INTRODUCTION

TO

LINEAR

ALGEBRA

Fifth Edition

MANUAL FOR INSTRUCTORS

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Problem Set 10.1, page 459

1 $A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$; nullspace contains $\begin{bmatrix} c \\ c \\ c \end{bmatrix}$; $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not orthogonal to that nullspace.

2 $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ for $\boldsymbol{y} = (1, -1, 1)$; current along edge 1, edge 3, back on edge 2 (full loop).

3 $[\,A \quad \boldsymbol{b}\,] = \begin{bmatrix} -1 & 1 & 0 & b_1 \\ -1 & 0 & 1 & b_2 \\ 0 & -1 & 1 & b_3 \end{bmatrix}$ leads to $[\,U \quad \boldsymbol{c}\,] = \begin{bmatrix} -1 & 1 & 0 & b_1 \\ 0 & -1 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - b_2 + b_1 \end{bmatrix}$.

The nonzero rows of U come from edges 1 and 3 in a tree. The zero row comes from the loop (all 3 edges).

4 For the matrix in Problem 3, $A\boldsymbol{x} = \boldsymbol{b}$ is solvable for $\boldsymbol{b} = (1, 1, 0)$ and not solvable for $\boldsymbol{b} = (1, 0, 0)$. For solvable $\boldsymbol{b}$ (in the column space), $\boldsymbol{b}$ must be orthogonal to $\boldsymbol{y} = (1, -1, 1)$; that combination of rows is the zero row, and $b_1 - b_2 + b_3 = 0$ is the third equation after elimination.

5 Kirchhoff's Current Law $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ is solvable for $\boldsymbol{f} = (1, -1, 0)$ and not solvable for $\boldsymbol{f} = (1, 0, 0)$; $\boldsymbol{f}$ must be orthogonal to $(1, 1, 1)$ in the nullspace: $f_1 + f_2 + f_3 = 0$.

6 $A^{\mathrm{T}}A\boldsymbol{x} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = \boldsymbol{f}$ produces $\boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ c \\ c \end{bmatrix}$; potentials $\boldsymbol{x} = 1, -1, 0$ and currents $-A\boldsymbol{x} = 2, 1, -1$; $\boldsymbol{f}$ sends 3 units from node 2 into node 1.

7 $A^{\mathrm{T}} \begin{bmatrix} 1 & & \\ & 2 & \\ & & 2 \end{bmatrix} A = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 3 & -2 \\ -2 & -2 & 4 \end{bmatrix}$; $\boldsymbol{f} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ yields $\boldsymbol{x} = \begin{bmatrix} 5/4 \\ 1 \\ 7/8 \end{bmatrix} +$ any $\begin{bmatrix} c \\ c \\ c \end{bmatrix}$; potentials $\boldsymbol{x} = \frac{5}{4}, 1, \frac{7}{8}$ and currents $-CA\boldsymbol{x} = \frac{1}{4}, \frac{3}{4}, \frac{1}{4}$.

8 $A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ leads to $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ solving $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$.

9 Elimination on $A\boldsymbol{x} = \boldsymbol{b}$ always leads to $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 0$ in the zero rows of U and R: $-b_1 + b_2 - b_3 = 0$ and $b_3 - b_4 + b_5 = 0$ (those $\boldsymbol{y}$'s are from Problem 8 in the left nullspace). This is Kirchhoff's ***Voltage*** Law around the two ***loops***.

10 The echelon form of A is $U = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ The nonzero rows of U keep edges 1, 2, 4. Other spanning trees from edges, 1, 2, 5; 1, 3, 4; 1, 3, 5; 1, 4, 5; 2, 3, 4; 2, 3, 5; 2, 4, 5.

11 $A^{\mathrm{T}}A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$ diagonal entry $=$ number of edges into the node
the trace is 2 times the number of nodes
off-diagonal entry $= -1$ if nodes are connected
$A^{\mathrm{T}}A$ is the **graph Laplacian**, $A^{\mathrm{T}}CA$ is **weighted** by C

12 (a) The nullspace and rank of $A^{\mathrm{T}}A$ and A are always the same (b) $A^{\mathrm{T}}A$ is always positive semidefinite because $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}A\boldsymbol{x} = \|A\boldsymbol{x}\|^2 \geq 0$. Not positive definite because rank is only 3 and $(1, 1, 1, 1)$ is in the nullspace (c) Real eigenvalues all ≥ 0 because positive semidefinite.

13 $A^{\mathrm{T}}CA\boldsymbol{x} = \begin{bmatrix} 4 & -2 & -2 & 0 \\ -2 & 8 & -3 & -3 \\ -2 & -3 & 8 & -3 \\ 0 & -3 & -3 & 6 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ gives four potentials $\boldsymbol{x} = (\frac{5}{12}, \frac{1}{6}, \frac{1}{6}, 0)$
I grounded $x_4 = 0$ and solved for $\boldsymbol{x}$
currents $\boldsymbol{y} = -CA\boldsymbol{x} = (\frac{2}{3}, \frac{2}{3}, 0, \frac{1}{2}, \frac{1}{2})$

14 $A^{\mathrm{T}}CA\boldsymbol{x} = \mathbf{0}$ for $\boldsymbol{x} = c(1, 1, 1, 1) = (c, c, c, c)$. If $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{f}$ is solvable, then $\boldsymbol{f}$ in the column space ($=$ row space by symmetry) must be orthogonal to $\boldsymbol{x}$ in the nullspace: $f_1 + f_2 + f_3 + f_4 = 0$.

15 The number of loops in this connected graph is $n - m + 1 = \mathbf{7} - \mathbf{7} + \mathbf{1} = \mathbf{1}$. What answer if the graph has two separate components (no edges between)?

16 Start from (4 nodes) $-$ (6 edges) $+$ (3 loops) $= 1$. If a new node connects to 1 old node, $5 - 7 + 3 = 1$. If the new node connects to 2 old nodes, a new loop is formed: $5 - 8 + 4 = 1$.

17 (a) 8 independent columns (b) $\boldsymbol{f}$ must be orthogonal to the nullspace so f's add to zero (c) Each edge goes into 2 nodes, 12 edges make diagonal entries sum to 24.

18 A *complete graph* has $5 + 4 + 3 + 2 + 1 = 15$ edges. With n nodes that count is $1 + \cdots + (n-1) = n(n-1)/2$. Tree has 5 edges.

Problem Set 10.2, page 472

1 Det $A_0^{\mathrm{T}} C_0 A_0 = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix}$ is by direct calculation. Set $c_4 = 0$ to find det $A_1^{\mathrm{T}} C_1 A_1 = c_1 c_2 c_3$.

2 $(A_1^{\mathrm{T}} C_1 A_1)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1^{-1} & & \\ & c_2^{-1} & \\ & & c_3^{-1} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} =$

$$\begin{bmatrix} c_1^{-1} & c_1^{-1} & c_1^{-1} \\ c_1^{-1} & c_1^{-1} + c_2^{-1} & c_1^{-1} + c_2^{-1} \\ c_1^{-1} & c_1^{-1} + c_2^{-1} & c_1^{-1} + c_2^{-1} + c_3^{-1} \end{bmatrix}.$$

3 The rows of the free-free matrix in equation (9) add to $[\,0 \quad 0 \quad 0\,]$ so the right side needs $f_1 + f_2 + f_3 = 0$. $\boldsymbol{f} = (-1, 0, 1)$ gives $c_2 u_1 - c_2 u_2 = -1, c_3 u_2 - c_3 u_3 = -1, 0 = 0$. Then $\boldsymbol{u}_{\text{particular}} = (-c_2^{-1} - c_3^{-1}, -c_3^{-1}, 0)$. Add any multiple of $\boldsymbol{u}_{\text{nullspace}} = (1, 1, 1)$.

4 $\displaystyle\int -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) dx = -\left[c(x)\frac{du}{dx}\right]_0^1 = 0$ (bdry cond) so we need $\displaystyle\int f(x)\,dx = 0$.

5 $-\dfrac{dy}{dx} = f(x)$ gives $y(x) = C - \displaystyle\int_0^x f(t)dt$. Then $y(1) = 0$ gives $C = \displaystyle\int_0^1 f(t)dt$ and $y(x) = \displaystyle\int_x^1 f(t)dt$. If the load is $f(x) = 1$ then the displacement is $y(x) = 1 - x$.

6 Multiply $A_1^{\mathrm{T}} C_1 A_1$ as columns of A_1^{T} times c's times rows of A_1. The first 3 by 3 "***element matrix***" $c_1 E_1 = [\,1 \;\; 0 \;\; 0\,]^{\mathrm{T}} c_1 [\,1 \;\; 0 \;\; 0\,]$ has c_1 in the top left corner.

7 For 5 springs and 4 masses, the 5 by 4 A has two nonzero diagonals: all $a_{ii} = 1$ and $a_{i+1,i} = -1$. With $C = \mathsf{diag}(c_1, c_2, c_3, c_4, c_5)$ we get $K = A^{\mathrm{T}} C A$, symmetric tridiagonal with diagonal entries $K_{ii} = c_i + c_{i+1}$ and off-diagonals $K_{i+1,i} = -c_{i+1}$. With $C = I$ this K is the $-1, 2, -1$ matrix and $K(2, 3, 3, 2) = (1, 1, 1, 1)$ solves $K\boldsymbol{u} = \mathsf{ones}(4, 1)$. ($K^{-1}$ will solve $K\boldsymbol{u} = \mathsf{ones}(4)$.)

8 The solution to $-u'' = 1$ with $u(0) = u(1) = 0$ is $u(x) = \frac{1}{2}(x - x^2)$. At $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ this gives $\boldsymbol{u} = 2, 3, 3, 2$ (discrete solution in Problem 7) times $(\Delta x)^2 = 1/25$.

9 $-u'' = mg$ has complete solution $u(x) = A + Bx - \frac{1}{2}mgx^2$. From $u(0) = 0$ we get $A = 0$. From $u'(1) = 0$ we get $B = mg$. Then $u(x) = \frac{1}{2}mg(2x - x^2)$ at $x = \frac{1}{3}, \frac{2}{3}, \frac{3}{3}$ equals $mg/6, 4mg/9, mg/2$. This $u(x)$ is ***not*** proportional to the discrete $u = (3mg, 5mg, 6mg)$ at the meshpoints. This imperfection is because the discrete problem uses a 1-sided difference, less accurate at the free end. Perfect accuracy is recovered by a centered difference (discussed on page 21 of my CSE textbook).

10 (added in later printing, changing **10-11** below into **11-12**). The solution in this fixed-fixed case is $(2.25, 2.50, 1.75)$ so the second mass moves furthest.

11 The two graphs of 100 points are "discrete parabolas" starting at $(0, 0)$: symmetric around 50 in the fixed-fixed case, ending with slope zero in the fixed-free case.

12 Forward/backward/centered for du/dx has a big effect because that term has the large coefficient. MATLAB: $E = \mathsf{diag}(\mathsf{ones}(6, 1), 1)$; $K = 64 * (2 * \mathsf{eye}(7) - E - E')$; $D = 80 * (E - \mathsf{eye}(7))$; $(K + D)\backslash\mathsf{ones}(7, 1)$; % forward; $(K - D')\backslash\mathsf{ones}(7, 1)$; % backward; $(K + D/2 - D'/2)\backslash\mathsf{ones}(7, 1)$; % centered is usually the best: more accurate

Problem Set 10.3, page 480

1 Eigenvalues $\lambda = 1$ and $.75$; $(A - I)\boldsymbol{x} = 0$ gives the steady state $\boldsymbol{x} = (.6, .4)$ with $A\boldsymbol{x} = \boldsymbol{x}$.

2 $A = \begin{bmatrix} .6 & -1 \\ .4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & .75 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix}$; $A^\infty = \begin{bmatrix} .6 & -1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$.

3 $\lambda = 1$ and $.8$, $\boldsymbol{x} = (1, 0)$; 1 and $-.8$, $\boldsymbol{x} = (\frac{5}{9}, \frac{4}{9})$; $1, \frac{1}{4}$, and $\frac{1}{4}$, $\boldsymbol{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

4 A^{T} always has the eigenvector $(1, 1, \ldots, 1)$ for $\lambda = 1$, because each row of A^{T} adds to 1. (Note again that many authors use row vectors multiplying Markov matrices. So they transpose our form of A.)

5 The steady state eigenvector for $\lambda = 1$ is $(0, 0, 1) =$ everyone is dead.

6 Add the components of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to find sum $s = \lambda s$. If $\lambda \neq 1$ the sum must be $s = 0$.

7 $(.5)^k \to 0$ gives $A^k \to A^\infty$; any $A = \begin{bmatrix} .6 + .4a & .6 - .6a \\ .4 - .4a & .4 + .6a \end{bmatrix}$ with $\begin{matrix} a \leq 1 \\ .4 + .6a \geq 0 \end{matrix}$

8 If $P =$ cyclic permutation and $\boldsymbol{u}_0 = (1, 0, 0, 0)$ then $\boldsymbol{u}_1 = (0, 0, 1, 0)$; $\boldsymbol{u}_2 = (0, 1, 0, 0)$; $\boldsymbol{u}_3 = (1, 0, 0, 0)$; $\boldsymbol{u}_4 = \boldsymbol{u}_0$. The eigenvalues $1, i, -1, -i$ are all ***on the unit circle***. This Markov matrix contains zeros; a ***positive*** matrix has ***one*** largest eigenvalue $\lambda = 1$.

9 M^2 is still nonnegative; $[\,1 \ \ \cdots \ \ 1\,]M = [\,1 \ \ \cdots \ \ 1\,]$ so multiply on the right by M to find $[\,1 \ \ \cdots \ \ 1\,]M^2 = [\,1 \ \ \cdots \ \ 1\,] \Rightarrow$ columns of M^2 add to 1.

10 $\lambda = 1$ and $a + d - 1$ from the trace; steady state is a multiple of $\boldsymbol{x}_1 = (b, 1 - a)$.

11 Last row $.2, .3, .5$ makes $A = A^{\mathrm{T}}$; rows also add to 1 so $(1, \ldots, 1)$ is also an eigenvector of A.

12 B has $\lambda = 0$ and $-.5$ with $\boldsymbol{x}_1 = (.3, \ .2)$ and $\boldsymbol{x}_2 = (-1, 1)$; A has $\lambda = 1$ so $A - I$ has $\lambda = 0$. $e^{-.5t}$ approaches zero and the solution approaches $c_1 e^{0t}\boldsymbol{x}_1 = c_1\boldsymbol{x}_1$.

13 $\boldsymbol{x} = (1, 1, 1)$ is an eigenvector when the row sums are equal; $A\boldsymbol{x} = (.9, .9, .9)$

14 $(I-A)(I+A+A^2+\cdots) = (I+A+A^2+\cdots)-(A+A^2+A^3+\cdots) = \boldsymbol{I}$. This says that $I + A + A^2 + \cdots$ is $(I - A)^{-1}$. When $A = \begin{bmatrix} 0 & .5 \\ 1 & 0 \end{bmatrix}$, $A^2 = \frac{1}{2}I$, $A^3 = \frac{1}{2}A$, $A^4 = \frac{1}{4}I$ and the series adds to $\begin{bmatrix} 1+\frac{1}{2}+\cdots & \frac{1}{2}+\frac{1}{4}+\cdots \\ 1+\frac{1}{2}+\cdots & 1+\frac{1}{2}+\cdots \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} = (I-A)^{-1}$.

15 The first two A's have $\lambda_{\max} < 1$; $\boldsymbol{p} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 130 \\ 32 \end{bmatrix}$; $I - \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix}$ has no inverse.

16 $\lambda = 1$ (Markov), 0 (singular), .2 (from trace). Steady state $(.3, .3, .4)$ and $(30, 30, 40)$.

17 *No*, A has an eigenvalue $\lambda = 1$ and $(I - A)^{-1}$ does not exist.

18 The Leslie matrix on page 435 has $\det(A-\lambda I) = \det \begin{bmatrix} F_1-\lambda & F_2 & F_3 \\ P_1 & -\lambda & 0 \\ 0 & P_2 & -\lambda \end{bmatrix} = -\lambda^3 + F_1\lambda^2 + F_2P_1\lambda + F_3P_1P_2$. This is negative for large λ. It is positive at $\lambda = 1$ provided that $F_1 + F_2P_1 + F_3P_1P_2 > 1$. Under this key condition, $\det(A - \lambda I)$ must be zero at some λ between 1 and ∞. That eigenvalue means that the population grows (under this condition connecting F's and P's reproduction and survival rates).

19 Λ times $X^{-1}\Delta X$ has the same diagonal as $X^{-1}\Delta X$ times Λ because Λ is diagonal.

20 If $B > A > 0$ and $A\boldsymbol{x} = \lambda_{\max}(A)\boldsymbol{x} > 0$ then $B\boldsymbol{x} > \lambda_{\max}(A)\boldsymbol{x}$ and $\lambda_{\max}(B) > \lambda_{\max}(A)$.

of C = four components of $F\boldsymbol{c}$. Circulants are special!

Problem Set 10.4, page 489

1 Feasible set = line segment $(6, 0)$ to $(0, 3)$; minimum cost at $(6, 0)$, maximum at $(0, 3)$.

2 Feasible set has corners $(0, 0), (6, 0), (2, 2), (0, 6)$. Minimum cost $2x - y$ at $(6, 0)$.

3 Only two corners $(4, 0, 0)$ and $(0, 2, 0)$; let $x_i \to -\infty$, $x_2 = 0$, and $x_3 = x_1 - 4$.

4 From $(0, 0, 2)$ move to $\boldsymbol{x} = (0, 1, 1.5)$ with the constraint $x_1 + x_2 + 2x_3 = 4$. The new cost is $3(1) + 8(1.5) = \$15$ so $r = -1$ is the reduced cost. The simplex method also checks $\boldsymbol{x} = (1, 0, 1.5)$ with cost $5(1) + 8(1.5) = \$17$; $r = 1$ means more expensive.

5 Cost $= 20$ at start $(4,0,0)$; keeping $x_1+x_2+2x_3 = 4$ move to $(3,1,0)$ with cost 18 and $r = -2$; or move to $(2,0,1)$ with cost 17 and $r = -3$. Choose x_3 as entering variable and move to $(0,0,2)$ with cost 14. Another step will reach $(0,4,0)$ with minimum cost 12.

6 If we reduce the Ph.D. cost to \$1 or \$2 (below the student cost of \$3), the job will go to the Ph.D. with cost vector $\boldsymbol{c} = (2,3,8)$ the Ph.D. takes 4 hours $(x_1 + x_2 + 2x_3 = 4)$ and charges \$8.

The teacher in the dual problem now has $y \leq 2, y \leq 3, 2y \leq 8$ as constraints $A^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}$ on the charge of y per problem. So the dual has maximum at $y = 2$. The dual cost is also \$8 for 4 problems and maximum $=$ minimum.

7 $\boldsymbol{x} = (2,2,0)$ is a corner of the feasible set with $x_1+x_2+2x_3 = 4$ and the new constraint $2x_1 + x_2 + x_3 = 6$. The cost of this corner is $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = (5,3,8)\cdot(2,2,0) = 16$. Is this the minimum cost?

Compute the reduced cost r if $x_3 = 1$ enters (x_3 was previously zero). The two constraint equations now require $x_1 = 3$ and $x_2 = -1$. With $\boldsymbol{x} = (3,-1,1)$ the new cost is $3.5 - 1.3 + 1.8 = 20$. This is higher than 16, so the original $\boldsymbol{x} = (2,2,0)$ was optimal.

Note that $x_3 = 1$ led to $x_2 = -1$ and a negative x_2 is not allowed. If x_3 reduced the cost (it didn't) we would not have used as much as $x_3 = 1$.

8 $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \leq \boldsymbol{y}^{\mathrm{T}}A\boldsymbol{x} = (A^{\mathrm{T}}\boldsymbol{y})^{\mathrm{T}}\boldsymbol{x} \leq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$. The first inequality needed $\boldsymbol{y} \geq 0$ and $A\boldsymbol{x}-\boldsymbol{b} \geq 0$.

Problem Set 10.5, page 494

1 $\int_0^{2\pi} \cos((j+k)x)\,dx = \left[\frac{\sin((j+k)x)}{j+k}\right]_0^{2\pi} = 0$ and similarly $\int_0^{2\pi} \cos((j-k)x)\,dx = 0$ Notice $j - k \neq 0$ in the denominator. If $j = k$ then $\int_0^{2\pi} \cos^2 jx\,dx = \pi$.

2 Three integral tests show that $1, x, x^2 - \frac{1}{3}$ are orthogonal on the interval $[-1,\ 1]$: $\int_{-1}^{1}(1)(x)\ \ dx = 0, \int_{-1}^{1}(1)(x^2 - \frac{1}{3})\ \ dx = 0, \int_{-1}^{1}(x)(x^2 - \frac{1}{3})\ \ dx = 0$. Then

$2x^2 = 2(x^2 - \frac{1}{3}) + 0(x) + \frac{2}{3}(1)$. Those coefficients $2, 0, \frac{2}{3}$ can come from integrating $f(x) = 2x^2$ times the 3 basis functions and dividing by their lengths squared—in other words using $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/a^{\mathrm{T}}a$ for functions (where $\boldsymbol{b}$ is $f(x)$ and $\boldsymbol{a}$ is 1 or x or $x^2 - \frac{1}{3}$) exactly as for vectors.

3 One example orthogonal to $\boldsymbol{v} = (1, \frac{1}{2}, \ldots)$ is $\boldsymbol{w} = (2, -1, 0, 0, \ldots)$ with $\|\boldsymbol{w}\| = \sqrt{5}$.

4 $\int_{-1}^{1}(1)(x^3 - cx)\,dx = 0$ and $\int_{-1}^{1}(x^2 - \frac{1}{3})(x^3 - cx)\,dx = 0$ for all c (odd functions). Choose c so that $\int_{-1}^{1} x(x^3 - cx)\,dx = [\frac{1}{5}x^5 - \frac{c}{3}x^3]_{-1}^{1} = \frac{2}{5} - c\frac{2}{3} = 0$. Then $c = \frac{3}{5}$.

5 The integrals lead to the Fourier coefficients $a_1 = 0,\ b_1 = 4/\pi,\ b_2 = 0$.

6 From eqn. (3) $a_k = 0$ and $b_k = 4/\pi k$ (***odd*** k). The square wave has $\|f\|^2 = 2\pi$. Then eqn. (6) is $2\pi = \pi(16/\pi^2)(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots)$. That infinite series equals $\pi^2/8$.

7 The $-1, 1$ odd square wave is $f(x) = x/|x|$ for $0 < |x| < \pi$. Its Fourier series in equation (8) is $4/\pi$ times $[\sin x + (\sin 3x)/3 + (\sin 5x/5) + \cdots]$. The sum of the first N terms has an interesting shape, close to the square wave except where the wave jumps between -1 and 1. At those jumps, the Fourier sum spikes the wrong way to ± 1.09 (the ***Gibbs phenomenon***) before it takes the jump with the true $f(x)$.

This happens for the Fourier sums of all functions with jumps. It makes shock waves hard to compute. You can see it clearly in a graph of the sum of 10 terms.

8 $\|\boldsymbol{v}\|^2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$ so $\|\boldsymbol{v}\| = \sqrt{2}$; $\|\boldsymbol{v}\|^2 = 1 + a^2 + a^4 + \cdots = 1/(1 - a^2)$ so $\|\boldsymbol{v}\| = 1/\sqrt{1 - a^2}$; $\int_0^{2\pi}(1 + 2\sin x + \sin^2 x)\,dx = 2\pi + 0 + \pi$ so $\|f\| = \sqrt{3\pi}$.

9 (a) $f(x) = (1 + \text{square wave})/2$ so the a's are $\frac{1}{2}$, 0, $0, \ldots$ and the b's are $2/\pi$, 0, $-2/3\pi, 0, 2/5\pi, \ldots$ (b) $a_0 = \int_0^{2\pi} x\,dx/2\pi = \pi$, all other $a_k = 0$, $b_k = -2/k$.

10 The integral from $-\pi$ to π or from 0 to 2π (or from any a to $a + 2\pi$) is over one complete period of the function. If $f(x)$ is periodic this changes $\int_0^{2\pi} f(x)\,dx$ to $\int_0^{\pi} f(x)\,dx + \int_{-\pi}^{0} f(x)\,dx$. If $f(x)$ is **odd**, those integrals cancel to give $\int f(x)\,dx = 0$ over one period.

11 $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$; $\cos(x + \frac{\pi}{3}) = \cos x \cos\frac{\pi}{3} - \sin x \sin\frac{\pi}{3} = \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$.

12 $\dfrac{d}{dx}\begin{bmatrix}1\\ \cos x\\ \sin x\\ \cos 2x\\ \sin 2x\end{bmatrix}=\begin{bmatrix}0\\ -\sin x\\ \cos x\\ -2\sin 2x\\ 2\cos 2x\end{bmatrix}=\begin{bmatrix}0&0&0&0&0\\0&0&-1&0&0\\0&1&0&0&0\\0&0&0&0&-2\\0&0&0&2&0\end{bmatrix}\begin{bmatrix}1\\ \cos x\\ \sin x\\ \cos 2x\\ \sin 2x\end{bmatrix}.$ This shows the differentiation matrix.

13 The square pulse with $F(x) = 1/h$ for $-x \le h/2 \le x$ is an even function, so all sine coefficients b_k are zero. The average a_0 and the cosine coefficients a_k are

$$a_0 = \frac{1}{2\pi}\int_{-h/2}^{h/2}(1/h)dx = \frac{1}{2\pi}$$

$$a_k = \frac{1}{\pi}\int_{-h/2}^{h/2}(1/h)\cos kx dx = \frac{2}{\pi k h}\left(\sin\frac{kh}{2}\right) \text{ which is } \frac{1}{\pi}\operatorname{sinc}\left(\frac{kh}{2}\right)$$

(introducing the sinc function $(\sin x)/x$). As h approaches zero, the number $x = kh/2$ approaches zero, and $(\sin x)/x$ approaches 1. So all those a_k approach $1/\pi$.

The limiting "delta function" contains an equal amount of all cosines: a very irregular function.

Problem Set 10.6, page 500

1 (x, y, z) has homogeneous coordinates (cx, cy, cz, c) for $c = 1$ and all $c \ne 0$.

2 For an affine transformation we also need T (origin), because $T(\mathbf{0})$ need not be $\mathbf{0}$ for affine T. Including this translation by $T(\mathbf{0})$, $(x, y, z, 1)$ is transformed to $xT(\boldsymbol{i}) + yT(\boldsymbol{j}) + zT(\boldsymbol{k}) + T(\mathbf{0})$.

3 $TT_1 = \begin{bmatrix}1&&&\\&1&&\\&&1&\\1&4&3&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\0&2&5&1\end{bmatrix}=\begin{bmatrix}1&&&\\&1&&\\&&1&\\1&6&8&1\end{bmatrix}$ is translation along $(1, 6, 8)$.

4 $S = \operatorname{diag}(c, c, c, 1)$; row 4 of ST and TS is $1, 4, 3, 1$ and $c, 4c, 3c, 1$; use $\boldsymbol{v}TS$!

5 $S = \begin{bmatrix} 1/8.5 & & \\ & 1/11 & \\ & & 1 \end{bmatrix}$ for a 1 by 1 square, starting from an 8.5 by 11 page.

6 $[\,x\ y\ z\ 1\,] \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -1 & -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 1 \end{bmatrix} = [\,x\ y\ z\ 1\,] \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ -2 & -2 & -4 & 1 \end{bmatrix}.$

The first matrix translates by $(-1,-1,-2)$. The second matrix rescales by 2.

7 The three parts of Q in equation (1) are $(\cos\theta)I$ and $(1-\cos\theta)\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}$ and $-\sin\theta(\boldsymbol{a}\times)$. Then $Q\boldsymbol{a} = \boldsymbol{a}$ because $\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = \boldsymbol{a}$(unit vector) and $\boldsymbol{a}\times\ \boldsymbol{a} = \mathbf{0}$.

8 If $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = 0$ and those three parts of Q (Problem 7) multiply $\boldsymbol{b}$, the results in $Q\boldsymbol{b}$ are $(\cos\theta)\boldsymbol{b}$ and $\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = \mathbf{0}$ and $(-\sin\theta)\boldsymbol{a}\times\ \boldsymbol{b}$. The component along $\boldsymbol{b}$ is $(\cos\theta)\boldsymbol{b}$.

9 $\boldsymbol{n} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$ has $P = I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} = \frac{1}{9}\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$. Notice $\|\boldsymbol{n}\| = 1$.

10 We can choose $(0,0,3)$ on the plane and multiply $T_- P T_+ = \frac{1}{9}\begin{bmatrix} 5 & -4 & -2 & 0 \\ -4 & 5 & -2 & 0 \\ -2 & -2 & 8 & 0 \\ 6 & 6 & 3 & 9 \end{bmatrix}$.

11 $(3,3,3)$ projects to $\frac{1}{3}\big(-1,-1,4\big)$ and $(3,3,3,1)$ projects to $\big(\frac{1}{3},\frac{1}{3},\frac{5}{3},1\big)$. Row vectors!

12 The projection of a square onto a plane is a parallelogram (or a line segment). The sides of the square are perpendicular, but their projections may not be ($\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = 0$ but $(P\boldsymbol{x})^{\mathrm{T}}(P\boldsymbol{y}) = \boldsymbol{x}^{\mathrm{T}}P^{\mathrm{T}}P\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}P\boldsymbol{y}$ may be nonzero).

13 That projection of a cube onto a plane produces a hexagon.

14 $(3,3,3)(I - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\begin{bmatrix} 1 & -8 & -4 \\ -8 & 1 & -4 \\ -4 & -4 & 7 \end{bmatrix} = \left(-\frac{11}{3}, -\frac{11}{3}, -\frac{1}{3}\right).$

15 $(3,3,3,1) \to (3,3,0,1) \to \left(-\frac{7}{3}, -\frac{7}{3}, -\frac{8}{3}, 1\right) \to \left(-\frac{7}{3}, -\frac{7}{3}, \frac{1}{3}, 1\right)$.

16 Just subtracting vectors would give $\boldsymbol{v} = (x,y,z,0)$ ending in 0 (not 1). In homogeneous coordinates, add a **vector** to a point.

17 Space is rescaled by $1/c$ because (x,y,z,c) is the same point as $(x/c, y/c, z/c, 1)$.

Problem Set 10.7, page 507

1 **Multiplying** n whole numbers gives an odd number only when ***all*** n ***numbers are odd***. This translates to multiplication (*mod* 2). Multiplying n 1's or 0's gives 1 only when all n numbers are 1.

2 **Adding** n whole numbers gives an odd number only when the n numbers include ***an odd number of odd numbers***. For addition of 1's and 0's (*mod* 2), the answer is odd when the number of 1's is odd.

3 (a) We are given that $y_1 - x_1$ and $y_2 - x_2$ are both divisible by p. Then their sum $y_1 + y_2 - x_1 - x_2$ is divisible by p.

(b) $5 \equiv 2$ (*mod* 3) and $8 \equiv 2$ (*mod* 3) add to $13 \equiv 4$ (*mod* 3). The number 1 is smaller than 4 and $13 \equiv 1$ (*mod* 3).

5 If $y - x$ is divisible by p then $x - y$ is also divisible by p. In other words, if $y - x = mp$ then $x - y = (-m)p$.

6 $A = \begin{bmatrix} 5 & 5 \\ 5 & 10 \end{bmatrix}$ is an invertible matrix but (*mod* 5) A becomes the zero matrix.

7 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ are invertible : 6 out of 16 possible 0-1 matrices.

8 Yes, $A\boldsymbol{x} = \mathbf{0}$ (*mod* 11) says that every row of A is orthogonal to every $\boldsymbol{x}$ in the nullspace (*mod* 11). But a basis for the usual $\mathbf{N}(A)$ could include vectors that are zero (*mod* 11).

9 For simplicity, number the letters as they appear in the message :

THISWHOLEBOOKISINCODE $= \quad 123/452/678/966/(10)34/3(11)(12)/6(13)8.$

Multiply each block by this L to obtain Hill's cipher.

$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ Cipher $= 1\ 3\ 6/4\ 9\ 11/6\ 13\ 21/9\ 15\ 21/10\ 13\ 17/3\ 14\ 26/6\ 19\ 27.$

If the cipher is ***mod*** p then replace each number by the correct number from 0 to $p-1$.

To decode, first multiply by L^{-1}. Then what to do??

10 First you have to discover the block size ($=$ matrix size) and also the matrix L itself. Start with a guess for the block size. Then the plaintext and the coded cipher tell you a series of matrix-vector products $L\boldsymbol{x} \equiv \boldsymbol{b}$. If the text is long enough (and the blocks are not too long) this is enough information to find L—or to show that the block size must be wrong, when there is no L that gets all correct blocks $L\boldsymbol{x} \equiv \boldsymbol{b}$.

The extra difficulty is to find the value of p.