18.440 Midterm 2, Fall 2009: 50 minutes, 100 points

1. **Carefully and clearly show your work on each problem (without writing anything that is technically not true).**
2. **No calculators, books, or notes may be used.**
3. **Put a box around each of your final computations.**

1. (20 points) Suppose that A_1 and A_2 are independent uniform random variables on $[0, 1]$. Let $X = \max\{A_1, A_2\}$ and $Y = \min\{A_1, A_2\}$. Compute the following:

(a) The probability density function f_X.

(b) The expectation $\mathbb{E}[X]$.

(c) The conditional expectation $\mathbb{E}[Y|X]$.

(d) The covariance $\mathrm{Cov}(X, Y)$.

2. (20 points) Suppose that in a certain town earthquakes occur as a Poisson point process with an average of 3 per decade, floods are a Poisson process with an average of 2 per decade, and meteor strikes are a Poisson process with an average of 1 per decade. Consider the present to be time zero, and write E, F and M for the time in decades between the present and the first earthquake, flood, and meteor strike (respectively). Compute the following:

(a) The probability mass function for the number of earthquakes before the first flood.

(b) The probability density function for the time (in decades) at which the second meteor strike occurs.

(c) The covariance $\mathrm{Cov}(\min\{E, F, M\}, M)$.

3. (20 points) Suppose that X is a standard normal random variable ($\mu = 0$, $\sigma^2 = 1$) and Y is uniform on $[0, 1]$ and that X and Y are independent.

1. Compute the variance of XY.

2. Compute the moment generating function of $X + Y$.

4. (20 points) Suppose that $X_1, X_2, \ldots, X_{100}$ are independent standard normal random variables.

1. Compute the correlation coefficient $\rho(X, Y)$ where $X = \sum_{i=1}^{60} X_i$ and $Y = \sum_{i=41}^{100} X_i$.

2. What is the conditional distribution of X given that $\sum_{i=1}^{100} X_i = x$?

5. (20 points) Harriet invests one dollar in a certain volatile stock. Each year her investment doubles in value with probability .4 and decreases in value by fifty percent (i.e., halves in value) with probability .6.

1. What is the expected value of Harriet's investment after 100 years?

2. Use a normal random variable to estimate the probability that Harriet's investment will be worth more than one dollar after 100 years. (You can use the function $\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answer.)