Inhomogeneous Refinement Equations†

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Abstract

Equations with two time scales (refinement equations or dilation equations) are central to wavelet theory. Several applications also include an inhomogeneous forcing term $F(t)$. We develop here a part of the existence theory for the inhomogeneous refinement equation

$$\phi(t) = \sum_{k\in\mathbb{Z}} a(k)\phi(2t-k) + F(t),$$

where $a(k)$ is a finite sequence and F is a compactly supported distribution on $\mathbb{R}$.

The existence of compactly supported distributional solutions to an inhomogeneous refinement equation is characterized in terms of conditions on the pair (a, F).

To have L_p solutions from $F \in L_p(\mathbb{R})$, we construct by the cascade algorithm a sequence of functions $\{\phi_n\}$ from a compactly supported initial function $\phi_0 \in L_p(\mathbb{R})$ as

$$\phi_n(t) = \sum_{k\in\mathbb{Z}} a(k)\phi_{n-1}(2t-k) + F(t).$$

A necessary and sufficient condition for the sequence $\{\phi_n\}$ to converge in $L_p(\mathbb{R})(1 \le p \le \infty)$ is given by the p-norm joint spectral radius of two matrices derived from the mask a. A convexity property of the p-norm joint spectral radius $(1 \le p \le \infty)$ is presented.

Finally, the general theory is applied to some examples and multiple refinable functions.

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Inhomogeneous Refinement Equations

§1. Introduction

The (homogeneous) refinement equation for the scaling function $\phi(t)$ is the fundamental equation in wavelet theory:

$$\phi(t) = \sum a(k)\phi(2t-k).$$

The existence, uniqueness and regularity of $\phi(t)$ have been studied by many authors. But the inclusion of an inhomogeneous term $F(t)$ changes the analysis, and this also deserves study. We have seen this term in two applications that will be mentioned again:

1. The construction of the Geronimo-Hardin-Massopust multiwavelets [5] from a "fractal equation".

2. The construction of boundary wavelets [14, p. 294].

In this paper we study the existence of solutions (first as distributions, then as functions in $L_p(\mathbb{R})$) to the **inhomogeneous refinement equation**

$$\phi(t) = \sum_{k\in\mathbb{Z}} a(k)\phi(2t-k) + F(t). \tag{1.1}$$

The **refinement mask** a is a finite sequence of complex numbers. The function $F(t)$ is compactly supported; we first assume only that it is a distribution. The special case $\phi(t) = a(0)\phi(2t) + F(t)$, with only a single coefficient in the mask, will be analyzed below in particular detail.

We begin with the Fourier transform of equation (1.1):

$$\hat{\phi}(\xi) = \tilde{a}(\xi/2)\hat{\phi}(\xi/2) + \hat{F}(\xi) \tag{1.2}$$

where $\tilde{a}$ is the symbol of the mask a given by

$$\tilde{a}(\xi) := \sum_{k\in\mathbb{Z}} a(k)e^{-ik\xi}/2, \qquad \xi \in \mathbb{C}.$$

The transform $\hat{F}(\xi) = \int_{\mathbb{R}} F(x)e^{-i\xi\cdot x}\,dx, \quad \xi \in \mathbb{C}$, has a natural extension to compactly supported distributions on $\mathbb{R}$.

Our first purpose is to characterize the existence of compactly supported distributional solutions of (1.1). A necessary and sufficient condition in terms of the mask a and the inhomogeneous term F will be given in Section 2. Our second purpose is to find L_p-solutions of the inhomogeneous refinement equation (1.1). One way to construct $\phi(t)$ is by the cascade algorithm. Define a sequence of functions $\{\phi_n\}$ by

$$\phi_n(t) = \sum_{k\in\mathbb{Z}} a(k)\phi_{n-1}(2t-k) + F(t).$$

Observe that if a, F and ϕ_0 are supported in $[N_1, N_2]$ with $N_1 \le 0 \le N_2$, then ϕ_n is supported in $[N_1, N_2]$ for any $n \in \mathbb{N}$. Therefore, if the sequence $\{\phi_n\}$ converges in $L_p(\mathbb{R})$ (that is, the cascade algorithm converges), then the limit function $\phi \in L_p(\mathbb{R})$ is supported in $[N_1, N_2]$ and solves (1.1). In Section 4 we shall present a necessary and sufficient condition for the cascade algorithm to converge. To this end, some properties of the joint spectral radius are needed, which we shall investigate in Section 3. Examples will be provided in Section 5 to illustrate the general theory. In Section 6 we apply our main results to a study of multiple refinable functions.

§2. Existence of Compactly Supported Distributional Solutions

We now characterize the existence of compactly supported distributional solutions to the inhomogeneous refinement equation (1.1). Note that if ϕ is a solution, then

$$(1-\tilde{a}(0))\hat{\phi}(0) = \hat{F}(0). \tag{2.1}$$

Let us discuss three different cases. In the first case, $\tilde{a}(0) \ne 2^n$ for any $n \in \mathbb{N} \cup \{0\}$. Then existence and uniqueness hold.

Theorem 1. *Let F be a compactly supported distribution on $\mathbb{R}$ and a be a finitely supported sequence such that $\tilde{a}(0) \ne 2^n$ for any $n \in \mathbb{N} \cup \{0\}$. Then (1.1) has a unique solution ϕ of compactly supported distribution on $\mathbb{R}$. Moreover, $\hat{\phi}(0) = \hat{F}(0)/(1-\tilde{a}(0))$.*

This theorem is also proved by Dinsenbacher and Hardin [3], in a preprint that we received after completing our paper. Their proof is different (and very interesting), and they allow matrix masks. For us the conditions in Theorems 1–3 are easy to test and easy to extend to the multivariate case.

The second case is $\tilde{a}(0) = 1$. In this case (2.1) tells us that a necessary condition for existence is $\hat{F}(0) = 0$.

Theorem 2. *Let F be a compactly supported distribution on $\mathbb{R}$ and a be a finite sequence such that $\tilde{a}(0) = 1$. Then (1.1) has a solution ϕ of compactly supported distribution on $\mathbb{R}$ if and only if $\hat{F}(0) = 0$. In this case, the solution ϕ is unique subject to $\hat{\phi}(0) = 1$.*

The third (more complicated) case has $\tilde{a}(0) = 2^n$ for some $n \in \mathbb{N}$. Then by (2.1),

$$\hat{\phi}(0) = \hat{F}(0)/(1 - 2^n),$$

and by (1.2), for $j = 1, \cdots, n$,

$$(2^j - 2^n)\hat{\phi}^{(j)}(0) = 2^j \hat{F}^{(j)}(0) + \sum_{k=0}^{j-1} \binom{j}{k} \tilde{a}^{(j-k)}(0)\hat{\phi}^{(k)}(0).$$

This shows the necessity of the following condition (2.2).

Theorem 3. *Let F be a compactly supported distribution on $\mathbb{R}$ and a be a finite sequence such that $\tilde{a}(0) = 2^n$ for some $n \in \mathbb{N}$. Then (1.1) has a solution ϕ of compactly supported distribution on $\mathbb{R}$ if and only if*

$$\hat{F}^{(n)}(0) = -2^{-n} \sum_{k=0}^{n-1} \binom{n}{k} \tilde{a}^{(n-k)}(0) b_k, \tag{2.2}$$

where $\{b_0, \cdots, b_{n-1}\}$ is defined inductively by $b_0 = \hat{F}(0)/(1 - 2^n)$ and

$$b_j = (2^j - 2^n)^{-1} \Big\{ 2^j \hat{F}^{(j)}(0) + \sum_{k=0}^{j-1} \binom{j}{k} \tilde{a}^{(j-k)}(0) b_k \Big\}, \qquad j = 1, \cdots, n-1.$$

Before proving these theorems let us introduce some preliminary results.

Recall that if f is a compactly supported distribution on $\mathbb{R}$, then

$$|\hat{f}(\xi)| \le C(1 + |\xi|)^B e^{A|\mathrm{Im}\xi|}, \qquad \xi \in \mathbb{C}, \tag{2.3}$$

where A, B and C are positive constants independent of ξ. Note that a finite sequence a can be regarded as a distribution compactly supported in $\mathbb{Z}$, the set of all integers. In this way, its symbol $\tilde{a}$ equals half of its Fourier transform.

Define a sequence $\{\Pi_n(\xi)\}$ by $\Pi_0(\xi) \equiv 1$ and

$$\Pi_n(\xi) = \Pi_{j=1}^{n} \tilde{a}(\xi/2^j), \qquad \xi \in \mathbb{C}.$$

Then we know (see *e.g.*, [17, Lemma]) that

$$|\Pi_n(\xi)| \leq C_0 \Lambda^n (1 + |\xi|)^{B_0} e^{A_0 |\mathrm{Im}\xi|}, \qquad \xi \in \mathbb{C}, \tag{2.4}$$

where A_0, B_0 and C_0 are positive constants independent of ξ and n, and

$$\Lambda := \begin{cases} |\tilde{a}(0)|, & \text{if } \tilde{a}(0) \neq 0, \\ 1/2, & \text{if } \tilde{a}(0) = 0. \end{cases}$$

Given $\{y_0, \cdots, y_m\} \subset \mathbb{C}$, there always exists a sequence d supported in $[0, m]$ such that

$$\tilde{d}^{(j)}(0) = y_j, \qquad j = 0, \cdots, m.$$

In fact, $\tilde{d}$ can be determined by

$$\tilde{d}(\xi) = \sum_{k=0}^{m} d_1(k)(1 - e^{-i\xi})^k$$

where $d_1(0) = y_0$, and recursively,

$$d_1(j) = i^{-j} \left\{ y_j - \left(\sum_{k=0}^{j-1} d_1(k)(1 - e^{-i\xi})^k \right)^{(j)} (0) \right\}.$$

It follows that if $2^n \leq |\tilde{a}(0)| < 2^{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$ and $\tilde{a}(0) \neq 2^n$, then there exists a sequence d supported in $[0, n]$ such that

$$\tilde{d}(2\xi) - \tilde{a}(\xi)\tilde{d}(\xi) - \hat{F}(2\xi) = O(|\xi|^{n+1}) \quad (\xi \to 0).$$

If $|\tilde{a}(0)| \leq 1$, and $\hat{F}(0) = 0$ if $\tilde{a}(0) = 1$, then there exists a sequence d supported at $\{0\}$ such that

$$\tilde{d}(2\xi) - \tilde{a}(\xi)\tilde{d}(\xi) - \hat{F}(2\xi) = O(|\xi|) \quad (\xi \to 0)$$

and $\tilde{d}(0) = 1$ if $\tilde{a}(0) = 1, \hat{F}(0) = 0$.

If $\tilde{a}(0) = 2^n$ for some $n \in \mathbb{N}$ and (2.2) holds, then there exists a sequence d supported in $[0, n]$ such that

$$\tilde{d}^{(j)}(0) = b_j, \qquad j = 0, \cdots, n-1,$$

which implies

$$\tilde{d}(2\xi) - \tilde{a}(\xi)\tilde{d}(\xi) - \hat{F}(2\xi) = O(|\xi|^{n+1}) \quad (\xi \to 0).$$

Now we are in a position to prove the main results on existence.

Proof of Theorems 1, 2 and 3.

Suppose that the conditions hold. We have either $|\tilde{a}(0)| < 1$ or $2^n \le |\tilde{a}(0)| < 2^{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$. If $|\tilde{a}(0)| < 1$, we set $n = 0$. Then, $\Lambda < 2^{n+1}$. By our previous discussion, there exists a sequence d supported in $[0, n]$ such that

$$\tilde{d}(2\xi) - \tilde{a}(\xi)\tilde{d}(\xi) - \hat{F}(2\xi) = O(|\xi|^{n+1}) \quad (\xi \to 0)$$

and $\tilde{d}(0) = 1$ if $\tilde{a}(0) = 1, \hat{F}(0) = 0$. It follows that

$$\tilde{d}(0) = \begin{cases} \hat{F}(0)/(1 - \tilde{a}(0)), & \text{if } \tilde{a}(0) \neq 1, \\ 1, & \text{if } \tilde{a}(0) = 1, \hat{F}(0) = 0. \end{cases}$$

The function $\tilde{d}(2\xi) - \tilde{a}(\xi)\tilde{d}(\xi) - \hat{F}(2\xi)$ can be regarded as the Fourier transform of a compactly supported distribution on $\mathbb{R}$. By the order of the zero at the origin and (2.3), we know that

$$|\tilde{d}(2\xi) - \tilde{a}(\xi)\tilde{d}(\xi) - \hat{F}(2\xi)| \le C_1 |\xi|^{n+1}(1 + |\xi|)^{B_1} e^{A_1|\mathrm{Im}\xi|}, \qquad \xi \in \mathbb{C}, \tag{2.5}$$

where A_1, B_1 and C_1 are positive constants independent of ξ.

Let us define a sequence $\{\hat{\Phi}_m\}_{m \in \mathbb{N}}$ of functions on $\mathbb{C}$ by

$$\hat{\Phi}_m(\xi) = \Pi_m(\xi)\tilde{d}(\xi/2^m) + \sum_{j=0}^{m-1} \Pi_j(\xi)\hat{F}(\xi/2^j), \qquad \xi \in \mathbb{C}. \tag{2.6}$$

Then

$$\hat{\Phi}_m(0) = \begin{cases} \hat{F}(0)/(1 - \tilde{a}(0)) = \tilde{d}(0), & \text{if } \tilde{a}(0) \neq 1, \\ \tilde{d}(0) = 1, & \text{if } \tilde{a}(0) = 1, \hat{F}(0) = 0. \end{cases}$$

For the refinement relation, we have

$$\hat{\Phi}_{m+1}(\xi) = \tilde{a}(\xi/2)\hat{\Phi}_m(\xi/2) + \hat{F}(\xi). \tag{2.7}$$

Let us give an estimate on $|\hat{\Phi}_m(\xi)|$ for $m \in \mathbb{N}, \xi \in \mathbb{C}$. By (2.6),

$$\hat{\Phi}_m(\xi) = \sum_{j=0}^{m-1} \left\{ \Pi_j(\xi)[\tilde{a}(\xi/2^{j+1})\tilde{d}(\xi/2^{j+1}) - \tilde{d}(\xi/2^j) + \hat{F}(\xi/2^j)] \right\} + \tilde{d}(\xi).$$

Applying (2.5) and (2.4), we obtain for $0 \le j \le m-1$,

$$\begin{aligned} &|\Pi_j(\xi)[\tilde{a}(\xi/2^{j+1})\tilde{d}(\xi/2^{j+1}) - \tilde{d}(\xi/2^j) + \hat{F}(\xi/2^j)]| \\ &\le C_0 C_1 (\Lambda/2^{n+1})^j (1+|\xi|)^{B_0+B_1} e^{(A_0+A_1)|\mathrm{Im}\xi|}. \end{aligned}$$

Thus, from the assumption that $\Lambda < 2^{n+1}$,

$$|\hat{\Phi}_m(\xi)| \le |\tilde{d}(\xi)| + C_0 C_1 \frac{2^{n+1}}{2^{n+1}-\Lambda}(1+|\xi|)^{B_0+B_1} e^{(A_0+A_1)|\mathrm{Im}\xi|}.$$

Applying (2.3) for d, we know that

$$|\hat{\Phi}_m(\xi)| \le C_2 (1+|\xi|)^{B_2} e^{A_2|\mathrm{Im}\xi|}, \tag{2.8}$$

where A_2, B_2 and C_2 are positive constants independent of ξ and m.

Now we show that the sequence $\{\hat{\Phi}_m\}$ converges uniformly on any bounded subset K of $\mathbb{C}$. Then by (2.6), for $\xi \in K$,

$$\hat{\Phi}_{m+1}(\xi) - \hat{\Phi}_m(\xi) = \Pi_m(\xi)[\tilde{a}(\xi/2^{m+1})\tilde{d}(\xi/2^{m+1}) - \tilde{d}(\xi/2^m) + \hat{F}(\xi/2^m)].$$

Together with (2.5) and (2.4) this implies that for $m \in \mathbb{N}, \xi \in K$,

$$\begin{aligned} |\hat{\Phi}_{m+1}(\xi) - \hat{\Phi}_m(\xi)| \le & C_0 \Lambda^m (1+|\xi|)^{B_0} e^{A_0|\mathrm{Im}\xi|} C_1 |\xi/2^{m+1}|^{n+1} \times \\ & \times (1+|\xi/2^{m+1}|)^{B_1} e^{A_1|\mathrm{Im}\xi/2^{m+1}|} \le C_0 C_1 C_K (\Lambda/2^{n+1})^m. \end{aligned}$$

Here C_K is the constant depending only on K given by

$$C_K = \sup_{\xi \in K}\{|\xi|^{n+1}(1+|\xi|)^{B_0+B_1} e^{(A_0+A_1)|\mathrm{Im}\xi|}\}.$$

Therefore, the sequence $\{\hat{\Phi}_m\}$ converges uniformly on K. This implies that the function $\lim_{m\to\infty}\hat{\Phi}_m(\xi)$ is analytic in $\mathbb{C}$.

By the Paley-Wiener Theorem, we conclude from (2.8) that there exists a compactly supported distribution ϕ on $\mathbb{R}$ such that

$$\hat{\phi}(\xi) = \lim_{m\to\infty}\hat{\Phi}_m(\xi), \qquad \xi\in\mathbb{C}.$$

By (2.7) we have

$$\hat{\phi}(\xi) = \tilde{a}(\xi/2)\hat{\phi}(\xi/2) + \hat{F}(\xi).$$

Hence (1.2) holds, which implies the refinement relation (1.1). Moreover, by our definition,

$$\hat{\phi}(0) = \lim_{m\to\infty}\hat{\Phi}_m(0) = \begin{cases} \hat{F}(0)/(1-\tilde{a}(0)), & \text{if } \tilde{a}(0)\neq 1, \\ 1, & \text{if } \tilde{a}(0)=1, \hat{F}(0)=0. \end{cases}$$

The necessity parts of Theorems 1, 2 and 3 are proved. The uniqueness in Theorems 1 and 2 follows from the condition for the existence and uniqueness of compactly supported distributional solutions of (1.1) with $F=0$. $\square$

§3. Joint Spectral Radius

In this section we review some results concerning the p-norm joint spectral radius. A convexity property is presented, which provides some interesting applications for the estimate of uniform joint spectral radius.

The joint spectral radius was introduced by Rota and Strang in [12]. This concept was used by Daubechies and Lagarias [2] to study the regularity of wavelets, which has initiated much further research in wavelets. The mean spectral radius ($p=1$) was introduced by Wang [16] who studied $L_1(\mathbb{R})$ refinable functions. The concept of p-norm joint spectral radius ($1\le p\le\infty$) was defined by Jia in [7] and was used implicitly by Lau and Wang [11] independently.

Let $\mathcal{A} := \{A_1,\cdots,A_d\}$ be a multiset of $M\times M$ complex matrices. Choose $\|\cdot\|$ as a matrix norm on $\mathbb{C}^{M\times M}$, the set of all $M\times M$ complex matrices. For $1\le p\le\infty, n\in\mathbb{N}$, define $\|\mathcal{A}^n\|_p$ as the ℓ_p norm of the sequence of all products (in all orders) of n matrices from $\mathcal{A}$:

$$\|\mathcal{A}^n\|_p := \begin{cases} \max\bigl\{\|A_{\varepsilon_1}\cdots A_{\varepsilon_n}\| : \varepsilon_1,\cdots,\varepsilon_n \in \{1,\cdots,d\}\bigr\}, & \text{if } p=\infty, \\ \bigl\{\sum_{\varepsilon_1,\cdots,\varepsilon_n\in\{1,\cdots,d\}} \|A_{\varepsilon_1}\cdots A_{\varepsilon_n}\|^p\bigr\}^{1/p}, & \text{if } 1\le p<\infty. \end{cases}$$

Then the p**-norm joint spectral radius** $\rho_p(\mathcal{A})$ is defined to be

$$\rho_p(\mathcal{A}) := \lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n}. \tag{3.1}$$

It is a classical fact that this limit exists, and equals the lim inf:

$$\lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n} = \inf_{n\in\mathbb{N}} \|\mathcal{A}^n\|_p^{1/n}.$$

Clearly, $\rho_p(\mathcal{A})$ is independent of the choice of the matrix norm $\|\cdot\|$. Recall that $\rho_\infty(\mathcal{A})$ is the uniform joint spectral radius introduced in [12].

If $\mathcal{A}$ consists of a single matrix A, then $\rho_p(\mathcal{A})$ equals the ordinary spectral radius $\rho(A)$. It is obvious that each $\rho(A_j)\le\rho_\infty(\mathcal{A})$, since $\rho_\infty(\mathcal{A})$ includes the powers $(A_j)^n$ when we choose the same matrix A_j each time by $\varepsilon_i=j$. Moreover, if $A_1=\cdots=A_d=A$, then $\rho_p(\mathcal{A})=d^{1/p}\rho(A)$ for $1\le p\le\infty$.

The Riesz-Thorin Theorem assures the convexity of ℓ_p operator norms as function of $1/p$ (see *e.g.*, [4]). This leads directly to the following **convexity property** for the p-norm joint spectral radius.

Theorem 4. *The function* $\ln\rho_{1/x}(\mathcal{A})$ *is convex on* $0\le x\le 1$.

What is disappointing about the joint spectral radius is that the limit in (3.1) is reached very slowly. We can hardly compute $\rho_p(\mathcal{A})$ using (3.1). Thus, it is quite desirable to find fast ways to compute the joint spectral radius. Theorem 4 provides a method to estimate $\rho_\infty(\mathcal{A})$ in terms of $\rho_p(\mathcal{A})$ (see [18]):

If $1\le p<\infty$ and $q>0$, then

$$\ln\rho_{p+q}(\mathcal{A}) \le \frac{p}{p+q}\ln\rho_p(\mathcal{A}) + \frac{q}{p+q}\ln\rho_\infty(\mathcal{A}).$$

Hence

$$\rho_\infty(\mathcal{A}) \ge \Bigl(\frac{\rho_{p+q}(\mathcal{A})}{\rho_p(\mathcal{A})}\Bigr)^{p/q}\rho_{p+q}(\mathcal{A}). \tag{3.2}$$

In particular, for $k, l \in \mathbb{N}$,

$$\rho_\infty(\mathcal{A}) \geq (\frac{\rho_{2k+2l}(\mathcal{A})}{\rho_{2k}(\mathcal{A})})^{k/l} \rho_{2k+2l}(\mathcal{A}).$$

Together with the following result from [18], this provides an effective method for estimating the uniform joint spectral radius $\rho_\infty(\mathcal{A})$. We can compute $\rho_p(\mathcal{A})$ explicitly when p is an even integer: For any $k \in \mathbb{N}$,

$$\rho_{2k}(\mathcal{A}) = \left\{ \rho\Big(\sum_{l=1}^{d} (\overline{A}_l \otimes A_l)^{[k]} \Big) \right\}^{1/2k}. \tag{3.3}$$

The **Kronecker product** of $A \in \mathbb{C}^{M\times M}$ and $B \in \mathbb{C}^{N\times N}$ is the block matrix

$$A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1M}B \\ a_{21}B & a_{22}B & \cdots & a_{2M}B \\ \vdots & \vdots & & \vdots \\ a_{M1}B & a_{M2}B & \cdots & a_{MM}B \end{bmatrix} \in \mathbb{C}^{MN\times MN}.$$

We set $A^{[1]} = A$ and $A^{[k+1]} := A \otimes A^{[k]}$ for $k \in \mathbb{N}$.

As an application, let us mention the following nice equivalence:

$$\rho_\infty(\mathcal{A}) = \rho_2(\mathcal{A}) \qquad \Longleftrightarrow \qquad \rho_4(\mathcal{A}) = \rho_2(\mathcal{A}).$$

§4. Convergence of the Cascade Algorithm

In this section we apply the p-norm joint spectral radius to the study of convergence of cascade algorithms associated with inhomogeneous refinement equations.

Let a be a finite sequence, F and ϕ_0 be compactly supported functions in $L_p(\mathbb{R})$ for $1 \leq p < \infty (C(\mathbb{R})$ if $p = \infty)$. Define a sequence of functions $\{\phi_n\}$ by

$$\phi_n(t) = \sum_{k\in\mathbb{Z}} a(k)\phi_{n-1}(2t-k) + F(t). \tag{4.1}$$

The **cascade algorithm** associated with (a, F) converges in $L_p(\mathbb{R})$ if there exists $\phi \in L_p(\mathbb{R})$ such that $\lim_{n\to\infty} \|\phi_n - \phi\|_p = 0$. In this case, ϕ solves the inhomogeneous refinement equation (1.1). Also, $\phi \in C(\mathbb{R})$ if $p = \infty$.

To consider the convergence of the cascade algorithm, we introduce a sequence $\{a_n\}$ by

$$a_1 = a \qquad \text{and} \qquad a_{n+1}(k) = \sum_{l\in\mathbb{Z}} a_n(l)a(k-2l), \qquad k \in \mathbb{Z}. \tag{4.2}$$

The relation between $\{a_n\}$ and $\{\phi_n\}$ can be seen from the following result.

Lemma 1. *Let a be a finite sequence, F and ϕ_0 be compactly supported distributions on* $\mathbb{R}$. *Define $\{\phi_n\}$ by (4.1) and $\{a_n\}$ by (4.2). Then*

$$a_{n+1}(k) = \sum_{l\in\mathbb{Z}} a(l)a_n(k-2^n l), \qquad k\in\mathbb{Z}, \tag{4.3}$$

and

$$\phi_n(t) = \sum_{k\in\mathbb{Z}} a_n(k)\phi_0(2^n t-k) + \sum_{j=1}^{n-1}\sum_{k\in\mathbb{Z}} a_j(k)F(2^j t-k) + F(t). \tag{4.4}$$

Proof. The proof proceeds by induction on n. The case $n=1$ is trivial by the definitions.

Suppose the lemma has been verified for $m = 1,\cdots,n-1$. Then by the induction hypothesis and (4.2), for $k\in\mathbb{Z}$,

$$\begin{aligned} a_{n+1}(k) &= \sum_{l\in\mathbb{Z}}\left[\sum_{j\in\mathbb{Z}} a(j)a_{n-1}(l-2^{n-1}j)\right]a(k-2l) \\ &= \sum_{j\in\mathbb{Z}} a(j)\left[\sum_{l\in\mathbb{Z}} a_{n-1}(l)a(k-2^n j-2l)\right] = \sum_{j\in\mathbb{Z}} a(j)a_n(k-2^n j). \end{aligned}$$

This in connection with the induction hypothesis and (4.1) implies

$$\begin{aligned} \phi_n(t) &= \sum_{k\in\mathbb{Z}} a(k)\left\{\sum_{l\in\mathbb{Z}} a_{n-1}(l)\phi_0(2^n t-2^{n-1}k-l)\right. \\ &\quad \left. + \sum_{j=1}^{n-2}\sum_{l\in\mathbb{Z}} a_j(l)F(2^{j+1}t-2^j k-l) + F(2t-k)\right\} + F(t) \\ &= \sum_{k\in\mathbb{Z}} a_n(k)\phi_0(2^n t-k) + \sum_{j=1}^{n-1}\sum_{k\in\mathbb{Z}} a_j(k)F(2^j t-k) + F(t) \end{aligned}$$

thereby completing the induction. $\square$

A consequence of Lemma 1 is a representation of $\phi_{n+1}-\phi_n$:

$$\phi_{n+1}(t)-\phi_n(t) = \sum_{k\in\mathbb{Z}} a_{n+1}(k)\phi_0(2^{n+1}t-k) - \sum_{k\in\mathbb{Z}} a_n(k)\phi_0(2^n t-k) + \sum_{k\in\mathbb{Z}} a_n(k)F(2^n t-k).$$

Thus

$$\phi_{n+1}(t)-\phi_n(t) = \sum_{k\in\mathbb{Z}} a_n(k)f(2^n t-k),$$

where

$$f(t) = \sum_{l\in\mathbb{Z}} a(l)\phi_0(2t-l) - \phi_0(t) + F(t) \in L_p(\mathbb{R}).$$

We assume that a, ϕ_0 and F are supported in $[0, N]$ for some positive integer N. Then f is also supported in $[0, N]$.

Let $\psi \in L_p(\mathbb{R})$ be supported in $[0, N]$ with linearly independent translates:

$$\sum_{k\in\mathbb{Z}} b(k)\psi(t-k) = 0 \quad \Longrightarrow \quad b(k) = 0 \quad \forall k \in \mathbb{Z}.$$

Suppose that for a sequence c,

$$f(t) = \sum_{k\in\mathbb{Z}} c(k)\psi(t-k). \tag{4.5}$$

Then we know that c is supported in $[0, N-1]$.

Now we can write $\phi_{n+1} - \phi_n$ in terms of the linearly independent generators ψ and the combination sequence c as

$$\phi_{n+1}(t) - \phi_n(t) = \sum_{k\in\mathbb{Z}} a_n{*}c(k)\psi(2^n t - k),$$

where

$$a_n{*}c(k) = \sum_{l\in\mathbb{Z}} a_n(k-l)c(l).$$

Taking the L_p-norms on both sides, we have

$$C_3^{-1}2^{-n/p}\|a_n{*}c\|_p \le \|\phi_{n+1} - \phi_n\|_p \le C_3 2^{-n/p}\|a_n{*}c\|_p,$$

where C_3 is a positive constant independent of n.

If $\{\phi_n\}$ converges in $L_p(\mathbb{R})$, then

$$2^{-n/p}\|a_n{*}c\|_p \to 0 \quad \text{as } n \to \infty.$$

Conversely, if

$$2^{-n/p}\|a_n{*}c\|_p \le C_4\rho^n$$

for some $\rho \in (0, 1)$ and a positive constant C_4 independent of n, then $\{\phi_n\}$ converges in $L_p(\mathbb{R})$.

Therefore, the convergence of $\{\phi_n\}$ is reduced to the behavior of the norms of the sequences $a_n{*}c$, which is closely related to the p-norm joint spectral radius according to the following result in [10].

Lemma 2. *Suppose that the sequence a is supported in $[0, N]$ and c is supported in $[0, N-1]$. Define two linear operators on $\mathbb{C}^N$ by*

$$A_0(k,l) = a(2k-l) \qquad \text{and} \qquad A_1(k,l) = a(2k-l+1), \qquad k,l = 0,\cdots,N-1. \quad (4.6)$$

Set $V(c)$ be the minimal common invariant subspace of A_0 and A_1 generated by c. Let $\mathcal{A}$ be the set of matrix representations of $A_0|_{V(c)}$ and $A_1|_{V(c)}$. Then there are positive constants M_0 and M_1 such that

$$M_0\|\mathcal{A}^n\|_p \leq \|a_n{*}c\|_p \leq M_1\|\mathcal{A}^n\|_p, \qquad 1 \leq p \leq \infty, \quad n \in \mathbb{N}.$$

Hence

$$\rho_p(\mathcal{A}) = \lim_{n\to\infty} \|a_n{*}c\|_p^{1/n}, \qquad 1 \leq p \leq \infty.$$

Combining the above discussions, we obtain our main result on the convergence of the cascade algorithm.

Theorem 5. *Let $1 \leq p \leq \infty$ and suppose that a, ϕ_0, F, ψ and c satisfy (4.5). For some positive integer N, the sequence a is supported in $[0, N]$ and c is supported in $[0, N-1]$. Define A_0 and A_1 by (4.6). Then the cascade algorithm associated with the pair (a, F) converges in $L_p(\mathbb{R})(C(\mathbb{R})$ if $p = \infty)$ if and only if*

$$\rho_p(A_0|_{V(c)}, A_1|_{V(c)}) < 2^{1/p}.$$

In particular, if $\rho_p(A_0, A_1) < 2^{1/p}$, the cascade algorithm converges in $L_p(\mathbb{R})$ for any compactly supported functions F and ϕ_0 in $L_p(\mathbb{R})(C(\mathbb{R})$ if $p = \infty)$.

Thus the convergence has been characterized by the p-norm joint spectral radius. The existence of ψ and c satisfying (4.5) is assured by a result of Jia [8].

§5. Examples

In this section we provide some examples for the case when a has short support. Let us begin with the case when a is supported at the origin. The inhomogeneous refinement equation takes the form

$$\phi(t) = a(0)\phi(2t) + F(t). \tag{5.1}$$

Such an equation with $a(0) = 1$ is satisfied by boundary scaling functions, see [14, p. 294].

If the mask has only one coefficient, then $A_0 = [a(0)]$ and $A_1 = [0]$. Hence for $1 \le p \le \infty$, we have $\rho_p(A_0, A_1) = |a(0)|$. By Theorem 5, if F and ϕ_0 are compactly supported and lie in $L_p(\mathbb{R})(C(\mathbb{R})$ if $p = \infty)$, and $|a(0)| < 2^{1/p}$, then the cascade algorithm associated with (a, F) converges in $L_p(\mathbb{R})$. Hence the equation (5.1) has a compactly supported solution $\phi \in L_p(\mathbb{R})(C(\mathbb{R})$ if $p = \infty)$. Moreover, an $L_p(\mathbb{R})$ solution ϕ can be found explicitly if $|a(0)| \ne 2^{1/p}$. However, when $|a(0)| > 2^{1/p}$, usually ϕ is not compactly supported. This shows that the second statement of Theorem 5 is sharp in some circumstances.

Theorem 6. *Let $1 \le p \le \infty, F$ be a compactly supported function in $L_p(\mathbb{R})(C(\mathbb{R})$ if $p = \infty)$ and $|a(0)| \ne 2^{1/p}$. Then the inhomogeneous refinement equation (5.1) has a unique solution ϕ in $L_p(\mathbb{R})(C(\mathbb{R}) \cap L_\infty(\mathbb{R})$ if $p = \infty)$ given by*

$$\phi(t) = \begin{cases} \sum_{j=0}^\infty a(0)^j F(2^j t), & \text{if } |a(0)| < 2^{1/p}, \\ -\sum_{j=1}^\infty a(0)^{-j} F(2^{-j} t), & \text{if } |a(0)| > 2^{1/p}. \end{cases}$$

Proof. Trivially, for $n \in \mathbb{Z}$,

$$\|a(0)^n F(2^n t)\|_p = (|a(0)|2^{-1/p})^n \|F\|_p.$$

If $|a(0)|2^{-1/p} < 1$, the series $\sum_{j=0}^\infty a(0)^j F(2^j t)$ converges in $L_p(\mathbb{R})$ to a function $\phi \in L_p(\mathbb{R})(C(\mathbb{R})$ if $p = \infty)$. This function ϕ is compactly supported and satisfies (5.1). To see the uniqueness, suppose that $\phi \in L_p(\mathbb{R})$ is a solution of (5.1). Then

$$\phi(t) - \sum_{j=0}^{n-1} a(0)^j F(2^j t) = a(0)^n \phi(2^n t) \to 0 \quad \text{in } L_p(\mathbb{R}) \quad (n \to \infty).$$

This tells that $\phi(t) = \sum_{j=0}^\infty a(0)^j F(2^j t)$.

If $|a(0)| > 2^{1/p}$, we can easily check that the series $-\sum_{j=1}^\infty a(0)^{-j} F(2^{-j} t)$ converges in $L_p(\mathbb{R})$ to a solution of (5.1). Suppose conversely that $\phi \in L_p(\mathbb{R})$ is a solution. Then

$$\phi(t) + \sum_{j=1}^{n} a(0)^{-j} F(2^{-j} t) = a(0)^{-n} \phi(2^{-n} t) \to 0 \quad \text{in } L_p(\mathbb{R}).$$

This shows that $\phi(t) = -\sum_{j=1}^\infty a(0)^{-j} F(2^{-j} t)$.

The proof of Theorem 6 is complete. □

When $|a(0)| = 2^{1/p}$, things are different. Let us consider the case when $p = \infty$ and $a(0) = 1$ only.

Theorem 7. *Let F be a compactly supported continuous function on* $\mathbb{R}$. *Then*

$$\phi(t) = \phi(2t) + F(t) \tag{5.2}$$

has a compactly supported continuous solution if and only if the series $\sum_{j=1}^{\infty} F(2^{-j}t)$ *converges uniformly on* $[-2, 2]$ *and there exists a constant* c *such that*

$$\sum_{j=-\infty}^{\infty} F(2^j t) = c, \qquad \forall t \in \mathbb{R} \setminus \{0\}. \tag{5.3}$$

In this case, the solution ϕ *is unique and given by*

$$\phi(t) = c - \sum_{j=1}^{\infty} F(2^{-j}t), \qquad t \in \mathbb{R}.$$

Proof. Necessity. Suppose that there exists a compactly supported continuous function ϕ such that (5.2) holds. Then, for $n \in \mathbb{N}, t \in \mathbb{R}$,

$$\sum_{j=1}^{n} F(2^{-j}t) = \phi(2^{-n}t) - \phi(t) \to \phi(0) - \phi(t) \qquad (n \to \infty).$$

Hence $\sum_{j=1}^{\infty} F(2^{-j}t) = \phi(0) - \phi(t)$ for $t \in \mathbb{R}$. Also, on the interval $[-2, 2]$, the series $\sum_{j=1}^{\infty} F(2^{-j}t)$ converges uniformly to $\phi(0) - \phi(t)$.

On the other hand, for $n \in \mathbb{N}, t \in \mathbb{R} \setminus \{0\}$,

$$\sum_{j=0}^{n-1} F(2^j t) = \phi(t) - \phi(2^n t) \to \phi(t) \qquad (n \to \infty),$$

which implies that for $t \in \mathbb{R} \setminus \{0\}$,

$$\sum_{j=0}^{\infty} F(2^j t) = \phi(t).$$

Combining the expressions involving $2^{-j}t$ and $2^j t$, we obtain

$$\sum_{j=-\infty}^{\infty} F(2^j t) = \phi(0), \qquad \forall t \in \mathbb{R} \setminus \{0\}.$$

Therefore, (5.3) holds with $c := \phi(0)$. Moreover,

$$\phi(t) = c - \sum_{j=1}^{\infty} F(2^{-j}t), \qquad \forall t \in \mathbb{R}.$$

Sufficiency. Suppose that $\sum_{j=1}^{\infty} F(2^{-j}t)$ converges uniformly on $[-2,2]$ and (5.3) holds. Then $\sum_{j=1}^{\infty} F(2^{-j}t)$ converges uniformly on any bounded interval. Define a function ϕ by

$$\phi(t) = c - \sum_{j=1}^{\infty} F(2^{-j}t), \qquad t \in \mathbb{R}.$$

Then ϕ is continuous on $\mathbb{R}$. Also,

$$\phi(2t) + F(t) = c - \sum_{j=0}^{\infty} F(2^{-j}t) + F(t) = \phi(t), \qquad \forall t \in \mathbb{R}.$$

Moreover, by (5.3), for $t \in \mathbb{R} \setminus \{0\}$,

$$\phi(t) = \sum_{j=0}^{\infty} F(2^{j}t).$$

If F is supported in $[N_1, N_2]$ with $N_1 \leq 0 \leq N_2$, then ϕ is supported in $[N_1, N_2]$. Therefore, ϕ is a compactly supported continuous solution of (5.2). $\square$

The condition that $\sum_{j=1}^{\infty} F(2^{-j}t)$ converges uniformly on $[-2,2]$ is a mild regularity requirement at the origin. The condition (5.3) which is typical in finding tight frames (see e.g., [1]) requires more information about the function F.

When $p = \infty$ and $|a(0)| > 1$, the combination of Theorems 6 and 7 provides a criterion for the continuous solution given in Theorem 6 to be compactly supported.

Now we turn to the two-coefficient case. Assume that $a(0) = a(1) = h$. Then the inhomogeneous refinement equation has the form

$$\phi(t) = h\phi(2t) + h\phi(2t-1) + F(t). \tag{5.4}$$

Note that the matrices in (4.6) are given by $A_0 = [h], A_1 = [h]$. Hence $\rho_p(A_0, A_1) = 2^{1/p}|h|$ for any $1 \leq p \leq \infty$. Applying Theorem 5, we know that if $|h| < 1$, the cascade algorithm associated with (a, F) converges. Moreover, if F is supported in $[0,1]$, we have an explicit formula for the solution.

Theorem 8. *Let $1 \leq p \leq \infty, |h| < 1, F$ be supported in $[0,1]$ and lie in $L_p(\mathbb{R})(C(\mathbb{R})$ if $p = \infty)$. Then equation (5.4) has a unique solution ϕ of compact support in $L_p(\mathbb{R})(C(\mathbb{R})$*

if $p = \infty$) given by

$$\phi(t) = \begin{cases} \sum_{n=0}^{\infty} h^n F(2^n t - [2^n t]), & \text{if } 0 \le t < 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $[x]$ denotes the greatest integer that is less than or equal to x.

Proof. For $n \geq 0$, we have

$$\int_0^1 |h^n F(2^n t - [2^n t])|^p dt = |h|^{np} \|F\|_p^p.$$

Since $|h| < 1$, we know that the series converges in $L_p(\mathbb{R})$. Thus, the function ϕ is supported in $[0,1]$ and lies in $L_p(\mathbb{R})$. It is easily seen that ϕ is continuous if $p = \infty$.

If $0 \le t < 1/2$, then

$$\phi(t) - h\phi(2t) - h\phi(2t-1) = \sum_{n=0}^{\infty} h^n F(2^n t - [2^n t]) - h \sum_{n=0}^{\infty} h^n F(2^{n+1} t - [2^{n+1} t]) = F(t).$$

If $1/2 \le t < 1$, then

$$\phi(t) - h\phi(2t) - h\phi(2t-1) = \sum_{n=0}^{\infty} h^n F(2^n t - [2^n t]) - h \sum_{n=0}^{\infty} h^n F(2^{n+1} t - 2^n - [2^{n+1} t - 2^n])$$

which is again $F(t)$. Therefore, (5.4) holds for the function ϕ which is unique since $|h| < 1$. $\square$

As an example, if $F(t) = \sin(2\pi t)$ for $t \in [0,1]$, $F(t) = 0$ elsewhere, then $F \in C(\mathbb{R})$. Choose $p = \infty$ and $0 < h < 1$. The solution ϕ of (5.4) derived in Theorem 8 is the well-known Weierstrass function:

$$\phi(t) = \sum_{n=0}^{\infty} h^n \sin(2^{n+1} \pi t), \qquad 0 \le t \le 1.$$

One essential difference between inhomogeneous and homogeneous refinement equations is that inhomogeneous equations may have C^∞ solutions of compact support. For example, choose Rvachev's up-function [13] which is supported in $[-1,1]$ and given by its Fourier transform as

$$\hat{up}(\xi) = \Pi_{j=1}^{\infty} \frac{\sin(\xi/2^j)}{\xi/2^j}, \qquad \xi \in \mathbb{C}.$$

If we set $F(t) = up(t) - up(2t)$, then $\phi(t) = up(t)$ is a compactly supported C^∞ solution of (5.2). This is trivial of course. However, it would be interesting to characterize, in terms of the pair (a, F), those inhomogeneous refinement equations (1.1) that have compactly supported C^∞ solutions. For the special equation (5.2), those conditions on existence in Theorem 7 and $F \in C^\infty(\mathbb{R})$ are necessary and sufficient.

§6. Multiple Refinable Functions

In this section we apply our main results to a study of some examples of multiple refinable functions. For the general theory and more examples of multiple refinable functions, we refer the reader to [5, 6, 9, 10, 15, 17].

The first example was introduced by Geronimo, Hardin and Massopust [5]. Consider the matrix refinement equation

$$\Phi(t) = \sum_{k=0}^{3} a_k \Phi(2t-k). \tag{6.1}$$

Here $\Phi = (\phi_1, \phi_2)^T$ and

$$a_0 = \begin{bmatrix} h_1 & 1 \\ h_2 & h_3 \end{bmatrix}, \qquad a_1 = \begin{bmatrix} h_1 & 0 \\ h_4 & 1 \end{bmatrix},$$

$$a_2 = \begin{bmatrix} 0 & 0 \\ h_4 & h_3 \end{bmatrix}, \qquad a_3 = \begin{bmatrix} 0 & 0 \\ h_2 & 0 \end{bmatrix},$$

where with a parameter s,

$$h_1 = -\frac{s^2-4s-3}{2(s+2)}, \qquad h_2 = -\frac{3(s^2-1)(s^2-3s-1)}{4(s+2)^2},$$

$$h_3 = \frac{3s^2+s-1}{2(s+2)}, \qquad h_4 = -\frac{3(s^2-1)(s^2-s+3)}{4(s+2)^2}.$$

When $|s| < 1$, the matrix refinement equation (6.1) has a continuous solution Φ with $\hat{\phi}_1(0) = 1$ and $\hat{\phi}_2(0) = (s-1)^2/(s+2)$. Moreover, $\mathrm{supp}\phi_1 = [0,1]$ and $\mathrm{supp}\phi_2 = [0,2]$.

Taking the first component of the matrix refinement equation (6.1), we obtain

$$\phi_1(t) = h_1\phi_1(2t) + h_1\phi_1(2t-1) + \phi_2(2t). \tag{6.2}$$

This is an inhomogeneous refinement equation, and $|h_1| < 1$ if $|s| < 1$. Applying Theorem 8 to (6.2), since $\mathrm{supp}\phi_2(2t) = [0,1]$, we have for $0 \le t < 1$,

$$\phi_1(t) = \sum_{n=0}^{\infty} h_1^n \phi_2(2^{n+1}t - 2[2^n t]). \tag{6.3}$$

This in connection with (6.2) tells us that each component of the solution Φ of (6.1) can be expressed explicitly by the other.

Our next example is taken from Jia, Riemenschneider and Zhou [10]. Consider the matrix refinement equation

$$\Phi(t) = \begin{bmatrix} 1 & 0 \\ \lambda & y \end{bmatrix} \Phi(2t) + \begin{bmatrix} 1 & 0 \\ -\lambda & y \end{bmatrix} \Phi(2t-1),$$

where $\Phi = (\phi_1, \phi_2)^T$ and λ, y are real parameters.

Take $\phi_1 = \chi_{[0,1)}$, the characteristic function of the interval $[0,1)$. Then the second component ϕ_2 satisfies the inhomogeneous refinement equation

$$\phi_2(t) = y\phi_2(2t) + y\phi_2(2t-1) + F(t),$$

where

$$F(t) = \lambda\phi_1(2t) - \lambda\phi_1(2t-1) = \begin{cases} \lambda, & \text{if } 0 \le t < 1/2, \\ -\lambda, & \text{if } 1/2 \le t < 1, \\ 0, & \text{otherwise.} \end{cases}$$

It was proved in [10] that when $|y| < 1, \phi_2 \in L_p(\mathbb{R})$ for any $1 \le p < \infty$. Here by Theorem 8, we obtain an explicit formula for ϕ_2:

$$\phi_2(t) = \begin{cases} \sum_{n=0}^{\infty} y^n F(2^n t - [2^n t]), & \text{if } 0 \le t < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let us write an irrational number t in $[0,1)$ as $t = \sum_{k=1}^{\infty} \frac{t_k}{2^k}$ with $t_k \in \{0,1\}$. Then

$$2^n t - [2^n t] = \sum_{k=n+1}^{\infty} \frac{t_k}{2^k}.$$

Hence

$$F(2^n t - [2^n t]) = (-1)^{t_{n+1}} \lambda.$$

Thus, for $t_k \in \{0,1\}$ with $k \in \mathbb{N}$, and $t_k = 0$ and 1 for infinitely many $k \in \mathbb{N}$, we have

$$\phi_2(\sum_{k=1}^{\infty} \frac{t_k}{2^k}) = \lambda \sum_{n=1}^{\infty} (-1)^{t_n} y^{n-1}.$$

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