Problem set 1, due February 20

This homework is graded on 4 points; the two first exercises are graded on 1 point each and the last on 2 points.

Throughout this problem set we take $X_0 = 0$ and $\varepsilon_n = X_n - X_{n-1}, n \geq 1$ to be mutually independent, identically distributed random variables with values in $\mathbb{R}$. We assume $\mathbb{E}[|\varepsilon_1|] < \infty$ and denote $m = \mathbb{E}[\varepsilon_1]$.

•**Exercise 1: The law of large numbers.**

(1) Show the weak law of large numbers: for any $\delta > 0$

$$\lim_{n\to\infty} \mathbb{P}[\left|\frac{X_n}{n} - m\right| > \delta] = 0$$

Hint: Show that

$$\lim_{n\to\infty} \mathbb{E}[\left|\frac{X_n}{n} - m\right|] = 0$$

(a) Assume that $\mathbb{E}[\varepsilon_1^2] < \infty$ and show that

$$\lim_{n\to\infty} \mathbb{E}[\left|\frac{X_n}{n} - m\right|^2] = 0\,.$$

(b) Set $\varepsilon_i^R = \varepsilon_i 1_{|\varepsilon_i|\leq R}$, that is $\varepsilon_i^R = \varepsilon_i$ if $|\varepsilon_i| \leq R$, and $\varepsilon_i^R = 0$ otherwise, and $X_n^R = \sum_{i=1}^n \varepsilon_i^R$. Show that

$$\lim_{R\to\infty} \lim_{n\to\infty} \mathbb{E}[\left|\frac{X_n}{n} - \frac{X_n^R}{n}\right|] = 0$$

and conclude.

(2) Show the strong law of large numbers for bounded variables: Assume that ε_1 is bounded uniformly. Show that for any $\delta > 0$

$$\mathbb{P}(\cup_{p\geq 0} \cap_{n\geq p} \{|\frac{X_n}{n} - \mathbb{E}[\varepsilon_1]| \leq \delta\}) = 1$$

Hint:

(a) Show that for any $\kappa > 0$ there exists a finite constant C_κ such that for all $t \in [-\kappa n, \kappa n]$

$$\mathbb{E}[e^{t(\frac{X_n}{n} - m)}] \leq C_\kappa e^{C_\kappa \frac{t^2}{n}}$$

and conclude that there exists a positive constant c and a finite constant C such for any $x \in [0, 1]$

$$\mathbb{P}(\frac{X_n}{n} - m \geq x) \leq Ce^{-cnx^2}$$

(b) Deduce that $\mathbb{P}(\cup_{n\geq p}\{\frac{X_n}{n} - \mathbb{E}[\varepsilon_1] > \delta\})$ goes to zero when p goes to infinity. Conclude.

The strong law of large numbers extends to variables in L^1, but this is more difficult to show.

•**Exercise 2: Recurrence for RW in $\mathbb{Z}^d$, $d \leq 2$** Assume that ε_i are symmetric (that is the law of $-\varepsilon$ is the same as the law of ε) and with values in $\mathbb{Z}^2$. Assume that $\mathbb{E}[\varepsilon_1^2] < \infty$. Show that the arguments developed in the course (see section 1.2.3 in Stroock book) to prove recurrence (that is $\mathbb{P}(\exists n \geq 1 : X_n = 0) = 1$) generalize to this setting.

•**Exercise 3: Queuing problem** We consider the case where ε_n belongs to $\{-1, +1\}$ as considered in the course, with probability p to be equal to $+1$. ε_n represents the number of people who arrive minus those who can be served in time $[n-1,n]$ (here assumed to be either $+1$ or -1 for simplicity). We consider the Markov chain which represents the number of people waiting

$$Q_0 = 0 \text{ and } Q_n = (Q_{n-1} + \varepsilon_n)^+$$

with $a^+ = \max\{x, 0\}$.

(1) Show that $Q_n = \max_{m \leq n}\{X_n - X_m\}$ and conclude that Q_n has the same law as $M_n = \max_{0 \leq m \leq n} X_m$.

(2) Show that

$$\lim_{n\to\infty} \mathbb{P}(Q_n = j) = \mathbb{P}(M_\infty = j) \quad \forall j \in \mathbb{N}$$

(3) Show that $\mathbb{P}(M_\infty = \infty) = 1$ when $m > 0$ by using the first exercise. That is the number of people waiting blows up as time goes in probability as long as $m > 0$.

(4) Assume $m < 0$. Using Exercise 1, show that M_∞ is finite with probability one. That is the number of people waiting stays finite almost surely.

(5) In the case $m = 0$, let $\eta_r = \inf\{n \geq 0; X_n \geq r\}$ and $\rho_0 = \inf\{n \geq 1 : X_n = 0\}$. Recall from the course that $\mathbb{P}(\rho_0 < \infty) = 1$. Show that

$$\delta := \mathbb{P}(\eta_1 > \rho_0) < 1$$

and deduce that $\mathbb{P}(\eta_1 < \infty) = 1$ Argue that $\mathbb{P}(\eta_{r+1} < \infty) \geq \mathbb{P}(\eta_r < \infty)\mathbb{P}(\eta_1 < \infty)$ and deduce that $\mathbb{P}(\eta_r < \infty) = 1$. Conclude that

$$\mathbb{P}(M_\infty = \infty) = 1\,.$$

(6) Use the computation of $\mathbb{P}(\zeta^a < \infty)$ done in the course to show that

$$\lim_{n\to\infty} \mathbb{P}(Q_n = j) = \begin{cases} \frac{q-p}{q}(\frac{p}{q})^j \text{ if } p < q \\ \\ 0 \text{ otherwise.} \end{cases}$$