

18.440 PROBLEM SET SEVEN, DUE APRIL 11

A. FROM TEXTBOOK CHAPTER FIVE:

1. Theoretical Exercise 26: If X is a beta random variable with parameters a and b show that

$$E[X] = \frac{a}{a+b},$$

$$\text{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)}.$$

2. Theoretical Exercise 28: Consider the beta distribution with parameters (a, b) . Show that
 - (a) when $a > 1$ and $b > 1$, the density is unimodal (that is, it has a unique mode) with mode equal to $(a-1)/(a+b-2)$;
 - (b) when $a \leq 1, b \leq 1$, and $a+b < 2$, the density is either unimodal with mode at 0 or 1 or U -shaped with modes at both 0 and 1;
 - (c) when $a = 1 = b$, all points in $[0, 1]$ are modes.

B. FROM TEXTBOOK CHAPTER SIX:

1. Problem 30: Jill's bowling scores are approximately normally distributed with mean 170 and standard deviation 20, while Jack's scores are approximately normally distributed with mean 160 and standard deviation 15. If Jack and Jill each bowl one game, then assuming that their scores are independent random variables, approximate the probability that
 - (a) Jack's score is higher;
 - (b) the total of their scores is above 350.
2. Theoretical Exercise 12: Show that the jointly continuous (discrete) random variables X_1, \dots, X_n are independent if and only if their joint probability density (mass) function $f(x_1, \dots, x_n)$ can be written as

$$f(x_1, \dots, x_n) = \prod_{i=1}^n g_i(x_i),$$

for nonnegative functions $g_i(x)$, $i = 1, \dots, n$.

C. FROM TEXTBOOK CHAPTER SEVEN:

1. Problems 34: If 10 wife-husband couples are randomly seated at a round table, compute (a) the expected number and (b) the variance of the number of wives who are seated next to their husbands.
2. Theoretical Exercise 19: Show that if X and Y are identically distributed (but not necessarily independent) then $\text{Cov}(X + Y, X - Y) = 0$.

D. The following is one formulation of a famous “two envelope” paradox. Jill is a money-loving individual who, given two options, invariably chooses the one that gives her the most money in expectation. One day Harry, a trusted (and capable of delivering) individual, offers her the following deal as a gift. He will secretly toss a fair coin until the first time that it comes up tails. If there are n heads before the first tails, he will place 10^n dollars in one envelope and 10^{n+1} dollars in the second envelope. (Thus, the probability that one envelope has 10^n dollars and the other has 10^{n+1} dollars is 2^{-n-1} for $n \geq 0$.) Harry will then hand Jill the pair of envelopes (randomly ordered, indistinguishable from the outside) and invite her to choose one. After Jill chooses an envelope she will be allowed to open it. Once she does, she will be allowed to either keep the money in the first envelope or switch to the second envelope and keep whatever amount of money is in the second envelope. However, if she decides to switch envelopes, she has to pay a one dollar “switching fee.”

1. If Jill finds 100 dollars in the first envelope she opens, what is the conditional probability that the other envelope contains 1000 dollars? What is the conditional probability that the other envelope contains 10 dollars?
2. If Jill finds 100 dollars in the first envelope she opens, how much money does Jill expect to win from the game if she does not switch envelopes? (Answer: 100 dollars.) How much does she expect to win (net, after the switching fee) if she *does* switch envelopes?
3. Generalize the answers above to the case that the first envelope contains 10^n dollars (for $n \geq 0$) instead of 100.
4. Jill concludes from the above that, no matter what she finds in the first envelope, she will expect to earn more money if she switches envelopes and pays the one dollar switching fee. This strikes Jill as a

bit odd. If she knows she will always switch envelopes, why doesn't she just take the second envelope first and avoid the envelope switching fee? How can she be maximizing her expected wealth if she spends an unnecessary "switching fee" dollar no matter what? How does one resolve this apparent paradox?