# Problem set 3, due March 13

This homework is graded on 4 points; the 3 first exercises are graded on 0.5 point each, the third on 1 point and the fourth on 1.5 points. The last exercise is optional, and is graded on 1 point. The final grade will be obtained by taking the minimum of 4 and the sum of the grades obtained in the 5 exercises.

Throughout the problem set, Doeblin condition is the weak condition, that is

$$\frac{1}{M}\sum_{k=0}^{M-1} P^k_{j,j_0} = A_M(j,j_0) \geq \epsilon > 0$$

for some $M$ finite, $j_0 \in \mathbb{S}$ and all $j \in \mathbb{S}$.

**•Classification of states**
Consider the Markov Chain on 5 states specified by the following matrix

$$P = \begin{pmatrix} \frac{9}{10} & \frac{1}{20} & 0 & \frac{1}{20} & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{4}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}.$$

Draw a directed graph with a vertex representing a state, and arrows representing possible transitions. Determine the decomposition of the chain in equivalence classes of recurrent and transient states.

**• Doeblin condition in finite state space** Take $P$ a Markov chain on a finite state space $\mathbb{S}$. Show that it satisfies Doeblin's condition if and only if $i \to j_0$ for all $i \in \mathbb{S}$.

**• Another version of Doeblin condition.** Assume that $P_{ij} \geq \epsilon_j$ for all $i, j$ and let $\epsilon = \sum_j \epsilon_j$. If $\epsilon > 0$ show that $P$ has a unique stationary measure $\pi$, and that for all probability vector $\mu$,

$$\|\mu P^n - \pi\|_{\mathrm{v}} \leq (1-\epsilon)^n \|\mu - \pi\|_v \,.$$

**• Doeblin condition in Galton-Watson models.** Take $\mathbb{S} = \mathbb{N}$. We let $(\mu_0, \ldots \mu_k, \ldots)$ be a probability vector so that $\mu_0 > 0$, $\mu_1 > 0$ and $\mu_0 + \mu_1 < 1$. The state $i$ represents the number of members in the population and the Markov chain $X_n$ evolves so that at step $n$

- With probability $p$, every $X_{n-1}$ individual, independently of the others, dies and is replaced by a random number of offsprings distributed according to a probability vector $\mu = (\mu_0, \mu_1, \ldots, \mu_k, \ldots)$ .
- With probability $1-p$ there is an epidemic and all the individuals die without progeny but one individual is born (namely $X_n = 1$ ).

(1) Show that if $0 < p < 1$, Doeblin's condition is satisfied. Show that every state is recurrent.
(2) Show that if $p = 1$, Doeblin's condition is not satisfied. *Hint*: use the fact that if the population is very large, it has a very tiny probability to have a small progeny.
Show that the origin is the only recurrent state *Hint:* Use that the probability to be extinct in one step is positive.

**• Doob's $h$-transformation** Let $P$ be a transition probability matrix on the state space $\mathbb{S}$. Let $\emptyset \neq \Gamma \subset \mathbb{S}$ be given and set

$$\rho_\Gamma = \inf\{n \geq 1 : X_n \in \Gamma\}\,.$$

Let the $h$-transform associated to a function $h$ be given by

$$\hat{P}_{ij} = \frac{1}{h(i)} P_{ij} h(j) \qquad \text{for } (i,j) \in \mathbb{S}^2$$

(A) Take

$$h(i) = \mathbb{P}(\rho_\Gamma = \infty | X_0 = i) \qquad \text{for all } i \in \hat{\mathbb{S}} = \mathbb{S} \backslash \Gamma \,,$$

and assume $h(i) > 0$ for all $i \in \hat{\mathbb{S}}$.

(1) Show that $h(i) = \sum_{j \in \hat{\mathbb{S}}} P_{ij} h(j)$ for all $i \in \hat{\mathbb{S}}$ and conclude that the matrix $\hat{P}$ is a transition probability matrix on $\hat{\mathbb{S}}$.

(2) For all $n \in \mathbb{N}$ and $(j_1, \ldots, j_n) \in (\hat{\mathbb{S}})^n$, show that for each $i \in \hat{S}$,

$$\hat{\mathbb{P}}(X_1 = j_1, \ldots, X_n = j_n | X_0 = i) = \mathbb{P}(X_1 = j_1, \ldots, X_n = j_n | \rho_\Gamma = \infty \text{ and } X_0 = i)$$

where $\hat{\mathbb{P}}$ is the probability computed from the transition probability matrix $\hat{P}$. Hence $\hat{P}$ is the Markov chain determined by $P$ conditioned to never hit $\Gamma$.

(B) Assume that $j_0 \in \mathbb{S}$ is transient but $i \to j_0$ for all $i \in \mathbb{S}$ and take

$$h(i) = P(\rho_{j_0} < \infty | X_0 = i) \qquad \text{for } i \neq j_0, \quad h(j_0) = 1 \,.$$

(1) Show that $h(i) > 0$ and set

$$\tilde{P}_{ij} = \hat{P}_{ij} \text{ if } i \neq j_0, \qquad \tilde{P}_{j_0 j} = P_{j_0 j}$$

Show that $\tilde{P}$ is a transition probability matrix.

(2) Denoting $\tilde{\mathbb{P}}$ the probability computed relative to the chain determined by $\tilde{P}$ show that

$$\tilde{\mathbb{P}}(\rho_{j_0} > n | X_0 = i) = \frac{1}{h(i)} \mathbb{P}(n < \rho_{j_0} < \infty | X_0 = i)$$

for all $n \in \mathbb{N}$ and $i \neq j_0$. *Hint:* Expand $\tilde{\mathbb{P}}(\rho_{j_0} > n | X_0 = i) h(i)$ in terms of the transition probability matrix $P$.

(3) Using the last result show that $j_0$ is recurrent for the chain determined by $\tilde{P}$.

• **Optional: Doeblin condition in card shuffling.** We consider a stack of 52 different cards and the following shuffling: at each time step, two cards are chosen uniformly randomly and exchanged. The two cards have to be different. Let $\mathbb{S}$ be the set of all possible ordering of the cards. Identifying the cards with a sequence of numbers $\{1, 2, \ldots, 52\}$, an ordering is a function $f : \{1, 2, \ldots, 52\} \to \{1, 2, \ldots, 52\}$ so that $f(i) \neq f(j)$ iff $i \neq j$. $\mathbb{S}$ is the set of all these functions.

(1) Show that the above shuffling can be interpreted as a random walk on $\mathbb{S}$; if $X_n$ is the ordering at time $n$ for all $i, j \in \mathbb{S}$

$$\mathbb{P}(X_n = i | X_{n-1} = j) = \frac{2}{51 \times 52}$$

if $i$and $j$ differs exactly at two sites, otherwise the probability of transition from $j$ to $i$ vanishes.

(2) Show that this Markov chain satisfies (the weak) Doeblin's condition. *Hint:* Show that you can transform a configuration $i$ into a configuration $j$ in less than 52 steps and deduce that for all $i, j \in \mathbb{S}$

$$A_{52}(i,j) \geq \frac{1}{52} \left( \frac{1}{51 \times 26} \right)^{52}$$

(3) Describe the stationary measure of this chain as well as the recurrence/transience of the states.

(4) Is the strong Doeblin condition $P^n_{jj_0} \geq \epsilon$ for all $j$ satisfied ?