Problem set 7, due April 22

This homework is graded on 4 points; the 2 first exercises are graded on 0.5 point each, the 3 next on 1 point each.

•(A) **Short time dependent Markov Chain** Let $\{X_n\}_{n\geq 0}$ be a stochastic process on a countable state space S. Suppose there exists a $k \in \mathbb{N}\backslash\{0\}$ such that

$$\mathbb{P}(X_n = j|X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}) = \mathbb{P}(X_n = j|X_{n-k} = i_{n-k}, \ldots, X_{n-1} = i_{n-1})$$

for all $n \geq k$ and all $i_0, \ldots, i_{n-1}, j \in S$ such that

$$P(X_0 = i_0, \ldots, X_{n-1} = i_{n-1}) > 0$$

Show that $Y_n = (X_n, \ldots, X_{n+k-1}), n \geq 0$ is a Markov chain.

• (B)**Patterns in coin tossing** You are tossing a coin repeatedly. Which pattern would you expect to see faster: HH or HT? For example, if you get the sequence $TTHHHTH\ldots$ then you see HH at the 4th toss and HT at the 6th. Letting N_1 and N_2 denote the times required to see HH and HT evaluate $\mathbb{E}[N_1]$ and $\mathbb{E}[N_2]$.

Hint: Construct a Markov chain with the 4 states $J = \{HT, HH, TT, TH\}$ and write down relations for $E_j[N_1]$ the expected time to see HH starting from $j \in J$.

•(C) **Bulb lifetime** Let $X_, X_2, ...$ be iid taking values in $\{1, ..., d\}$. You might, for example, think of these random variables as lifetimes of light bulbs. Define $S_k = X_1 + +X_k$, $\tau(n) = \inf\{k : S_k \geq n\}$, and $R_n = S_{\tau(n)} - n$. Then R_n is called the residual lifetime at time N. This is the amount of lifetime remaining in the light bulb that is in operation at time n.

(1) Show that the sequence $R_0, R_1, \ldots$ is a Markov chain. Describe its transition matrix and its stationary distribution
(2) Define the total lifetime L_n at time n by $L_n = X_{\tau(n)}$. This is the total lifetime of the light bulb in operation at time n. Show that $L_0, L_1, \ldots$ is not a Markov chain. But L_n still has a limiting distribution, and we would like to find it. We do this by constructing a Markov chain by enlarging the state space and considering the sequence of random vectors $(R_0, L_0), (R_1, L_1), \ldots$ Show that this sequence is a Markov chain, describe its transition probability and stationary distribution.

•(D) **Lazy random walks and the bottleneck**

(1) Let G be a complete graph with n vertices (that is all vertices have an edge in between them). Estimate the mixing time from above and below of the lazy random walk on G. Get bounds independent of n.
(2) Let G_1, G_2 be two complete graphs with n vertices sharing exactly one vertex. Had loops to each edge so that each vertex has degree $2n - 1$. We proved in the course that the mixing time$t_{\mathrm{mix}}(1/4)$ is at most of order $8n$. Show that it is bounded below by $n/2(1 + o(1))$.
(3) Let G_1, G_2 be two complete graphs with n vertices and connect them by a one dimensional graph $G' = (v_1, v_2, \ldots, v_p)$ so that $v_1 \in G_1, v_p \in G_2$, (v_i, v_{i+1}) is an edge but $\{v_2, \ldots, v_{p-1}\}$ has no other edges and do not belong to $G_1 \cup G_2$. Give lower bounds on the mixing time of the lazy random walk on $G' \cup G_1 \cup G_2$.

•(E) **Stationary time for the Ising model** Consider the Glauber dynamics for the Ising model on a finite graph $G = (V, E)$. The state space configuration is $\Omega = \{-1, +1\}^{|V|}$ and the Markov chain is describe as follows: at each time t we pick a vertex v at random and then update the spin at v according to the distribution

$$P(x,y) = \frac{\pi(y)}{\pi(\Omega(x,v)}$$

with

$$\Omega(x,v) = \{y \in \Omega : y(w) = x(w) \quad \forall w \neq v\} = \{y^+_{x,v}\} \cup \{y^-_{x,v}\}$$

where $y^+_{x,v}(v) = +1$ (resp. $y^-_{x,v}(v) = -1$) and otherwise $y^\pm_{x,v}(w) = x(w), w \neq v$. and π the maesure on Ω

$$\pi(x) = \frac{e^{\beta H(x)}}{\sum_{y\in\Omega} e^{\beta H(y)}}$$

if $H(x) = \sum_{w\sim v} x_v x_w$. We assume $\beta \geq 0$.

(1) Coupling: Pick $v \in V$ and $r \in [0,1]$ uniformly at random and we take the spin at this vertex as follows. We take $X_{t+1} \in \Omega(X_t, v)$ so that $X_{t+1}(v) = +1$ iff $r \leq P(X_t, y^+_{X_t,v})$.

Show that this defines a coupling for the Glauber dynamics $\{X_t(x), t \geq 0, x \in \Omega\}$.

(2) Show that the previous coupling preserves monotonicity, that is $x_v \leq y_v$ for all v implies $X_t(x)_v \leq X_t(y)_v$ for all t and v.

(3) Let $\tau = \min\{t : X_t((-1)^{|V|}) = X_t((+1)^{|V|})$, that is the first time where the configuration initially filled with $+1$ equals that initially filled with -1. Show that τ is finite almost surely and that

$$\bar{d}(t) \leq \mathbb{P}(\tau > t)\,.$$