Problem set 4, due March 20

This homework is graded on 4 points; the first exercise is graded on 0.5 point, the second and third on 1 point, the fourth on 2 point. The final grade will be obtained by taking the minimum of 4 and the sum of the grades obtained in the 4 exercises.

•(A) **Hitting times in a discrete Markov chain** Consider the Markov chain with transition matrix

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Compute $\mathbb{P}(\rho_1 < \infty | X_0 = 1), \mathbb{P}(\rho_1 < \infty | X_0 = 2), \mathbb{P}(\rho_2 < \infty | X_0 = 1), \mathbb{P}(\rho_3 < \infty | X_0 = 1)$. *Hint:* Obtain equations on the above numbers by conditioning by the first step.

•(B)**Decomposition in irreducible classes**

For an integer $m \geq 2$, let $m = a_k a_{k-1} \cdots a_0$ denote its expansion in base 10. Let $0 < p < 1$ and $q = 1 - p$. Let $\mathbb{Z}_{\geq 2} = \{2, 3, \ldots\}$ be the set of integers greater or equal than 2. Consider the Markov chain with state space $\mathbb{Z}_{\geq 2}$ defined by the following rule:

$$\mathbb{P}(X_{n+1} = \max\{2, a_0^2 + a_1^2 + \cdots + a_k^2\} | X_n = a_k a_{k-1} \cdots a_0) = q, \qquad \mathbb{P}(X_{n+1} = 2 | X_n = a_k a_{k-1} \cdots a_0) = p\,.$$

Show that

(1)
$$C = \{2, 4, 16, 20, 37, 42, 58, 89, 145\}$$
is an irreducible closed set of recurrent states.

(2) All $j \notin C$ are transient.

•(C)**Transience for the random walk in $\mathbb{Z}^3$.** Let P be the transition probability for the random walk on $\mathbb{Z}^3$:

$$P_{\mathbf{k}\ell} = 0 \text{ if } |\mathbf{k} - \ell| \neq 1\,, \qquad \frac{1}{6} \text{ otherwise.}$$

Here $\mathbf{k} = (k_1, k_2, k_3)$ and $|\mathbf{k}|^2 = \sum_{i=1}^3 k_i^2$. Let for $\alpha \geq 1$ and for $\mathbf{k} = (k_1, k_2, k_3)$

$$u(\mathbf{k}) = (\alpha^2 + \sum_{i=1}^3 k_i^2)^{-\frac{1}{2}} \qquad \text{for } \mathbf{k} \in \mathbb{Z}^3$$

then show that if α is sufficiently large

(1)
$$(Pu)_{\mathbf{k}} \leq u(\mathbf{k}) \leq u(\mathbf{0})$$

(2) Deduce that $\mathbf{0}$ is transient. What can you say about the other sites of $\mathbb{Z}^3$?

Hints to prove (1):

(1) Let $\mathbf{k} \in \mathbb{Z}^3$ be given and set

$$M = 1 + \alpha^2 + |\mathbf{k}|^2, \qquad x_i = \frac{k_i}{M} \quad \text{for } 1 \leq i \leq 3\,.$$

Show that $(Pu)_{\mathbf{k}} \leq u(\mathbf{k})$ if and only if

$$(1 - \frac{1}{M})^{-\frac{1}{2}} \geq \frac{1}{3} \sum_{i=1}^3 \frac{(1 + 2x_i)^{\frac{1}{2}} + (1 - 2x_i)^{\frac{1}{2}}}{2(1 - 4x_i^2)^{\frac{1}{2}}}$$

(2) Show that $(1-\frac{1}{M})^{-\frac{1}{2}} \geq 1+\frac{1}{2M}$ and that

$$\frac{(1+\xi)^{\frac{1}{2}}+(1-\xi)^{\frac{1}{2}}}{2} \leq 1-\frac{\xi^2}{8} \text{ for } |\xi|<1,$$

and conclude that $(Pu)_{\mathbf{k}} \leq u(\mathbf{k})$ if

$$1+\frac{1}{2M} \geq \frac{1}{3}\sum_{i=1}^{3}\frac{1}{(1-4x_i^2)^{\frac{1}{2}}} - \frac{\sum_{i=1}^{3} x_i^2}{6}$$

(3) Show that there is a constant $C<\infty$ such that as long as $\alpha \geq 1$,

$$\frac{1}{3}\sum_{i=1}^{3}\frac{1}{(1-4x_i^2)^{\frac{1}{2}}} \leq 1+\frac{2}{3}(\sum_{i=1}^{3} x_i^2) + C\left(\sum_{i=1}^{3} x_i^2\right)^2,$$

and conclude that we can take $\alpha \geq 1$ so that $\alpha^2+1 \geq 2C$ to obtain the desired bound.

•(D) **Positive recurrence** We let X_n be a Markov chain constructed from a transition probability P and denote $\mathbb{E}$ the expectation constructed from P. For a set $B \subset \mathbb{S}$, let

$$\tau_B = \inf\{n \geq 1 : X_n \in B\}.$$

B is said to be positive recurrent if $\sup_{x\in B} \mathbb{E}[\tau_B | X_o = x] < \infty$.

(1) Assume that there exists a Lyapounov function $V : \mathbb{S} \to \mathbb{R}^+$, $c>0$ and C, M finite such that

$$\mathbb{E}[V(X_1)-V(x)|X_0=x] \leq -c \quad \text{if } V(x) \geq M \qquad \mathbb{E}[V(X_1)-V(x)|X_0=x] \leq C \text{ if } V(x) \leq M$$

Then show that the set $B=\{x : V(x) \leq M\}$ is positive recurrent. *Hint:* Consider $E_n = \sum_{i=0}^{n} V(X_i) 1_{\tau_B \geq i}$ and

$$\mathbb{E}[E_n - E_0|X_0=x] \leq V(x)+C+c+\mathbb{E}[E_n-E_0|X_0=x] - c\mathbb{E}[n\wedge\tau_B|X_0=x]$$

Conclude that

$$c\mathbb{E}[n\wedge\tau_B|X_0=x] \leq V(x)+C+c \qquad \text{for any } n$$

and complete the proof.

(2) Let ξ_n be independent centered equidistributed integer-valued variables so that $\mathbb{E}[|\xi_1|^2]<\infty$. Let a real so that $|a|<1$. Show that for B large enough $[-B,B]$ is positive recurrent for the Markov chain

$$X_{n+1} = aX_n + \xi_n$$

by exhibiting a convenient Lyapounov function (note here and below that ξ_n is independent from X_n)

(3) Let ξ_n be independent equidistributed real-valued variables so that $\mathbb{E}[|\xi_1|]<\infty$ and $\mathbb{E}[\xi_1]<0$. Show that for B large enough $[0,B]$ is positive recurrent for the Markov chain

$$X_{n+1} = (X_n+\xi_n)^+$$

by exhibiting a convenient Lyapounov function.