THE FIRST MOMENT OF WAVELET RANDOM VARIABLES

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SUMMARY. The squares $\phi^2(x)$ and $\psi^2(x)$ of orthonormal scaling functions and wavelets are interesting probability densities. Their moments can be computed in terms of the filter coefficients h_n and g_n in the dilation equation and wavelet equation. One particular case gives a remarkable result that is independent of the values of those coefficients. The first moment $\int x\psi^2(x)dx$ equals the midpoint of the support of $\psi(x)$.

KEY WORDS AND PHRASES: Halfband Polynomial, Scaling Equations, Moments, Wavelets Squared.

1. Introduction

In this note we consider random variables whose densities are squares of wavelet basis functions. Such densities can be viewed as a special case of random densities described in Vidakovic (1996)[6].

Let ϕ and ψ be the scaling function and wavelet (father and mother wavelets) generated by an orthonormal multiresolution analysis. The functions $\phi^2(x)$ and $\psi^2(x)$ are probability densities, nonnegative with integral one. The scaling equations that recursively connect $\phi(x)$ and $\psi(x)$ with $\phi(2x)$ give an algorithm for finding moments. Shann and Yan(1994)[4] derived a recursive relation described below. Closely related results can be found in Dahmen and Micchelli (1993)[1]. For some probabilistic applications and extensions of Shann-Yan's recursive relation also see Vidakovic (1997)[7]. Here, we find an *exact expression* for the first moment $\int x\psi^2(x)dx$.

2. Moments of Wavelet Densities

We first give the result of Shann and Yan as well as several important definitions. The main result of the note states that the first moment (the mean) of $\psi^2(x)$ is the center of its support. The proof uses only elementary properties of wavelet bases. However, the properties of the associated halfband polynomial are used in a novel way.

The scaling equations are

$$\phi(x) = \sum_n h_n \sqrt{2}\phi(2x-n) \tag{1}$$

$$\psi(x) = \sum_n g_n \sqrt{2}\phi(2x-n).$$

The "generalized moments" of $\phi(x)$ are defined by

$$\mu_{k,t} = \int_{\mathbb{R}} x^k \phi(x)\phi(x-t)dx. \tag{2}$$

Table 1: Expectations and variances for Daubechies' family $\phi^2(x)$

	mean $\mu_{1,0}$	variance
DAUB1	0.5	0.083333
DAUB2	0.770948	0.097718
DAUB3	1.022422	0.132056
DAUB4	1.266408	0.172921
DAUB5	1.506244	0.219200
DAUB6	1.743334	0.270688
DAUB7	1.978412	0.327253
DAUB8	2.211921	0.388751
DAUB9	2.444157	0.455041
DAUB10	2.675332	0.525993

The low-pass filter coefficients $(h_0, \ldots, h_{2N-1})$ are associated with $\phi(x)$ via (1). The length of h is always even for an orthogonal filter bank. Let $T = 2N - 2$.

Theorem 1. (Shann and Yan, 1994) [4] *The vector $\mu_k = (\mu_{k,t})$, $|t| \leq T$, is a solution of the system*

$$(I - \frac{1}{2^k}A)\mu_k = b_k \ , \tag{3}$$

where

$$A_{ij} = \sum_n h_n h_{n+i-2j} \ , \quad -T \leq i, j \leq T \tag{4}$$

is the transition matrix (or Lawton matrix). The vector b_k has components

$$b_{k,t} = \frac{1}{2^k} \sum_n \sum_l h_n h_l \sum_{j=1}^{k} \binom{k}{j} n^j \mu_{k-j,l-n+2t} \ , \ -T \leq t \leq T. \tag{5}$$

From the definition of $\mu_{k,t}$ given by (2) and orthogonality of the $\phi(t-k)$, we have $\mu_{1,t} = \mu_{1,-t}$.

The recursion starts with $\mu_{0,t} = \delta(t)$. The values $\mu_{k,0}$ represent the moments of a random variable with the density $\phi^2(x)$. As an illustration we give means and variances of random variables having the Daubechies ϕ^2 distribution. The numbers in Table 1 are obtained by solving equation (3) recursively. Once these $\mu_{k,t}$ are known, it is straightforward to show that the corresponding generalized moments $\xi_{k,t}$ for the wavelet $\psi(x)$ come from (1) :

$$\begin{aligned} \xi_{k,t} &= \int_{\mathbb{R}} x^k \psi(x)\psi(x-t)dx \\ &= \frac{1}{2^k} \sum_n \sum_l g_n g_l \sum_{r=0}^{k} n^{k-r} \binom{k}{r} \mu_{r,2t+l-n} \ . \end{aligned} \tag{6}$$

Though the general relation (6) provides an effective way to calculate any moment $\xi_{k,0}$, it requires pre-calculation of many generalized moments $\mu_{k,t}$. A surprisingly simple result holds for the mean of any compactly supported wavelet associated with an orthogonal multiresolution analysis.

Theorem 2. The mean $\xi_{1,0} = \int x\psi^2(x)dx$ is at the center of support of $\psi(x)$.

Proof. We will place the support of $\phi(x)$ and $\psi(x)$ on $[0, 2N-1]$. This comes with the construction $g_k = (-1)^k h_{2N-1-k}$. Shifting the g_k to $(-1)^k h_{1-k}$ moves the support of ψ to $[1-N, N]$. Then the mean of ψ^2 moves to the new center point $\xi_{1,0} = \frac{1}{2}$.

Let $H_0(z) = \sum_{i=0}^{2N-1} h_i z^i$, where h_i are the low-pass filter coefficients from (1). The polynomial $P(z) = H(z)H(z^{-1})$ is called "halfband" because of the requirements for orthogonality(see Strang and Nguyen, 1996[5]):

The coefficients

$$p_n = \sum_i h_i h_{i+n}$$

satisfy

$$p_{2k} = \delta(k) \quad \text{and} \quad p_n = p_{-n} \; . \tag{7}$$

We first prove that $\mu_{1,0} = \frac{1}{2}\sum_i p_i \mu_{1,i} + \frac{1}{2}\sum_i i h_i^2$. Indeed by using (1) and changing integration and summation variables we obtain:

$$\begin{aligned}
\int x\phi^2(x)dx &= \int x\, 2\sum_k \sum_s h_k h_s \phi(2x-k)\phi(2x-s)dx \\
&= \frac{1}{2}\sum_k \sum_s h_k h_s \int x\phi(x-k)\phi(x-s)dx \\
&= \frac{1}{2}\sum_k \sum_i h_k h_{k-i} \int (x+k)\phi(x)\phi(x+i)dx \qquad (s = k-i) \\
&= \frac{1}{2}\sum_i p_i \mu_{1,i} + \frac{1}{2}\sum_i i h_i^2 \; . \qquad \text{(by orthogonality)}
\end{aligned} \tag{8}$$

Let $g_k = (-1)^k h_{2N-1-k}$ be the coefficients of the high-pass filter corresponding to the low-pass filter $\{h_0, \ldots, h_{2N-1}\}$. Then

$$p_i = \sum_k h_k h_{k+i} = (-1)^i \sum_k g_k g_{k+i} \; . \tag{9}$$

The relation of $\sum_i i g_i^2$ to $\sum_i i h_i^2$ is straightforward:

$$\sum_i i g_i^2 = \sum_i i h_{2N-1-i}^2 = (2N-1)\sum_i h_{2N-1-i}^2 - \sum_i (2N-1-i)h_{2N-1-i}^2 = 2N-1-\sum_i i h_i^2 \; . \tag{10}$$

By imitating the steps in (8) we obtain

$$\xi_{1,0} = \frac{1}{2}\sum_i (-1)^i p_i \mu_{1,i} + \frac{1}{2}\sum_i i g_i^2 \; . \tag{11}$$

From (8) we express $\frac{1}{2}\sum_i i h_i^2$ as $\mu_{1,0} - \frac{1}{2}\sum_i p_i \mu_{1,i}$. Then (7), (9), and (10) imply that the first moment falls halfway along the support:

$$\xi_{1,0} = \frac{1}{2}\sum_i (-1)^i p_i \mu_{1,i} + \frac{2N-1}{2} - \frac{1}{2}\sum_i i h_i^2 \qquad \text{by(10)}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_i (-1)^i p_i \mu_{1,i} + \frac{2N-1}{2} - (\mu_{1,0} - \frac{1}{2}\sum_i p_i \mu_{1,i}) && \text{by}(8) \\
&= \sum_{i \text{ even}} p_i \mu_{1,i} + \frac{2N-1}{2} - \mu_{1,0} \\
&= \mu_{1,0} + \frac{2N-1}{2} - \mu_{1,0} && \text{because } P_{2k} = \delta(k) \\
&= \frac{2N-1}{2}. \quad \square
\end{aligned}
$$

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