}^+$ in $\mathbf{L}_\lambda$. It is known that $\mathbf{L}_\lambda$ is an irreducible $\mathbf{U}$-module in which E_i acts by left multiplication, F_i maps x_0 to 0 and K_i maps x_0 to $v^{-\lambda(i)} x_0$.

There is a well defined $\mathbf{Q}$-algebra isomorphism ("bar involution")$^-$: $\mathbf{U}^+ \to \mathbf{U}^+$ such that $\bar{E}_i = E_i, \bar{v} = v^{-1}$.

Let $\mathcal{A}' = \mathbf{Q}[v, v^{-1}]$. In [L88] I defined an $\mathcal{A}'$-form $\mathbf{U}^+_{\mathcal{A}'}$ of $\mathbf{U}^+$.

A.2. We now assume that C is symmetric, positive definite. Let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$. In [L90a], I defined an $\mathcal{A}$-submodule $\mathbf{U}^+_{\mathcal{A}}$ of $\mathbf{U}^+$ which was an $\mathcal{A}$-subalgebra and defined an $\mathcal{A}$-basis for it. This is one of several $\mathcal{A}$-bases (PBW bases) which can be defined using the braid group action [L88] on $\mathbf{U}$ (there is one PBW basis for each reduced expression of the longest element of the Weyl group).

A.3. The paper [L90] contains the definition of the canonical basis of $\mathbf{U}^+$. The definition is as follows.

We show that the $\mathbf{Z}[v^{-1}]$-submodule $\mathcal{L}$ of $\mathbf{U}^+$ generated by any PBW basis is independent of the PBW basis and that the image of any PBW basis under the obvious map $\pi : \mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L}$ is a $\mathbf{Z}$-basis β of $\mathcal{L}/v^{-1}\mathcal{L}$ independent of the PBW-basis.

We show that

(a) for any $b \in \beta$ there is a unique $\tilde{b} \in \mathcal{L}$ such that $\bar{\tilde{b}} = b, \pi(\tilde{b}) = b$;

(b) $\mathbf{B} := \{\tilde{b}; b \in \beta\}$ is a $\mathbf{Q}(v)$-basis of $\mathbf{U}^-$, an $\mathcal{A}$-basis of $\mathbf{U}^+_{\mathcal{A}}$, a $\mathbf{Z}[v^{-1}]$-basis of $\mathcal{L}$ and a $\mathbf{Z}$-basis of $\mathcal{L} \cap \bar{\mathcal{L}}$.

This is the canonical basis of $\mathbf{U}^+$.

A.4. Let $\lambda : I \to \mathbf{N}$. In [L90] it is shown that the nonzero vectors in the image of $\mathbf{B}$ under the obvious map $\mathbf{U}^+ \to \mathbf{L}_\lambda$ form a basis $\mathbf{B}_\lambda$ of $\mathbf{L}_\lambda$. This is the canonical basis of $\mathbf{L}_\lambda$. By specializing v to 1 one obtains a canonical basis of any irreducible representation of the Lie algebra defined by C.

A.5. We return to a general C. Let $\lambda : I \to N$. Let A be the ring consisting of the elements in $\mathbf{Q}(v)$ which have no pole at $v = 0$. In [K90] an explicit collection X_λ of vectors in $\mathbf{L}_\lambda$ is defined. Let $L(\lambda)$ be the A-submodule of $\mathbf{L}_\lambda$ spanned by X_λ and let $B(\lambda)$ be the set of nonzero elements in the image of X_λ under the obvious map $L(\lambda) \to L(\lambda)/vL(\lambda)$. In [K90] it is conjectured that $B(\lambda)$ is a $\mathbf{Q}$-basis of $L(\lambda)/vL(\lambda)$. This conjecture is proved in [K90] in the case where C is of finite, classical type.

A.6. In [K90a], Kashiwara announced a proof of his conjecture in A.5 and a proof of the following version of that conjecture.

(a) There is an explicit collection X_∞ of vectors in $\mathbf{U}^+$ such that if $L(\infty)$ is the A-submodule of $\mathbf{U}^+$ spanned by X_∞ and $B(\infty)$ is the set of nonzero elements in the image of X_∞ under the obvious map $\pi_\infty : L(\infty) \to L(\infty)/vL(\infty)$ then $B(\infty)$ is a $\mathbf{Q}$-basis of $L(\infty)/vL(\infty)$.

Now $L(\infty), B(\infty)$ did not appear in [K90]. They did appear in [L90] for type ADE; indeed in that case $L(\infty)$ is the same as $A \otimes \mathcal{L}$ (with $\mathcal{L}$ as in A.3, after changing v to v^{-1}) and $B(\infty)$ is similarly the same as β in A.3. But Kashiwara does not say that the idea to consider $L(\infty), B(\infty)$ (which was missing in [K90]) comes from [L90].

A.7. The paper [K90a] announces a proof of statements (a),(b) below.

(a) For any $b \in B(\infty)$ there is a unique $\tilde{b} \in L(\infty) \cap \mathbf{U}^-_{\mathcal{A}'}$ such that $\bar{\tilde{b}} = \tilde{b}, \pi_\infty(\tilde{b}) = b$.

(b) The set $\mathbf{B}' = \{\tilde{b}; b \in B(\infty)\}$ is a $\mathbf{Q}(v)$-basis of $\mathbf{U}^+$ and an $\mathcal{A}'$-basis of $\mathbf{U}^+_{\mathcal{A}'}$.

$\mathbf{B}'$ is called the global crystal basis of $\mathbf{U}^+$. There is also a statement about a global crystal basis $\mathbf{B}'_\lambda$ of $\mathbf{L}_\lambda$ for $\lambda : I \to \mathbf{N}$.

A.8. The idea to lift b to $\tilde{b}$ as in A.7(a) is copied in [K90a] from the analogous idea in [L90], see A.3(a), without reference.

A.9. The statements of [K90a] were proved in [K91]. A geometric construction of the canonical basis $\mathbf{B}$ for symmetric C (which has very strong positivity properties not seen in the approach of [K91]) is given in [L91] extending one of the two approaches of [L90]. The case of symmetrizable C is deduced from the symmetric case in [L93]. It is known that $\mathbf{B} = \mathbf{B}'$ (see [L90b] for the case where C is symmetric, positive definite and [GL] for the other cases).

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