**PROBLEM SET 7 – 18.725 FALL 2017**

GIULIA SACCÀ

**Exercise 1.** Dual variety. (For the sake of this exercise we will say that an affine scheme \( Z = \text{Spec}(k[x_1, \ldots, x_N]/(f_1, \ldots, f_k)) \subset \mathbb{A}^N \) of dimension \( n \) is singular at a closed point \( x \in Z \) if the Jacobian matrix \( (\frac{\partial f_i}{\partial x_j}) (x) \) has rank \( < N - n \).

1. Let \( X \subset \mathbb{A}^N \) be a smooth variety of dimension \( n \) and consider the tangent space \( T_xX \) of \( X \) at \( x \) (viewed naturally as an affine subspace in \( \mathbb{A}^n \), passing through \( x \)). Show that if \( X \) is not a linear space then the scheme theoretic intersection \( Y = X \cap T_xX \) is singular at \( x \).

2. Let \( W = \{ F = 0 \} \subset \mathbb{P}^n \) be a smooth hypersurface of degree \( d \geq 2 \) and let \( \mathbb{P}^n \) be the dual projective space (i.e. the projective space whose points parametrize hyperplanes \( H \subset \mathbb{P}^n \)). Show that

\[ Z = \{(w, H) \mid w \in H \cap W \text{ is a singular point} \} \subset W \times \mathbb{P}^n \subset \mathbb{P}^n \times \mathbb{P}^n \]

is closed (find equations).

3. Let \( p : Z \to \mathbb{P}^n \) be the morphism induced by the second projection and set \( W^\vee := p(W) \). Show that \( W^\vee \) (which is called the dual variety of \( W \)) is also the image of the morphism \( \Phi : W \to \mathbb{P}^n \) defined by \( w \mapsto [\frac{\partial F}{\partial x_0}(w), \ldots, \frac{\partial F}{\partial x_n}(w)] \). Deduce that \( \dim W^\vee \leq \dim W \).

4. Let \( [H] \in W^\vee \) be a point. What is \( \Phi^{-1}([H]) \)?

5. What happens to \( \Phi \) for \( d = 1 \)?

6. Describe \( W^\vee \), when \( W \subset \mathbb{P}^2 \) is a plane conic.

**Exercise 2.** The following problems are from Vakil’s notes (numbering refers to the June 4, 2017 Version, available online at [http://math.stanford.edu/~vakil/216blog/FOAGjun0417public.pdf](http://math.stanford.edu/~vakil/216blog/FOAGjun0417public.pdf))

1. Ex. 3.2.Q
2. Ex. 3.6.J-K
3. Ex. 3.7.F

Mathematics Department, MIT, 77 Massachusetts Avenue, Cambridge, MA, 02139-4307

E-mail address: gsacca@mit.edu