Supper Convergence of Finite Element Method Based on Approximate Inertial Manifold *

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Abstract: In this paper we study finite element approximation to Navier-Stokes equations by means of approximate inertial manifold (AIM). Main idea is to analyse the error of lower components of genue solution to finite element approximation solution, then to construct an AIM to improve the error between higher frequence components of genue solution and projection of finite element solution on AIM. Such that new approximate solution possesses more than double convergent rate comparing with finite element solution.
This proceedure can be made on several level of meshes.

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1 Introduction

In order to improve the rate of convergence for Galerkin finite element solution, a lot of authors derived special technique, for example, Lin Qun[?], J.Novo and E. Titi[?] and W. Layton[?], J. Xu[?] use entrapotation, post-Galerkin method and two level mesh respectively.

Assume that (u,p) is a solution of Navier-Stokes equation, Y_h is a suitable finite element subspace, $Y_h \subset Y$, Y is a Sobolev space. Then

$$(u,p) = (Q_h(u,p), R_h(u,p)) + (\hat{u},\hat{p}) \tag{1.1}$$

where

$$(Q_h(u,p), R_h(u,p)) \in Y_h, (\hat{u},\hat{p}) \in \hat{Y}_h, Y = Y_h \oplus \hat{Y}_h \tag{1.2}$$

$$(Q_h, R_h) \text{ is a projection from } Y \text{ onto } Y_h \tag{1.3}$$

$(Q_h(u,p), R_h(u,p))$ is called lower frequence components and $(\hat{u},\hat{p})$ is called higher frequence components of genue solution (u,p). In addition, we suppose that (u_h,p_h)

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is a Galerkin finite element solution of Navier-Stokes equations. Analysis of error shows that $|||(Q_h(u,p) - u_h, R_h(u,p) - p_h)|||$ possesses higher accuracy comparing with $|||(u_h - u, p_h - p)|||$. However,

$$|||(u_h - u, p_h - p)||| \leq |||(u_h - Q_h(u,p), R_h(u,p) - p_h)||| + |||(\hat{u}, \hat{p})|||$$

$|||(\hat{u}, \hat{p})|||$ is error of projection, is a error of approximation theory (EAT).

Our main idea is to construct a mapping $\Phi = (\phi, \xi)$: $Y_h \to \hat{Y_h}$ such that

$$|||(\hat{u} - \phi, \hat{p} - \xi)||| \text{ and } |||(u_h - Q_h(u,p), R_h(u,p) - p_h)|||$$

has a some accuracy. So, two problems must be resolved: (1) we have to construct a suitable projection (Q_h, R_h): $Y \to Y_h$; (2) we have to construct a Lipschitz mapping $\Phi : Y_h \to \hat{Y}_h$.

Then content of this paper is managed as following. In section 2, we consider Navier-Stokes equation and its Galerkin finite element approximation; in section 3, the suitable projection (Q_h, R_h): $Y \to Y_h$ is constructed and the norm of higher frequence components

$$|||(\hat{u}, \hat{p})||| = |||(u - Q_h(u,p), p - R_h(u,p))|||$$

will be derived; in section 4, we give an estimation of error between Galerkin finite element solution (u_h, p_h) and $(Q_h(u,p), R_h(u,p))$; in section 5, we construct the mapping Φ and make estimate of $|||(\hat{u} - \phi, \hat{p} - \xi)|||$.

2 Preliminary

Letr us consider stationary Navier-Stokes equation

$$\begin{cases} -\lambda\Delta u + (u\nabla)u + \nabla p = f & \text{in } \Omega \\ \mathrm{div} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega = \Gamma \end{cases} \tag{2.1}$$

where Ω is a bounded domain with Lipschitz boundary Γ in R^d, $d = 2$ or 3. $u : \Omega \to R^d$ is the flow field, $p : \Omega \to R$ is the pressure, f represents the exterior forces which drive the flow, $\lambda = 1/Re$, Re is the Reynolds number of the flow, ∇ is gradient operator, Δ is Laplace operator in Cartain coordinates system.

The Velocity–Pressure variational formulation for (2.1) reads

$$\begin{cases} \text{find } (u,p) \in X \times M \text{ such that} \\ a(u,v) + b(u;u,v) - (p, \mathrm{div} v) + (q, \mathrm{div} u) = (f, v), \quad \forall (v,q) \in X \times M \end{cases} \tag{2.2}$$

where

$X = H_0^1(\Omega)^d, \quad M = L_0^2(\Omega) = \{q | q \in L^2(\Omega), \int_\Omega q dx = 0\}.$
Later on, we set $Y = X \times M$.

The inner product and norm of X and M are respectively denoted by

$$((u,v)) = (\nabla u, \nabla v), \quad \|u\|^2 = ((u,u)),$$
$$(q, \phi) = \int_\Omega q\phi dx \quad \forall q, \phi \in L_0^2(\Omega, \quad |q|^2 = (q,q).$$

The bilinear form $a(\cdot,\cdot)$ and trilinear form $b(\cdot;\cdot,\cdot)$ are given by

$$a(u,v) = \lambda((u,v)) \quad \forall u,v \in X \tag{2.3}$$

$$b(u;w,v) = ((u\nabla)w,v) \quad \forall u,v,w \in X \tag{2.4}$$

It is obvious that

$$|a(u,v)| \le \lambda\|u\|\,\|v\|, \quad a(u,u) \ge \lambda\|u\|^2 \quad \forall u \in X \tag{2.5}$$

i.e. $a(\cdot,\cdot)$ is symmetry positive definite and continuous. Whereas, $b(\cdot;\cdot,\cdot)$ is continuous on $X\times X\times X$ and symmetry.

$$|b(u;w,v)| \le c\|u\|\,\|v\|\,\|w\| \quad \forall u,w,v \in X \tag{2.6}$$

$$b(u;w,v) = -b(u;v,w) \quad \forall u \in V, v,w \in X \tag{2.7}$$

Note(see [?]), $b(\cdot;\cdot,\cdot)$ possess following properties.

$$|b(u;w,v)| \le c\|u\|_{s_1}\|w\|_{s_2+1}\|v\|_{s_1} \tag{2.9}$$

$$\forall u \in H^{s_1}(\Omega)^d, v \in H^{s_2}(\Omega), w \in H^{s_2+1}(\Omega)^d$$

$$s_1+s_2+s_3 \ge \frac{d}{2}, \quad s_1 \ge 0, s_2 \ge 0, s_3 \ge 0,$$

$$(s_1,s_2,s_3) \ne (0,0,\frac{d}{2}) \ne (0,\frac{d}{2},0) \ne (\frac{d}{2},0,0)$$

and

$$|b(u;w,v)| \le \|u\|_{0,r}\|w\|\,\|v\|_{0,s}, \quad \frac{1}{r}+\frac{1}{s}=\frac{1}{2}. \tag{2.10}$$

In sequel, we will use (2.9), (2.10) frequently.

Equivaletly, (2.1) can be rewritten by

$$\begin{cases} \text{find } u \in V \text{ such that} \\ a(u,v)+b(u;u,v) = (f,f), \quad \forall v \in V \end{cases} \tag{2.11}$$

Let us introduce finite element subspace $Y_h = X_h \times M_h$, where $X_h \subset X$, $M_h \subset M$. So $Y_h \subset Y$.

It is well known that (X,M) satisfies compatibility of velocity–pressure space

$$\inf_{q\in M}\sup_{v\in X}\frac{(q,\mathrm{div}v)}{|q|\,\|v\|} \ge \beta_0 > 0 \tag{2.12}$$

In order to guarantee the solvability of problem, finite element subspace Y_h is required to possess the following properties:

(H1) The approximation properties of piecewise polynomials of degree $(k,k-1)$

$$\forall (u,p) \in Y \bigcap (H^{k+1}(\Omega)^d \times H^k(\Omega))$$

$$\inf_{(v_h,q_h)\in Y_h}\{h\|u-u_h\|+\|v-v_h\|+h|p-q_h|\} \le ch^{k+1}\{\|u\|_{k+1}+\|p\|_k\} \tag{2.15}$$

(H2) Interpolant properties: assume that (I_h, J_h) is interpolant operator in Y_h

$$\|v - I_h v\| + \|q - J_h q| \leq ch^k(\|v\|_{k+1} + \|q\|_k)$$
$$\forall (v,q) \in Y \bigcap (H^{k+1}(\Omega)^d \times H^k(\Omega)) \tag{2.14}$$

(H3) Inverse inequality

$$\|v_h\| \leq ch^{-1}|v_h| \quad \forall v_h \in X_h$$

(H4) inf-sup condition for the compatibility of velocity–pressure (discrete LBB conditions)

$$\inf_{q \in M_h} \sup_{v \in X_h} \frac{(q, \mathrm{div} v)}{|q| \, \|v\|} \geq \beta > 0.$$

where β is a constant independent of h.

It is well known that (see [?]) classical Galerkin finite element approximation for stationary Navier-Stokes equation reach following optional results:

Theorem 2.1 If $(u,p) \in Y \bigcap (H^{k+1}(\Omega)^d, H^k(\Omega))$ is a nonsingular solution of (2.2) (see section 3) and finite element subspace Y_h satisfies assumption (H1)~(H4); (u_h, p_h) is a solution of Galerkin equation:

$$\begin{cases} \text{find } (u_h, p_h) \in Y_h \text{ such that} \\ a(u_h, v) + b(u_h; u_h, v) + (q, \mathrm{div} u_h) - (p_h, \mathrm{div} v) = (f, v) \quad \forall (v,q) \in Y_h \end{cases} \tag{2.16}$$

Then following estimation of error is valid

$$h\|u - u_h\| + |u - u_h| + h|p - p_h| \leq ch^{k+1}(\|u\|_{k+1} + \|p\|_k) \tag{2.17}$$

3 Projection

Later on, we ues the graph norm

$$|||(v,q)|||^2 = \|v\|^2 + |q|^2 \quad \forall (v,q) \in Y \tag{3.1}$$

Naver-Stokes operator $F(\cdot,\cdot)$ is a mapping$Y \to Y^*$ (dual space) defined by:

$\forall (u,p) \in Y,$

$$< F(u,p), (v,q) >= a(u,v) + b(u;u,v) + (q, \mathrm{div} u) - (p, \mathrm{div} v) \quad \forall (v,q) \in Y \tag{3.2}$$

Then Navier-Stokes equation (2.2) is equivalent to

$$F(u,p) = 0 \tag{3.3}$$

The Frechte derivative operator of $F(u,p)$ at (u,p) is denoted by $DF(u,p)$, which is linear mapping from Y to Y and defined by

$$(D_u F(u,p)(w,v), (v,q)) = a(w,v) + b(u;w,v) + b(w;u,v) + (q, \mathrm{div} w) - (r, \mathrm{div} v) \tag{3.4}$$

For sake of simplicity, we introduce bilinear form

$$Y \times Y \to R : \forall (w,r) \in Y, (v,q) \in Y$$

$$\mathcal{L}((w,v),(v,q)) = (D_u F(u,p)(w,v),(v,q)), \tag{3.5}$$

where (u,p) is supposed to be solution of (3.3). If we denote

$$C_u(w,v) = a(w,v) + b(u;w,v) + b(w;u,v) \tag{3.6}$$

then

$$\mathcal{L}((w,v),(v,q)) = C_u(w,v) + (q, \mathrm{div} w) - (r, \mathrm{div} v) \tag{3.7}$$

In similar manner, we define mapping $F(u) : V \to V^*$ by

$$< F(u), v >= a(u,v) + b(u;u,v) \quad \forall v \in V \tag{3.8}$$

It is obvious that

$$< D_u F(u)w, v >= C_u(w,v) \tag{3.9}$$

It is well known that u is a nonsingular solution of (2.11) if and only if $D_u F(u)$ is an isomorphism on V. Similarly, (u,p) is a nonsigular solution of (2.2) if and only if $DF(u,p)$ is an isomorphism on Y.

By virtue of general Lax-Milgram theorem, u is a nonsigular solution of (2.11) if and only if $C_u(\cdot,\cdot)$ satisfies inf-sup conditions

$$\inf_{w \in V} \sup_{v \in V} \frac{C_u(w,v)}{\|w\| \, \|v\|} \geq \alpha_0 > 0 \tag{3.10}$$

$$\inf_{v \in V} \sup_{w \in V} \frac{C_u(w,v)}{\|w\| \, \|v\|} \geq \alpha_0 > 0 \tag{3.11}$$

Therefore, variational problem

$$\begin{array}{l} \text{find } w \in V \text{ such that} \\ C_u(w,v) =< f, v > \quad \forall v \in V \end{array}$$

has an unique solution for any $f \in V'$.

In similar manner, (u,p) is a nonsigular solution of (2.2) if and only if $\mathcal{L}(\cdot,\cdot)$ satisfies inf-sup conditions

$$\inf_{(w,r) \in Y} \sup_{(v,q) \in Y} \frac{\mathcal{L}((w,r),(v,q))}{|||(w,r)||| \, |||(v,q)|||} \geq \beta > 0 \tag{3.12}$$

$$\inf_{(v,q) \in Y} \sup_{(w,r) \in Y} \frac{\mathcal{L}((w,r),(v,q))}{|||(w,r)||| \, |||(v,q)|||} \geq \beta > 0 \tag{3.13}$$

Therefore, the variational problem

$$\left\{ \begin{array}{l} \text{find } (w,r) \in Y \text{ such that} \\ \mathcal{L}((w,r),(v,q)) = (f,v) \quad \forall (v,q) \in Y \end{array} \right. \tag{3.14}$$

has a unique solution.

Following lemma indicates the relationship between (3.10), (3.11) and (3.12),(3.13).

Lemma 3.1 Suppose that (2.12) is satisfied and the bilinear form $C_u(\cdot,\cdot)$ satisfies (3.10),(3.11). Then the bilinear form $\mathcal{L}(\cdot,\cdot)$ satisfies (3.12),(3.13).

The proof see in [**?**][**?**].

Next lemma describes the conditions which ensure one point $(\tilde{u},\tilde{p})$ close to nonsigular point (u,p) is a nonsingular point, too.

Lemma 3.2 Assume $\tilde{Y}\subset Y$ is a finite dimensional subspace, and $\tilde{F}$ is a smooth mapping $\tilde{Y}\to Y^*$. Let (u,p) be a nonsigular point of $F(u,p)$. Let

$$\sigma(u,p)=\|DF(u,p)^{-1}\|_{\mathcal{L}(Y,Y)} \tag{3.15}$$

$$\mu(\tilde{u},\tilde{p})=\|DF(u,p)-D\tilde{F}(\tilde{u},\tilde{p})\|_{\mathcal{L}(Y,Y)} \tag{3.16}$$

If $(\tilde{u},\tilde{p})$ is closed to (u,p) such that

$$\sigma(u,p)\mu(\tilde{u},\tilde{p})<1 \tag{3.17}$$

Then $D\tilde{F}(\tilde{u},\tilde{p})$ is an isomorphism on $\tilde{Y}$. Hence $(\tilde{u},\tilde{p})$ is a nonsigular point of $\tilde{F}$.

Proof. See [**?**].

By similar manner, (2.16) can be rewritten by operator form

$$\begin{aligned}<\tilde{F}(u_h,p_h),(v,q)>=&a(u_h,v)+b(u_h;u_h,v)+(q,\mathrm{div}u_h)\\&-(p_h,\mathrm{div}v)-(f,v)=0,\quad\forall(v,q)\in Y_h\end{aligned}$$

and

$$<D\tilde{F}(u_h,p_h)(w,r),(v,q)>=a(w,v)+b(u_h;w,v)+b(w;u_h,v)+(q,\mathrm{div}w)-(r,\mathrm{div}v).$$

If (u,p) is a nonsigular solution of (2.2), inf-sup conditions (3.12), (3.13) and general Lax-Milgram theorem show that

$$\sigma(u,p)\equiv\|DF(u,p)^{-1}\|_{\mathcal{L}(Y,Y)}\le\beta^{-1}$$

on the otherhand, set $\eta=u-u_h$, then

$$<(DF(u,p)-D\tilde{F}(u_h,p_h))(w,r),(v,q)>=b(\eta;w,v)+b(w;\eta,v)$$

Therefore

$$\begin{aligned}\mu(\tilde{u},\tilde{p})=&\|DF(u,p)-D\tilde{F}(u_h,p_h)\|_{\mathcal{L}(Y_h,Y_h)}\\=&\sup_{(w,r)\in Y_h,(v,q)\in Y_h}\frac{<(D_uF(u,p)-D\tilde{F}(u_h,p_h))(w,r),(v,q)>}{|||(u,p)|||\,|||(w,r)|||}\\\le&c\|\eta\|=c\|u-u_h\|\le ch(\|u\|_2+\|p\|)\end{aligned}$$

¿From (3.17) we conclude

Collorary 3.3 Assume that (u,p) is a nonsingular solution and h is chosen such that

$$c(\|u\|_2 + \|p\|)h\beta^{-1} < 1$$

Then (u_h, p_h) is a nonsingular solution of (2.16).

Later on, set

$$C_h(w,v) = a(w,v) + b(u_h; w, v) + b(w; u_h, v) \tag{3.18}$$

$$\mathcal{L}_h((w,r),(v,q)) = C_h(w,v) + (q, \mathrm{div} w) - (r, \mathrm{div} v) \tag{3.19}$$

Therefore

$$< D\tilde{F}(u_h,p_h)(w,r),(v,q) >= \mathcal{L}_h((w,r),(v,q)) \tag{3.20}$$

Collorary yields that $\mathcal{L}_h(\cdot,\cdot)$ satisfies inf-sup condition: there exists $\gamma > 0$ such that

$$\inf_{(w,r)\in Y_h,(v,q)\in Y_h} \frac{\mathcal{L}((w,r),(v,q))}{|||(w,r)|||\,|||(v,q)|||} \geq \gamma > 0 \tag{3.21}$$

$$\inf_{(v,q)\in Y_h,(w,r)\in Y_h} \frac{\mathcal{L}((w,r),(v,q))}{|||(w,r)|||\,|||(v,q)|||} \geq \gamma > 0 \tag{3.22}$$

in sequence we will frequently use

$$\begin{aligned} \mathcal{L}_h((w,r),(v,q)) =& \mathcal{L}((w,r),(v,q)) + b(u_h - u; w, v) \\ &+ b(w; u_h - u, v) \end{aligned} \tag{3.23}$$

Now, we define projection $(Q_h, R_h) : Y \to Y_h$ by

$$\begin{cases} \forall (w,r) \in Y, (Q_h(w,r), R_h(w,r)) \in Y_h \text{ such that} \\ \mathcal{L}_h((w - Q_h, r - R_h),(v,q)) = 0 \quad \forall (v,q) \in Y_h. \end{cases} \tag{3.24}$$

$\forall (w,r) \in Y$, we can make decomposition

$$w = Q_h(w,r) + \hat{w}, \quad r = R_h(w,r) + \hat{r}, \quad (\hat{w},\hat{r}) \in \hat{Y}_h \tag{3.25}$$

$$Y = Y_h \oplus \hat{Y}_h \tag{3.26}$$

Consequently, (3.24) can be rewritten

$$\mathcal{L}_h((\hat{w},\hat{r}),(v,q)) = 0 \quad \forall (v,q) \in Y_h \tag{3.27}$$

i.e. Y_h and $\hat{Y}_h$ are orthogonal with respect to $\mathcal{L}_h(\cdot,\cdot)$.

Furthermore, we define adjoint bilinear form $\mathcal{L}_h^*(\cdot,\cdot)$ by

$$\begin{aligned} \mathcal{L}_h^*((w,r),(v,q)) =& C_h(v,w) - (q, \mathrm{div} w) + (r, \mathrm{div} v) \\ & \forall (w,r) \in Y, (v,q) \in Y \end{aligned} \tag{3.28}$$

From this definition, it is easy to show

$$\mathcal{L}_h^*((w,r),(v,q)) = \mathcal{L}_h^*((v,q),(w,r))$$

$$\forall(w,r)\in Y,(v,q)\in Y \tag{3.29}$$

in view of (3.27), we have: $\forall(\hat{w},\hat{r})\in\hat{Y}_h$,

$$\mathcal{L}_h^*((v,q),(\hat{w},\hat{r}))=0,\quad \forall(v,q)\in Y_h \tag{3.30}$$

Y_h is a finite dimensional subspace which consists of lower frequence components in Y and $\hat{Y}_h$ is a infinite dimensional subspace which consists of higher frequence components in Y. Following lemma shows norm of higher frequence components is very smaller.

Lemma 3.4 Assume that u is a nonsigualr solution of (2.11) and the adjoint linearized Navier-Stokes equations:

$$\left\{\begin{array}{l}\text{find } (w,r)\in Y \text{ such that}\\ \mathcal{L}^*((w,r),(v,q))=(f,v)\quad \forall(v,q)\in Y\end{array}\right. \tag{3.31}$$

is $H^2(\Omega)$ –regular. Furthermore, finite element subspace Y_h satisfies assumptions (H1)~(H4).

Then projection (Q_h,R_h) defined by (3.26) satisfies:

$$|||(w-Q_h(w,r),r-R_h(w,r))|||\le c|||(w,r)|||\quad \forall(w,r)\in Y \tag{3.32}$$

$$|||(\hat{w},\hat{r})|||\le ch^k(\|w\|_{k+1}+\|r\|_k),\quad \forall(w,r)\in Y\bigcap(H^{k+1}(\Omega)^d\times H^k(\Omega)) \tag{3.33}$$

$$|\hat{w}|\le ch\|\hat{w}\|\quad \forall w\in X \tag{3.34}$$

$$\begin{aligned}\|\hat{w}\|_\theta\le& ch^{k+1-\theta}(\|w\|_{k+1}+\|r\|_k),\\ &\forall(w,r)\in Y\bigcap(H^{k+1}(\Omega)^d\times H^k(\Omega)),\theta\in[0,1]\end{aligned} \tag{3.35}$$

Proof. Owing to nonsigurality of u, by virtue of lemma 3.1 and 3.3, we have: $\forall(Q_h(w,r),R_h(v,q))$

$$\begin{aligned}|||(Q_h,R_h)|||\le&\gamma^{-1}\sup_{(v,q)\in Y_h}\frac{\mathcal{L}_h((Q_h,R_h),(v,q))}{|||(v,q)|||}\\ \le&\gamma^{-1}\sup_{(v,q)\in Y_h}\frac{\mathcal{L}_h((w,r),(v,q))}{|||(v,q)|||}\le\gamma^{-1}c|||(w,r)|||\end{aligned}$$

This derives (3.33) by triangle inequality.

Assume that (ϕ,λ) is a solution of (3.31) with right side $f=\hat{w}=w-Q_h(w,r)$. Setting $(v,q)=(\hat{w},\hat{r})$ and using (3.23) we obtain

$$\mathcal{L}_h^*((\phi,\lambda),(\hat{w},\hat{r}))=|\hat{w}|^2+b(u_h-u;\hat{w},\phi)+b(\hat{w};u_h-u,\phi)$$

Taking arbitrary $(\phi_h,\lambda_h)\in Y_h$ and using (3.30), we have

$$|\mathcal{L}_h^*((\phi-\phi_h,\lambda-\lambda_h),(\hat{w},\hat{r}))|\le c|||(\phi-\phi_h,\lambda-\lambda_h)|||\;|||(\hat{w},\hat{r})|||$$

$$\leq ch(\|\phi\|_2 + \|\lambda\|)|||(w,r)||| \leq ch|\hat{w}|\,|||(\hat{w},\hat{r})|||$$

Here we used H^2–regurlarity of (3.31):

$$\|\phi\|_2 + \|\lambda\| \leq c|f|$$

In addition, by applying (2.9) with $s_1 = 1$, $s_2 = 0$, $s_3 = 2$,

$$|b(u_h - u; \hat{w}, p)| \leq c\|u_h - u\|\|\phi\|_2\|\hat{w}\| \leq ch\|u\|\,|\hat{w}|\,\|\hat{w}\| \tag{3.37}$$

$$|b(\hat{w}; u - u_h, \phi)| \leq c\|\hat{w}\|\,\|u - u_h\|\,\|\phi\|_2 \leq ch\|u\|\,|\hat{w}|\,\|\hat{w}\|$$

Therefore, from (3.36) and above inequalities, if follows that

$$|\hat{w}| \leq ch|||(\hat{w},\hat{r})|||$$

Owing to arbitrary of (v,q), this derive (3.34) which is called enhanced Poincare inequality.

It is remainder to consider $|||(\hat{w},\hat{r})|||$. Let (I_h, J_h) be interpolation operator in Y, then

$$\begin{aligned}
|||(I_h w - Q_h(w,r), J_h r - R_h(w,r))||| &\leq \gamma^{-1} \sup_{(v,q)\in Y_h} \frac{\mathcal{L}_h((I_h w - Q_h, J_h r - R_h),(v,q))}{|||(v,q)|||} \\
&\leq \gamma^{-1} \sup_{(v,q)\in Y_h} \frac{\mathcal{L}_h((I_h w - w, J_h r - r),(v,q))}{|||(v,q)|||} \\
&\leq c\gamma^{-1}|||(I_h w - w, J_h r - r)||| \leq c\gamma^{-1}h^k(\|w\|_{k+1} + \|r\|_k)
\end{aligned} \tag{3.38}$$

By triangle inequality and (3.38)

$$\begin{aligned}
|||(\hat{w},\hat{r})||| &\leq |||(w - I_h w, r - J_h r)||| + |||(I_h w - Q_h, J_h r - R_h)||| \\
&\leq ch^k(\|w\|_{k+1} + \|r\|_k) + c\gamma^{-1}h^k(\|w\|_{k+1} + \|r\|_k)
\end{aligned}$$

This derive (3.33). By interpolation inequality of norm in sobolev space and (3.33), (3.34), we obtain (3.35). □

4 Lower frequence components

Assume that (u,p) is a solution of (2.2). It can be splitted by

$$u = Q_h(u,p) + \hat{u}, \quad p = R_h(u,p) + \hat{p}, \quad (Q_h, R_h) \in Y_h, \quad (\hat{u},\hat{p}) \in \hat{Y} \tag{4.1}$$

Set

$$e = Q_h(u,p) - u_h, \quad \varepsilon = R_h(u,p) - p_h \tag{4.2}$$

where (u_h, p_h) is a solution of Galerkin finite element equation (2.16). It is easy to show that

$$\eta = u - u_h = e + \hat{u}, \quad \xi = p - p_h = \varepsilon + \hat{p} \tag{4.3}$$

We rewrite Navier-Stokes equations

$$a(Q_h+\hat{u},v)+b(Q_h+\hat{u};Q_h+\hat{u},v)+(q,\mathrm{div}(Q_h+\hat{u}))-(R_h+\hat{p},\mathrm{div}v)=(f,v).$$

Using (3.19), it yields that

$$\begin{aligned}\mathcal{L}_h((Q_h,R_h),(v,q))+\mathcal{L}_h((\hat{u},\hat{p}),(v,q))+b(u;u,v)-b(Q_h;u_h,v)\\ -b(u_h;Q_h,v)-b(u_h;\hat{u},v)-b(\hat{u};u_h,v)=(f,v).\end{aligned}\tag{4.4}$$

$$\begin{aligned}b(u;u,v)-&b(Q_h;u_h,v)-b(\hat{u};u_h,v)-b(u_h;Q_h,v)-b(u_h;\hat{u},v)\\ =&b(u;u,v)-b(u;u_h,v)-b(u_h;u,v)=b(u-u_h;u-u_h,v)-b(u_h;u_h,v).\end{aligned}$$

(4.4) yields that

$$\mathcal{L}_h((Q_h,R_h),(v,q))+\mathcal{L}_h((\hat{u},\hat{p}),(v,q))+b(\eta;\eta,v)-b(u_h;u_h,v)=(f,v).\tag{4.5}$$

This is an other form of Naver-Stokes equations. On the other hand, Galerkin finite element equations (2.16) can be rewritten by

$$\mathcal{L}_h((u_h,p_h),(v,q))-b(u_h;u_h,v)=(f,v)\quad\forall(v,q)\in Y_h.\tag{4.6}$$

Substracting (4.5) from (4.6), we using (3.30), we have

$$\mathcal{L}_h((e,\varepsilon),(v,q))+b(\eta;\eta,v)=0,\quad\forall(v,q)\in Y_h.\tag{4.7}$$

In view of (3.21),

$$\begin{aligned}|||(e,\varepsilon)|||\le&\gamma^{-1}\sup_{(v,q)\in Y_h}\frac{\mathcal{L}_h((e,\varepsilon),(v,q))}{|||(v,q)|||}\\ \le&\gamma^{-1}\sup_{(v,q)\in Y_h}\frac{-b(\eta;\eta,v)}{|||(v,q)|||}.\end{aligned}\tag{4.8}$$

By virtue of (2.9)

$$|b(\eta;\eta,v)|\le c\|\eta\|_{s_1}\|\eta\|_{s_2+1}\|v\|.$$

$$s_1+s_2+1\ge\frac{d}{2}$$

When $d=2$, $s_1+s_2\ge0$. Hence, taking $s_1=s_2=0$, we have

$$|b(\eta;\eta,v)|\le c|\eta|\,\|\eta\|\,\|v\|.$$

When $d=3$, $s_1+s_2\ge\frac{1}{2}$. Hence, taking $s_1=0$, $s_2=\frac{1}{2}$ and using interpolation inequality

$$\|\eta\|_{\frac{1}{2}}\le|\eta|^{\frac{1}{2}}\|\eta\|^{\frac{1}{2}}.$$

we obtain

$$|b(\eta;\eta,v)|\le c|\eta|^{\frac{1}{2}}\|\eta\|^{\frac{3}{2}}\|v\|.$$

Finally,

$$|b(\eta;\eta,v)|\le c|\eta|^{1-\varepsilon}\|\eta\|^{1+\varepsilon}. \tag{4.9}$$

$$\varepsilon=\begin{cases}0, & d=2\\ \dfrac{1}{2}, & d=3\end{cases} \tag{4.10}$$

Combining (4.8), (4.9) and (4.10) we obtain

$$|||(e,\varepsilon)|||\le\gamma^{-1}c|\eta|^{1-\varepsilon}\|\eta\|^{1+\varepsilon}. \tag{4.11}$$

where

$$\varepsilon=\begin{cases}0, & \text{if } d=2\\ \dfrac{1}{2}, & \text{if } d=3\end{cases}$$

Next we derive estimate for $|e|$. To do this, suppose (3.31) is H^2–regular and (ϕ,λ) is a solution of (3.31) with $f=e$, therefore

$$\|\phi\|_2+\|\lambda\|\le c|e| \tag{4.12}$$

Using (3.23), we rewrite (3.31) by

$$\mathcal{L}_h^*((\phi,\lambda),(v,q))=(e,v)+b(u_h-u;v,\phi)+b(v;u_h-u,\phi)$$

Setting $(v,q)=(e,\varepsilon)$, it yields

$$\begin{aligned}|e|^2=&\mathcal{L}_h^*((\phi,\lambda),(e,\varepsilon))+b(\eta;e,\phi)+b(e;\eta,\phi)\\ =&\mathcal{L}_h((\phi-\phi_h,\lambda-\lambda_h),(e,\varepsilon))+b(\eta;e,\phi)+b(e;\eta,\phi)\\ \le&c|||(\phi-\phi_h,\lambda-\lambda_h)|||\;|||(e,\varepsilon)|||+c(|\eta|\;\|e\|+|e|\;\|\eta\|)\|\phi\|_2\\ \le&ch\{|||(e,\varepsilon)|||+\|e\|\}(\|\phi\|_2+\|\lambda\|)\\ \le&ch|||(e,\varepsilon)|||\;|e|\end{aligned}$$

i.e.

$$|e|\le ch|||(e,\varepsilon)||| \tag{4.13}$$

Consequently, we obtain

Theorem 4.1 Suppose that (2.2) is $H^{k+1}(\Omega)$ regular and (3.31) is $H^2(\Omega)$ regular. Finite element subspace Y_h satisfies assumptions (H1)$\sim$(H4). Assume (u,p) and (u_h,p_h) are solution of (2.2) and (2.16) respectively. (u,p) is nonsingular. (Q_h,R_h) is projection on Y_h defined by (3.24). Then following estimates of error are valid

$$|||(e,\varepsilon)|||\le ch^{2k+1-\varepsilon}(\|u\|_{k+1}+\|p\|_k) \tag{4.14}$$

$$|e|\le ch^{2k+2-\varepsilon}(\|u\|_{k+1}+\|p\|_k) \tag{4.15}$$

where

$$\varepsilon=\begin{cases}0, & \text{for } d=2\\ \dfrac{1}{2}, & \text{for } d=3\end{cases}$$

5 Higher Frequence Components

Follows the section 4,

$$\|u - e\| \leq \|Q_h(u,p) - e + \hat{u}\| \leq \|e\| + \|\hat{u}\|.$$

It is obvious that though $\|e\|$ possess higher accuracy, but $\|\hat{u}\|$ do not possess higher accuracy, it is the same accuracy as in approximation theory. In order impove the rate of convergence for (e,ε) we have to correct $(\hat{u},\hat{p})$. To do this we introduce the approximate inertial manifold which is the Graph of mapping $\Phi : Y_h \to \hat{Y}_h$ satisfying following properties

(H1) Φ is Lipschitz continuous with Lipschitz constant l

$$|||(\Phi(w_1,r_1) - \Phi(w_2,r_2)||| \leq l|||(w_1 - w_2, r_1 - r_2)||| \qquad (5.1)$$

$$\forall (w_i, r_i) \in Y_h \bigcap B$$

$$B = \{|||(w,r)||| \leq \rho, \forall (w,r) \in Y\}, \text{ and } l \text{ is dependent upon } \rho$$

(H2) Φ is attracting all solution (u,p) of (2.2)

$$|||(\hat{u},\hat{p}) - \Phi(Q_h(u,p), R_h(u,p))||| \leq \delta \qquad (5.2)$$

$\delta > 0$ is a small constant.

$\mathcal{M} = \text{Graph}(\Phi)$ is called approximate inertial manifold with δ–thinkness because all solution (u,p) of (2.2) is at the δ– neighborhood of $\mathcal{M}$. Indeed

$$\begin{aligned}\text{dist}((u,p),\mathcal{M}) =& \inf_{\forall (w,r)\in\mathcal{M}} |||(u,p) - (w,r)|||\\ \leq& |||(u - ((Q_h(u,p), R_h(u,p)) + \Phi(Q_h, R_h)))|||\\ =& |||(\hat{u},\hat{p}) - \Phi(Q_h, R_h)||| \leq \delta\end{aligned}$$

i.e.

$$\text{dist}((u,p),\mathcal{M}) \leq \delta \qquad (5.3)$$

In order to construct Φ, we rewrite Navier-Stokes equations into a form of (4.5)

$$\begin{aligned}&\mathcal{L}_h((Q_h,R_h),(v,q)) + \mathcal{L}_h((\hat{u},\hat{p}),(v,q))\\ &\qquad + b(\eta;\eta,v) - b(u_h;u_h,v) = (f,v) \quad \forall (v,q) \in Y\end{aligned}$$

However

$$\begin{aligned}&\mathcal{L}_h((Q_h,R_h),(v,q)) = -\mathcal{L}_h^*((Q_h,R_h),(v,q))\\ &\quad + 2a(Q_h,v) + b(u_h;Q_h,v) + b(Q_h;u_h,v) + b(u_h;v,Q_h) + b(v;u_h,Q_h)\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{L}_h((\hat{u},\hat{p}),(v,q)) &- \mathcal{L}_h^*((Q_h,R_h),(v,q)) + b(\eta;\eta,v) - b(u_h;u_h,v)\\ &+ 2a(Q_h,v) + b(u_h;Q_h,v) + b(Q_h;u_h,v) + b(u_h;v,Q_h) + b(v;u_h,Q_h)\\ =&(f,v) \quad \forall (v,q)\in Y.\end{aligned}$$

Note

$$b(u_h;Q_h,v) - b(u_h;u_h,v) = -b(\eta;e,v) + b(u;e,v).$$

and

$$\mathcal{L}^*((Q_h,R_h),(v,q)) = 0 \quad \forall (v,q)\in \hat{Y}_h.$$

then

$$\begin{aligned}\mathcal{L}_h((\hat{u},\hat{p}),(v,q)) &+ b(\eta;\eta,v) + b(u;e,v) - b(\eta;e,v)\\ &+ 2a(Q_h,v) + b(Q_h;u_h,v) + b(u_h;v,Q_h) + b(v;u_h,Q_h)\\ =&(f,v) \quad \forall (v,q)\in \hat{Y}_h. \end{aligned} \tag{5.4}$$

It is clear that we suppose $\Phi = (\phi,\xi)$ satisfies

$$\begin{aligned}&\text{find } (\phi,\xi)\in \hat{Y}_h \text{ such that } \forall f\in X^*,\\ &\mathcal{L}_h((\phi,\xi),(v,q)) + 2a(u_h,v) + b(u_h;u_h,v) + b(u_h;v,u_h) + b(v;u_h,u_h)\\ &\quad = (f,v) \quad \forall (v,q)\in \hat{Y}_h.\end{aligned} \tag{5.5}$$

Owing inf-sup condition for $\mathcal{L}_h(\cdot,\cdot)$, we assert that there exists unique solution (ϕ,ξ) of (5.5).

Substracting (5.5) from (5.4), we obtain

$$\begin{aligned}\mathcal{L}_h((\hat{u}-\phi,\hat{p}-\xi),(v,q)) &+ b(\eta;\eta,v) + b(u;e,v) - b(\eta;e,v)\\ &+ 2a(e,v) + b(e;u_h,v) + b(u_h;v,e) + b(v;u_h,e) = 0.\end{aligned}$$

Notice

$$\begin{aligned}&b(e;u_h,v) + b(u_h;v,e) + b(v;u_h,e)\\ =&-b(e;e,v) + b(e;u,v) - b(e;v,e) + b(u;v,e) - b(v;e,e) + b(v;u,e)\\ =&b(u;v,e) + b(e;u,v) + b(v;u,e) - b(e;e,v) - b(e;v,e) - b(v;e,e).\end{aligned}$$

Therefore, using

$$b(u;e,v) + b(u;v,e) = 0,$$

we derive

$$\begin{aligned}\mathcal{L}_h((\hat{u}-\phi,\hat{p}-\xi),(v,q)) &+ b(\eta;\eta,v) - b(\eta;e,v)\\ &+ 2a(e,v) + b(e;u,v) + b(v;u,e) - b(e;e,v) - b(e;v,e) - b(v;e,e) = 0.\end{aligned} \tag{5.6}$$

By virtue of inf-sup condition and lemma 3.2 and 3.3,

$$\gamma \le \inf_{(w,r)\in Y} \sup_{(v,q)\in Y} \frac{\mathcal{L}_h((w,r),(v,q))}{|||(w,r)|||\,|||(v,q)|||}$$

$$\le \sup_{(v,q)\in Y} \frac{\mathcal{L}_h((\hat{u}-\phi,\hat{p}-\xi),(v,q))}{|||(\hat{u}-\phi,\hat{p}-\xi)|||\,|||(v,q)|||}$$

Using (3.28), it follows that

$$|||(\hat{u}-\phi,\hat{p}-\xi)||| \le \gamma^{-1} \sup_{(v,q)\in \hat{Y}_h} \frac{\mathcal{L}_h((\hat{u}-\phi,\hat{p}-\xi),(v,q))}{|||(v,q)|||} \tag{5.7}$$

Now we estimate right hand side of (5.6) term by term.

$$\begin{aligned}
&|a(e,v)| \le c\|e\|\,\|v\| \\
&|b(\eta;\eta,v)| \le c|\eta|^{1-\varepsilon}\|\eta\|^{1+\varepsilon}\|v\| \\
&|b(\eta;e,v)| \le c\|\eta\|\,\|e\|\,\|v\| \\
&|b(e;u,v)| \le c|e|\,\|u\|_2\|v\| \\
&|b(v;u,e)| \le c|e|\,\|u\|_2\|v\| \\
&|b(e;e,v)| \le c|e|^{1-\varepsilon}\|e\|^{1+\varepsilon}\|v\| \\
&|b(e;v,e)| \le c\|v\|\,|e|^{1-\varepsilon}\|e\|^{1+\varepsilon} \\
&|b(v;e,e)| \le c\|v\|\,|e|^{1-\varepsilon}\|e\|^{1+\varepsilon}
\end{aligned} \tag{5.8}$$

where

$$\varepsilon = \begin{cases} 0, & d=2 \\ \frac{1}{2}, & d=3. \end{cases}$$

Taking (5.6)$\sim$(5.8) into account, we assert that

$$\begin{aligned}
|||(\hat{u}-\phi,\hat{p}-\xi)||| \le& c\gamma^{-1}\{\|e\| + |e| + \|\eta\|\,\|e\| + |e|^{1-\varepsilon}\|e\|^{1+\varepsilon} + |\eta|^{1+\varepsilon}\|\eta\|^{1+\varepsilon}\} \\
\le& c\gamma'\|e\|
\end{aligned}$$

Repeating the arguments for making estimate $|\hat{u}-\phi|$ as in section 4, we conclude

Theorem 5.1 Suppose that assumptions of theorem 4.1 are satisfied. Then there exists unique solution (ϕ,ξ) of (5.5) which is lipschitz continuous with respective to u_h. Furthermore, following estimates hold.

$$\begin{aligned}
&|||(\hat{u}-\phi,\hat{p}-\xi)||| \le ch^{2k+1-\varepsilon}(\|u\|_{k+1} + \|p\|_k) \\
&|\hat{u}-\phi| \le ch^{2k+2-\varepsilon}(\|u\|_{k+1} + \|p\|_k)
\end{aligned} \tag{5.9}$$

where

$$\varepsilon = \begin{cases} 0, & d=2 \\ \frac{1}{2}, & d=3 \end{cases} \tag{5.10}$$

Proof. The remainder of proof has to prove existence and Lipschitz continuous. Indeed, if (u,p) is a nonsingular solution of (2.2) therefore $\mathcal{L}(\cdot,\cdot)$ satisfies inf-sup condition (3.12) and (3.13) on space Y. In view of (3.23), (2.6) and (2.17)

$$\inf_{(w,r)\in Y}\sup_{(v,q)\in Y}\frac{\mathcal{L}_h((w,r),(v,q))}{|||(w,r)|||\,|||(v,q)|||}\geq\inf_{(w,r)\in Y}\sup_{(v,q)\in Y}\frac{\mathcal{L}((w,r),(v,q))}{|||(w,r)|||\,|||(v,q)|||}-c\|u-u_h\|$$

$$\geq -c\gamma h(\|u\|_2+\|p\|)$$

If we choose h such that

$$h\leq c^{-1}(\|u\|_2+\|p\|)^{-1}\gamma/2$$

then

$$\inf_{(w,r)\in Y}\sup_{(v,q)\in Y}\frac{\mathcal{L}_h((w,r),(v,q))}{|||(w,r)|||\,|||(v,q)|||}\geq\frac{\gamma}{2}$$

Moreover,

$$\frac{\gamma}{2}\leq\inf_{(w,r)\in Y}\sup_{(v,q)\in Y}\frac{\mathcal{L}_h((w,r),(v,q))}{|||(w,r)|||\,|||(v,q)|||}$$

$$\leq\inf_{(w,r)\in\hat{Y}_h}\sup_{(v,q)\in Y}\frac{\mathcal{L}_h((w,r),(v,q))}{|||(w,r)|||\,|||(v,q)|||}$$

(by (3.27))

$$=\inf_{(w,r)\in\hat{Y}_h}\sup_{(v,q)\in\hat{Y}_h}\frac{\mathcal{L}_h((w,r),(v,q))}{|||(w,r)|||\,|||(v,q)|||}$$

By similar manner, we can prove like (3.13) but on space $\hat{Y}_h$. Therefore, we have

$$\inf_{(w,r)\in\hat{Y}_h}\sup_{(v,q)\in\hat{Y}_h}\frac{\mathcal{L}_h((w,r),(v,q))}{|||(w,r)|||\,|||(v,q)|||}\geq\frac{\gamma}{2}\tag{5.11}$$

$$\inf_{(v,q)\in\hat{Y}_h}\sup_{(w,r)\in\hat{Y}_h}\frac{\mathcal{L}_h((w,r),(v,q))}{|||(w,r)|||\,|||(v,q)|||}\geq\frac{\gamma}{2}\tag{5.12}$$

By using general Lax-Milgram theorem, we assert that there exists unique solution (ϕ,ξ) of (5.5) and

$$|||(\phi,\xi)|||\leq c(\|f\|^*+\|u_h\|+\|u_h\|^2)$$

Assume that (ϕ,ξ) and $(\tilde{\phi},\tilde{\xi})$ are solution of (5.5) with respect to u_h and $\tilde{u}_h$. By elemental calculation from (5.5), setting

$$\Delta\phi=\phi-\tilde{\phi},\quad \Delta\xi=\xi-\tilde{\xi},\quad \Delta u_h=u_h-\tilde{u}_h$$

we obtain

$$\mathcal{L}_h((\Delta\phi,\Delta\xi),(v,q))=E$$

$$E=-\,b(\Delta u_h;\tilde{\phi}+u_h,v)-b(\tilde{\phi}+\tilde{u}_h;\Delta u_h,v)-2a(\Delta u_h,v)$$

$$- b(\Delta u_h; v, u_h) - b(\tilde{u}_h; v, \Delta u_h) - b(v; \Delta u_h, u_h) - b(v; \tilde{u}_h, \Delta u_h)$$

However

$$\|u_h\| \leq \|u - u_h\| + \|u\| \leq ch(\|u\|_2 + \|p\|) + \|u\|,$$

Consequently, using inf-sup condition (5.11)

$$|||(\Delta\phi, \Delta\xi)||| \leq 2\gamma^{-1} \sup_{(v,q)\in\hat{Y}_h} \frac{E}{|||(v,q)|||} \leq c(u)\|u_h - \tilde{u}_h\|$$

This is Lipschitz continuous of Φ. Proof is completed.

Following theorem is obvious.

Theorem 5.2 Suppose that assumptions of theorem 5.1 are valid. Then following estimates hold

$$|||(u - u_*, p - p_*)||| \leq ch^{2k+1-\varepsilon}$$
$$|u - u_*| \leq ch^{2k+2-\varepsilon}$$

where

$$u_* = u_h + \phi(u_h)$$
$$p_* = p_h + \xi(u_h)$$

(u_h, p_h) is a Galerkin finite element approximation

$$\varepsilon = \begin{cases} 0, & d = 2 \\ \frac{1}{2}, & d = 3 \end{cases}$$

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