18440: Probability and Random variables
Quiz 1, Version 2
Wednesday, October 22, 2014

- You will have 50 minutes to complete this test.
- No calculators, notes, or books are permitted.
- If a question calls for a numerical answer, you do not need to multiply everything out. (For example, it is fine to write something like $(0.9)7!/(3!2!)$ as your answer.)
- Don't forget to write your name on the top of every page.
- Please show your work and explain your answer. We will not award full credit for the correct numerical answer without proper explanation. Good luck!

Problem 1 (20 points) Let $A, B, C \subset S$ be three non-empty events such that A and B are disjoint, and the union of all three events is the entire sample space S. We have the following: $P(A) = p, P(B) = 2/3, P(C) = 1/2$, and $P(A|C) = 3/4$.

(1) (5 points) Compute $P(A \cup B)$ and find the largest possible value of p.
(2) (5 points) Compute $P(A \cap C)$.
(3) (5 points) Compute $P(B \cap C)$ and find the smallest possible value of p.
(4) (5 points) Let $p = 1/4$. Compute $P(C|A \cup B)$.

Proof. (1): $P(A \cup B) = P(A) + P(B) = 2/3 + p$, since A and B are disjoint. Since $1 \geq P(A \cup B)$ we conclude $p \leq 1/3$.
(2): $P(A \cap C) = P(A|C)P(C) = 3/4 \times 1/2 = 3/8$.
(3): $P(B \cap C) = P(C) - P(A \cap C) - (1 - P(A \cup B)) = 1/2 - 3/8 - 1 + 2/3 + p = 1/6 - 3/8 + p = p - 5/24$. Latter needs to be nonnegative so we conclude $p \geq 5/24$.
(4):

$$P(C|A \cup B) = \frac{P((C \cap A) \cup (C \cap B))}{P(A \cup B)} = \frac{P(C \cap A) + P(C \cap B)}{P(A \cup B)} =$$
$$\frac{3/8 - 5/24 + p}{2/3 + p} = \frac{9 - 5 + 6}{16 + 6} = \frac{5}{11}.$$

□

Problem 2 (15 points) On a given day 500 customers visited an ice-cream stand and independently chose among 100 flavors of ice-cream (one of them is vanilla) with equal probability. Let X be the number of people who ordered vanilla.

(1) Find $E[X]$.
(2) Using the Poisson approximation estimate $P(X = 2)$.

Proof. (1):We observe that the random variable X has the binomial distribution with parameter $(\frac{1}{100}, 500)$. We thus conclude that

$$E[X] = 500 \times \frac{1}{100} = 5.$$

(2): We know that for small values the binomial distribution X ($Binom(p, n)$) can be approximated by a Poisson distribution with parameter $\lambda = E[X] = np$, provided n is large and p is small but np is of order 1. Thus we conclude that

$$P(X = 2) \approx e^{-5}\frac{25}{2}.$$

$\square$

Problem 3 (10 points) Two cards are randomly selected without replacement from a deck of 52 cards. What is the probability that the first card is a Queen, given that the second card is a heart?

Proof. Let A be the event that the first card is a Queen, B the probability that the second card is a heart. Let Q be the event that the first card chosen is a heart. Then

$$P(A|B) = \frac{P(A,B)}{P(B)}.$$

There are 52×51 ways to pick two cards with order, and from those 3×13 involve a non-heart Queen followed by a heart and 1×12 involve the first being a Queen of hearts and followed by a heart. So $39 + 12 = 51$ in total. On the other hand we have,

$$P(B) = P(B|Q)P(Q) + P(B|Q^c)P(Q^c) = \frac{12}{51}\frac{1}{4} + \frac{13}{51}\frac{3}{4} = \frac{12 + 3 \times 13}{204} = \frac{1}{4}.$$

Thus we have

$$P(A|B) = \frac{51}{52 \times 51}\frac{1}{1/4} = \frac{4}{52} = \frac{1}{13}.$$

$\square$

Problem 4 (35 points) The color of a person's eyes is determined by a single pair of genes. If they are both blue-eyed genes, then the person will have blue eyes; if they are both brown-eyed genes, then the person will have brown eyes; and if one of them is a blue-eyed gene and the other a brown-eyed gene, then the person will have brown eyes. (Because of the latter fact, we say that the brown-eyed gene is dominant over the blue-eyed one.) A newborn child independently receives one eye gene from each of its parents, and the gene it receives from a parent is equally likely to be either of the two eye genes of that parent. Suppose John and his parents both have brown eyes, but John's sister Jane has blue eyes.

(1) (5 points) What is the gene type of John's parents.

(2) (10 points) What is the probability that John has two dominant genes.

(3) (10 points) Suppose John married Mary who has blue eyes. What is the probability that their first child has brown eyes?

(4) (10 points) What is the probability that John and Mary's second child has blue eyes, if their first one had brown eyes.

Proof. (1): Let the dominant gene be A and the blue-eyed gene a. Since Jane has blue eyes, while both her parents have brown we conclude the parents both have gene type Aa.

(2): Let J be John's gene type, and E the event that John has brown eyes. Then we have

$$P(J = AA|E) = \frac{P(J = AA, E)}{P(E)} = \frac{P(J = AA)}{P(J = AA \text{ or } J = Aa)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

(3): We let M be Mary's gene type, and F the event that the child has brown eyes. We assume Mary and John's gene types are independent. Then we have

$$P(F|M = aa, E) = \frac{P(F, M = aa, E)}{P(M = aa, E)} = \frac{P(F, M = aa, J = AA) + P(F, M = aa, J = Aa)}{P(M = aa, E)} =$$

$$\frac{P(M = aa, J = AA) + P(F, M = aa, J = Aa)}{P(M = aa, E)} =$$

$$\frac{P(M = aa)P(J = AA) + P(F|M = aa, J = Aa)P(M = aa)P(J = Aa)}{P(M = aa)P(E)} =$$

$$\frac{P(J = AA) + (1/2)P(J = Aa)}{P(E)} = P(J = AA|E) + (1/2)P(J = Aa|E) = 1/3 + 1/3 = 2/3.$$

(4): Let S be the gene type of John and Mary's second child. Then we have

$$P(S = aa|M = aa, F, E) = \frac{P(S = aa, F|M = aa, E)}{P(F|M = aa, E)}.$$

Now

$$P(S = aa, F|M = aa, E) =$$

$$\frac{P(S = aa, F|M = aa, J = AA)P(J = AA) + P(S = aa, F|M = aa, J = Aa)P(J = Aa)}{P(E)}$$

$$= 0 + 1/4 \times 2/3 = \frac{1}{6}.$$

Thus we conclude

$$P(S = aa|M = aa, F, E) = \frac{1/6}{2/3} = \frac{1}{4}.$$

$\square$

Problem 5 (20 points) Roll ten dice independently. Let X be the number of ways of choosing a two dice that have the same value. Also let Y be the number of ways of choosing three dice with the same value (if they all have the same value then $X = 10 \times 9/2$ and $Y = 10 \times 9 \times 8/6$).

(1) (10 points) Compute $\mathbb{E}[X]$ and $E[Y]$.

(2) (10 points) Find the probabilities of $\{X = 5, Y = 1\}$ and $\{X = 4, Y = 1\}$.

Proof. (1): Number the dice from 1 to 10. Let E_j^i be the random variable that equals 1 if die i has the same value as die j and 0 otherwise. Then we have that

$$X = \sum_{i<j} E_i^j, \text{ while } Y = \sum_{i<j<k} E_i^j E_j^k.$$

We now observe that

$$E[E_i^j] = \frac{1}{6}, \text{ while } E[E_i^j E_j^k] = 1/36.$$

Thus we conclude that

$$E[X] = \frac{1}{6} \times \binom{10}{2} = \frac{1}{6} \times 45 = \frac{15}{2}.$$

$$E[Y] = \frac{1}{36} \times \binom{10}{3} = \frac{1}{36} \times 120 = 5.$$

(2): We have that if $Y = 1$ then there are exactly three dice that have the same value v_1. This produces 3 pairs, and so we must have additional two pairs of dice that have values v_2 and v_3. This leaves 3 unused values and 3 dice, so each gets its own value. Now we have that there are

$$6 \times \binom{5}{2},$$

ways to pick which value $1, \ldots, 6$ gets how many dice. Then there are

$$\binom{10}{3} \times \binom{7}{2} \times \binom{5}{2} \times 3 \times 2 \times 1$$

ways to choose which of the 10 dice has which value. So in probability is

$$\frac{6 \times \binom{5}{2} \times \binom{10}{3} \times \binom{7}{2} \times \binom{5}{2} \times 6}{6^{10}} = \frac{\binom{5}{2} \times \binom{10}{3} \times \binom{7}{2} \times \binom{5}{2}}{6^8}.$$

We next consider $\{X = 4, Y = 1\}$. As above we conclude there are exactly three dice that have the same value v_1. This produces 3 pairs, and so we must have an additional single pair of dice that have value v_2. This leaves 5 dice and 4 possible values. So another pair must be present. This implies $P(\{X = 4, Y = 1\}) = 0$. □