The Interplay of Ranks of Submatrices

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Abstract

A banded invertible matrix T has a remarkable inverse. All "upper" and "lower" submatrices of T^{-1} have low rank (depending on the bandwidth in T). The exact rank condition is known, and it allows fast multiplication by full matrices that arise in the boundary element method.

We look for the "right" proof of this property of T^{-1}. Ultimately it reduces to a fact that deserves to be better known: Complementary submatrices of any T and T^{-1} have the same nullity. The last figure in the paper (when T is tridiagonal) shows two submatrices with the same nullity $n-3$. Then C has rank 1. On and above the diagonal of T^{-1}, all rows are proportional.

Key words: Band matrix, low rank submatrix, fast multiplication.

AMS subject classification: 15A09, 15A15, 65F05

1. Introduction

An n by n real tridiagonal matrix T is specified by $3n-2$ parameters—its entries on the three central diagonals. In some way T^{-1} must also be specified by $3n-2$ parameters (if T is invertible). It will be especially nice if those parameters can be the "tridiagonal part" of T^{-1}. Fortunately they can. T^{-1} can be built outwards from that part, diagonal by diagonal (and without knowing T), because the inverse of a tridiagonal matrix possesses these two properties:

Upper: On and above the main diagonal, every 2 by 2 minor of T^{-1} is zero.

Lower: On and below the main diagonal, every 2 by 2 minor of T^{-1} is zero.

Expressed in another way, those properties are statements of *low rank*. Of course the whole matrix T^{-1} has full rank! But it has many submatrices of rank ≤ 1 (since rank 2 would imply that some 2 by 2 minor is nonzero):

Upper: On and above the main diagonal, every submatrix of T^{-1} has rank ≤ 1.

Lower: On and below the main diagonal, every submatrix of T^{-1} has rank ≤ 1.

When the subdiagonal and superdiagonal of T are nonzero, we can specify $3n-2$ parameters for T^{-1}:

$$(T^{-1})_{ij} = \left\{ \begin{array}{ll} a_i b_j & \text{for} \quad j \geq i \\ c_i d_j & \text{for} \quad j \leq i \end{array} \right\} \qquad \begin{array}{ll} \text{with} & a_i b_i = c_i d_i \\ \text{and} & a_1 = c_1 = 1 \end{array}$$

The $4n$ parameters a, b, c, d are reduced by $n+2$ constraints to $3n-2$.

The friendly but impatient reader will quickly ask for generalizations and proofs. A natural generalization allows T to have a wider band. Suppose $T_{ij} = 0$ for $|i-j| > p$, so that $p = 1$ means tridiagonal. The corresponding conditions on T^{-1} involve zero subdeterminants of order $p+1$. The key is to know which submatrices of T^{-1} are involved when T is banded:

Upper: Above the pth subdiagonal, every submatrix of T^{-1} has rank $\leq p$.

Lower: Below the pth superdiagonal, every submatrix of T^{-1} has rank $\leq p$.

Equivalently, all upper and lower minors of order $p+1$ are zero. Assuming nondegeneracy, T^{-1} can be completed starting from its "banded part". The count of parameters in T and T^{-1} still agrees.

A next small step would allow the lower triangular part of T to be full. Only the *upper* condition will apply to T^{-1}, coming from the upper condition $T_{ij} = 0$ for $j - i > p$. *We will pursue this one-sided formulation from now on.* Upper conditions on T (above the pth superdiagonal) will be equivalent to upper conditions on T^{-1} (above the pth subdiagonal).

Here is a significant extension. Requiring zero entries in T is a statement about 1 by 1 minors. We could ask instead for all k by k minors of T to vanish (above the pth diagonal). Equivalently, all "upper submatrices" B would have $\text{rank}(B) < k$. *What property does this imply for T^{-1}?* The answer is neat and already known. But this relation between submatrices of T and T^{-1} is certainly not well known. The goal of this paper is to try for a new proof (and in the end we present two proofs). Here is the main result:

Theorem (for invertible T)

All submatrices B above the pth superdiagonal of T have $\text{rank}(B) < k$

if and only if

All submatrices C above the pth subdiagonal of T^{-1} have $\text{rank}(C) < p+k$.

Our tridiagonal case (or Hessenberg case, which is the one-sided version) had $p = 1$ and $k = 1$. Even more special is $p = 0$ and $k = 1$. Then T is lower triangular if and only if T^{-1} is lower triangular (extremely well known!). The case $p = 0$ and $k = 2$ is not so familiar—if T is "rank 1 above the main diagonal" then so is T^{-1}. This also comes from the Woodbury-Morrison formula [20, p. 82], [16, p. 19], which exhibits the rank one effect on T^{-1} of a rank one change in T.

We comment in advance on the two proofs. One works directly with rank, the other works with nullity (dimension of the nullspace). For pairs of submatrices B and C of T and T^{-1}, the goal is to show that

$$\text{rank}(B) < k \quad \text{if and only if} \quad \text{rank}(C) < p + k\,. \tag{1}$$

Our original proof uses a simple inequality for ranks of products. Wayne Barrett observed that this lemma is a special case of the Frobenius Rank Inequality

$$\text{rank}(AB) + \text{rank}(BC) \leq \text{rank}(B) + \text{rank}(ABC)\,. \tag{2}$$

Then we noticed that (2) follows quickly from our special case. Conceivably this provides a new proof of (2).

Barrett also pointed out that (1) follows immediately from the beautiful observation by Fiedler and Markham [10] that (in this notation)

$$\text{nullity}(B) = \text{nullity}(C)\,. \tag{3}$$

Barrett's approach surely gives the best proof of the Theorem. It has the great merit that (3) is an *equality* (of nullities) instead of an inequality (of ranks). We will present this proof first. After the second proof, the Theorem is applied to *fast multiplication* by these full "semiseparable" matrices T^{-1}.

Notice that there are an equal number of free parameters in T and T^{-1}. Figure 1 shows how the entry in position $(1, p+2k)$ of both T and T^{-1} is generically the first to be determined from the (equivalent) conditions in the theorem. (This entry is in the upper corner of a square singular submatrix for both matrices. There will be degenerate cases when that entry is not determined, in the same way that specifying three entries of a singular 2 by 2 matrix does not always determine the fourth.) The entries on all earlier diagonals, before position $(1, p+2k)$, can be the free parameters for T and also for T^{-1}.

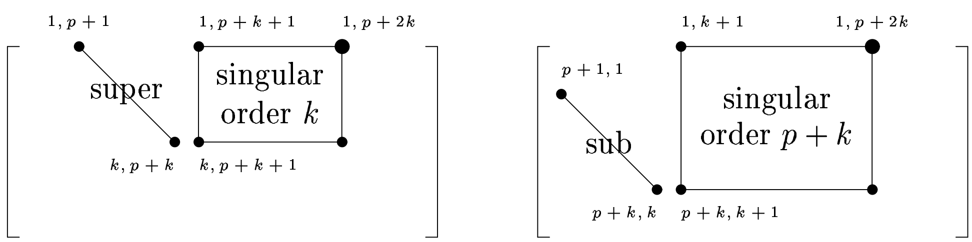


Figure 1: For both T and T^{-1}, the first entry to be determined from earlier diagonals is in position $(1, p+2k)$. The earlier diagonals are free.

2. The Nullity Theorem

The Nullity Theorem was given in 1984 by Gustafson [13] in the language of modules over a ring and in 1986 by Fiedler and Markham [10] in matrix language. It may have an earlier history (and it should!). Here is an equivalent statement:

Nullity Theorem

Complementary submatrices of a square matrix and its inverse have the same nullity.

Two submatrices are "*complementary*" when the row numbers not used in one are the column numbers used in the other. If the first submatrix A is M by N, the other submatrix D is $n-N$ by $n-M$. Suppose A is the upper left corner of an invertible matrix T, so that D is the lower right corner of the inverse:

$$M \text{ rows} \quad \begin{bmatrix} A & * \\ * & * \end{bmatrix}^{-1} = \begin{bmatrix} * & * \\ * & D \end{bmatrix} \quad \text{has} \quad \text{nullity}(A) = \text{nullity}(D)\,. \tag{4}$$
$$\qquad\qquad N \text{ columns} \qquad\quad n{-}M \text{ columns}$$

Note that all blocks can be rectangular. One partitioning is the "transpose" of the other, to allow block multiplication. A permutation will put both submatrices into the upper right corner (appropriately for our application):

$$M \text{ rows} \quad \begin{bmatrix} * & B \\ * & * \end{bmatrix}^{-1} = \begin{bmatrix} * & C \\ * & * \end{bmatrix} \quad \text{has} \quad \text{nullity}(B) = \text{nullity}(C)\,. \tag{5}$$
$$\qquad\qquad N \text{ columns} \qquad\quad n{-}M \text{ columns}$$

The submatrices A^{T} and D^{T} (as well as B^{T} and C^{T}) are again complementary, after the entire block matrices are transposed. So the Nullity Theorem applies also to the transposed submatrices (but we don't use it):

$$\text{nullity}(A^{\mathrm{T}}) = \text{nullity}(D^{\mathrm{T}})\,. \tag{6}$$

Following Fiedler and Markham, the direct proof begins with multiplication of the block matrix and its inverse to produce I.

Proof. Suppose the 2 by 2 block matrix T has $T_{11} = A$, and the inverse block matrix has $(T^{-1})_{22} = D$. Put a basis for the nullspace of A into the columns of a matrix N_1 (so $AN_1 = 0$). We will show that the (equal number of) columns of $N_2 = T_{21}N_1$ are a basis for the nullspace of D.

The second row of $T^{-1}T = I$ gives $(T^{-1})_{21}A + DT_{21} = 0$. Multiplying on the right by N_1 yields $DN_2 = 0$. To prove that the columns of N_2 are independent, suppose $N_2 y = 0$. Then

$$\begin{bmatrix} A & * \\ T_{21} & * \end{bmatrix} \begin{bmatrix} N_1 y \\ 0 \end{bmatrix} = \begin{bmatrix} AN_1 y \\ N_2 y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The invertibility of T forces $N_1 y = 0$. Since N_1 has independent columns, y must be zero. No other y is in the nullspace of N_2, so its columns are independent. This shows that $\text{nullity}(D) \geq \text{nullity}(A)$. Reversing the reasoning and using $TT^{-1} = I$, the nullities must be equal.

This neat proof will be in Section 0.7.5 of the forthcoming new edition of [16]. We thank our referee for expressing it so clearly. After two examples, we show how this Nullity Theorem (applied to B and C) leads instantly to the desired submatrix ranks in T^{-1}.

In the first special case, D (or C) is a 1 by 1 matrix. If its single entry is zero the nullity is 1. By the standard cofactor formula for this entry of the inverse matrix, the determinant of A (or B) of order $n-1$ is also zero. The nullity is therefore at least 1—but why not greater? Because if A (or B) had nullity greater than 1, and therefore had rank less than $n-2$, the whole n by n matrix T could not be invertible.

A particular form of block matrix arises frequently in applications, with a square zero block on the diagonal. It is interesting to see the Nullity Theorem in action once more:

$$T^{-1} = \begin{bmatrix} I_M & U \\ L & 0_N \end{bmatrix}^{-1} = \begin{bmatrix} I_M - U(LU)^{-1}L & U(LU)^{-1} \\ (LU)^{-1}L & -(LU)^{-1} \end{bmatrix}. \tag{7}$$

T is invertible only if the N columns of U and also the N rows of L are independent (requiring $M \geq N$). The nullity of 0_N is N. The complementary submatrix $I_M - U(LU)^{-1}L$ must have the same nullity. In this case the N columns of U are a basis for the nullspace of that submatrix:

$$(I_M - U(LU)^{-1}L)U = \text{ zero matrix}.$$

3. Proof of the Main Theorem

We restate the main theorem and prove it using the Nullity Theorem.

Every submatrix B above the pth superdiagonal of T has $\text{rank}(B) < k$ IFF every submatrix C above the pth subdiagonal of T^{-1} has $\text{rank}(C) < p+k$.

The proof comes directly from $TT^{-1} = I$. Look at the first M rows of T (with $k \leq M \leq n-p-k$). They multiply the last $n-M$ columns of T^{-1} to give a zero submatrix of I:

$$\begin{array}{l} M \text{ rows} \\ \text{of } T \\ \\ \end{array} \left[\begin{array}{cc} \boxed{A} & \boxed{B} \\ * & * \end{array}\right] \left[\begin{array}{cc} * & \boxed{C} \\ * & \boxed{D} \end{array}\right] = \left[\begin{array}{cc} I_M & \boxed{0} \\ * & * \end{array}\right]$$

$$M+p \quad n-M-p \qquad M \quad n-M$$

The lower left entry of B is in row M and column $M+p+1$, so this submatrix B is immediately above the pth superdiagonal of T. The lower left entry of

C is in row $M+p$ and column $M+1$. The difference $p-1$ means that C is immediately above the pth subdiagonal of T^{-1}.

These B's and C's are "maximal". They contain all the square submatrices required for the theorem (all B of size k and C of size $p+k$). As M varies from k to $n-p-k$, C captures all those submatrices of T^{-1} above the pth subdiagonal. And the B's contain all k by k submatrices of T above the pth superdiagonal.

Since B and C are complementary, the Nullity Theorem applies:

$$\text{nullity}(B) = \text{nullity}(C)\,.$$

The matrix B has $n-M-p$ columns, and C has $n-M$ columns. Therefore

$$n - M - p - \text{rank}(B) = n - M - \text{rank}(C)\,. \tag{8}$$

Then (exactly!) $\text{rank}(C) = p + \text{rank}(B)$. This means that $\text{rank}(B) < k$ if and only if $\text{rank}(C) < p+k$, and the main theorem is proved.

This proof and the next extend immediately to *block matrices* T and T^{-1}, because they deal one at a time with complementary pairs B and C. Suppose that T has block entries T_{ij} (of any compatible sizes), with $T_{ij} =$ zero block for $j-i>p$. The inverse of this block band matrix T has low rank submatrices C above the pth block subdiagonal. The proof is the same except that the choices of M respect the block form of T. The Mth row of B comes at the end of a block in T.

This block case is of utmost importance in applications.

4. The Rank Lemma and Second Proof

Lemma. Suppose the matrices A and C are M by N and N by L. Then

$$\text{rank}(A) + \text{rank}(C) \le N + \text{rank}(AC)\,. \tag{9}$$

Proof. If $AC =$ zero matrix, the column space of C is contained in the nullspace of A. The dimensions of those spaces give $\text{rank}(C) \le N - \text{rank}(A)$. This is (9) in the case $\text{rank}(AC) = 0$.

To reduce every other case to that one, suppose AC has rank $R > 0$. Then AC has a "full rank factorization" as the product $A'C'$ of an M by R matrix and an R by L matrix. (The singular value decomposition $AC = U\Sigma V^{\mathrm{T}}$

yields A' from the first R columns of U and C' from the first R rows of ΣV^{T}.) Now two block matrices multiply to give zero:

$$\begin{bmatrix} A & A' \end{bmatrix} \begin{bmatrix} C \\ -C' \end{bmatrix} = AC - A'C' = \text{ zero matrix}\,.$$

This returns us to the first case, with $N+R$ columns and rows instead of N:

$$\operatorname{rank}(A) + \operatorname{rank}(C) \le \operatorname{rank}(\begin{bmatrix} A & A' \end{bmatrix}) + \operatorname{rank}(\begin{bmatrix} C \\ -C' \end{bmatrix}) \le N+R, \text{ which is (9)}\,.$$

Note. This lemma is a key to our proof of the main theorem, but (on such a basic subject!) it could not possibly be new. It is the special case $B = I_N$ of the *Frobenius rank inequality* [16, p. 13] for three matrices:

$$\operatorname{rank}(AB) + \operatorname{rank}(BC) \le \operatorname{rank}(B) + \operatorname{rank}(ABC)\,. \tag{10}$$

We noticed that our weaker result (9) leads quickly to (10). Suppose B is any N_1 by N_2 matrix with $\operatorname{rank}(B) = N$. Then B has a full rank factorization into $B_1 B_2 = (N_1 \text{ by } N)(N \text{ by } N_2)$. Applying (9) to the matrices AB_1 and B_2C gives

$$\operatorname{rank}(AB_1) + \operatorname{rank}(B_2C) \le N + \operatorname{rank}(ABC)\,. \tag{11}$$

Always $\operatorname{rank}(AB) = \operatorname{rank}(AB_1B_2) \le \operatorname{rank}(AB_1)$. Similarly $\operatorname{rank}(BC) = \operatorname{rank}(B_1B_2C) \le \operatorname{rank}(B_2C)$. So (11) implies (10).

Our second proof of the main theorem follows quickly. The submatrices A, B in T and C, D in T^{-1} still have $AC + BD =$ zero submatrix of I. Suppose $\operatorname{rank}(B) < k$, and therefore $\operatorname{rank}(AC) = \operatorname{rank}(BD) < k$. Since A has $M+p$ columns, the lemma gives

$$\operatorname{rank}(A) + \operatorname{rank}(C) < M + p + k\,. \tag{12}$$

If A has full rank M, then $\operatorname{rank}(C) < p + k$ as the theorem requires.

In case A fails to have full rank M, perturb it a little. The new T is still invertible and the theorem applies. We have proved that the new submatrix C in T^{-1} has rank less than $p+k$. That rank cannot jump as the perturbation goes to zero, so the actual C also has rank less than $p+k$.

The proof of the converse is similar, but with a little twist. (We need a nullspace matrix N.) Suppose $\operatorname{rank}(C) < p + k$. Since C has $n - M$ columns, its nullspace has dimension at least $L = n - M - p - k + 1$. Put

L linearly independent nullvectors of C into the columns of a matrix N, so that $CN = 0$. Then $ACN = 0$ implies $BDN = 0$ (since $AC + BD = 0$) and we can apply the rank lemma:

$$\text{rank}(B) + \text{rank}(DN) \leq \text{ (number of columns of } B) \ = n - M - p\,. \quad (13)$$

If DN has full column rank L, our conclusion follows:

$$\text{rank}(B) \leq n - M - p - L = k - 1, \text{ as desired}\,.$$

Notice that DN has $n - M - p$ rows, which is at least L. If it happens that $\text{rank}(DN) < L$, perturb D a little to achieve full rank. We don't change C or N, so our proof applies and the new submatrix B (in the new T) has rank less than k. As the perturbation of T^{-1} goes to zero, the new T approaches the actual T. So the actual submatrix B has rank less than k (since the rank can't suddenly increase).

This completes the second proof, when M takes all values from k to $n-p-k$. That largest M captures the submatrix B in the last k columns of T, above the pth superdiagonal. We have proved that all k by k submatrices of T above that diagonal are singular.

Note on the proof: To see where the matrix N is needed (as in the Nullity Theorem!) write out $AC + BD = 0$ in the 3 by 3 case for $p = k = M = 1$:

$$\begin{bmatrix} T_{11} & T_{12} \end{bmatrix} \begin{bmatrix} C \text{ with} \\ \text{rank} < 2 \end{bmatrix} + B \begin{bmatrix} T^{-1}_{32} & T^{-1}_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

We must prove $\text{rank}(B) < 1$, in other words $B = [\,T_{13}\,] = 0$. If N contains a 2 by 1 nullvector of C, the first term ACN disappears to leave

$$BDN = B \begin{bmatrix} T^{-1}_{32} & T^{-1}_{33} \end{bmatrix} N = 0\,.$$

This proves $B = 0$, after possible perturbation of D to make $DN \neq 0$.

5. References and Alternative Proofs

Normally we would comment on the existing literature before adding to it! That is the proper order, especially for a theorem to which many earlier authors have contributed. The first reference we know is to Asplund [1] in 1959. He discovered the banded case: bandwidth p in T and rank p

submatrices in T^{-1}. Rereading his proof, we see that he anticipated many of the ideas that came later.

Combining his theorem with the Woodbury-Morrison formula for a rank k perturbation would complete the proof. Remarkably, it was a paper by his father S. O. Asplund (in the same journal) that led Edgar Asplund to the problem. We thank Gene Golub and Vidar Thomée for this history of the theorem.

The crucial tridiagonal case appears independently in the famous 1960 text of Gantmacher and Krein [11, p. 95]. There is a close connection to second-order differential equations, noted in our final section below. Karlin's 1968 book *Total Positivity* [18] refers to T^{-1} as a "Green's matrix". Subsequently, Barrett [2] proved from explicit formulas that tridiagonality of T is equivalent to rank 1 submatrices in T^{-1}. The formulas allowed him to clarify all cases of zero entries in T.

The natural extension to block tridiagonal matrices was given in the same year by Ikebe [17]. Then Cao and Stewart [6] allowed $p > 1$ and blocks of varying sizes. The crucial step to $k > 1$ was taken by Barrett and Feinsilver [3]. Their proof of the Theorem is based on the formula for the minors of T^{-1}. Meurant [19] has provided an extremely helpful survey of the literature, and a stable algorithm for computing T^{-1} in the tridiagonal case.

It is interesting to recognize these two approaches: *determinant formulas* or *rank and nullity formulas*. The former come ultimately from Jacobi and Sylvester. They led in [3] to the conditions of unique completion of T^{-1} starting from its central band (for $k = 1$ and any p). Rank and nullity are free of coordinates. The Nullity Theorem applies to $A = P_1TP_2$ and $D = (I - P_2)T^{-1}(I - P_1)$ for all orthogonal projections P_1 and P_2. A key question will be the generalization to infinite dimensions.

6. Fast Matrix Multiplication and Applications

For a tridiagonal matrix T, the upper blocks C in T^{-1} (and also the lower blocks) have $\text{rank}(C) \leq 1$. How quickly can we multiply a full n by n matrix T^{-1} of this "semiseparable" form by a vector x? We take this tridiagonal case ($p = 1$ and $k = 0$) as our model, and approach the complexity of $T^{-1}x$ in the most direct way.

If $C = uv^{\mathrm{T}}$ is an M by N matrix of rank one, the product $Cy = u(v^{\mathrm{T}}y)$ requires $M + N$ individual multiplications (rather than MN). The natural partition of T^{-1} is to begin with square blocks C_{12} and C_{21} of size $n/2$, in the upper right and lower left corners of T^{-1}. This leaves blocks C_{11} and C_{22} of size $n/2$ on the diagonal of T^{-1}, to be partitioned (recursively) in the same way. The multiplication count $m(n) = m(C_{11}) + m(C_{22}) + m(C_{12}) + m(C_{21})$ obeys a rule much like the FFT:

$$m(n) = 2\,m\Big(\frac{n}{2}\Big) + 2\left(\frac{n}{2} + \frac{n}{2}\right) . \tag{14}$$

This recursion is satisfied by $m(n) = 2n \log_2 n$.

The true applications of the Theorem in this paper are not to tridiagonal matrices but to integral equations—often with space dimension greater than one. Now a model problem is the approximate computation of a single or double integral:

$$\int K(s,x) f(x)\, dx \quad \text{or} \quad \iint K(s,t,x,y) f(x,y)\, dx\, dy \, .$$

The kernel K may be the Green's function of an underlying differential equation. We expect to see blocks, rather than scalar entries, when approximating double integrals. The parameters p (for bandwidth) and k (for off-diagonal rank) have "continuous" analogs for an integral operator:

Decay rate: Fast decay away from the diagonal $K(x,x)$ corresponds to low bandwidth.

Smoothness: A slowly varying kernel corresponds to low rank submatrices.

In both cases the word "approximate" should be included. We have matrix analysis rather than matrix algebra.

To summarize the applications to fast solution of integral equations, our best plan is to point to several active groups (with apologies to others). The first group has emphasized the connections to earlier "panel methods" and the delicate partitioning that can sometimes reduce the operation count to $O(n)$—for matrix inversion as well as multiplication. We hope these names and references will help the reader:

1. Hackbusch and Khoromskij [14, 15]

2. Chandrasekaran and Gu [7]

3. Tyrtyshnikov, Gorainov, and Yeremin [12, 21]

4. Eidelman and Gohberg [8, 9]

7. Tridiagonal Matrices and Differential Equations

The tridiagonal case is the simplest and most important. If we look at the first two columns of T^{-1}, below the first row, then the "lower" statement at the start of our paper means: *Those columns are proportional.* We want to discuss this conclusion directly, and also to recognize the analogous statement for second-order differential equations and their Green's functions.

In the matrix case, the tridiagonal T multiplies the first column of T^{-1} to give zeros below the diagonal. When the second column of T^{-1} is proportional to the first, multiplication by T gives those zeros again (below the $2,2$ entry). Somehow that small observation is the key to rank 1 (below the diagonal). Roughly speaking, these columns of T^{-1} contain a solution of the "second-order difference equation $Ty = 0$". It is the solution that satisfies the "boundary condition" at the right endpoint (the end of the column).

We see multiples of this homogeneous solution in all columns of T^{-1}, below the diagonal. They meet multiples of the other solution on the main diagonal. That other solution of $Ty = 0$ satisfies the boundary condition at the left endpoint (the top of the column, above the diagonal).

Since T can be any invertible tridiagonal matrix, $Ty = 0$ may not look like a second-order difference equation. (Actually we could reproduce row k of Ty by choosing three numbers a_k, b_k, c_k to multiply $\Delta^2 y_k, \Delta y_k, y_k$.) It may be useful to compare with the standard approach to second-order differential equations, where the analog of column j in $TT^{-1} = I$ is

$$Ly = a(x)y'' + b(x)y' + c(x)y = \delta(x - a)$$

with boundary conditions at $x = 0$ and $x = 1$. The solution $y(x) = G(x, a)$ is the Green's function and it corresponds to $(T^{-1})_{ij}$.

A good text like [5, pp. 15–18] notes the equivalence between computing G and variation of parameters. The latter begins with two independent solutions $y_1(x)$ and $y_2(x)$ of $Ly = 0$ and finds a particular solution to $Ly = \delta$ of the form

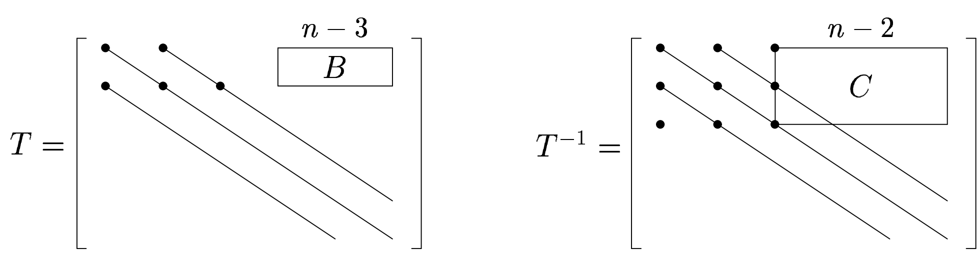
$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)\,. \tag{15}$$

A few remarks will connect this continuous problem with the discrete $TT^{-1} = I$. Suppose $y_2(x)$ is chosen to satisfy the boundary condition at the right endpoint $x = 1$, as well as $Ly_2 = 0$. Similarly $y_1(x)$ satisfies $Ly_1 = 0$ and the boundary condition at $x = 0$. With those special choices, the Green's function $G(x, a)$ will be a multiple of $y_2(x)$ for $x \geq a$ and a multiple of $y_1(x)$ for

$x \leq a$. Those multiples $u_1(x)$ and $u_2(x)$ correspond to the "proportionality constants" that connect the columns of T^{-1}, above and below its main diagonal. And those constants are given by the first and last *rows* of T^{-1}, which solve the *adjoint* problem based on the transpose of T.

Perhaps we can summarize the discrete case in this way. The matrix T^{-1} is determined by its first and last columns and rows. It has rank 1 above and below the diagonal. Those parameters are reduced to the correct number $3n-2$ by equality along the diagonal.

To close the circle, we specialize the matrices B and C to this tridiagonal case. For $M = 2$ rows, B is a 2 by $n-3$ submatrix of zeros and its nullity is $n-3$. Then C is a 3 by $n-2$ matrix with the same nullity. Therefore its rank is 1! As M varies, all the submatrices C of T^{-1} have rank 1.



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References

[1] E. Asplund, Inverses of matrices $\{a_{ij}\}$ which satisfy $a_{ij} = 0$ for $j > i+p$, *Math. Scand.*, **7** (1959) 57–60.

[2] W. W. Barrett, C. R. Johnson, D. D. Olesky and P. van den Driessche, Inherited matrix entries: principal submatrices of the inverse, *SIAM J. Alg. Disc. Meth.*, **8** (1987) 313–322.

[3] W. W. Barrett and P. J. Feinsilver, Inverses of banded matrices, *Linear Alg. Appl.*, **41** (1981) 111–130.

[4] W. W. Barrett, A theorem on inverse of tridiagonal matrices, *Linear Alg. Appl.*, **27** (1979) 211–217.

[5] C. Bender and S. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill, 1978; Springer, 1999.

[6] W. L. Cao and W. J. Stewart, A note on inverses of Hessenberg-like matrices, *Linear Alg. Appl.*, **76** (1986) 233–240.

[7] S. Chandrasekaran and M. Gu, Fast and stable algorithms for banded plus semiseparable systems of linear equations, *SIAM J. Matrix Anal. Appl.*, **25** (2003) 373–384.

[8] Y. Eidelman and I. Gohberg, On a new class of structured matrices, *Integral Eq. and Oper. Theory*, **34** (1999) 293–324.

[9] Y. Eidelman and I. Gohberg, Inversion formulas and linear complexity algorithm for diagonal plus semiseparable matrices, *Computers and Math. with Appl.*, **33** (1997) 69–79.

[10] M. Fiedler and T. L. Markham, Completing a matrix when certain entries of its inverse are specified, *Linear Alg. Appl.*, **74** (1986) 225–237.

[11] F. R. Gantmacher and M. G. Krein, *Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme*, Akademie-Verlag, 1960. *Oscillation matrices and kernels and small vibrations of mechanical systems*, Amer. Math. Society, 2002.

[12] S. Gorainov, E. Tyrtyshnikov and A. Yeremin, Matrix-free iterative solution strategies for large dense linear systems, *Numerical Linear Alg. and Appl.*, **4** (1997) 273–294.

[13] W. H. Gustafson, A note on matrix inversion, *Linear Alg. Appl.*, **57** (1984) 71–73.

[14] W. Hackbusch, A sparse $\mathcal{H}$-matrix arithmetic, *Computing*, **62** (1999) 89–108.

[15] W. Hackbusch and B. N. Khoromskij, *Towards $\mathcal{H}$-matrix approximation of linear complexity*, Problems and Methods in Mathematical Physics, J. Elschner, I. Gohberg and B. Silbermann, eds., Birkhäuser, Basel, 2001.

[16] R. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge Univ. Press, 1985.

[17] Y. Ikebe, On inverses of Hessenberg matrices, *Linear Alg. Appl.*, **24** (1979) 93–97.

[18] S. Karlin, *Total Positivity*, Stanford Univ. Press, 1968.

[19] G. Meurant, A review of the inverse of symmetric tridiagonal and block tridiagonal matrices, *SIAM J. Matrix Anal. Appl.*, **13** (1992) 707–728.

[20] G. Strang, *Introduction to Linear Algebra*, 3rd Edition, Wellesley-Cambridge Press, 2003.

[21] E. E. Tyrtyshnikov, Mosaic-skeleton approximations, *Calcolo*, **33** (1996) 47–57.