The Limits of Refinable Functions

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Abstract

A function $\phi(x)$ is refinable ($\phi \in S$) if it is in the closed span of $\{\phi(2x-k)\}$. This set S is not closed in $L_2(\mathbb{R})$, and we characterize its closure. A necessary and sufficient condition for a function to be refinable is presented without any information on the refinement mask. The Fourier transform of every $f \in \overline{S} \setminus S$ vanishes on a set of positive measure. As an example, we show that all functions with Fourier transform supported in $[-\frac{4}{3}\pi, \frac{4}{3}\pi]$ are the limits of refinable functions. The relation between a refinable function and its mask is studied, and nonuniqueness is proved. For inhomogeneous refinement equations we determine when a solution is refinable. This result is used to investigate refinable components of multiple refinable functions. Finally, we investigate fully refinable functions for which all translates (by any real number) are refinable.

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The Limits of Refinable Functions

§1. Introduction and Main Results

The central equation in wavelet analysis is the **refinement equation** for the scaling function $\phi(x)$:

$$\phi(x) = \sum_{k=0}^{N} a(k)\phi(2x-k). \tag{1.1}$$

In approximation theory, the sequence $\{a(k)\}$ is the **mask**. In signal processing these are the coefficients of a lowpass filter.

For a given mask $\{a(k)\}$, wavelet theory yields the properties of the family $\{\phi(x-k)\}$. We can determine whether these translates form a Riesz basis of a subspace in $L_2(\mathbb{R})$, whether this basis is orthogonal, and which polynomials $1, x, \cdots, x^{p-1}$ are linear combinations of the translates. This theory is summarized in [3] and [13]. What we *do not know* is how to choose the mask $\{a(k)\}$ so that $\phi(x)$ is close to a given function $f(x)$.

This "inverse problem" arises naturally in applications. We want to recognize objects whose shape is indicated by $f(x)$. We hope that a scaling function of similar shape will allow us to identify a good match. The thesis of Chapa [2] made a start on this problem using band-limited scaling functions. In that case the Fourier transform $\hat{\phi}(\xi)$ has compact support and the sequence $\{a(k)\}$ is infinite.

We want to start again, by answering this preliminary question: *What is the closure of the set S of all refinable functions in $L_2(\mathbb{R})$?* A solution to (1.1) is a refinable function. More generally, we say that

$$\phi \text{ is } \textbf{refinable } (\phi \in S) \quad \text{if and only if} \quad \phi(x) \in \overline{\text{span}}\{\phi(2x-k): \quad k \in \mathbb{Z}\}. \tag{1.2}$$

Thus an infinite mask is allowed. We wondered at first whether this set S is closed. We will show that it is not closed, and Theorem 1 will describe its closure $\overline{S}$.

The crucial questions involve the *zeros of the Fourier transform*. The transform of the refinement equation (1.1) is

$$\hat{\phi}(\xi) = (\frac{1}{2}\sum_{k\in\mathbb{Z}} a(k)e^{-ik\xi/2})\hat{\phi}(\frac{\xi}{2}) \equiv \tilde{a}(\frac{\xi}{2})\hat{\phi}(\frac{\xi}{2}). \tag{1.3}$$

For convenience we rescale to $\hat{\phi}(2\xi) = \tilde{a}(\xi)\hat{\phi}(\xi)$. The function $\tilde{a}(\xi)$ is clearly 2π-periodic. In the inverse direction, $f(x)$ will be refinable if $\hat{f}(2\xi)/\hat{f}(\xi)$ happens to be 2π-periodic, and $\hat{f}(\xi)$ is never zero. Then $f(x)$ will solve equation (1.1) with mask given by

$$\tilde{a}(\xi) = \frac{\hat{f}(2\xi)}{\hat{f}(\xi)}.$$

But if $\hat{f}(\xi)$ has zeros (which is typical!), we have to consider their relation to the zeros of $\hat{f}(2\xi)$. This eventually leads to our characterization of the closure of S:

Theorem 1. *A function $f(x)$ lies in $\overline{S}$, the closure of S in $L_2(\mathbb{R})$, if and only if for any positive integers j and k,*

$$\hat{f}(2^j(\xi + 2k\pi))\hat{f}(\xi) = \hat{f}(2^j\xi)\hat{f}(\xi + 2k\pi) \quad \text{for almost every } \xi. \tag{1.4}$$

As an example, the function $f(x) \in L_2(\mathbb{R})$ given by

$$\hat{f}(\xi) = \begin{cases} 1, & \text{if } \xi \in (-\frac{4}{3}\pi, -\frac{2}{3}\pi) \cup (\frac{2}{3}\pi, \frac{4}{3}\pi), \\ \\ 0, & \text{otherwise} \end{cases}$$

is in $\overline{S}$, but is not refinable. The condition (1.4) certainly holds in the band-limited case when $\hat{f}(\xi)$ is supported in $[-\frac{4}{3}\pi, \frac{4}{3}\pi]$, because for ξ in this interval we have

$$|2^j(\xi + 2k\pi)| \geq 2(2\pi - |\xi|) \geq \frac{4}{3}\pi.$$

Then both sides of (1.4) are identically zero and $f(x) \in \overline{S}$. Section 3 will show that if $b > \frac{4}{3}\pi$, there are functions with $\hat{f}(\xi)$ supported on $[-b, b]$ for which (1.4) does not hold.

Our second main result is a lower bound on the distance $d(f, S)$ from f to S:

$$d(f, S) = \inf\{\|f - \phi\|_2 : \quad \phi \in S\}.$$

From the characterization of Theorem 1, it is natural to measure this distance in terms of the family of functions

$$D_{j,k}(f)(\xi) := \hat{f}(2^j(\xi + 2k\pi))\hat{f}(\xi) - \hat{f}(2^j\xi)\hat{f}(\xi + 2k\pi).$$

Theorem 2. *Let $f(x)$ be a nonzero function in $L_2(\mathbb{R})$. Then*

$$d(f, S) \geq \frac{\sqrt{2} - 1}{12\pi\|f\|_2} \sup_{k \in \mathbb{N}} \left\{ \sum_{j=1}^{\infty} \|D_{j,k}(f)\|_1 \right\}. \tag{1.5}$$

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¿From the proof of Theorem 1 given in Section 2, we shall easily see the following characterization of refinable functions.

Theorem 3. *Let $f(x) \in L_2(\mathbb{R})$ and define $K_j(f), j = 0, 1, \cdots,$ by*

$$K_j(f) := \{\xi \in [-\pi, \pi) : \quad \hat{f}(2^j(\xi + 2l\pi)) \neq 0 \text{ for some } l \in \mathbb{Z}\}. \tag{1.6}$$

Then $f(x)$ lies in S if and only if $f(x) \in \overline{S}$ and the set $\cup_{j=1}^{\infty} K_j(f) \setminus K_0(f)$ has measure zero.

Corollary. *If $f(x) \in \overline{S} \setminus S$, then $\hat{f}(\xi)$ vanishes on a set of positive measure.*

The final sections of the paper deal with smaller points in the theory of refinable functions:

Section 4: Nonuniqueness of the mask.

Section 5. Refinable solutions to inhomogeneous refinement equations.

Section 6. Multiple refinable functions (leading to multiwavelets).

Section 7. Fully refinable functions (all translates $\phi(x - t)$ are refinable).

§2. Proof of the Main Results

We need the following characterization of closed shift-invariant subspaces in $L_2(\mathbb{R})$ given by de Boor, DeVore, and Ron [1]. Each such subspace is associated with a function ϕ in $L_2(\mathbb{R})$. The subspace is

$$S_2(\phi) = \{f(x) \in L_2(\mathbb{R}) : \quad \hat{f}(\xi) = \tau(\xi)\hat{\phi}(\xi) \text{ for a } 2\pi\text{-periodic function } \tau(\xi)\}. \tag{2.1}$$

Proof of Theorem 1.

Necessity of (1.4). Suppose that there is a sequence $\{\phi_n(x)\} \subset S$ such that $\|\phi_n - f\|_2 \to 0$ as $n \to \infty$. Then $\|\hat{\phi}_n - \hat{f}\|_2 \to 0$. Hence there is a subsequence $\{\hat{\phi}_{n_k}(\xi)\}$ such that

$$\lim_{k\to\infty} \hat{\phi}_{n_k}(\xi) = \hat{f}(\xi)$$

almost everywhere. By replacing $\{\phi_n(x)\}$ with this subsequence, we may assume that

$$\lim_{n\to\infty} \hat{\phi}_n(\xi) = \hat{f}(\xi), \qquad \forall \xi \in \mathbb{R} \setminus T, \tag{2.2}$$

where T is a set of measure zero (null set).

Since $\phi_n(x)$ is refinable, by (2.1) there is a 2π-periodic function $\tilde{a}_n(\xi)$ such that

$$\hat{\phi}_n(2\xi) = \tilde{a}_n(\xi)\hat{\phi}_n(\xi) \tag{2.3}$$

almost everywhere. By recursion this implies for all n and all $j = 1, 2, \cdots$ that

$$\hat{\phi}_n(2^j\xi) = \tilde{a}_n(2^{j-1}\xi)\cdots\tilde{a}_n(\xi)\hat{\phi}_n(\xi), \qquad \forall\xi \in \mathbb{R}\setminus T', \tag{2.4}$$

where T' is another null set. Then $T_j = (T' + 2\pi\mathbb{Z}) \cup (2^{-j}T + 2\pi\mathbb{Z}) \cup (T + 2\pi\mathbb{Z})$ is also a null set. Suppose $\xi \in \mathbb{R} \setminus T_j$.

Let $k \in \mathbb{N}$. If $\hat{f}(\xi + 2k\pi) = \hat{f}(\xi) = 0$, then (1.4) holds trivially.

If $\hat{f}(\xi + 2l\pi) \neq 0$ for some $l \in \{0, k\}$, then (2.2) and (2.4) imply that

$$\lim_{n\to\infty} \tilde{a}_n(2^{j-1}\xi)\cdots\tilde{a}_n(\xi) = \hat{f}(2^j(\xi + 2l\pi))/\hat{f}(\xi + 2l\pi).$$

By the 2π-periodicity, taking the limits in (2.2) and (2.4) again, we have

$$\hat{f}(2^j(\xi + 2p\pi)) = \frac{\hat{f}(2^j(\xi + 2l\pi))}{\hat{f}(\xi + 2l\pi)}\hat{f}(\xi + 2p\pi), \qquad \forall p \in \mathbb{Z}.$$

In particular, the choice $p \in \{0, k\} \setminus \{l\}$ implies (1.4).

Thus (1.4) is true for every $k \in \mathbb{N}$ and this ξ.

Since the set T_j has measure zero, (1.4) holds almost everywhere. This proves the necessity of (1.4).

Sufficiency. Suppose that (1.4) is true. It is still true if we replace ξ by $2^m(\xi + 2l\pi)$ and k by $2^m k$, for $m \in \mathbb{N}$ and $l \in \mathbb{Z}$. Now change $j + m$ back to j and $k + l$ back to k. The result is

$$\hat{f}(2^j(\xi + 2k\pi))\hat{f}(2^m(\xi + 2l\pi)) = \hat{f}(2^j(\xi + 2l\pi))\hat{f}(2^m(\xi + 2k\pi)) \tag{2.5}$$

for any $j, m \in \mathbb{N}, k, l \in \mathbb{Z}$ and every $\xi \in \mathbb{R} \setminus T$, where T is a null set.

Let us define a sequence $\{\phi_n(x)\}$ of refinable functions tending to $f(x)$.

Let $M_j(f)$ be the union of the sets $K_0(f), \cdots, K_{j-1}(f)$ defined in (1.6):

$$M_j(f) := \cup_{i=0}^{j-1} K_i(f) = \{\xi \in [-\pi, \pi) : \hat{f}(2^i(\xi + 2l\pi)) \neq 0 \text{ for some } 0 \leq i < j \text{ and } l \in \mathbb{Z}\}.$$

For $\xi \in K_0(f)$, define

$$\hat{\phi}_n(\xi + 2k\pi) = \hat{f}(\xi + 2k\pi), \qquad \forall k \in \mathbb{Z}. \tag{2.6}$$

For $j \in \mathbb{N}$ and $\xi \in K_j(f) \setminus M_j(f)$, define

$$\hat{\phi}_n(\xi + 2k\pi) = \hat{f}(2^j(\xi + 2k\pi))/n, \qquad \forall k \in \mathbb{Z}. \tag{2.7}$$

For $\xi \in [-\pi, \pi) \setminus M_\infty(f)$, define

$$\hat{\phi}_n(\xi + 2k\pi) = 0, \qquad \forall k \in \mathbb{Z}. \tag{2.8}$$

Thus, $\hat{\phi}_n(\xi)$ has been defined for all ξ. We first show that $\hat{\phi}_n(\xi) \to \hat{f}(\xi)$ in $L_2(\mathbb{R})$. By equations (2.6) and (2.8),

$$\hat{\phi}_n(\xi + 2k\pi) - \hat{f}(\xi + 2k\pi) = 0, \qquad \forall \xi \in K_0(f) \cup \Big([-\pi, \pi) \setminus M_\infty(f)\Big), \quad k \in \mathbb{Z}.$$

Hence

$$\begin{aligned}
\|\hat{\phi}_n - \hat{f}\|_2^2 &= \sum_{j=1}^{\infty} \int_{K_j(f)\setminus M_j(f)} \sum_{k\in\mathbb{Z}} |\hat{\phi}_n(\xi + 2k\pi) - \hat{f}(\xi + 2k\pi)|^2 d\xi \\
&= \sum_{j=1}^{\infty} \int_{K_j(f)\setminus M_j(f)} \sum_{k\in\mathbb{Z}} |\hat{f}(2^j(\xi + 2k\pi))/n|^2 d\xi \\
&\le \sum_{j=1}^{\infty} \int_{\mathbb{R}} |\hat{f}(2^j\xi)|^2 d\xi/n^2 \\
&= \|\hat{f}\|_2^2/n^2 \to 0.
\end{aligned}$$

Therefore, $\phi_n(x) \in L_2(\mathbb{R})$ and $\lim_{n\to\infty} \|\phi_n - f\|_2 = 0$.

Next we show that $\phi_n(x)$ is refinable, by constructing $\tilde{a}_n(\xi)$ on $[-\pi, \pi)$ such that

$$\hat{\phi}_n(2(\xi + 2k\pi)) = \tilde{a}_n(\xi)\hat{\phi}_n(\xi + 2k\pi), \qquad \forall k \in \mathbb{Z}, \quad \xi \in [-\pi, \pi) \setminus T. \tag{2.9}$$

Let $j \in \mathbb{N} \cup \{0\}$ and $\xi \in K_j(f) \setminus M_j(f)$, we choose $k_\xi \in \mathbb{Z}$ such that $\hat{f}(2^j(\xi + 2k_\xi\pi)) \neq 0$. Define

$$\tilde{a}_n(\xi) = \begin{cases} \hat{\phi}_n(2(\xi + 2k_\xi\pi))/\hat{f}(\xi + 2k_\xi\pi), & \text{if } j = 0, \\ n\hat{\phi}_n(2(\xi + 2k_\xi\pi))/\hat{f}(2^j(\xi + 2k_\xi\pi)), & \text{if } j \in \mathbb{N}. \end{cases}$$

For $\xi \in [-\pi, \pi) \setminus M_\infty(f)$, we can define $\tilde{a}_n(\xi)$ arbitrarily.

Let us now verify the refinement relation (2.9), first for $\xi \in M_\infty(f) \setminus T$.

Let $j \in \mathbb{N} \cup \{0\}$ and $\xi \in K_j(f) \setminus M_j(f) \setminus T$. For every $k \in \mathbb{Z}$,

$$\tilde{a}_n(\xi)\hat{\phi}_n(\xi + 2k\pi) = \frac{\hat{\phi}_n(2(\xi + 2k_\xi\pi))}{\hat{f}(2^j(\xi + 2k_\xi\pi))}\hat{f}(2^j(\xi + 2k\pi)). \tag{2.10}$$

To see that this equals $\hat{\phi}_n(2(\xi + 2k\pi))$, write 2ξ as $\eta + 2s\pi$ with $\eta \in [-\pi, \pi)$ and $s \in \mathbb{Z}$. Then if $\eta \notin M_\infty(f)$,

$$\hat{\phi}_n(2(\xi + 2l\pi)) = \hat{\phi}_n(\eta + 2s\pi + 4l\pi) = 0, \qquad \forall l \in \mathbb{Z}.$$

Hence (2.10) equals $\hat{\phi}_n(2(\xi + 2k\pi))$ in this case.

If $\eta \in K_0(f)$, then

$$\hat{\phi}_n(2(\xi + 2l\pi)) = \hat{\phi}_n(\eta + 2s\pi + 4l\pi) = \hat{f}(\eta + 2s\pi + 4l\pi) = \hat{f}(2(\xi + 2l\pi)).$$

Hence (2.10) equals $\hat{\phi}_n(2(\xi + 2k\pi))$ again by the condition (2.5).

If $\eta \in K_m(f) \setminus M_m(f)$ for some $m \in \mathbb{N}$, then

$$\hat{\phi}_n(2(\xi + 2l\pi)) = \hat{\phi}_n(\eta + 2s\pi + 4l\pi) = \hat{f}(2^m(\eta + 2s\pi + 4l\pi))/n = \hat{f}(2^{m+1}(\xi + 2l\pi))/n.$$

Hence (2.10) equals $\hat{\phi}_n(2(\xi + 2k\pi))$ in this final case by the condition (2.5).

Thus, the refinement relation (2.9) has been proved for $\xi \in M_\infty(f) \setminus T$.

Next we consider $\xi \in [-\pi, \pi) \setminus M_\infty(f) \setminus T$. Here we have

$$\hat{\phi}_n(\xi + 2l\pi) = 0, \qquad \forall l \in \mathbb{Z}.$$

Let us show that $\hat{\phi}_n(2(\xi + 2l\pi)) = 0$ for every $l \in \mathbb{Z}$. Write 2ξ as $\eta + 2s\pi$ again with $\eta \in [-\pi, \pi)$ and $s \in \mathbb{Z}$.

If $\eta \notin M_\infty(f)$, then

$$\hat{\phi}_n(2(\xi + 2l\pi)) = \hat{\phi}_n(\eta + 2s\pi + 4l\pi) = 0, \qquad \forall l \in \mathbb{Z}.$$

If $\eta \in K_0(f)$, then $\xi \notin K_1(f)$ implies that for every $l \in \mathbb{Z}$,

$$\hat{\phi}_n(2(\xi + 2l\pi)) = \hat{f}(\eta + 2s\pi + 4l\pi) = \hat{f}(2(\xi + 2l\pi)) = 0.$$

If $\eta \in K_j(f) \setminus M_j(f)$ for some $j \in \mathbb{N}$, then

$$\begin{aligned}\hat{\phi}_n(2(\xi + 2l\pi)) = \hat{\phi}_n(\eta + 2s\pi + 4l\pi) &= \hat{f}(2^j(\eta + 2s\pi + 4l\pi))/n \\ &= \hat{f}(2^{j+1}(\xi + 2l\pi))/n = 0, \qquad \forall l \in \mathbb{Z},\end{aligned}$$

since $\xi \notin K_{j+1}(f)$.

Thus, in all the three cases,

$$\hat{\phi}_n(2(\xi + 2l\pi)) = 0, \qquad \forall l \in \mathbb{Z}.$$

Therefore, the refinement relation (2.9) holds true on $[\pi, \pi) \setminus T$. Hence $\phi_n(x)$ is refinable; it lies in S. Then $\lim \|\phi_n - f\|_2 = 0$ tells us that $f(x)$ lies in the closure of S. □

Once we have proved Theorem 1, the proof of Theorem 2 follows quickly.

Proof of Theorem 2.

For $\phi(x) \in S$, Theorem 1 gives $D_{j,k}(\phi)(\xi) = 0$ almost everywhere. Then

$$\begin{aligned}\|D_{j,k}(f)\|_1 &= \|D_{j,k}(f) - D_{j,k}(\phi)\|_1 \\ &= \int_{\mathbb{R}} \big| \big[\hat{f}(2^j(\xi + 2k\pi)) - \hat{\phi}(2^j(\xi + 2k\pi))\big]\hat{f}(\xi) \\ &\quad + \hat{\phi}(2^j(\xi + 2k\pi))\big[\hat{f}(\xi) - \hat{\phi}(\xi)\big] \\ &\quad + \hat{\phi}(2^j\xi)\big[\hat{\phi}(\xi + 2k\pi) - \hat{f}(\xi + 2k\pi)\big] \\ &\quad + \hat{f}(\xi + 2k\pi)\big[\hat{\phi}(2^j\xi) - \hat{f}(2^j\xi)\big] \big| d\xi.\end{aligned}$$

Applying the Schwarz inequality, we get

$$\begin{aligned}\|D_{j,k}(f)\|_1 &\leq 2\Big(\int |\hat{f}(2^j\xi) - \hat{\phi}(2^j\xi)|^2 d\xi\Big)^{1/2} \|\hat{f}\|_2 + 2\Big(\int |\hat{\phi}(2^j\xi)|^2 d\xi\Big)^{1/2} \|\hat{f} - \hat{\phi}\|_2 \\ &= 2^{1-j/2}\|\hat{f} - \hat{\phi}\|_2(\|\hat{f}\|_2 + \|\hat{\phi}\|_2).\end{aligned}$$

For each k we sum over $j \in \mathbb{N}$:

$$\|f - \phi\|_2(\|f\|_2 + \|\phi\|_2) \geq \frac{\sqrt{2} - 1}{4\pi} \sum_{j=1}^{\infty} \|D_{j,k}(f)\|_1. \tag{2.11}$$

In computing the distance $d(f, S)$ we may restrict to $\phi \in S$ with $\|\phi\|_2 \leq 2\|f\|_2$, since otherwise $\|\phi - f\| \geq \|0 - f\|$. Then (2.11), for each k, yields the lower bound on $d(f, S)$ in Theorem 2:

$$d(f, S) \quad 3\|f\|_2 \geq \frac{\sqrt{2} - 1}{4\pi} \sum_{j=1}^{\infty} \|D_{j,k}(f)\|_1. \tag{2.12}$$

The proof of Theorem 2 is complete. □

Remark on Condition (1.4)

If the Fourier transform of a refinable function were never zero, division would be allowed and everything becomes easy:

$$\frac{\hat{f}(2\xi)}{\hat{f}(\xi)} \quad \text{is periodic by the refinement equation.} \tag{2.13}$$

$$\frac{\hat{f}(2^j\xi)}{\hat{f}(\xi)} = \frac{\hat{f}(2^j\xi)}{\hat{f}(2^{j-1}\xi)} \cdots \frac{\hat{f}(2\xi)}{\hat{f}(\xi)} \quad \text{is periodic by induction.} \tag{2.14}$$

Condition (1.4) is simply the periodicity of $\hat{f}(2^j\xi)/\hat{f}(\xi)$ after multiplication to clear out the (possibly zero!) denominators.

Since the periodicity (2.14) for all j follows from (2.13) for $j=1$, it is natural to ask whether this is also true in Condition (1.4). *Must we impose this condition for all $j \in \mathbb{N}$?*

The following example shows that we must.

Example 1. *Let $\hat{f}(\xi) = 1$ for $\xi \in (-\pi, -\pi/2) \cup (-\pi/2^{m+1}, -\pi/2^{m+2}) \cup (2\pi - \pi/2^{m+1}, 2\pi - \pi/2^{m+2})$ and zero elsewhere. Then (1.4) holds for $j = 1, \cdots, m$ and all k, but not for $j = m+1$ and $k = 1$.*

Proof. If $\xi > 0$, then

$$\hat{f}(\xi + 2k\pi) = \hat{f}(2^j(\xi + 2k\pi)) = 0 \quad \text{for all } j, k \in \mathbb{N}.$$

If $\xi < -\pi$, then

$$\hat{f}(\xi) = \hat{f}(2^j\xi) = 0 \quad \text{for all } j \in \mathbb{N}.$$

If $-\pi < \xi < 0$, then

$$\hat{f}(2^j(\xi + 2k\pi)) = 0 \quad \text{for all } j, k \in \mathbb{N}.$$

If $-\pi < \xi < -\pi/2^{m+1}$ or $-\pi/2^{m+2} < \xi < 0$, then

$$\hat{f}(\xi + 2k\pi) = 0 \quad \text{for all } k \in \mathbb{N}.$$

If $-\pi < \xi < 0$, then

$$\hat{f}(\xi + 2k\pi) = 0 \quad \text{for all } k \geq 2.$$

Thus we only need to check Condition (1.4) for $-\pi/2^{m+1} < \xi < -\pi/2^{m+2}$ and $k = 1$. In this case, $\hat{f}(\xi + 2\pi) = 1$. For $j = 1, \cdots, m$, $\hat{f}(2^j\xi) = 0$ which implies (1.4). However, $\hat{f}(2^{m+1}\xi) = 1$ which contradicts (1.4) for $k = 1$. $\square$

§3. Band-limited Functions

Let X_b be the set of band-limited functions with frequencies ξ restricted to the band $[-b, b]$:

$$X_b := \left\{ f(x) \in L_2(\mathbb{R}) : \quad \mathrm{supp}\hat{f} \subset [-b, b] \right\}.$$

We observed in the introduction that $X_b \subset \overline{S}$ for $b \le \frac{4}{3}\pi$. The converse is also true.

Theorem 4. *$X_b \subset \overline{S}$ if and only if $b \le \frac{4}{3}\pi$.*

Proof. If $b > \frac{4}{3}\pi$, let $f(x) \in L_2(\mathbb{R})$ be given by its Fourier transform as

$$\hat{f}(\xi) = \begin{cases} 1, & \text{if } |\xi| < B := \min\{b, 2\pi\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then for $j = 1, k = 1, -B < \xi < -\frac{4}{3}\pi$, we have

$$\hat{f}(2(\xi + 2\pi)) = \hat{f}(\xi) = 1 \quad \text{but} \quad \hat{f}(2\xi) = 0.$$

Hence (1.4) does not hold on the interval $(-B, -\frac{4}{3}\pi)$ for $j = 1, k = 1$. Thus, $f(x) \in X_b \setminus \overline{S}$.
$\square$

Now we show that some functions are not refinable but are the limits of refinable functions. The example in the introduction was

$$\hat{f}(\xi) = \begin{cases} 1, & \text{if } \xi \in (-\frac{4}{3}\pi, -\frac{2}{3}\pi) \cup (\frac{2}{3}\pi, \frac{4}{3}\pi), \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 5. *Suppose $f(x)$ is band-limited: $f(x) \in X_{2\pi}$. Then $f(x) \in S$ if and only if $\hat{f}(2\xi) = 0$ almost everywhere on the sets*

$$A = \{\xi \in [-\pi, \pi) : \hat{f}(\xi) = 0\} \quad \textit{and} \quad C = \{\xi \in [-\pi, \pi) : |\hat{f}(\xi + 2\pi)| + |\hat{f}(\xi - 2\pi)| > 0\}.$$

Proof. Note that for $f(x) \in X_{2\pi}$,

$$K_j(f) = \{\xi \in [-\pi, \pi) : \quad \hat{f}(2^j\xi) \neq 0\}, \qquad j \in \mathbb{N}.$$

Suppose $f(x) \in S$. Then (1.4) holds for $j = 1, k = \pm 1$. This tells that $\hat{f}(2\xi) = 0$ almost everywhere on the set C, since $|2(\xi \pm 2\pi)| > 2\pi$ and $\hat{f}(2(\xi \pm 2\pi)) = 0$.

The set $\{\xi \in [-\pi, \pi) : \hat{f}(\xi) = \hat{f}(\xi+2\pi) = \hat{f}(\xi-2\pi) = 0\}$ is a subset of $[-\pi, \pi) \setminus K_0(f)$. Hence Theorem 3 tells that this set is contained in $[-\pi, \pi) \setminus K_1(f)$ up to a null set, that is, $\hat{f}(2\xi) = 0$ almost everywhere on this set. This proves the necessity.

For the sufficiency, suppose $\hat{f}(2\xi) = 0$ on A and C. We first prove that $f(x) \in \overline{S}$.

Let $j, k \in \mathbb{N}$. The equation (1.4) is trivially true for $\xi \notin [-2\pi, \pi]$ or $k \geq 2$.

Let $k = 1$, $s \in \{0, 1\}$. For $\xi \in (-\pi, \pi) - 2s\pi$, we have $\hat{f}(2^j(\xi + 2(1-s)\pi)) = 0$, and

$$\hat{f}(2^j(\xi + 2(1-s)\pi))\hat{f}(\xi + 2s\pi) = 0.$$

If $\hat{f}(\xi + 2(1-s)\pi) \neq 0$, then $\xi + 2s\pi \in C$. By our condition, $\hat{f}(2(\xi + 2s\pi)) = 0$. Hence $\hat{f}(2^j(\xi + 2s\pi)) = \cdots = \hat{f}(2(\xi + 2s\pi)) = 0$ if $|2^j(\xi + 2s\pi)| < 2\pi$ by our condition; while $\hat{f}(2^j(\xi + 2s\pi)) = 0$ if $|2^j(\xi + 2s\pi)| > 2\pi$.

Therefore, almost everywhere

$$\hat{f}(2^j(\xi + 2s\pi))\hat{f}(\xi + 2(1-s)\pi) = 0.$$

Thus, for any $j, k \in \mathbb{N}$, (1.4) is true almost everywhere. By Theorem 1, $f(x) \in \overline{S}$.

Next, we show that $M_\infty(f) \setminus K_0(f)$ has measure zero.

By our condition, the set $T := \{\xi \in [-\pi, \pi] : \hat{f}(\xi) = 0, \hat{f}(2\xi) \neq 0\}$ is a null set.

For $j \in \mathbb{N}$ and $\xi \in K_j(f) \setminus M_j(f) \setminus \{-\pi\}$, we have $\hat{f}(2^j\xi) \neq 0$ and $\hat{f}(2^{j-1}\xi) = 0$, which implies $|2^j\xi| \leq 2\pi$ and $2^{j-1}\xi \in T$. Hence, $M_\infty(f) \setminus K_0(f)$ is a null set, since

$$M_\infty(f) \setminus K_0(f) = \cup_{j=1}^{\infty}\Big(K_j(f) \setminus M_j(f)\Big) \subset \cup_{j=1}^{\infty}\Big(2^{1-j}T \cup \{-\pi\}\Big).$$

Thus, by Theorem 3, $f(x)$ is refinable. $\square$

Combining Theorems 4 and 5, we know that every nonzero function in $X_{\frac{4}{3}\pi}$ whose Fourier transform vanishes on $[-\frac{2}{3}\pi, \frac{2}{3}\pi]$ lies in $\overline{S} \setminus S$.

The following result follows from Theorems 4 and 5, or directly from Theorem 3.

Theorem 6. *A function $f(x) \in X_\pi$ is refinable if and only if the set $\{\xi \in [-\pi/2, \pi/2] : \hat{f}(\xi) = 0, \hat{f}(2\xi) \neq 0\}$ has measure zero.*

§4. Refinable Functions and Masks: Nonuniqueness

We apply the characterization of refinable functions in Theorem 3 to show that the function may not determine the mask (and vice versa). Observe from (2.1) that a refinable function in $L_2(\mathbb{R})$ satisfies a **refinement equation** of the form

$$\hat{\phi}(2\xi) = \tilde{a}(\xi)\hat{\phi}(\xi). \tag{4.1}$$

First, we show that the 2π-periodic symbol of the mask $\tilde{a}(\xi)$ is sometimes not unique.

Theorem 7. *Let $\phi(x)$ be a nonzero refinable function in $L_2(\mathbb{R})$ and $K_0(\phi)$ be defined by (1.6). Then the refinement mask $\tilde{a}(\xi)$ satisfying (4.1) is unique (up to a null set) if and only if meas$(K_0(\phi)) = 2\pi$, i.e., for almost every $\xi \in [-\pi, \pi)$, there is some $k_\xi \in \mathbb{Z}$ such that $\hat{\phi}(\xi + 2k_\xi\pi) \neq 0$.*

Proof. The sufficiency is clear, since $\tilde{a}(\xi)$ is determined for $\xi \in K_0(\phi)$ by

$$\tilde{a}(\xi) = \hat{\phi}(2(\xi + 2k_\xi\pi))/\hat{\phi}(\xi + 2k_\xi\pi),$$

which defines $\tilde{a}(\xi)$ uniquely up to a null set.

For the necessity, suppose to the contrary that meas$(K_0(\phi)) < 2\pi$, then meas$([-\pi, \pi) \setminus M_\infty(\phi)) > 0$ by Theorem 3.

Let $\tilde{a}(\xi)$ be the symbol of a refinement mask satisfying (4.1). Choose $\tilde{b}(\xi)$ to be a 2π-periodic function satisfying

$$\tilde{b}(\xi) = \tilde{a}(\xi), \qquad \forall \xi \in K_0(\phi).$$

Then we can see that for almost every $\xi \in [-\pi, \pi)$,

$$\hat{\phi}(2(\xi + 2k\pi)) = \tilde{b}(\xi)\hat{\phi}(\xi + 2k\pi), \qquad \forall k \in \mathbb{Z}.$$

In fact, for $\xi \in [-\pi, \pi) \setminus M_\infty(\phi)$,

$$\hat{\phi}(2(\xi + 2k\pi)) = \hat{\phi}(\xi + 2k\pi) = 0, \qquad \forall k \in \mathbb{Z},$$

while $\tilde{b}(\xi) = \tilde{a}(\xi)$ for $\xi \in K_0(\phi)$. Hence the refinement relation is reduced to (4.1). Note that meas$(([-\pi, \pi) \setminus M_\infty(\phi)) \cup K_0(\phi)) = 2\pi$ by Theorem 3. The function $\tilde{b}(\xi)$ is also the symbol of a refinement mask for $\phi(x)$. Therefore the mask is not unique. $\square$

Second, we show that the refinable function is never unique, given a refinement mask.

The classical approach begins with a sequence $\{a(k)\}$ satisfying

$$\sum_{k\in\mathbb{Z}} |a(k)||k|^{\delta} < \infty \qquad \text{for some } \delta > 0.$$

Then the refinement equation (1.1) has at most one integrable solution up to a constant multiplication, see Daubechies and Lagarias [4].

However, when we consider solutions in $L_2(\mathbb{R})$, this uniqueness never holds. For the more general refinement equation (4.1), we have the following result whose proof is easy and omitted here.

Theorem 8. *Let $\phi(x) \in L_2(\mathbb{R})$ satisfy the refinement equation (4.1) for some 2π-periodic function $\tilde{a}(\xi)$. If $\tau(\xi)$ is an arbitrary measurable bounded function on $[-2\pi, 2\pi]$, then the function $\psi(x)$ defined by its Fourier transform as*

$$\hat{\psi}(\xi) = \tau(2^j\xi)\hat{\phi}(\xi), \qquad \xi \in [2^{-j}\pi, 2^{1-j}\pi) \cup (-2^{1-j}\pi, -2^{-j}\pi], \quad j \in \mathbb{Z}$$

satisfies the refinement equation (4.1).

However, if we require that $\hat{\phi}(\xi)$ is continuous at the origin, which is the case when $\phi(x) \in L_1(\mathbb{R})$ and $\hat{\phi}(0) \neq 0$, then the solution is unique up to a constant multiplication.

Theorem 9. *If $\phi(x) \in L_2(\mathbb{R})$ satisfies (4.1) and $\lim_{\xi\to 0}\hat{\phi}(\xi) = \hat{\phi}(0) \neq 0$, then any solution $\psi(x) \in L_2(\mathbb{R})$ of (4.1) with $\lim_{\xi\to 0}\hat{\psi}(\xi) = \hat{\psi}(0)$ can be written as*

$$\psi(x) = \frac{\hat{\psi}(0)}{\hat{\phi}(0)}\phi(x).$$

Proof. By our assumption, for almost every ξ,

$$\lim_{n\to\infty} \tilde{a}(\xi/2)\cdots\tilde{a}(\xi/2^n) = \frac{\hat{\phi}(\xi)}{\hat{\phi}(0)}.$$

Therefore, for almost every ξ,

$$\hat{\psi}(\xi) = \lim_{n\to\infty} \tilde{a}(\xi/2)\cdots\tilde{a}(\xi/2^n) \lim_{n\to\infty} \hat{\psi}(\xi/2^n) = \frac{\hat{\psi}(0)}{\hat{\phi}(0)}\hat{\phi}(\xi).$$

The proof of Theorem 9 is complete. $\square$

As a consequence, if there is a solution $\phi(x) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ to (4.1) with $\hat{\phi}(0) \neq 0$, then all the other solutions in $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ are $c\phi(x)$. This extends the result of Daubechies and Lagarias [4].

§5. Inhomogeneous Refinement Equations

In this section we study inhomogeneous refinement equations and characterize those solutions which are (homogeneously) refinable.

The inhomogeneous refinement equation was introduced in [15] as

$$\phi(x) = \sum_{k\in\mathbb{Z}} a(k)\phi(2x-k) + F(x), \tag{5.1}$$

where $\{a(k)\}$ is a finitely supported sequence and F is a compactly supported distribution. Here we are interested in compactly supported L_2 solutions of (5.1), so we assume that $F(x)$ is a nonzero function in $L_2(\mathbb{R})$. Denote

$$\tilde{a}(\xi) = \frac{1}{2}\sum_{k\in\mathbb{Z}} a(k)e^{-ik\xi}, \qquad \xi \in \mathbb{R}.$$

Let $\phi(x)$ be a compactly supported square-integrable solution of (5.1), which is guaranteed by the convergence of the cascade algorithm in $L_2(\mathbb{R})$, see [15]. Our purpose here is to find out when $\phi(x)$ can be refinable (and thus solve a *homogeneous* equation).

Recall the description (2.1) of shift-invariant subspace $S_2(\phi)$. Note that if $g(x) \in S_2(\phi)$ and $\hat{g}(\xi) \neq 0$ (hence $\tau(\xi) \neq 0$ in (2.1)) almost everywhere, then $\hat{\phi}(\xi) = \hat{g}(\xi)/\tau(\xi)$. This implies that $\phi(x) \in S_2(g)$ and $S_2(\phi) = S_2(g)$. In particular, this is the case if $g(x)$ is compactly supported.

Suppose that a is supported on $[0, N]$, F is supported on $[0, N/2]$, and $\phi(x)$ is a compactly supported L_2 solution of (5.1). Then by [15], $\phi(x)$ is supported in $[0, N]$.

According to the analysis of Jia [8], for the function $F(x/2)$ there exists a unique function $\psi(x) \in L_2(\mathbb{R})$ (up to a constant multiplication) compactly supported in $[0, N]$ but not in $[1, N]$ such that its integer translates are linearly independent and for some sequence $\{c(k)\}$,

$$F(x/2) = \sum_{k=0}^{N-1} c(k)\psi(x-k). \tag{5.2}$$

By what we just mentioned, $S_2(F(x/2)) = S_2(\psi)$.

If $\phi(x)$ is refinable, then (5.1) implies

$$F(x/2) = \phi(x/2) - \sum_{k=0}^{N} a(k)\phi(x-k) \in S_2(\phi).$$

Once again,

$$S_2(F(x/2)) = S_2(\phi) \quad \text{and thus} \quad S_2(\phi) = S_2(\psi).$$

In particular, from the linear independence of ψ and the supports, there are sequences $\{b(k)\}$ and $\{d(k)\}$ such that

$$\phi(x) = \sum_{k=0}^{N-1} b(k)\psi(x-k),$$

$$\psi(x) = \sum_{k=0}^{N} d(k)\psi(2x-k).$$

Taking Fourier transforms and using (5.1), (5.2), we have

$$2\tilde{b}(2\xi)\tilde{d}(\xi)\hat{\psi}(\xi) = 2\tilde{a}(\xi)\tilde{b}(\xi)\hat{\psi}(\xi) + \tilde{c}(\xi)\hat{\psi}(\xi),$$

which implies

$$\tilde{b}(2\xi)\tilde{d}(\xi) = \tilde{a}(\xi)\tilde{b}(\xi) + \tilde{c}(\xi)/2, \qquad \forall \xi \in \mathbb{R}. \tag{5.3}$$

Moreover, since $\psi(x) \in L_2(\mathbb{R})$ is compactly supported and refinable, we know (as in [7, Theorem 2.4]) that

$$\hat{\psi}(2k\pi) = 0, \qquad \forall k \in \mathbb{Z} \setminus \{0\}.$$

But the integer translates of ψ are linearly independent, hence

$$\hat{\psi}(0) \neq 0, \qquad \tilde{d}(0) = 1, \qquad \text{and} \qquad \tilde{d}(\pi) = 0.$$

Conversely, we have

Theorem 10. *Assume that the sequence $\{a(k)\}$ is supported in $[0, N]$ and $F(x) \in L_2(\mathbb{R})$ is supported in $[0, N/2]$. Let $\psi(x) \in L_2(\mathbb{R})$ be a function compactly supported in $[0, N]$ but not in $[1, N]$ such that its integer translates are linearly independent, and (5.2) holds. Then (5.1) has a refinable solution $\phi(x) \in L_2(\mathbb{R})$ if and only if $\psi(x)$ is refinable, $\hat{\psi}(0) \neq 0$ and the equation (5.3) is solvable for some sequence $\{b(k)\}$ supported in $[0, N-1]$, where $\tilde{d}(\xi)$ is the symbol of the refinement mask of the function $\psi(x), \tilde{d}(0) = 1$ and $\tilde{d}(\pi) = 0$.*

Observe that (5.3) is a system of linear equations whose solvability can be easily checked.

§6. Applications to Multiple Refinable Functions

In this section we apply Theorem 10 to a study of some examples of **multiple refinable functions**. For the general theory and more examples of multiple refinable functions, we refer the reader to [5, 6, 9, 10, 11, 14, 17].

The first example was introduced by Geronimo, Hardin and Massopust [5]. Consider the matrix refinement equation

$$\Phi(x) = \sum_{k\in\mathbb{Z}} a_k \Phi(2x-k). \tag{6.1}$$

Here $\Phi(x) = (\phi_1(x), \phi_2(x))^T$ and $\{a_k\}$ is supported in $[0,3]$ with

$$a_0 = \begin{bmatrix} h_1 & 1 \\ h_2 & h_3 \end{bmatrix}, \qquad a_1 = \begin{bmatrix} h_1 & 0 \\ h_4 & 1 \end{bmatrix},$$

$$a_2 = \begin{bmatrix} 0 & 0 \\ h_4 & h_3 \end{bmatrix}, \qquad a_3 = \begin{bmatrix} 0 & 0 \\ h_2 & 0 \end{bmatrix}.$$

The matrix entries involve a parameter s:

$$h_1 = -\frac{s^2-4s-3}{2(s+2)}, \qquad h_2 = -\frac{3(s^2-1)(s^2-3s-1)}{4(s+2)^2},$$

$$h_3 = \frac{3s^2+s-1}{2(s+2)}, \qquad h_4 = -\frac{3(s^2-1)(s^2-s+3)}{4(s+2)^2}.$$

When $|s| < 1$, the matrix refinement equation (6.1) has a continuous solution Φ with $\hat{\phi}_1(0) = 1$ and $\hat{\phi}_2(0) = (s-1)^2/(s+2)$. Moreover, $\mathrm{supp}\phi_1 = [0,1]$ and $\mathrm{supp}\phi_2 = [0,2]$.

Applying Theorem 10, we conclude that neither $\phi_1(x)$ nor $\phi_2(x)$ is refinable.

Example 2. *Let $|s| < 1$ and $\Phi(x) = (\phi_1(x), \phi_2(x))^T$ be the continuous solution of (6.1) with $\hat{\phi}_1(0) = 1$ and $\hat{\phi}_2(0) = (s-1)^2/(s+2)$. Then neither $\phi_1(x)$ nor $\phi_2(x)$ is refinable.*

Proof. We first show that $\phi_1(x)$ is not refinable. The first component of (6.1) is an inhomogeneous refinement equation:

$$\phi_1(x) = h_1\phi_1(2x) + h_1\phi_1(2x-1) + \phi_2(2x). \tag{6.2}$$

It is proved in [11] that the integer translates of ϕ_1 are linearly independent. Hence we can take $N = 2, F(x) = \phi_2(2x)$ and $\psi(x) = \phi_2(x)$.

If $\phi_1(x)$ were refinable, then Theorem 10 would provide a nonzero $\tilde{b}(\xi) = (b(0) + b(1)e^{-i\xi})/2$ such that (5.3) is true for some $\tilde{d}(\xi) = (d(0) + d(1)e^{-i\xi} + d(2)e^{-i2\xi})/2$ with $\tilde{d}(0) = 1$ and $\tilde{d}(\pi) = 0$. Analyzing the degrees of $\tilde{b}$ and $\tilde{d}$, we see that $d(2) = 0$. (If $b(1) = 0$ this implies $d(2) = 0$ again.) This in turn implies that $d(0) = d(1) = 1$. Then the equation (5.3) becomes

$$(\tilde{b}(2\xi) - h_1\tilde{b}(\xi))(1 + e^{-i\xi})/2 = 1/2, \qquad \forall \xi \in \mathbb{R},$$

which can not hold. Therefore, $\phi_1(x)$ is not refinable.

The proof for $\phi_2(x)$ is easier. The second component of (6.1) is an inhomogeneous refinement equation

$$\phi_2(x) = h_3\phi_2(2x) + \phi_2(2x-1) + h_3\phi_2(2x-2) + F(x),$$

where

$$F(x/2) = h_2\phi_1(x) + h_4\phi_1(x-1) + h_4\phi_1(x-2) + h_2\phi_1(x-3).$$

Take $N = 4$ and $\psi(x) = \phi_1(x)$, since the integer translates of ϕ_1 are linearly independent [11]. If $\phi_2(x)$ is refinable, then Theorem 10 shows that $\phi_1(x) = \psi(x)$ is also refinable, which is a contradiction.

Thus, $\phi_2(x)$ is not refinable, either. $\square$

Our second example is taken from [9, 10, 11]. Let $\{a_k\}$ be supported in $[0, 2]$ with

$$a_0 = \begin{bmatrix} \frac{1}{2} & \frac{s}{2} \\ t & \lambda \end{bmatrix}, \qquad a_1 = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}, \quad \text{and} \quad a_2 = \begin{bmatrix} \frac{1}{2} & -\frac{s}{2} \\ -t & \lambda \end{bmatrix}. \tag{6.3}$$

Here s, t, λ, μ are real parameters. We assume that $|2\lambda + \mu| < 2$. Then there exists a unique distributional solution $\Phi(x) = (\phi_1(x), \phi_2(x))^T$ of (6.1) with $\hat{\Phi}(0) = (1, 0)^T$ supported in $[0, 2]$. The distribution $\phi_1(x)$ is symmetric about 1, and $\phi_2(x)$ is anti-symmetric about 1. It was proved in [9, Example 4.3] that the shifts of ϕ_1 and ϕ_2 reproduce all quadratic polynomials if and only if

$$t \neq 0, \quad \mu = 1/2, \quad \text{and} \quad \lambda = 1/4 + 2st. \tag{6.4}$$

In this case, the condition $|2\lambda + \mu| < 2$ reduces to $-3/4 < st < 1/4$, and it is verified in [10, 11] that the solution is continuous.

Example 3. *Let $\{a_k\}$ be the mask given in (6.3) and (6.4) with $-3/4 < st < 1/4$. Let $\Phi(x) = (\phi_1(x), \phi_2(x))^T$ be the continuous solution of (6.1) with $\hat{\Phi}(0) = (1, 0)^T$. Then $\phi_1(x)$ is refinable if and only if $s = 0$, while $\phi_2(x)$ is never refinable.*

Proof. First, we consider the case $s \neq 0$. In this case, it is proved in [11] that the integer translates of ϕ_1 and ϕ_2 are linearly independent.

For $\phi_1(x)$, the first component of (6.1) is an inhomogeneous refinement equation

$$\phi_1(x) = \phi_1(2x)/2 + \phi_1(2x-1) + \phi_1(2x-2)/2 + F(x),$$

where $F(x/2) = s\phi_2(x)/2 - s\phi_2(x-2)/2$.

Let $N = 4$ and $\psi(x) = \phi_2(x)$. If $\phi_1(x)$ is refinable, then Theorem 10 shows that $\hat{\phi}_2(0) = \hat{\psi}(0) \neq 0$, which is a contradiction.

The function $\phi_2(x)$ is trivially not refinable, since otherwise $\hat{\phi}_2(0) \neq 0$ by [7, Theorem 2.4].

Second, we investigate the case $s = 0$. Then $\phi_1(x)$ is refinable, since the first component of (6.1) reduces to a homogeneous equation for $\phi_1(x)$. In fact, $\phi_1(x)$ is the hat function on $[0, 2]$.

To consider $\phi_2(x)$, the second component of (6.1) is

$$\phi_2(x) = \phi_2(2x)/4 + \phi_2(2x-1)/2 + \phi_2(2x-2)/4 + F(x),$$

where $F(x/2) = t\phi_1(x) - t\phi_1(x-2)$. By Theorem 1 in [15], the solution to this equation is unique.

Let $N = 4$ and $\psi(x) = \phi_1(x)$. Then $\tilde{d}(\xi) = (1 + e^{-i\xi})^2/4$ and $\tilde{c}(\xi) = t(1 - e^{-i2\xi})/2$. Theorem 10 shows that $\phi_2(x)$ is refinable if and only if there exists a trigonometric polynomial of degree 3 such that (5.3) holds. If this is the case, analyzing the degrees of the trigonometric polynomials, we can see that $\tilde{b}(\xi) \equiv b(0)/2$ and the equation (5.3) implies $b(0) = t = 0$, which is a contradiction.

Therefore, $\phi_2(x)$ is never refinable. $\square$

The explicit formula for the solution $\Phi(x)$ in the special case $s = 3/2$, $t = -1/8$, $\lambda = -1/8$, and $\mu = 1/2$ was given by Heil, Strang, and Strela [6]. In this case, $\Phi(x)$ is supported on $[0, 2]$:

$$\phi_1(x) = \begin{cases} x^2(-2x+3) & \text{for } 0 \le x \le 1, \\ (2-x)^2(2x-1) & \text{for } 1 < x \le 2, \end{cases}$$

and

$$\phi_2(x) = \begin{cases} x^2(x-1) & \text{for } 0 \le x \le 1, \\ (2-x)^2(x-1) & \text{for } 1 < x \le 2. \end{cases}$$

§7. Fully Refinable Functions

A function $\phi(x) \in L_2(\mathbb{R})$ is **fully refinable** if for every $t \in \mathbb{R}$, the shifted function $\phi_t(x) := \phi(x-t)$ is refinable. It is shown in [16] that Meyer's well-known scaling function [12] is fully refinable.

Let $\phi(x)$ be a refinable function in $L_2(\mathbb{R})$ and $t \in \mathbb{R}$. Then $K_j(\phi_t) = K_j(\phi)$ for $j \in \mathbb{N} \cup \{0\}$. Theorem 3 tells us that $\phi_t(x)$ is refinable if and only if $\phi_t(x) \in \overline{S}$. By Theorem 1, this is equivalent to

$$\hat{\phi}(2^j(\xi + 2k\pi))\hat{\phi}(\xi)(e^{-it(2^j-1)2k\pi} - 1) = 0$$

for any $j, k \in \mathbb{N}$, almost everywhere in ξ.

Thus, a translate $\phi_t(x)$ of a compactly supported refinable function $\phi(x) \in L_2(\mathbb{R})$ is not refinable unless t is an integer.

Moreover, if $\phi(x)$ is fully refinable, then up to a null set $\hat{\phi}(\xi) \neq 0$ implies $\hat{\phi}(\xi/2) \neq 0$ and hence

$$\hat{\phi}(\xi + 4k\pi) = \hat{\phi}(2(\xi/2 + 2k\pi)) = 0, \quad \forall k \in \mathbb{Z} \setminus \{0\}.$$

This shows that the measure of the support of $\hat{\phi}$ is not greater than 4π.

However, Theorem 4 tells that every refinable function in $X_{\frac{4}{3}\pi}$ is fully refinable.

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