Compactly Supported Refinable Functions with Infinite Masks

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Abstract

A compactly supported scaling function can come from a refinement equation with infinitely many nonzero coefficients (an infinite mask). In this case we prove that the symbol of the mask must have the special rational form $\tilde{a}(Z) = \tilde{b}(Z^2)\tilde{c}(Z)/\tilde{b}(Z)$. Any finite combination of the shifts of a refinable function will have such a mask, and will be refinable.

We also study compactly supported solutions of vector refinement equations with infinite masks. Our characterization is based on the two-scale similarity transform which plays an essential role in the investigation of multiple wavelets. This concept is used to characterize refinable subspaces of refinable shift-invariant spaces. One advantage of our approach is to provide the refinement masks for generators of refinable subspaces.

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§1. Introduction and Main Results

The central equation in wavelet analysis is the **refinement equation** for the scaling function ϕ:

$$\phi(x) = \sum_{k\in\mathbb{Z}} a(k)\phi(2x-k). \tag{1.1}$$

In approximation theory, the sequence $a := \{a(k)\}$ is the **mask**. In signal processing these $a(k)$ are the coefficients of a lowpass filter.

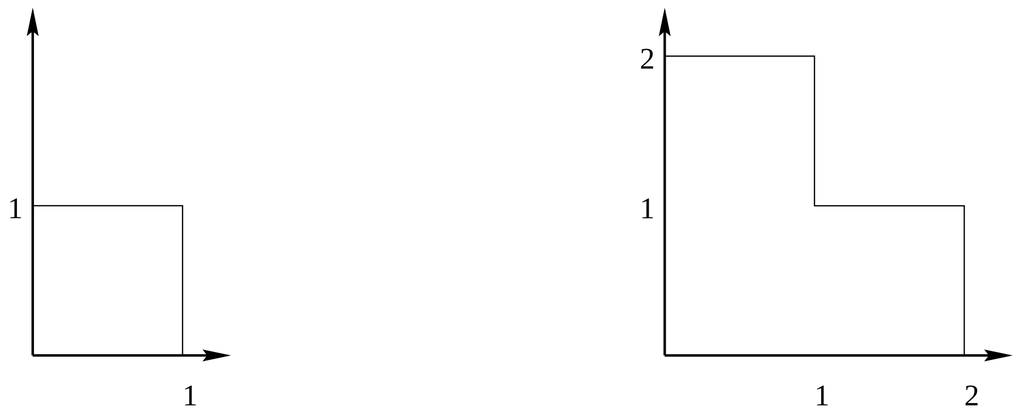
A solution ϕ of (1.1) is called a **refinable function** (or distribution) associated with the mask a. Usually in wavelet analysis, we assume that the mask is finitely supported. Then ϕ is compactly supported and its properties can be determined from the mask [3, 10].

The simplest refinement equation (or dilation equation) has only two coefficients: $\beta(x) = \beta(2x) + \beta(2x-1)$. It is certain that the solution will be supported on $[0,1]$. In this case $\beta(x)$ is just Haar's box function, $\beta(x) = 1$ for $0 \le x < 1$. We are interested in the following example which is supported on $[0,2]$ but its mask is infinite.

Consider a combination $\gamma(x) = 2\beta(x) + \beta(x-1)$ of the Haar function and its shift (see Figure 1). This two-box function is also refinable, but with infinite mask:

$$\gamma(x) = \gamma(2x) + \frac{1}{2}\gamma(2x-1) + \frac{1}{4}\gamma(2x-2) + \sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^k \frac{3}{8}\gamma(2x-k-3).$$

This shows that the refinement equation (1.1) may have a compactly supported solution while the mask is infinitely supported. Another such example appeared in [13, p. 897].



$\beta(x) = \beta(2x) + \beta(2x-1)$ $\qquad$ $\gamma(x) = 2\beta(x) + \beta(x-1)$

Figure 1. Haar and two-box scaling functions

Define the symbol $\tilde{a}(Z)$ corresponding to the mask a as the formal Laurent series

$$\tilde{a}(Z) := \sum_{k\in\mathbb{Z}} a(k)Z^k.$$

The symbol for Haar is $\tilde{\beta}(Z) = 1 + Z$, while the two-box case has an infinite symbol:

$$\tilde{\gamma}(Z) = 1 + \frac{1}{2}Z + \frac{1}{4}Z^2 + \sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^k \frac{3}{8}Z^{k+3} = (2+Z^2)(1+Z)/(2+Z).$$

The simple ratio in the last formula is no surprise. Since $\beta(x)$ and $\gamma(x)$ are refinable, their Fourier transforms must satisfy two-scale relations involving the masks $\tilde{\beta}$ and $\tilde{\gamma}$:

$$\hat{\beta}(2\xi) = \frac{1}{2}\tilde{\beta}(e^{-i\xi})\hat{\beta}(\xi), \qquad \hat{\gamma}(2\xi) = \frac{1}{2}\tilde{\gamma}(e^{-i\xi})\hat{\gamma}(\xi).$$

By construction $\gamma(x)$ is a combination of translates of $\beta(x)$, so $\hat{\gamma}(\xi) = (2 + e^{-i\xi})\hat{\beta}(\xi)$ or $\hat{\beta}(\xi) = \hat{\gamma}(\xi)/(2 + e^{-i\xi})$. Substituting this formula and comparing the two-scale relations reveals that

$$\tilde{\gamma}(e^{-i\xi}) = (2 + e^{-i2\xi})\tilde{\beta}(e^{-i\xi})/(2 + e^{-i\xi}) = (2 + e^{-i2\xi})(1 + e^{-i\xi})/(2 + e^{-i\xi}).$$

Our purpose is to show that this example is typical. When ϕ is finitely supported, its symbol is rational and of a special form. This fact was proved in [9] (for scalar coefficients $a(k)$) and was pointed out to the third author by Amos Ron while we were writing the paper. We analyze the case of matrix coefficients also. Moreover, if ϕ_1 is a finite linear combination of the translates of ϕ then we confirm that ϕ_1 is also refinable.

Theorem 1. *If ϕ is a nontrivial compactly supported distribution satisfying (1.1), then there are two finite Laurent polynomials $\tilde{b}(Z)$ and $\tilde{c}(Z)$ such that*

$$\tilde{a}(Z)\tilde{b}(Z) = \tilde{b}(Z^2)\tilde{c}(Z). \tag{1.2}$$

Hence $\tilde{a}(Z)$ is rational: $\tilde{a}(Z) = \tilde{b}(Z^2)\tilde{c}(Z)/\tilde{b}(Z)$.

The two-box example has $\tilde{b}(Z) = (2 + Z)$ and $\tilde{c}(Z) = (1 + Z)$.

Theorem 1 is a corollary of the characterization (given in Theorem 2) of existence of compactly supported refinable distributions in terms of the masks. Actually, we are able to provide this characterization for vector refinement equations.

A **vector refinement equation** takes the same form as (1.1). But the coefficients $a(k)$ are $r \times r$ matrices, and $\phi = (\phi_1, \cdots, \phi_r)^T$ is an r-vector of functions or distributions. For a sequence $d := \{d(k)\}_{k\in\mathbb{Z}}$ of $m \times n$ matrices, we define the symbol $\tilde{d}(Z)$ as the matrix of formal Laurent series

$$\tilde{d}(Z) := \sum_{k\in\mathbb{Z}} d(k)Z^k.$$

If d is finitely supported, $d \in (\ell_0(\mathbb{Z}))^{m\times n}$, then $\tilde{d}(Z)$ becomes a Laurent polynomial.

The existence of refinable vectors of compactly supported distributions can be characterized in terms of the mask as follows.

Theorem 2. *Let $r \in \mathbb{N}$ and $a := \{a(k)\}_{k\in\mathbb{Z}}$ be a nontrivial sequence of $r \times r$ matrices. Then the vector refinement equation (1.1) has a nontrivial compactly supported distributional solution $\phi = (\phi_1, \cdots, \phi_r)^T$ if and only if there are $m \in \{1, \cdots, r\}$, $b \in (\ell_0(\mathbb{Z}))^{r\times m}$, $c \in (\ell_0(\mathbb{Z}))^{m\times m}$ such that $\tilde{c}(1)$ has an eigenvalue of the form 2^n, $n \in \mathbb{N}$, $\tilde{b}(z)$ has rank m except at finitely many points, and*

$$\tilde{a}(Z)\tilde{b}(Z) = \tilde{b}(Z^2)\tilde{c}(Z). \tag{1.3}$$

The condition (1.3) is a generalization of the two-scale similarity transform which corresponds to the case $m = r$ and can be found in [12].

Definition. *Let a and c be sequences of $r \times r$ and $m \times m$ matrices, respectively. We say that $\tilde{a}$ and $\tilde{c}$ are* **two-scale similar** *if there is some nonzero sequence $b \in (\ell_0(\mathbb{Z}))^{r\times m}$ such that (1.3) holds.*

This concept of two-scale similarity can be used for different purposes in multiple wavelets. In Section 4 we shall provide such an example and show how to characterize nontrivial refinable subspaces for a refinable shift-invariant space.

§2. Compactly supported refinable distributions

In this section we shall prove the main result (Theorem 2) on compactly supported refinable distributions. The proof of Theorem 1 then follows by setting $r = 1$.

The following result of Jia [5] on shift-invariant spaces plays an essential role in our proof. The shift-invariant space $S(\phi)$ contains all (infinite) combinations of the shifts of $\phi_1, \cdots, \phi_r$:

$$S(\phi) = \{\sum_{j=1}^{r} \sum_{k\in\mathbb{Z}} f_j(k)\phi_j(\cdot - k) : \quad f_j(k) \in \mathbb{C}\}.$$

Jia's Lemma. *Let $\phi = (\phi_1, \cdots, \phi_r)^T$ be a nontrivial vector of compactly supported distributions. Then there exists another vector $\psi = (\psi_1, \cdots, \psi_m)^T$ of compactly supported distributions with the following properties:*

(**a**) The shifts of $\psi_1, \cdots, \psi_m$ are linearly independent;

(**b**) $m \le r$;

(**c**) $S(\phi) = S(\psi)$;

(**d**) $\phi(x) = \sum_{k\in\mathbb{Z}} b(k)\psi(x - k)$, where $\{b(k)\} \in (\ell_0(\mathbb{Z}))^{r\times m}$.

The linear independence was characterized by Jia and Micchelli in [6]: the shifts of $\psi_1, \cdots, \psi_m$ are linearly independent if and only if $(\hat{\psi}(\xi + 2k\pi))_{k\in\mathbb{Z}}$ has rank m for every $\xi \in \mathbb{C}$. The linear independence implies the existence of duals [1]. Hence if $f \in S(\psi)$ is compactly supported and

$$f(x) = \sum_{k\in\mathbb{Z}} f(k)\psi(x - k),$$

where $f(k) \in \mathbb{C}^{1\times m}$ for each k, then the sequence $\{f(k)\}$ is finitely supported. Therefore, a compactly supported distributional solution ϕ of (1.1) with the mask being not finite can never be linearly independent, but can be stable, see the example in [13]. We are now in a position to prove Theorem 2.

Proof of Theorem 2. Necessity. Suppose that $\phi = (\phi_1, \cdots, \phi_r)^T$ is a nontrivial compactly supported distributional solution of (1.1). Applying the Lemma, we find some ψ satisfying all the properties (**a**) - (**d**).

The combination of (**d**) and (1.1) tells us that

$$\phi(x) = \sum_k a(k) \sum_l b(l)\psi(2x - k - l) := \sum_{k\in\mathbb{Z}} f(k)\psi(2x - k), \tag{2.1}$$

where $f(k) = \sum_l a(k - l)b(l) \in \mathbb{C}^{r\times m}$ for each $k \in \mathbb{Z}$. Since ϕ is compactly supported, $\{f(k)\}$ is finitely supported.

By (**c**), $\psi_1, \cdots, \psi_m \in S(\phi)$. Hence there is a sequence $\{g(k)\}$ of $m \times r$ matrices such that

$$\psi(x) = \sum_{k\in\mathbb{Z}} g(k)\phi(x - k).$$

Since $\{f(k)\}$ is finitely supported, this in connection with (2.1) tells

$$\psi(x) = \sum_{k\in\mathbb{Z}} \{\sum_{l\in\mathbb{Z}} g(l)f(k - 2l)\}\psi(2x - k).$$

Set c as the sequence $\{\sum_l g(l)f(k - 2l)\}_{k\in\mathbb{Z}}$. Then

$$\psi(x) = \sum_{k\in\mathbb{Z}} c(k)\psi(2x - k). \tag{2.2}$$

Since ψ is compactly supported, the sequence c is finitely supported.

The property (**d**) and (2.2) show that

$$\phi(x) = \sum_{k\in\mathbb{Z}} \{\sum_{l\in\mathbb{Z}} b(l)c(k - 2l)\}\psi(2x - k).$$

On the other hand, (1.1) and (**d**) tell us that

$$\phi(x) = \sum_{k\in\mathbb{Z}} \{\sum_{l\in\mathbb{Z}} a(l)b(k - l)\}\psi(2x - k).$$

These two expressions for ϕ in connection with the linear independence of ψ imply that

$$\sum_{l\in\mathbb{Z}} a(l)b(k - l) = \sum_{l\in\mathbb{Z}} b(l)c(k - 2l), \quad \forall k \in \mathbb{Z}.$$

Hence as formal Laurent series,

$$\tilde{a}(Z)\tilde{b}(Z) = \tilde{b}(Z^2)\tilde{c}(Z).$$

This proves (1.3).

By (**d**),

$$\hat{\phi}(\xi) = \tilde{b}(e^{-i\xi})\hat{\psi}(\xi).$$

It follows from the equality $S(\phi) = S(\psi)$ that $\tilde{b}(z)$ has rank m except at finitely many points.

Finally, taking the Fourier transform in (2.2), we have

$$\hat{\psi}(2\xi) = \frac{1}{2}\tilde{c}(e^{-i\xi})\hat{\psi}(\xi).$$

Since $\hat{\psi}$ is a vector of analytic functions, there is some $n \in \mathbb{N}$ such that $\hat{\psi}(0) = \cdots = \hat{\psi}^{(n-2)}(0) = 0$ and $\hat{\psi}^{(n-1)}(0) \neq 0$. Hence 2^{n-1} is an eigenvalue of $\tilde{c}(1)/2$ corresponding to the eigenvector $\hat{\psi}^{(n-1)}(0)$. This proves the necessity.

Sufficiency. Suppose that all the conditions hold. Since $\tilde{c}(1)$ has an eigenvalue 2^n for some $n \in \mathbb{N}$, we know from [7, 14] that there exists a nontrivial vector $\psi = (\psi_1, \cdots, \psi_m)^T$ of compactly supported distributions such that

$$\psi(x) = \sum_{k \in \mathbb{Z}} c(k)\psi(2x - k).$$

Define a vector $\phi = (\phi_1, \cdots, \phi_r)^T$ of compactly supported distributions by

$$\phi(x) = \sum_{k \in \mathbb{Z}} b(k)\psi(x - k).$$

Then

$$\hat{\phi}(\xi) = \tilde{b}(e^{-i\xi})\hat{\psi}(\xi), \qquad \xi \in \mathbb{C}.$$

Since $\tilde{b}(e^{-i\xi})$ has rank m for $\xi \in \mathbb{C}$ except at finitely many points, we know that ϕ is nontrivial.

Let us check the refinement relation for ϕ. By the definition of ϕ and the refinement equation for ψ,

$$\phi(x) = \sum_l b(l) \sum_k c(k)\psi(2x - 2l - k) = \sum_{k \in \mathbb{Z}} \Big\{ \sum_{l \in \mathbb{Z}} b(l)c(k - 2l) \Big\} \psi(2x - k).$$

The two-scale similarity (1.3) tells us that $\sum_l a(l)b(k-l) = \sum_l b(l)c(k-2l)$ for each $k \in \mathbb{Z}$. Hence

$$\phi(x) = \sum_{k\in\mathbb{Z}} \big\{ \sum_{l\in\mathbb{Z}} a(l)b(k-l) \big\} \psi(2x-k) = \sum_{k\in\mathbb{Z}} a(k)\phi(2x-k).$$

Therefore, the vector refinement equation (1.1) associated with the mask a has a nontrivial compactly supported distributional solution ϕ. This completes the proof of Theorem 2. $\square$

The proof shows that if ϕ is a compactly supported distributional solution of (1.1) associated with an arbitrary mask, then any distribution f in $S(\phi)$ can be written as

$$\begin{aligned} f(x/2) &= \sum_{k\in\mathbb{Z}} f(k)^T \phi(x/2-k) = \sum_{k\in\mathbb{Z}} \big\{ \sum_{l\in\mathbb{Z}} f(l)^T b(k-l) \big\} \psi(x/2-k) \\ &= \sum_{k\in\mathbb{Z}} \big\{ \sum_{l\in\mathbb{Z}} \big[\sum_{n\in\mathbb{Z}} f(n)^T b(l-n) \big] \big\} c(k-2l)\psi(x-k) \in S(\psi) = S(\phi). \end{aligned}$$

Hence $S(\phi)$ is refinable in the sense that $f(x/2) \in S(\phi)$ for any $f \in S(\phi)$.

Moreover, the following result holds.

Theorem 3. *Suppose ϕ is finitely supported and refinable. If ϕ_1 is a finite linear combination of the translates of ϕ then ϕ_1 is also refinable and finitely supported.*

Proof. By Jia's Lemma, there exist compactly supported distributions ψ and ψ_1 such that $S(\psi) = S(\phi)$ and $S(\psi_1) = S(\phi_1)$ (with $\phi \in S_0(\psi)$ and $\phi_1 \in S_0(\psi_1)$) and the integer translates of ψ (ψ_1) are linearly independent. Since $\phi_1 \in S(\phi) = S(\psi)$, we know that $S(\phi_1) = S(\psi_1) \subset S(\psi)$. By the linear independence,

$$\hat{\psi}_1(\xi) = \tilde{d}(e^{-i\xi})\hat{\psi}(\xi),$$

where d is a finitely supported sequence. Since ψ_1 is also linearly independent, the characterization of Jia and Micchelli [6] tells us that $\tilde{d}(e^{-i\xi}) \neq 0$ for any $\xi \in \mathbb{C}$. Therefore, the only zero that $\tilde{d}(Z)$ may have is the origin. Hence, $\tilde{d}(Z) = \text{const} \cdot Z^l$ for some $l \in \mathbb{Z}$. This implies that

$$\psi_1(x) = \text{const} \cdot \psi(x-l).$$

It follows that $S(\phi_1) = S(\psi_1) = S(\psi) = S(\phi)$ is refinable. That means

$$\phi_1(x/2) \in S(\phi_1) \quad \text{and} \quad \phi_1(x) = \sum_{k\in\mathbb{Z}} b(k)\phi_1(2x-k)$$

for some sequence b. $\square$

§3. Examples with infinite masks

Before discussing the applications of two-scale similarity transforms in Section 4, let us provide some examples of compactly supported distributional solutions of the refinement equations with infinite masks.

We consider the scalar case $r = 1$ only. Theorem 2 tells us that the existence of nontrivial compactly supported ϕ is equivalent to the existence of nonzero Laurent polynomials $\tilde{b}$ and $\tilde{c}$ such that $\tilde{c}(1) = 2^n$ for some $n \in \mathbb{N}$ and (1.3) holds. We may assume in (1.3) that $\tilde{b}(z)$ is a polynomial with $\tilde{b}(0) \neq 0$, $\tilde{b}(1) \neq 0$; $\tilde{c}(z) = z^s \tilde{c}_0(z)$ for some $s \in \mathbb{Z}$ and a polynomial $\tilde{c}_0$ with $\tilde{c}_0(0) \neq 0$; and that $\tilde{b}$ and $\tilde{c}_0$ are coprime. Then the degree of $\tilde{b}(z^2)\tilde{c}_0(z)$ is greater than the degree of $\tilde{b}(z)$ (unless both $\tilde{b}$ and $\tilde{c}_0$ are constants).

Our first example corresponds to the transfer functions of the Butterworth filters [8]. For the regularity of these refinable functions, see Cohen and Daubechies [2].

Example 1. *Let $N > 1$ and the sequence a_N be given by*

$$\tilde{a}_N(z) = \frac{2(z+1)^{2N}}{(z+1)^{2N} + (-1)^N (z-1)^{2N}}.$$

Then the scalar refinement equation (1.1) associated with the mask a_N has no compactly supported distributional solution.

Proof. Write $P_N(z) = (z+1)^{2N} + (-1)^N (z-1)^{2N}$. When N is even, P_N is a polynomial of exact degree $2N$ with $P_N(0) \neq 0$. When N is odd, $P_N(z) = zQ_N(z)$ where $Q_N(z)$ is a polynomial of exact degree $2N-2$ with $Q_N(0) \neq 0$. Note that $P_N(z)$ and $(z+1)^{2N}$ are coprime.

The conclusion for even N is trivial, since (1.3) in connection with $\deg P_N = \deg(z+1)^{2N}$ would imply that both $\tilde{b}$ and $\tilde{c}$ are constants, which is a contradiction.

When N is odd ($N \geq 3$), (1.3) means that $s = -1$ and

$$\frac{\tilde{b}(z^2)}{\tilde{b}(z)}\ \tilde{c}_0(z) = \frac{2(z+1)^{2N}}{Q_N(z)}.$$

This implies that $\deg\{\tilde{b}(z^2)\tilde{c}_0(z)\} - \deg\tilde{b}(z) = \deg\tilde{b} + \deg\tilde{c}_0 = 2$ and Q_N divides $\tilde{b}$. Hence $\deg\tilde{b} \geq \deg Q_N = 2N-2 \geq 2$. Therefore, we must have $\deg\tilde{c}_0 = 0$, $\deg\tilde{b} = 2$ and $N = 3$. Thus, $(z+1)^6$ divides $\tilde{b}(z^2)$, which is again a contradiction. $\square$

The simplest example in the scalar case should be $\tilde{a}(z) = \tilde{q}(z)/(z-\lambda^2)$ with $\lambda \neq 0$ and $\tilde{q}(\lambda^2) \neq 0$. If $\deg\tilde{q} = 2$, then it can be easily seen that the existence of compactly supported distributional solution is equivalent to $\tilde{q}(z) = 2^n(z^2-\lambda^2)$ for some $n \in \mathbb{N}$.

When $\deg\tilde{q} = 3$, we have

Example 2. *Let $\lambda \neq 0$, $\tilde{a}(z) = \tilde{q}(z)/(z-\lambda^2)$, where $\tilde{q}$ is a polynomial of exact degree 3 with $\tilde{q}(\lambda^2) \neq 0$ and $\tilde{q}(0) \neq 0$. Then the refinement equation (1.1) associated with the mask a has a nontrivial compactly supported distributional solution if and only if $\tilde{q}(z)$ is one of the following three types: $2^n(z-\lambda)(z^2+\lambda)$; $2^n(z+\lambda)(z^2-\lambda)$; $(z^2-\lambda^2)(2^n+t(z-1))$ with $t \neq 0$, where n is a positive integer.*

Proof. By Theorem 2, the existence ϕ is equivalent to

$$\frac{\tilde{q}(z)}{z-\lambda^2} = \frac{(z^2-\lambda^2)\tilde{d}(z^2)}{(z-\lambda^2)\tilde{d}(z)}\tilde{c}_0(z),$$

where $\tilde{d}, \tilde{c}_0$ are polynomials and $\tilde{d}(0) \neq 0$, $\tilde{d}(1) \neq 0$, $\tilde{c}_0(1) = 2^n$ for some $n \in \mathbb{N}$. Since $\deg\tilde{q} = 3$, this means either $\deg\tilde{d} = 1$ and $\deg\tilde{c}_0 = 0$; or $\deg\tilde{d} = 0$ and $\deg\tilde{c}_0 = 1$. In the first case, the equivalent condition is that $\tilde{d}(z) = d_0(z \pm \lambda)$ and $\tilde{c}_0(z) \equiv 2^n$. Hence $\tilde{q}(z) = 2^n(z-\lambda)(z^2+\lambda)$ or $2^n(z+\lambda)(z^2-\lambda)$. In the second case, the equivalent statement is that $\tilde{q}(z) = (z^2-\lambda^2)\tilde{c}_0(z)$ where $\tilde{c}_0(1) = 2^n$. Hence our conclusion holds. $\square$

§4. Refinable subspaces

The two-scale similarity transform plays an essential role in our characterization of refinable vectors of compactly supported distributions with infinite masks. In this section, we apply this transform to study the inclusion of refinable subspaces. This problem was considered by Hardin and Hogan in [4]. The special case of refinability of components (of refinable vectors) was studied by Strang and Zhou in [11].

By Theorem 2, for any refinable vector ϕ of compactly supported distributions with an arbitrary mask, there always exists another refinable vector ψ with a finite mask and linearly independent shifts such that $S(\phi) = S(\psi)$. So we may assume that the shifts of ψ are linearly independent (hence the associated mask c is finitely supported) when we consider the subspaces of $S(\psi)$.

Theorem 4. *Let $\psi = (\psi_1, \cdots, \psi_m)^T$ be a vector of compactly supported distributions with linearly independent shifts satisfying the vector refinement equation*

$$\psi(x) = \sum_{k \in \mathbb{Z}} c(k)\psi(2x - k).$$

Let $r \in \{1, \cdots, m-1\}$. If $S(\phi)$ is a nontrivial refinable subspace of $S(\psi)$, generated by a vector $\phi = (\phi_1, \cdots, \phi_r)^T$ of compactly supported distributions with linearly independent shifts, then there exist nonzero sequences $a \in (\ell_0(\mathbb{Z}))^{r \times r}$ and $b \in (\ell_0(\mathbb{Z}))^{r \times m}$ such that (1.3) holds, and

$$\phi(x) = \sum_{k \in \mathbb{Z}} b(k)\psi(x - k). \tag{4.1}$$

Conversely, if nonzero sequences $a \in (\ell_0(\mathbb{Z}))^{r \times r}$, $b \in (\ell_0(\mathbb{Z}))^{r \times m}$ satisfy (1.3), then ϕ defined by (4.1) generates a nontrivial refinable subspace $S(\phi)$ of $S(\psi)$. Moreover, the sequence a is the refinement mask for ϕ in both statements.

Proof. Necessity. Suppose that $\phi = (\phi_1, \cdots, \phi_r)^T$ generates a nontrivial refinable subspace $S(\phi)$ and the shifts of ϕ are linearly independent. Since $\phi \in S(\phi) \subset S(\psi)$, there is a nonzero sequence $b \in (\ell_0(\mathbb{Z}))^{r \times m}$ such that (4.1) holds. The refinement equation for ψ tells that

$$\phi(x) = \sum_{k \in \mathbb{Z}} \big[\sum_{l \in \mathbb{Z}} b(l)c(k - 2l)\big]\psi(2x - k).$$

Since $S(\phi)$ is refinable and the shifts of ϕ are linearly independent, there exists a nonzero sequence $a \in (\ell_0(\mathbb{Z}))^{r \times r}$ such that

$$\phi(x) = \sum_k a(k)\phi(2x - k) = \sum_k \big[\sum_l a(l)b(k - l)\big]\psi(2x - k).$$

The above two expressions for ϕ in connection with the linear independence of ψ imply that

$$\sum_{l \in \mathbb{Z}} a(l)b(k - l) = \sum_{l \in \mathbb{Z}} b(l)c(k - 2l), \qquad k \in \mathbb{Z}.$$

Hence

$$\tilde{a}(Z)\tilde{b}(Z) = \tilde{b}(Z^2)\tilde{c}(Z).$$

This proves (1.3) and the first statement.

To see the second statement, suppose that (1.3) holds for nonzero sequences a and b. Let

$$\phi(x) = \sum_{k\in\mathbb{Z}} b(k)\psi(x-k).$$

Then $S(\phi)$ is a nontrivial subspace of $S(\psi)$.

To see that $S(\phi)$ is refinable, we use the two-scale similarity transform. By our definition of ϕ and the refinement equation for ψ,

$$\phi(x) = \sum_{k\in\mathbb{Z}} \big[\sum_{l\in\mathbb{Z}} b(l)c(k-2l)\big]\psi(2x-k).$$

Then (1.3) tells that

$$\phi(x) = \sum_{k\in\mathbb{Z}} \big[\sum_{l\in\mathbb{Z}} a(l)b(k-l)\big]\psi(2x-k) = \sum_{k\in\mathbb{Z}} a(k)\phi(2x-k).$$

Hence ϕ is refinable. This completes the proof of Theorem 4. $\square$

Let us show how to apply Theorem 4 in the special case when $\mathrm{supp} c = [0,1]$, *i.e.*, $\tilde{c}(z) = c(0) + c(1)z$. In this case, under the assumption that $c(0)$ is invertible, Hardin and Hogan gave a characterization of refinable subspaces in terms of (left) invariant subspaces of $c(0)$. Our result here is constructive; we give the refinement mask a and the combination coefficients b for ϕ. Also, we do not assume that $c(0)$ is invertible.

By changing the generator ϕ by its shifts, we may assume that a is supported on $[0,N]$, hence $\mathrm{supp}\phi \subset [0,N]$. Then (4.1) tells that $\mathrm{supp} b \subset [0,N-1]$. Thus the two-scale similarity transform (1.3) is reduced to a system of quadratic equations. We should not expect to bound the length N by the support of c. For example, if we take $\psi_j(x) = x^{j-1}, j = 1,\cdots,m$, then the cardinal B-spline ϕ of order m generates a refinable subspace of $S(\psi)$, while $N = m$ can be arbitrarily large. However, it is possible to bound the length N by m and the support of c. Let us give such an example with $r = 1$.

Theorem 5. *Let $\psi = (\psi_1,\cdots,\psi_m)^T$ be a vector of compactly supported distributions with linearly independent shifts satisfying the vector refinement equation*

$$\psi(x) = c(0)\psi(2x) + c(1)\psi(2x-1).$$

If V is a nontrivial refinable subspace of $S(\psi)$, generated by a compactly supported distribution with linearly independent shifts, then there exist nonzero sequences $a \in (\ell_0(\mathbb{Z}))^{r\times r}$ and $b \in (\ell_0(\mathbb{Z}))^{r\times m}$, supported on $[0, m]$ and $[0, m-1]$ respectively, such that (1.3) holds, and ϕ defined by (4.1) is a generator of $V = S(\phi)$.

Proof. Let ϕ be a generator of V such that ϕ has minimal support of length not greater than $N \in \mathbb{N}$ and $\mathrm{supp}\phi \subset [0, N]$. Then ϕ is linearly independent and refinable. These two properties imply the well-known fact that the refinement mask a of ϕ has support $[0, N]$ with $a(0) \neq 0$ and $a(N) \neq 0$; $\phi|_{[l,l+1)} \neq 0$ for each $l \in \{0, \cdots, N-1\}$ and $\{\phi|_{[l,l+1)}\}_{l=0}^{N-1}$ are linearly independent. The linear independence of ψ tells that

$$\phi(x) = \sum_{k=0}^{N-1} b(k)\psi(x-k)$$

for some $b \in (\ell_0(\mathbb{Z}))^{1\times m}$ supported in $[0, N-1]$. Also, $b(k) \neq 0$ for each $0 \leq k \leq m-1$. The proof of Theorem 4 shows that (1.3) holds.

We state that $N \leq m$. Suppose to the contrary that $N > m$. Then $\{b(k)\}_{k=0}^{N-1}$ are linearly dependent. There exist numbers $\lambda_k, k = 0, \cdots, N-1$, not all zero, such that

$$\sum_{k=0}^{N-1} \lambda_k b(k) = 0.$$

Observe that

$$\sum_{k=0}^{N-1} \lambda_k \phi(x+k) = \sum_{n\in\mathbb{Z}} [\sum_{k=0}^{N-1} \lambda_k b(k+n)]\psi(x-n).$$

Therefore, as a distribution on $[0, 1)$,

$$\sum_{k=0}^{N-1} \lambda_k \phi(x+k)|_{[0,1)} = [\sum_{k=0}^{N-1} \lambda_k b(k)]\psi(x)|_{[0,1)} = 0.$$

Hence

$$\sum_{k=0}^{N-1} \lambda_k \phi|_{[k,k+1)} = 0.$$

This contradicts the linear independence of $\{\phi|_{[k,k+1)}\}_{k=0}^{N-1}$, and shows that $N \leq m$. The proof of Theorem 5 is complete. $\square$

The above mentioned example of cardinal B-splines tells us that the bound $N \le m$ given in Theorem 5 is sharp.

Let us finish our discussion with an example in the case $m = 2$. We may assume a canonical form for $\tilde{c}(1) = c(0) + c(1) = \begin{bmatrix} 2 & 0 \\ 0 & \lambda \end{bmatrix}$ with $|\lambda| < 2$.

Example 3. *Let $0 \ne u, s, t \in \mathbb{C}$ with $|s+t| < 2$. Consider the vector refinement equation*

$$\psi(x) = \begin{bmatrix} 1 & 0 \\ u & s \end{bmatrix} \psi(2x) + \begin{bmatrix} 1 & 0 \\ -u & t \end{bmatrix} \psi(2x-1).$$

The compactly supported distributional solution $\psi = (\psi_1, \psi_2)^T$ with $\hat{\psi}(0) = (1,0)^T$ lies in $(L_1(\mathbb{R}))^2$ and has linearly independent shifts if and only if $|s| + |t| < 2$. The subspace $S(\psi_1)$, $\psi_1 = \chi_{[0,1)}$, is refinable. $S(\psi)$ contains another nontrivial refinable subspace $S(\phi)$ generated by a compactly supported function ϕ if and only if $-1/2 < t < 3/2, t \ne 0, 1$, and $s = 1 - t$. In this case, the generator ϕ satisfies the refinement equation

$$\phi(x) = (1-t)\phi(2x) + \phi(2x-1) + t\phi(2x-2). \tag{4.2}$$

Proof. The first statement and the refinability of $S(\psi_1)$ are trivial.

Suppose that $S(\phi)$ is another nontrivial refinable subspace of $S(\psi)$. By Theorem 5 we may assume that ϕ has linearly independent shifts and is given by

$$\phi(x) = b(0)\psi(x) + b(1)\psi(x-1),$$

where $\tilde{b}$ satisfies (1.3) for some nonzero sequence a supported in $[0,2]$. Since a is the refinement mask of ϕ, $\tilde{a}(1) = 2$.

The two-scale similarity transform (1.3) here is equivalent to the following relations:

$$a(0)b(0) = b(0)c(0), \qquad a(2)b(1) = b(1)c(1),$$

and

$$a(1)b(0) + a(0)b(1) = b(0)c(1), \qquad a(2)b(0) + a(1)b(1) = b(1)c(0).$$

Note that $(1,0)$ is the common left eigenvector of $c(0)$ and $c(1)$ with eigenvalue 1. If either $b(0)$ or $b(1)$ equals $\alpha(1,0)$ for some $\alpha \in \mathbb{C}$, then $S(\phi) = S(\psi_1)$.

If neither $b(0)$ nor $b(1)$ is $\alpha(1,0)$ for any $\alpha \in \mathbb{C}$, then

$$a(0) = s, \qquad b(0) = \alpha(1, (s-1)/u) \quad \text{with } \alpha \neq 0$$

and

$$a(2) = t, \qquad b(1) = \beta(1, (1-t)/u) \quad \text{with } \beta \neq 0.$$

Under this condition, (1.3) is equivalent to that

$$a(1)\alpha(1, (s-1)/u) = \alpha(2-s, t(s-1)/u) - s\beta(1, (1-t)/u)$$

and

$$a(1)\beta(1, (1-t)/u) = \beta(2-t, s(1-t)/u) - t\alpha(1, (s-1)/u).$$

This yields

$$a(1) = 2 - s - s\beta/\alpha = 2 - t - t\alpha/\beta \qquad \Longrightarrow \qquad \alpha/\beta = -1 \quad \text{or} \quad t\alpha/\beta = s.$$

If $\alpha/\beta = -1$, then $a(1) = 2$. The second condition would imply $s + t = 2$, which is a contradiction.

If $t\alpha/\beta = s$, then $a(1) = 2 - s - t$. The second condition implies that $s + t = 1$ or 2. Since $|s+t| < 2$, we must have $s+t = 1$. In this case, if $t = 0$ or $s = 0$, then $S(\phi) = S(\psi_1)$. Therefore, we must have $s + t = 1, t \neq 0, 1$. The requirement $|s| + |t| < 2$ tells us that $-1/2 < t < 3/2$.

Conversely, suppose that $-1/2 < t < 3/2, t \neq 0, 1, s = 1 - t$. Let $0 \neq \alpha, \beta \in \mathbb{C}$ such that $\alpha/\beta = s/t$. Define

$$\tilde{a}(z) = (1-t) + z + tz^2, \qquad \tilde{b}(z) = (\beta(1-t)/t, -t\alpha/u) + (\beta, (1-t)/u)z.$$

Then the two-scale similarity relation (1.3) holds. By Theorem 4, $S(\psi)$ contains a nontrivial refinable subspace $S(\phi)$, and ϕ satisfies (4.2). Hence $S(\phi) \neq S(\psi_1)$. Also, $\phi \in L_1(\mathbb{R})$, and it has linearly independent shifts. This completes the proof of the statements in the example. $\square$

In Example 3, we provide the explicit refinement mask for ϕ which is unknown in [4, Example 4.2]. From this refinement mask we know that for $1 \leq p < \infty$, $\phi \in L_p(\mathbb{R})$ if and only if $|t|^p + |1-t|^p < 2$, while ϕ is continuous if and only if $0 < t < 1$. Here, we do not assume that $c(0)$ is invertible.

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