INTRODUCTION

TO

LINEAR

ALGEBRA

Fifth Edition

MANUAL FOR INSTRUCTORS

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Problem Set 6.1, page 298

1 The eigenvalues are 1 and 0.5 for A, 1 and 0.25 for A^2, 1 and 0 for A^∞. Exchanging the rows of A changes the eigenvalues to 1 and -0.5 (the trace is now $0.2 + 0.3$). Singular matrices stay singular during elimination, so $\lambda = 0$ does not change.

2 A has $\lambda_1 = -1$ and $\lambda_2 = 5$ with eigenvectors $x_1 = (-2, 1)$ and $x_2 = (1, 1)$. The matrix $A + I$ has the same eigenvectors, with eigenvalues increased by 1 to $\mathbf{0}$ and $\mathbf{6}$. That zero eigenvalue correctly indicates that $A + I$ is singular.

3 A has $\lambda_1 = 2$ and $\lambda_2 = -1$ (check trace and determinant) with $\boldsymbol{x}_1 = (1, 1)$ and $\boldsymbol{x}_2 = (2, -1)$. A^{-1} has the same eigenvectors, with eigenvalues $1/\lambda = \frac{1}{2}$ and -1.

4 $\det(A - \lambda I) = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$. Then A has $\lambda_1 = -3$ and $\lambda_2 = 2$ (check trace $= -1$ and determinant $= -6$) with $\boldsymbol{x}_1 = (3, -2)$ and $\boldsymbol{x}_2 = (1, 1)$. A^2 has the *same eigenvectors* as A, with eigenvalues $\lambda_1^2 = 9$ and $\lambda_2^2 = 4$.

5 A and B have eigenvalues 1 and 3 (their diagonal entries : triangular matrices). $A + B$ has $\lambda^2 + 8\lambda + 15 = 0$ and $\lambda_1 = 3$, $\lambda_2 = 5$. Eigenvalues of $A + B$ *are not equal* to eigenvalues of A plus eigenvalues of B.

6 A and B have $\lambda_1 = 1$ and $\lambda_2 = 1$. AB and BA have $\lambda^2 - 4\lambda + 1$ and the quadratic formula gives $\lambda = 2 \pm \sqrt{3}$. Eigenvalues of AB *are not equal* to eigenvalues of A times eigenvalues of B. Eigenvalues of AB and BA ***are equal*** (this is proved at the end of Section 6.2).

7 The eigenvalues of U (on its diagonal) are the *pivots* of A. The eigenvalues of L (on its diagonal) are all 1's. The eigenvalues of A *are not* the same as the pivots.

8 (a) Multiply $A\boldsymbol{x}$ to see $\lambda\boldsymbol{x}$ which reveals λ (b) Solve $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ to find $\boldsymbol{x}$.

9 (a) Multiply by A: $A(A\boldsymbol{x}) = A(\lambda\boldsymbol{x}) = \lambda A\boldsymbol{x}$ gives $A^2\boldsymbol{x} = \boldsymbol{\lambda}^2\boldsymbol{x}$

(b) Multiply by A^{-1}: $\boldsymbol{x} = A^{-1}A\boldsymbol{x} = A^{-1}\lambda\boldsymbol{x} = \lambda A^{-1}\boldsymbol{x}$ gives $A^{-1}\boldsymbol{x} = \frac{1}{\boldsymbol{\lambda}}\boldsymbol{x}$

(c) Add $I\boldsymbol{x} = \boldsymbol{x}$: $(A + I)\boldsymbol{x} = (\boldsymbol{\lambda} + \mathbf{1})\boldsymbol{x}$.

10 $\det(A-\lambda I)=d^2-1.4\lambda+0.4$ so A has $\lambda_1=1$ and $\lambda_2=0.4$ with $\boldsymbol{x}_1=(1,2)$ and $\boldsymbol{x}_2=(1,-1)$. A^∞ has $\lambda_1=1$ and $\lambda_2=0$ (same eigenvectors). A^{100} has $\lambda_1=1$ and $\lambda_2=(0.4)^{100}$ which is near zero. So A^{100} is very near A^∞: same eigenvectors and close eigenvalues.

11 Columns of $A-\lambda_1 I$ are in the nullspace of $A-\lambda_2 I$ because $M=(A-\lambda_2 I)(A-\lambda_1 I)$ is the zero matrix **[**this is the *Cayley-Hamilton Theorem* in Problem 6.2.30**]**. Notice that M has *zero eigenvalues* $(\lambda_1-\lambda_2)(\lambda_1-\lambda_1)=0$ and $(\lambda_2-\lambda_2)(\lambda_2-\lambda_1)=0$. So those columns solve $(A-\lambda_2 I)\,\boldsymbol{x}=\mathbf{0}$, they are eigenvectors.

12 The projection matrix P has $\lambda=1,0,1$ with eigenvectors $(1,2,0)$, $(2,-1,0)$, $(0,0,1)$. Add the first and last vectors: $(1,2,1)$ also has $\lambda=1$. The whole column space of P contains eigenvectors with $\lambda=1$! *Note* $P^2=P$ leads to $\lambda^2=\lambda$ so $\lambda=0$ or 1.

13 (a) $P\boldsymbol{u}=(\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u})=\boldsymbol{u}$ so $\lambda=1$ (b) $P\boldsymbol{v}=(\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{v}=\boldsymbol{u}(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v})=\mathbf{0}$
(c) $\boldsymbol{x}_1=(-1,1,0,0)$, $\boldsymbol{x}_2=(-3,0,1,0)$, $\boldsymbol{x}_3=(-5,0,0,1)$ all have $P\boldsymbol{x}=0\boldsymbol{x}=\mathbf{0}$.

14 $\det(Q-\lambda I)=\lambda^2-2\lambda\cos\theta+1=0$ when $\lambda=\cos\theta\pm i\sin\theta=e^{i\theta}$ and $e^{-i\theta}$. Check that $\lambda_1\lambda_2=1$ and $\lambda_1+\lambda_2=2\cos\theta$. Two eigenvectors of this rotation matrix are $\boldsymbol{x}_1=(1,i)$ and $\boldsymbol{x}_2=(1,-i)$ (more generally $c\boldsymbol{x}_1$ and $d\boldsymbol{x}_2$ with $cd\neq 0$).

15 The other two eigenvalues are $\lambda=\frac{1}{2}(-1\pm i\sqrt{3})$. The three eigenvalues are $1,1,-1$.

16 Set $\lambda=0$ in $\det(A-\lambda I)=(\lambda_1-\lambda)\ldots(\lambda_n-\lambda)$ to find $\det A=(\lambda_1)(\lambda_2)\cdots(\lambda_n)$.

17 $\lambda_1=\frac{1}{2}(a+d+\sqrt{(a-d)^2+4bc})$ and $\lambda_2=\frac{1}{2}(a+d-\sqrt{})$ add to $a+d$. If A has $\lambda_1=3$ and $\lambda_2=4$ then $\det(A-\lambda I)=(\lambda-3)(\lambda-4)=\lambda^2-7\lambda+12$.

18 These 3 matrices have $\lambda=4$ and 5, trace 9, $\det 20$: $\begin{bmatrix}4&0\\0&5\end{bmatrix},\begin{bmatrix}3&2\\-1&6\end{bmatrix},\begin{bmatrix}2&2\\-3&7\end{bmatrix}$.

19 (a) rank $=2$ (b) $\det(B^{\mathrm{T}}B)=0$ (d) eigenvalues of $(B^2+I)^{-1}$ are $1,\frac{1}{2},\frac{1}{5}$.

20 $A=\begin{bmatrix}0&1\\\mathbf{-28}&\mathbf{11}\end{bmatrix}$ has trace 11 and determinant 28, so $\lambda=4$ and 7. Moving to a 3 by 3 companion matrix, for eigenvalues $1,2,3$ we want $\det(C-\lambda I)=(1-\lambda)(2-\lambda)(3-\lambda)$. Multiply out to get $-\lambda^3+6\lambda^2-11\lambda+6$. To get those numbers $6,-11,6$ from a companion matrix you just put them into the last row:

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mathbf{6} & \mathbf{-11} & \mathbf{6} \end{bmatrix}$$ Notice the trace $6 = 1 + 2 + 3$ and determinant $6 = (1)(2)(3)$.

21 $(A - \lambda I)$ has the same determinant as $(A - \lambda I)^{\mathrm{T}}$ because every square matrix has $\det M = \det M^{\mathrm{T}}$. Pick $M = A - \lambda I$.

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$ have *different eigenvectors*.

22 The eigenvalues must be $\lambda = \mathbf{1}$ (because the matrix is Markov), $\mathbf{0}$ (for singular), $-\frac{\mathbf{1}}{\mathbf{2}}$ (so sum of eigenvalues = trace = $\frac{1}{2}$).

23 $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$. Always A^2 is the zero matrix if $\lambda = 0$ and 0, by the Cayley-Hamilton Theorem in Problem 6.2.30.

24 $\lambda = \mathbf{0, 0, 6}$ (*notice rank* 1 *and trace* 6). Two eigenvectors of $\boldsymbol{uv}^{\mathrm{T}}$ are perpendicular to $\boldsymbol{v}$ and the third eigenvector is $\boldsymbol{u}$: $\boldsymbol{x}_1 = (0, -2, 1)$, $\boldsymbol{x}_2 = (1, -2, 0)$, $\boldsymbol{x}_3 = (1, 2, 1)$.

25 When A and B have the same n λ's and $\boldsymbol{x}$'s, look at any combination $\boldsymbol{v} = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$. Multiply by A and B: $A\boldsymbol{v} = c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n\boldsymbol{x}_n$ **equals** $B\boldsymbol{v} = c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n\boldsymbol{x}_n$ **for all vectors** $\boldsymbol{v}$. So $A = B$.

26 The block matrix has $\lambda = 1, 2$ from B and $\lambda = 5, 7$ from D. All entries of C are multiplied by zeros in $\det(A - \lambda I)$, so C has no effect on the eigenvalues of the block matrix.

27 A has rank 1 with eigenvalues $0, 0, 0, 4$ (the 4 comes from the trace of A). C has rank 2 (ensuring two zero eigenvalues) and $(1, 1, 1, 1)$ is an eigenvector with $\lambda = 2$. With trace 4, the other eigenvalue is also $\lambda = 2$, and its eigenvector is $(1, -1, 1, -1)$.

28 Subtract from $0, 0, 0, 4$ in Problem 27. $B = A - I$ has $\lambda = -1, -1, -1, 3$ and $C = I - A$ has $\lambda = 1, 1, 1, -3$. Both have $\det = -3$.

29 A is triangular: $\lambda(A) = 1, 4, 6$; $\lambda(B) = 2, \sqrt{3}, -\sqrt{3}$; C has rank one: $\lambda(C) = 0, 0, 6$.

30 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = (\boldsymbol{a}+\boldsymbol{b})\begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\lambda_2 = \boldsymbol{d}-\boldsymbol{b}$ to produce the correct trace $(a+b)+(d-b)=a+d$.

31 Eigenvector $(1,3,4)$ for A with $\lambda = 11$ and eigenvector $(3,1,4)$ for PAP^{T} with $\lambda = 11$. Eigenvectors with $\lambda \neq 0$ must be in the column space since $A\boldsymbol{x}$ is always in the column space, and $\boldsymbol{x} = A\boldsymbol{x}/\lambda$.

32 (a) $\boldsymbol{u}$ is a basis for the nullspace (we know $A\boldsymbol{u} = 0\boldsymbol{u}$); $\boldsymbol{v}$ and $\boldsymbol{w}$ give a basis for the column space (we know $A\boldsymbol{v}$ and $A\boldsymbol{w}$ are in the column space).

(b) $A(\boldsymbol{v}/3+\boldsymbol{w}/5) = 3\boldsymbol{v}/3+5\boldsymbol{w}/5 = \boldsymbol{v}+\boldsymbol{w}$. So $\boldsymbol{x} = \boldsymbol{v}/3+\boldsymbol{w}/5$ is a particular solution to $A\boldsymbol{x} = \boldsymbol{v}+\boldsymbol{w}$. Add any $c\boldsymbol{u}$ from the nullspace

(c) If $A\boldsymbol{x} = \boldsymbol{u}$ had a solution, $\boldsymbol{u}$ would be in the column space: wrong dimension 3.

33 Always $(\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}})\,\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u})$ so $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ with $\lambda = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{u}$. (watch numbers $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u}$, vectors $\boldsymbol{u}$, matrices $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$!!) If $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = 0$ then $A^2 = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u})\boldsymbol{v}^{\mathrm{T}}$ is the zero matrix and $\lambda^2 = 0, 0$ and $\lambda = 0, 0$ and trace $(A) = 0$. This zero trace also comes from adding the diagonal entries of $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$:

$$A = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix} \quad \text{has trace } \; u_1v_1+u_2v_2 = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = 0$$

34 $\det(P-\lambda I) = 0$ gives the equation $\lambda^4 = 1$. This reflects the fact that $P^4 = I$. The solutions of $\lambda^4 = 1$ are $\lambda = 1, i, -1, -i$. The real eigenvector $\boldsymbol{x}_1 = (1,1,1,1)$ is not changed by the permutation P. Three more eigenvectors are $(1, i, i^2, i^3)$ and $(1,-1,1,-1)$ and $(1,-i,(-i)^2,(-i)^3)$.

35 The six 3 by 3 permutation matrices include $P = I$ and three single row exchange matrices P_{12}, P_{13}, P_{23} and two double exchange matrices like $P_{12}P_{13}$. Since $P^{\mathrm{T}}P = I$ gives $(\det P)^2 = 1$, the determinant of P is 1 or -1. The pivots are always 1 (but there may be row exchanges). The trace of P can be 3 (for $P = I$) or 1 (for row exchange) or 0 (for double exchange). The possible eigenvalues are 1 and -1 and $e^{2\pi i/3}$ and $e^{-2\pi i/3}$.

36 $AB - BA = I$ can happen only for infinite matrices. If $A^{\mathrm{T}} = A$ and $B^{\mathrm{T}} = -B$ then

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}(AB - BA)\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}B + B^{\mathrm{T}}A)\boldsymbol{x} \le ||A\boldsymbol{x}||\,||B\boldsymbol{x}|| + ||B\boldsymbol{x}||\,||A\boldsymbol{x}||.$$

Therefore $||A\boldsymbol{x}||\,||B\boldsymbol{x}|| \ge \frac{1}{2}||\boldsymbol{x}||^2$ and $(||A\boldsymbol{x}||/||\boldsymbol{x}||)\,(||B\boldsymbol{x}||/||\boldsymbol{x}||) \ge \frac{1}{2}$.

37 $\lambda_1 = e^{2\pi i/3}$ and $\lambda_2 = e^{-2\pi i/3}$ give $\det \lambda_1\lambda_2 = 1$ and trace $\lambda_1 + \lambda_2 = -1$. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ with $\theta = \dfrac{2\pi}{3}$ has this trace and det. So does every $M^{-1}AM$!

38 (a) Since the columns of A add to 1, one eigenvalue is $\lambda = 1$ and the other is $c - 0.6$ (to give the correct trace $c + 0.4$).

(b) If $c = 1.6$ then both eigenvalues are 1, and all solutions to $(A - I)\,\boldsymbol{x} = \boldsymbol{0}$ are multiples of $\boldsymbol{x} = (1, -1)$. In this case A has rank 1.

(c) If $c = 0.8$, the eigenvectors for $\lambda = 1$ are multiples of (1, 3). Since all powers A^n also have column sums $= 1$, A^n will approach $\dfrac{1}{4}\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ = rank-1 matrix A^{∞} with eigenvalues $1, 0$ and correct eigenvectors. $(1, 3)$ and $(1, -1)$.

Problem Set 6.2, page 314

1 Eigenvectors in $\boldsymbol{X}$ and eigenvalues in Λ. Then $A = X\Lambda X^{-1}$ is $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

The second matrix has $\lambda = 0$ (rank 1) and $\lambda = 4$ (trace $= 4$). Then $A = X\Lambda X^{-1}$ is $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$.

2 Put the eigenvectors in X and eigenvalues $2, 5$ in Λ. $A = X\Lambda X^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$.

3 If $A = X\Lambda X^{-1}$ then the eigenvalue matrix for $A + 2I$ is $\Lambda + 2I$ and the eigenvector matrix is still X. So $A + 2I = S(\Lambda + 2I)X^{-1} = X\Lambda X^{-1} + X(2I)X^{-1} = A + 2I$.

4 (a) False: We are not given the λ's (b) True (c) True (d) False: For this we would need the eigenvectors of X

5 With $X = I, A = X\Lambda X^{-1} = \Lambda$ is a diagonal matrix. If X is triangular, then X^{-1} is triangular, so $X\Lambda X^{-1}$ is also triangular.

6 The columns of S are nonzero multiples of (2,1) and (0,1): either order. The same eigenvector matrices diagonalize A and A^{-1}.

7 $A = X\Lambda X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} /2 = \begin{bmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix} /2.$

These are the matrices $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$, their eigenvectors are $(1, 1)$ and $(1, -1)$.

8 $A = X\Lambda X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \dfrac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}.$

$X\Lambda^k X^{-1} = \dfrac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

The second component is $F_k = (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2)$.

9 (a) The equations are $\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = A \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$ with $A = \begin{bmatrix} .5 & .5 \\ 1 & 0 \end{bmatrix}$. This matrix has $\lambda_1 = 1,\ \lambda_2 = -\frac{1}{2}$ with $\boldsymbol{x}_1 = (1, 1),\ \boldsymbol{x}_2 = (1, -2)$

(b) $A^n = X\Lambda^n X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (-.5)^n \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \to A^\infty = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

10 The rule $F_{k+2} = F_{k+1} + F_k$ produces the pattern: even, odd, odd, even, odd, odd, . . .

11 (a) *True* (no zero eigenvalues) (b) *False* (repeated $\lambda = 2$ may have only one line of eigenvectors) (c) *False* (repeated λ may have a full set of eigenvectors)

12 (a) False: don't know if $\lambda = 0$ or not.

(b) True: an eigenvector is missing, which can only happen for a repeated eigenvalue.

(c) True: We know there is only one line of eigenvectors.

13 $A = \begin{bmatrix} 8 & 3 \\ -3 & 2 \end{bmatrix}$ (or other), $A = \begin{bmatrix} 9 & 4 \\ -4 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 10 & 5 \\ -5 & 0 \end{bmatrix}$; only eigenvectors are $\boldsymbol{x} = (c, -c)$.

14 The rank of $A - 3I$ is $r = 1$. Changing any entry except $a_{12} = 1$ makes A diagonalizable (the new A will have two different eigenvalues)

15 $A^k = X\Lambda^k X^{-1}$ approaches zero **if and only if every** $|\boldsymbol{\lambda}| < \mathbf{1}$; A_1 is a Markov matrix so $\lambda_{\max} = 1$ and $A_1^k \to A_1^\infty$, A_2 has $\lambda = .6 \pm .3$ so $A_2^k \to 0$.

16 A_1 is $X\Lambda X^{-1}$ with $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & .2 \end{bmatrix}$ and $X = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$; $\Lambda^k \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Then $A_1 k = X\Lambda^k X^{-1} \to \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$: *steady state*.

17 A_2 is $X\Lambda X^{-1}$ with $\Lambda = \begin{bmatrix} .9 & 0 \\ 0 & .3 \end{bmatrix}$ and $X = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$; $A_2^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = (.9)^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$A_2^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = (.3)^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Then $A_2^{10} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = (.9)^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + (.3)^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ because $\begin{bmatrix} 6 \\ 0 \end{bmatrix}$ is the sum of $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

18 $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = X\Lambda X^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and

$A^k = X\Lambda^k X^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Multiply those last three matrices to get $A^k = \frac{1}{2} \begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}$.

19 $B^k = X\Lambda^k X^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}$.

20 $\det A = (\det X)(\det \Lambda)(\det X^{-1}) = \det \Lambda = \lambda_1 \cdots \lambda_n$. This proof ($\det$ = product of λ's) works when A is *diagonalizable*. The formula is always true.

21 trace $XY = (aq + bs) + (cr + dt)$ is equal to $(qa + rc) + (sb + td) =$ trace YX. Diagonalizable case: the trace of $X\Lambda X^{-1}$ = trace of $(\Lambda X^{-1})X = \Lambda$: *sum of the* λ's.

22 $AB - BA = I$ is impossible since trace AB − trace BA = *zero* $\neq$ trace I. $AB - BA = C$ is possible when trace $(C) = 0$. For example $E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ has $EE^{\mathrm{T}} - E^{\mathrm{T}}E = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = C$ with trace zero.

23 If $A = X\Lambda X^{-1}$ then $B = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & 2\Lambda \end{bmatrix} \begin{bmatrix} X^{-1} & 0 \\ 0 & X^{-1} \end{bmatrix}$. So B has the original λ's from A and the additional eigenvalues $2\lambda_1, \ldots, 2\lambda_n$ from $2A$.

24 The A's form a subspace since cA and $A_1 + A_2$ all have the same X. When $X = I$ the A's with those eigenvectors give the subspace of **diagonal matrices**. The dimension of that matrix space is 4 since the matrices are 4 by 4.

25 If A has columns $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ then column by column, $A^2 = A$ means every $A\boldsymbol{x}_i = \boldsymbol{x}_i$. All vectors in the column space (combinations of those columns $\boldsymbol{x}_i$) are eigenvectors with $\lambda = 1$. Always the nullspace has $\lambda = 0$ (A might have dependent columns, so there could be less than n eigenvectors with $\lambda = 1$). Dimensions of those spaces $\boldsymbol{C}(A)$ and $\boldsymbol{N}(A)$ add to n by the Fundamental Theorem, so A *is diagonalizable* (n independent eigenvectors altogether).

26 Two problems: The nullspace and column space can overlap, so $\boldsymbol{x}$ could be in both. There may not be r independent eigenvectors in the column space.

27 $R = X\sqrt{\Lambda}X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} /2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $R^2 = A$.

$\sqrt{B}$ needs $\lambda = \sqrt{9}$ and $\sqrt{-1}$, trace (their sum) is not real so $\sqrt{B}$ cannot be real. Note that $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ has *two* imaginary eigenvalues $\sqrt{-1} = i$ and $-i$, real trace 0, real square root $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

28 The factorizations of A and B into $X\Lambda X^{-1}$ are the same. So $A = B$. (This is the same as Problem 6.1.25, expressed in matrix form.)

29 $A = X\Lambda_1 X^{-1}$ and $B = X\Lambda_2 X^{-1}$. Diagonal matrices always give $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$. Then $AB = BA$ from

$$X\Lambda_1 X^{-1} X\Lambda_2 X^{-1} = X\boldsymbol{\Lambda_1\Lambda_2} X^{-1} = X\boldsymbol{\Lambda_2\Lambda_1} X^{-1} = X\Lambda_2 X^{-1} X\Lambda_1 X^{-1} = BA.$$

30 (a) $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ has $\lambda = a$ and $\lambda = d$: $(A-aI)(A-dI) = \begin{bmatrix} 0 & b \\ 0 & d-a \end{bmatrix} \begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix}$ $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. (b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has $A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $A^2 - A - I = 0$ is true, matching $\lambda^2 - \lambda - 1 = 0$ as the Cayley-Hamilton Theorem predicts.

31 When $A = X\Lambda X^{-1}$ is diagonalizable, the matrix $A - \lambda_j I = X(\Lambda - \lambda_j I)X^{-1}$ will have 0 in the j, j diagonal entry of $\Lambda - \lambda_j I$. The product $p(A)$ becomes

$$p(A) = (A - \lambda_1 I) \cdots (A - \lambda_n I) = X(\Lambda - \lambda_1 I) \cdots (\Lambda - \lambda_n I)X^{-1}.$$

That product is the zero matrix because the factors produce a zero in each diagonal position. Then $p(A) =$ zero matrix, which is the Cayley-Hamilton Theorem. (If A is not diagonalizable, one proof is to take a sequence of diagonalizable matrices approaching A.)

Comment I have also seen the following Cayley-Hamilton proof but I am not convinced:

Apply the formula $AC^{\mathrm{T}} = (\det A)I$ from Section 5.3 to $A - \lambda I$ with variable λ. Its cofactor matrix C will be a polynomial in λ, since cofactors are determinants:

$$(A - \lambda I)C^{\mathrm{T}} = \det(A - \lambda I)I = p(\lambda)I.$$

"For fixed A, this is an identity between two matrix polynomials." Set $\lambda = A$ to find the zero matrix on the left, so $p(A) =$ zero matrix on the right—which is the Cayley-Hamilton Theorem.

I am not certain about the key step of substituting a matrix for λ. If other matrices B are substituted for λ, does the identity remain true? If $AB \neq BA$, even the order of multiplication seems unclear . . .

32 If $AB = BA$, then B has the same eigenvectors $(1,0)$ and $(0,1)$ as A. So B is also diagonal $b = c = 0$. The nullspace for the following equation is 2-dimensional:

$$AB - BA = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -\boldsymbol{b} \\ \boldsymbol{c} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Those 4 equations $0 = 0, -b = 0, c = 0, 0 = 0$ have a 4 by 4 coefficient matrix with rank $4 - 2 = 2$.

33 B has $\lambda = i$ and $-i$, so B^4 has $\lambda^4 = 1$ and 1 and $B^{1024} = I$.

C has $\lambda = (1 \pm \sqrt{3}i)/2$. This λ is $\exp(\pm\pi i/3)$ so $\lambda^3 = -1$ and -1. Then $C^3 = -I$ which leads to $C^{1024} = (-I)^{341}C = -C$.

34 The eigenvalues of $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ are $\lambda = e^{i\theta}$ and $e^{-i\theta}$ (trace $2\cos\theta$ and determinant $= 1$). Their eigenvectors are $(1,-i)$ and $(1,i)$:

$$A^n = X\Lambda^n X^{-1} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{in\theta} & \\ & e^{-in\theta} \end{bmatrix} \begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} /2i$$

$$= \begin{bmatrix} (e^{in\theta} + e^{-in\theta})/2 & \cdots \\ (e^{in\theta} - e^{-in\theta})/2i & \cdots \end{bmatrix} = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Geometrically, n rotations by θ give one rotation by $n\theta$.

35 Columns of X times rows of ΛX^{-1} gives a sum of r rank-1 matrices ($r =$ rank of A).

36 Multiply $\mathsf{ones}(n) * \mathsf{ones}(n) = n * \mathsf{ones}(n)$. This leads to $C = \boldsymbol{-1/(n+1)}$.

$$AA^{-1} = (\mathsf{eye}(n) + \mathsf{ones}(n)) * (\mathsf{eye}(n) + C * \mathsf{ones}(n))$$

$$= \mathsf{eye}(n) + (1 + C + Cn) * \mathsf{ones}(n) = \mathsf{eye}(n).$$

Problem Set 6.3, page 332

1 Eigenvalues 4 and 1 with eigenvectors $(1,0)$ and $(1,-1)$ give solutions $\boldsymbol{u}_1 = e^{4t}\begin{bmatrix}1\\0\end{bmatrix}$

and $\boldsymbol{u}_2 = e^t\begin{bmatrix}1\\-1\end{bmatrix}$. If $\boldsymbol{u}(0) = \begin{bmatrix}5\\-2\end{bmatrix} = 3\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}1\\-1\end{bmatrix}$, then

$\boldsymbol{u}(t) = 3e^{4t}\begin{bmatrix}1\\0\end{bmatrix} + 2e^t\begin{bmatrix}1\\-1\end{bmatrix}$.

2 $z(t) = 2e^t$ solves $dx/dt = z$ with $z(0) = 2$. Then $dy/dt = 4y - 6e^t$ with $y(0) = 5$ gives $y(t) = 3e^{4t} + 2e^t$ as in Problem 1.

3 (a) If every column of A adds to zero, this means that the rows add to the zero row. So the rows are dependent, and A is singular, and $\lambda = 0$ is an eigenvalue.

(b) The eigenvalues of $A = \begin{bmatrix}-2 & 3\\ 2 & -3\end{bmatrix}$ are $\lambda_1 = 0$ with eigenvector $\boldsymbol{x}_1 = (3,2)$ and $\lambda_2 = -5$ (to give trace $= -5$) with $\boldsymbol{x}_2 = (1,-1)$. Then the usual 3 steps:

1. Write $u(0) = \begin{bmatrix}4\\1\end{bmatrix}$ as $\begin{bmatrix}3\\2\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix} = \boldsymbol{x}_1 + \boldsymbol{x}_2 =$ combination of eigenvectors
2. The solutions follow those eigenvectors: $e^{0t}\boldsymbol{x}_1$ and $e^{-5t}\boldsymbol{x}_2$
3. The solution $\boldsymbol{u}(t) = \boldsymbol{x}_1 + e^{-5t}\boldsymbol{x}_2$ has steady state $\boldsymbol{x}_1 = (3,2)$ since $e^{-5t} \to 0$.

4 $d(v+w)/dt = (w-v) + (v-w) = 0$, so the total $v + w$ is constant. $A = \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$ has $\begin{matrix}\lambda_1 = 0\\ \lambda_2 = -2\end{matrix}$ with $\boldsymbol{x}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\boldsymbol{x}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$.

$\begin{bmatrix}v(0)\\w(0)\end{bmatrix} = \begin{bmatrix}30\\10\end{bmatrix} = 20\begin{bmatrix}1\\1\end{bmatrix} + 10\begin{bmatrix}1\\-1\end{bmatrix}$ leads to $\begin{matrix}v(1) = 20 + 10e^{-2} & v(\infty) = 20\\ w(1) = 20 - 10e^{-2} & w(\infty) = 20\end{matrix}$

5 $\dfrac{d}{dt}\begin{bmatrix}v\\w\end{bmatrix} = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$ has $\lambda = 0$ and $\lambda = +2$: $v(t) = \boldsymbol{20 + 10e^{2t}} \to -\infty$ as $t \to \infty$.

6 $A = \begin{bmatrix}a & 1\\ 1 & a\end{bmatrix}$ has real eigenvalues $a+1$ and $a-1$. These are both negative if $\boldsymbol{a < -1}$. In this case the solutions of $\boldsymbol{u}' = A\boldsymbol{u}$ approach zero.

$B = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix}$ has complex eigenvalues $b+i$ and $b-i$. These have negative real parts if $\boldsymbol{b} < \mathbf{0}$. In this case and all solutions of $\boldsymbol{v}' = B\boldsymbol{v}$ approach zero.

7 A projection matrix has eigenvalues $\lambda = 1$ and $\lambda = 0$. Eigenvectors $P\boldsymbol{x} = \boldsymbol{x}$ fill the subspace that P projects onto: here $\boldsymbol{x} = (1, 1)$. Eigenvectors with $P\boldsymbol{x} = \mathbf{0}$ fill the perpendicular subspace: here $\boldsymbol{x} = (1, -1)$. For the solution to $\boldsymbol{u}' = -P\boldsymbol{u}$,

$$\boldsymbol{u}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \boldsymbol{u}(t) = e^{-t}\begin{bmatrix} 2 \\ 2 \end{bmatrix} + e^{0t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ approaches } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

8 $\begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$ has $\lambda_1 = 5,\ \boldsymbol{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \lambda_2 = 2,\ \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; rabbits $r(t) = 20e^{5t} + 10e^{2t}$, $w(t) = 10e^{5t} + 20e^{2t}$. The ratio of rabbits to wolves approaches $20/10$; e^{5t} dominates.

9 (a) $\begin{bmatrix} 4 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ i \end{bmatrix} + 2\begin{bmatrix} 1 \\ -i \end{bmatrix}$. (b) Then $u(t) = 2e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + 2e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 4\cos t \\ 4\sin t \end{bmatrix}$.

10 $\dfrac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}$. This correctly gives $y' = y'$ and $y'' = 4y + 5y'$. $A = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix}$ has $\det(A - \lambda I) = \lambda^2 - 5\lambda - 4 = 0$. Directly substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4y$ also gives $\lambda^2 = 5\lambda + 4$ and the same two values of λ. Those values are $\frac{1}{2}(5 \pm \sqrt{41})$ by the quadratic formula.

11 The series for e^{At} is $e^{At} = I + t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \text{ zeros } = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$.

Then $\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} \begin{bmatrix} y(0) + y'(0)t \\ y'(0) \end{bmatrix}$. This $y(t) = y(0) + y'(0)t$ solves the equation—the factor t tells us that A had only one eigenvector: not diagonalizable.

12 $A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}$ has trace 6, det 9, $\lambda = 3$ and 3 with *one* independent eigenvector $(1, 3)$. Substitute $y = te^{3t}$ to show that this gives the needed second solution ($y = e^{3t}$ is the first solution).

13 (a) $y(t) = \cos 3t$ and $\sin 3t$ solve $y'' = -9y$. It is $\mathbf{3}\cos 3t$ that starts with $y(0) = 3$ and $y'(0) = 0$. (b) $A = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}$ has det $= 9$: $\lambda = 3i$ and $-3i$ with eigenvectors $x = \begin{bmatrix} 1 \\ 3i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -3i \end{bmatrix}$. Then $\boldsymbol{u}(t) = \frac{3}{2}e^{3it}\begin{bmatrix} 1 \\ 3i \end{bmatrix} + \frac{3}{2}e^{-3it}\begin{bmatrix} 1 \\ -3i \end{bmatrix} = \begin{bmatrix} 3\cos 3t \\ \mathbf{-9\sin 3t} \end{bmatrix}$.

14 When A is skew-symmetric, the derivative of $||u(t)||^2$ is *zero*. Then $\|\boldsymbol{u}(t)\| = \|e^{At}\boldsymbol{u}(0)\|$ stays at $\|\boldsymbol{u}(0)\|$. So e^{At} is matrix *orthogonal*.

15 $\boldsymbol{u}_p = 4$ and $\boldsymbol{u}(t) = ce^t + 4$. For the matrix equation, the particular solution $\boldsymbol{u}_p = A^{-1}\boldsymbol{b}$ is $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\boldsymbol{u}(t) = c_1e^t\begin{bmatrix} 1 \\ t \end{bmatrix} + c_2e^t\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

16 Substituting $\boldsymbol{u} = e^{ct}\boldsymbol{v}$ gives $ce^{ct}\boldsymbol{v} = Ae^{ct}\boldsymbol{v} - e^{ct}\boldsymbol{b}$ or $(A - cI)\boldsymbol{v} = \boldsymbol{b}$ or $\boldsymbol{v} = (A - cI)^{-1}\boldsymbol{b} =$ particular solution. If c is an eigenvalue then $A - cI$ is not invertible.

17 (a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. These show the unstable cases
(a) $\lambda_1 < 0$ and $\lambda_2 > 0$ (b) $\lambda_1 > 0$ and $\lambda_2 > 0$ (c) $\lambda = a \pm ib$ with $a > 0$

18 $d/dt(e^{At}) = A + A^2t + \frac{1}{2}A^3t^2 + \frac{1}{6}A^4t^3 + \cdots = A(I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \cdots)$. This is exactly Ae^{At}, the derivative we expect.

19 $e^{Bt} = I + Bt$ (short series with $B^2 = 0$) $= \begin{bmatrix} \mathbf{1} & \mathbf{-4t} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$. Derivative $= \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix} = B$.

20 The solution at time $t + T$ is $e^{A(t+T)}\boldsymbol{u}(0)$. Thus e^{At} times e^{AT} equals $e^{A(t+T)}$.

21 $\begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{0} \end{bmatrix}$ diagonalizes $A = X\Lambda X^{-1}$.

Then $e^{At} = Xe^{\Lambda t}X^{-1} = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}$; $\begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \boldsymbol{e^t} & 0 \\ 0 & \mathbf{1} \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} e^t & 4e^t - 4 \\ 0 & 1 \end{bmatrix}$.

22 $A^2 = A$ gives $e^{At} = I + At + \frac{1}{2}\boldsymbol{At^2} + \frac{1}{6}At^3 + \cdots = I + (e^t - 1)A = \begin{bmatrix} e^t & 4e^t - 4 \\ 0 & 1 \end{bmatrix}$.

23 $e^A = \begin{bmatrix} e & 4(e-1) \\ 0 & 1 \end{bmatrix}$ from **21** and $e^B = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$ from **19**. By direct multiplication $e^A e^B \neq e^B e^A \neq e^{A+B} = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix}$.

24 $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$. Then $e^{At} = \begin{bmatrix} \boldsymbol{e^t} & \frac{1}{2}(e^{3t} - e^t) \\ 0 & \boldsymbol{e^{3t}} \end{bmatrix}$.

At $t = 0, e^{At} = I$ and $\Lambda e^{At} = A$.

25 The matrix has $A^2 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} = A$. *Then all* $A^n = A$. So $e^{At} = I + (t + t^2/2! + \cdots)A = I + (e^t - 1)A = \begin{bmatrix} e^t & 3(e^t - 1) \\ 0 & 0 \end{bmatrix}$ as in Problem 22.

26 (a) The inverse of e^{At} is e^{-At} (b) If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $e^{At}\boldsymbol{x} = e^{\lambda t}\boldsymbol{x}$ and $e^{\lambda t} \neq 0$.

To see $e^{At}\boldsymbol{x}$, write $(I + At + \frac{1}{2}A^2t^2 + \cdots)\boldsymbol{x} = (1 + \lambda t + \frac{1}{2}\lambda^2 t^2 + \cdots)\boldsymbol{x} = e^{\lambda t}\boldsymbol{x}$.

27 $(x, y) = (e^{4t}, e^{-4t})$ is a growing solution. The correct matrix for the exchanged $\boldsymbol{u} = \begin{bmatrix} y \\ x \end{bmatrix}$ is $\begin{bmatrix} 2 & -2 \\ -4 & 0 \end{bmatrix}$. It *does* have the same eigenvalues as the original matrix.

28 Invert $\begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix}$ to produce $\boldsymbol{U}_{n+1} = \begin{bmatrix} 1 & 0 \\ -\Delta t & 1 \end{bmatrix} \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \boldsymbol{U}_n = \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 - (\Delta t)^2 \end{bmatrix} \boldsymbol{U}_n$.

At $\Delta t = 1$, $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ has $\lambda = e^{i\pi/3}$ and $e^{-i\pi/3}$. Both eigenvalues have $\lambda^6 = 1$ so $\boldsymbol{A^6 = I}$. Therefore $\boldsymbol{U}_6 = A^6\boldsymbol{U}_0$ comes exactly back to $\boldsymbol{U}_0$.

29 First A has $\lambda = \pm i$ and $A^4 = I$. Second A has $\lambda = -1, -1$ and $A^n = (-1)^n \begin{bmatrix} 1 - 2n & -2n \\ 2n & 2n + 1 \end{bmatrix}$ Linear growth.

30 With $a = \Delta t/2$ the trapezoidal step is $\boldsymbol{U}_{n+1} = \dfrac{1}{1 + a^2} \begin{bmatrix} 1 - a^2 & 2a \\ -2a & 1 - a^2 \end{bmatrix} \boldsymbol{U}_n$.

That matrix has orthonormal columns $\Rightarrow$ orthogonal matrix $\Rightarrow \|\boldsymbol{U}_{n+1}\| = \|\boldsymbol{U}_n\|$

31 (a) If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then the infinite cosine series gives $(\cos A)\boldsymbol{x} = (\cos\lambda)\boldsymbol{x}$

(b) $\lambda(A) = 2\pi$ and 0 so $\cos\lambda = 1$ and 1 which means that $\cos A = I$

(c) $\boldsymbol{u}(t) = 3(\cos 2\pi t)(1,1) + 1(\cos 0t)(1,-1)$ [$\boldsymbol{u}' = A\boldsymbol{u}$ has **exp**, $\boldsymbol{u}'' = A\boldsymbol{u}$ has **cos**]

32 For proof 2, square the start of the series to see $(I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3)^2 = I + 2A + \frac{1}{2}(2A)^2 + \frac{1}{6}(2A)^3 + \cdots$. The diagonalizing proof is easiest when it works (needing diagonalizable A).

Problem Set 6.4, page 345

Note A way to complete the proof at the end of page 334, (perturbing the matrix to produce distinct eigenvalues) is now on the course website: "*Proofs of the Spectral Theorem*." **math.mit.edu/linearalgebra**.

1 The first is ASA^{T}: symmetric but eigenvalues are different from 1 and -1 for S.

The second is ASA^{-1}: same eigenvalues as S but not symmetric.

The third is $ASA^{\mathrm{T}} = ASA^{-1}$: **symmetric with the same eigenvalues as S**.

This needed $B = A^{\mathrm{T}} = A^{-1}$ to be an **orthogonal matrix**.

2 (a) ASB stays symmetric like S when $B = A^{\mathrm{T}}$

(b) ASB is similar to S when $B = A^{-1}$

To have both (a) and (b) we need $B = A^{\mathrm{T}} = A^{-1}$ to be an **orthogonal matrix**

3 $A = \begin{bmatrix} 1 & 3 & 6 \\ 3 & 3 & 3 \\ 6 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}$ $= \frac{1}{2}(A + A^{\mathrm{T}}) + \frac{1}{2}(A - A^{\mathrm{T}})$ $=$ **symmetric** $+$ **skew-symmetric**.

4 $(A^{\mathrm{T}}CA)^{\mathrm{T}} = A^{\mathrm{T}}C^{\mathrm{T}}(A^{\mathrm{T}})^{\mathrm{T}} = A^{\mathrm{T}}CA$. When A is 6 by 3, C will be 6 by 6 and the triple product $A^{\mathrm{T}}CA$ is 3 by 3.

5 $\lambda = 0, 4, -2$; unit vectors $\pm(0,1,-1)/\sqrt{2}$ and $\pm(2,1,1)/\sqrt{6}$ and $\pm(1,-1,-1)/\sqrt{3}$.

6 $\lambda = 10$ and -5 in $\Lambda = \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ have to be normalized to unit vectors in $Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

7 $Q = \frac{1}{3} \begin{bmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ -1 & -2 & 2 \end{bmatrix}$. The columns of Q are unit eigenvectors of S. Each unit eigenvector could be multiplied by -1

8 $S = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$ has $\lambda = 0$ and 25 so the columns of Q are the two eigenvectors: $Q = \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}$ or we can exchange columns or reverse the signs of any column.

9 (a) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ has $\lambda = -1$ and 3 (b) The pivots $1, 1 - b^2$ have the same signs as the λ's

(c) The trace is $\lambda_1 + \lambda_2 = 2$, so S can't have two negative eigenvalues.

10 If $A^3 = 0$ then all $\lambda^3 = 0$ so all $\lambda = 0$ as in $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. If A is *symmetric* then $A^3 = Q\Lambda^3 Q^{\mathrm{T}} = 0$ requires $\Lambda = 0$. The only symmetric A is $Q\,0\,Q^{\mathrm{T}} =$ zero matrix.

11 If λ is complex then $\overline{\lambda}$ is also an eigenvalue ($A\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}$). Always $\lambda + \overline{\lambda}$ is real. The trace is real so the third eigenvalue of a 3 by 3 real matrix must be real.

12 If $\boldsymbol{x}$ is not real then $\lambda = \boldsymbol{x}^{\mathrm{T}} A\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ is *not* always real. Can't assume real eigenvectors!

13 $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = 2 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + 4 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$; $\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = 0 \begin{bmatrix} .64 & -.48 \\ -.48 & .36 \end{bmatrix} + 25 \begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$

14 $\begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 \end{bmatrix}$ is an Q matrix so $P_1 + P_2 = \boldsymbol{x}_1\boldsymbol{x}_1^{\mathrm{T}} + \boldsymbol{x}_2\boldsymbol{x}_2^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1^{\mathrm{T}} \\ \boldsymbol{x}_2^{\mathrm{T}} \end{bmatrix} = I$; also $P_1P_2 = \boldsymbol{x}_1(\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{x}_2)\boldsymbol{x}_2^{\mathrm{T}} =$ zero matrix.

Second proof: $P_1P_2 = P_1(I - P_1) = P_1 - P_1 = 0$ since $P_1^2 = P_1$.

15 $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ has $\lambda = ib$ and $-ib$. The block matrices $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ and $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ are also skew-symmetric with $\lambda = ib$ (twice) and $\lambda = -ib$ (twice).

16 M is skew-symmetric and **orthogonal**; λ's must be i, i, $-i$, $-i$ to have trace zero.

17 $A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}$ has $\lambda = 0, 0$ and only one independent eigenvector $\boldsymbol{x} = (i, 1)$. The good property for complex matrices is not $A^{\mathrm{T}} = A$ (symmetric) but $\overline{A}^{\mathrm{T}} = A$ (Hermitian with real eigenvalues and orthogonal eigenvectors: see Problem 22 and Section 9.2).

18 (a) If $A\boldsymbol{z} = \lambda\boldsymbol{y}$ and $A^{\mathrm{T}}\boldsymbol{y} = \lambda\boldsymbol{z}$ then $B[\,\boldsymbol{y};\ -\boldsymbol{z}\,] = [\,-A\boldsymbol{z};\ A^{\mathrm{T}}\boldsymbol{y}\,] = -\lambda[\,\boldsymbol{y};\ -\boldsymbol{z}\,]$. So $-\lambda$ is also an eigenvalue of B. (b) $A^{\mathrm{T}}A\boldsymbol{z} = A^{\mathrm{T}}(\lambda\boldsymbol{y}) = \lambda^2\boldsymbol{z}$. (c) $\lambda = -1, -1, 1, 1$; $\boldsymbol{x}_1 = (1, 0, -1, 0)$, $\boldsymbol{x}_2 = (0, 1, 0, -1)$, $\boldsymbol{x}_3 = (1, 0, 1, 0)$, $\boldsymbol{x}_4 = (0, 1, 0, 1)$.

19 The eigenvalues of $S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ are $0, \sqrt{2}, -\sqrt{2}$ by Problem 16 with

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix}, \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 1 \\ -\sqrt{2} \end{bmatrix}.$$

20 **1.** $\boldsymbol{y}$ is in the nullspace of S and $\boldsymbol{x}$ is in the column space (that is also row space because $S = S^{\mathrm{T}}$). The nullspace and row space are perpendicular so $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{x} = 0$.

2. If $S\boldsymbol{x} = \lambda\boldsymbol{x}$ and $S\boldsymbol{y} = \beta\boldsymbol{y}$ then shift S by βI to have a zero eigenvalue that matches Step **1**.$(S - \beta I)\boldsymbol{x} = (\lambda - \beta)\boldsymbol{x}$ and $(S - \beta I)\boldsymbol{y} = \boldsymbol{0}$ and again $\boldsymbol{x}$ is perpendicular to y.

21 S has $X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; B has $X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2d \end{bmatrix}$. Perpendicular for A. Not perpendicular for S since $B^{\mathrm{T}} \neq B$

22 $S = \begin{bmatrix} 1 & 3+4i \\ 3-4i & 1 \end{bmatrix}$ is a *Hermitian matrix* ($\overline{S}^{\mathrm{T}} = S$). Its eigenvalues 6 and -4 are *real*. Adjust equations (1)–(2) in the text to prove that λ is always real when $\overline{S}^{\mathrm{T}} = S$:

$S\boldsymbol{x} = \lambda\boldsymbol{x}$ leads to $\overline{S}\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}$. Transpose to $\overline{x}^{\mathrm{T}}S = \overline{x}^{\mathrm{T}}\overline{\lambda}$ using $\overline{S}^{\mathrm{T}} = S$.

Then $\overline{\boldsymbol{x}}^{\mathrm{T}}S\boldsymbol{x} = \overline{\boldsymbol{x}}^{\mathrm{T}}\lambda\boldsymbol{x}$ and also $\overline{\boldsymbol{x}}^{\mathrm{T}}S\boldsymbol{x} = \overline{\boldsymbol{x}}^{\mathrm{T}}\overline{\lambda}\boldsymbol{x}$. So $\lambda = \overline{\lambda}$ is real.

23 (a) False. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) True from $A^{\mathrm{T}} = Q\Lambda Q^{\mathrm{T}} = A$ (c) True from $S^{-1} = Q\Lambda^{-1}Q^{\mathrm{T}}$ (d) False!

24 A and A^{T} have the same λ's but the *order* of the $\boldsymbol{x}$'s can change. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $\lambda_1 = i$ and $\lambda_2 = -i$ with $\boldsymbol{x}_1 = (1, i)$ first for A but $\boldsymbol{x}_1 = (1, -i)$ is first for A^{T}.

25 A is invertible, orthogonal, permutation, diagonalizable, Markov; B is projection, diagonalizable, Markov. A allows $QR, X\Lambda X^{-1}, Q\Lambda Q^{\mathrm{T}}$; B allows $X\Lambda X^{-1}$ and $Q\Lambda Q^{\mathrm{T}}$.

26 Symmetry gives $Q\Lambda Q^{\mathrm{T}}$ if $b = 1$; repeated λ and no X if $b = -1$; singular if $b = 0$.

27 Orthogonal and symmetric requires $|\lambda| = 1$ and λ real, so $\lambda = \pm 1$. Then $S = \pm I$ or $S = Q\Lambda Q^{\mathrm{T}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.

28 Eigenvectors $(1, 0)$ and $(\mathbf{1}, \mathbf{1})$ give a $45°$ angle even with A^{T} very close to A.

29 The roots of $\lambda^2 + b\lambda + c = 0$ are $\frac{1}{2}(-b \pm \sqrt{b^2 - 4ac})$. Then $\lambda_1 - \lambda_2$ is $\sqrt{b^2 - 4c}$. For $\det(A + tB - \lambda I)$ we have $b = -3 - 8t$ and $c = 2 + 16t - t^2$. The minimum of $b^2 - 4c$ is $1/17$ at $t = 2/17$. Then $\lambda_2 - \lambda_1 = 1/\sqrt{17}$: close but not equal !

30 $S = \begin{bmatrix} 4 & 2+i \\ 2-i & 0 \end{bmatrix} = \overline{S}^{\mathrm{T}}$ has real eigenvalues $\lambda = 5$ and -1 with trace $= 4$ and $\det = -5$. The solution to **20** proves that λ is real when $\overline{S}^{\mathrm{T}} = S$ is Hermitian.

31 (a) $A = Q\Lambda\overline{Q}^{\mathrm{T}}$ times $\overline{A}^{\mathrm{T}} = Q\overline{\Lambda}^{\mathrm{T}}\overline{Q}^{\mathrm{T}}$ equals $\overline{A}^{\mathrm{T}}$ times A because $Q = \overline{Q}^{\mathrm{T}}$ and $\Lambda\overline{\Lambda}^{\mathrm{T}} = \overline{\Lambda}^{\mathrm{T}}\Lambda$ (diagonal!) (b) Step 2: The 1, 1 entries of $\overline{T}^{\mathrm{T}}\,T$ and $T\overline{T}^{\mathrm{T}}$ are $|a|^2$ and $|a|^2 + |b|^2$. Equally makes $b = 0$ and $T = \Lambda$.

32 a_{11} is $\begin{bmatrix} q_{11} \dots q_{1n} \end{bmatrix} \begin{bmatrix} \lambda_1 \overline{q}_{11} \dots \lambda_n \overline{q}_{1n} \end{bmatrix}^{\mathrm{T}} \le \lambda_{\max} \left(|q_{11}|^2 + \cdots + |q_{1n}|^2 \right) = \lambda_{\max}$.

33 (a) $\boldsymbol{x}^{\mathrm{T}}(A\boldsymbol{x}) = (A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}\boldsymbol{x} = -\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$. (b) $\overline{\boldsymbol{z}}^{\mathrm{T}}A\boldsymbol{z}$ is pure imaginary, its real part is $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} + \boldsymbol{y}^{\mathrm{T}}A\boldsymbol{y} = 0 + 0$ (c) $\det A = \lambda_1 \dots \lambda_n \ge 0$: pairs of λ's $= ib, -ib$.

34 Since S is diagonalizable with eigenvalue matrix $\Lambda = 2I$, the matrix S itself has to be $X\Lambda X^{-1} = X(2I)X^{-1} = 2I$. (The unsymmetric matrix $[2\ 1\ ;\ 0\ 2]$ also has $\lambda = 2, 2$.)

35 (a) $S^{\mathrm{T}} = S$ and $S^{\mathrm{T}}S = I$ lead to $S^2 = I$.

(b) The only possible eigenvalues of S are 1 and -1.

(c) $\Lambda = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ so $S = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \Lambda \begin{bmatrix} Q_1^{\mathrm{T}} \\ Q_2^{\mathrm{T}} \end{bmatrix} = Q_1Q_1^{\mathrm{T}} - Q_2Q_2^{\mathrm{T}}$ with $Q_1^{\mathrm{T}}Q_2 = 0$.

36 $(A^{\mathrm{T}}SA)^{\mathrm{T}} = A^{\mathrm{T}}S^{\mathrm{T}}A^{\mathrm{TT}} = A^{\mathrm{T}}SA$. This matrix $A^{\mathrm{T}}SA$ may have different eigenvalues from S, but the "inertia theorem" says that the two sets of eigenvalues have the same signs. The inertia = number of (positive, zero, negative) eigenvalues is the same for S and $A^{\mathrm{T}}SA$.

37 Substitute $\lambda = a$ to find $\det(S - aI) = a^2 - a^2 - ca + ac - b^2 = -b^2$ (negative). The parabola crosses at the eigenvalues λ because they have $\det(S - \lambda I) = 0$.

Problem Set 6.5, page 358

1 Suppose $a > 0$ and $ac > b^2$ so that also $c > b^2/a > 0$.

(i) The eigenvalues have the *same sign* because $\lambda_1\lambda_2 = \det = ac - b^2 > 0$.

(ii) That sign is *positive* because $\lambda_1 + \lambda_2 > 0$ (it equals the trace $a + c > 0$).

2 Only $S_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$ has two positive eigenvalues since $101 > 10^2$.

$\boldsymbol{x}^{\mathrm{T}}S_1\boldsymbol{x} = 5x_1^2 + 12x_1x_2 + 7x_2^2$ is negative for example when $x_1 = 4$ and $x_2 = -3$: A_1 is not positive definite as its determinant confirms; S_2 has trace c_0; S_3 has $\det = 0$.

3 Positive definite for $-3 < b < 3$ $\begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}\begin{bmatrix}1 & b\\ 0 & 9-b^2\end{bmatrix} = \begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 9-b^2\end{bmatrix}\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix} = LDL^{\mathrm{T}}$

Positive definite for $c > 8$ $\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}2 & 4\\ 0 & c-8\end{bmatrix} = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & c-8\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix} = LDL^{\mathrm{T}}.$

Positive definite for $c > b$ $L = \begin{bmatrix}1 & 1\\ -b/c & 0\end{bmatrix}$ $D = \begin{bmatrix}c & 0\\ 0 & c-b/c\end{bmatrix}$ $S = LDL^{\mathrm{T}}.$

4 $f(x,y) = x^2 + 4xy + 9y^2 = (x+2y)^2 + 5y^2;\ x^2 + 6xy + 9y^2 = (x+3y)^2.$

5 $x^2 + 4xy + 3y^2 = (x+2y)^2 - y^2 =$ *difference of squares* is negative at $x = 2$, $y = -1$, where the first square is zero.

6 $A = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ produces $f(x,y) = \begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \boldsymbol{2xy}$. A has $\lambda = 1$ and -1. Then A is an *indefinite matrix* and $f(x,y) = 2xy$ has a *saddle point*.

7 $A^{\mathrm{T}}A = \begin{bmatrix}1 & 2\\ 2 & 13\end{bmatrix}$ and $A^{\mathrm{T}}A = \begin{bmatrix}6 & 5\\ 5 & 6\end{bmatrix}$ are positive definite; $A^{\mathrm{T}}A = \begin{bmatrix}2 & 3 & 3\\ 3 & 5 & 4\\ 3 & 4 & 5\end{bmatrix}$ is singular (and positive semidefinite). The first two A's have independent columns. The 2 by 3 A cannot have full column rank 3, with only 2 rows; $A^{\mathrm{T}}A$ is singular.

8 $S = \begin{bmatrix}3 & 6\\ 6 & 16\end{bmatrix} = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$. Pivots $3, 4$ outside squares, ℓ_{ij} inside. $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} = 3(x+2y)^2 + 4y^2$

9 $S = \begin{bmatrix}4 & -4 & 8\\ -4 & 4 & -8\\ 8 & -8 & 16\end{bmatrix}$ has only one pivot $= 4$, rank $S = 1$, eigenvalues are $24, 0, 0$, $\det S = 0$.

10 $S = \begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}$ has pivots $2, \frac{3}{2}, \frac{4}{3}$; $T = \begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}$ is singular; $T\begin{bmatrix}1\\ 1\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$.

11 Corner determinants $|S_1| = 2$, $|S_2| = 6$, $|S_3| = 30$. The pivots are $2/1, 6/2, 30/6$.

12 S is positive definite for $c > 1$; determinants $c, c^2 - 1$, and $(c-1)^2(c+2) > 0$. T is *never* positive definite (determinants $d - 4$ and $-4d + 12$ are never both positive).

13 $S = \begin{bmatrix} 1 & 5 \\ 5 & 10 \end{bmatrix}$ is an example with $a + c > 2b$ but $ac < b^2$, so not positive definite.

14 The eigenvalues of S^{-1} are positive because they are $1/\lambda(S)$. Also the entries of S^{-1} pass the determinant tests. And $\boldsymbol{x}^{\mathrm{T}} S^{-1} \boldsymbol{x} = (S^{-1}\boldsymbol{x})^{\mathrm{T}} S (S^{-1}\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \mathbf{0}$.

15 Since $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} > 0$ and $\boldsymbol{x}^{\mathrm{T}} T \boldsymbol{x} > 0$ we have $\boldsymbol{x}^{\mathrm{T}}(S + T)\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}} T \boldsymbol{x} > 0$ for all $\boldsymbol{x} \neq \mathbf{0}$. Then $S + T$ is a positive definite matrix. The second proof uses the test $S = A^{\mathrm{T}} A$ (independent columns in A): If $S = A^{\mathrm{T}} A$ and $T = B^{\mathrm{T}} B$ pass this test, then $S + T = \begin{bmatrix} A & B \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} A \\ B \end{bmatrix}$ also passes, and must be positive definite.

16 $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x}$ is zero when $(x_1, x_2, x_3) = (0, 1, 0)$ because of the zero on the diagonal. Actually $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x}$ goes *negative* for $\boldsymbol{x} = (1, -10, 0)$ because the second pivot is *negative*.

17 If a_{jj} were smaller than all λ's, $S - a_{jj} I$ would have all eigenvalues > 0 (positive definite). But $S - a_{jj} I$ has a *zero* in the (j, j) position; impossible by Problem 16.

18 If $S\boldsymbol{x} = \lambda \boldsymbol{x}$ then $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} = \lambda \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x}$. If S is positive definite this leads to $\lambda = \boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} / \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x} > 0$ (ratio of positive numbers). So positive energy $\Rightarrow$ positive eigenvalues.

19 All cross terms are $\boldsymbol{x}_i^{\mathrm{T}} \boldsymbol{x}_j = 0$ because symmetric matrices have orthogonal eigenvectors. So positive eigenvalues $\Rightarrow$ positive energy.

20 (a) The determinant is positive; all $\lambda > 0$ (b) All projection matrices except I are singular (c) The diagonal entries of D are its eigenvalues (d) $S = -I$ has $\det = +1$ when n is even.

21 S is positive definite when $s > 8$; T is positive definite when $t > 5$ by determinants.

22 $A = \dfrac{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}{\sqrt{2}} \begin{bmatrix} \sqrt{9} & \\ & \sqrt{1} \end{bmatrix} \dfrac{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; $A = Q \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} Q^{\mathrm{T}} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

23 $x^2/a^2 + y^2/b^2$ is $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x}$ when $S = \mathsf{diag}(1/a^2, 1/b^2)$. Then $\lambda_1 = 1/a^2$ and $\lambda_2 = 1/b^2$ so $a = 1/\sqrt{\lambda_1}$ and $b = 1/\sqrt{\lambda_2}$. The ellipse $9x^2 + 16y^2 = 1$ has axes with half-lengths $a = \frac{1}{3}$ and $b = \frac{1}{4}$. The points $(\frac{1}{3}, 0)$ and $(0, \frac{1}{4})$ are at the ends of the axes.

24 The ellipse $x^2 + xy + y^2 = 1$ has axes with half-lengths $1/\sqrt{\lambda} = \sqrt{2}$ and $\sqrt{2/3}$.

25 $S = C^{\mathrm{T}}C = \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix}$; $\begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$

26 The Cholesky factors $C = \left(L\sqrt{D}\right)^{\mathrm{T}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$ have *square roots* of the pivots from D. Note again $C^{\mathrm{T}}C = LDL^{\mathrm{T}} = S$.

27 Writing out $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}LDL^{\mathrm{T}}\boldsymbol{x}$ gives $ax^2 + 2bxy + cy^2 = a(x + \frac{b}{a}y)^2 + \frac{ac-b^2}{a}y^2$. So the LDL^{T} from elimination is exactly the same as *completing the square*. The example $2x^2 + 8xy + 10y^2 = 2(x+2y)^2 + 2y^2$ with pivots $2, 2$ outside the squares and multiplier 2 inside.

28 $\det S = (1)(10)(1) = 10$; $\lambda = 2$ and 5; $\boldsymbol{x}_1 = (\cos\theta, \sin\theta)$, $\boldsymbol{x}_2 = (-\sin\theta, \cos\theta)$; the λ's are positive. So S is positive definite.

29 $S_1 = \begin{bmatrix} 6x^2 & 2x \\ 2x & 2 \end{bmatrix}$ is semidefinite; $f_1 = (\frac{1}{2}x^2 + y)^2 = 0$ on the curve $\frac{1}{2}x^2 + y = 0$; $S_2 = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is indefinite at $(0, 1)$ where first derivatives $= 0$. Then $x = 0, y = 1$ is a saddle point of the function $f_2(x, y)$.

30 $ax^2 + 2bxy + cy^2$ has a saddle point if $ac < b^2$. The matrix is *indefinite* ($\lambda < 0$ and $\lambda > 0$) because the determinant $ac - b^2$ is *negative*.

31 If $c > 9$ the graph of z is a bowl, if $c < 9$ the graph has a saddle point. When $c = 9$ the graph of $z = (2x + 3y)^2$ is a "trough" staying at zero along the line $2x + 3y = 0$.

32 Orthogonal matrices, exponentials e^{At}, matrices with $\det = 1$ are groups. Examples of subgroups are orthogonal matrices with $\det = 1$, exponentials e^{An} for integer n. Another subgroup: lower triangular elimination matrices E with diagonal 1's.

33 A product ST of symmetric positive definite matrices comes into many applications. The "*generalized*" eigenvalue problem $K\boldsymbol{x} = \lambda M\boldsymbol{x}$ has $ST = M^{-1}K$. (Often we use

eig(K, M) without actually inverting M.) All eigenvalues λ are positive:

$$ST\boldsymbol{x} = \lambda\boldsymbol{x} \text{ gives } (T\boldsymbol{x})^{\mathrm{T}} ST\boldsymbol{x} = (T\boldsymbol{x})^{\mathrm{T}}\lambda x. \text{ Then } \lambda = \boldsymbol{x}^{\mathrm{T}} T^{\mathrm{T}} ST\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}} T\boldsymbol{x} > 0.$$

34 The five eigenvalues of K are $2 - 2\cos\frac{k\pi}{6} = 2 - \sqrt{3}, 2 - 1, 2, 2 + 1, 2 + \sqrt{3}$. The product of those eigenvalues is $6 = \det K$.

35 Put parentheses in $\boldsymbol{x}^{\mathrm{T}} A^{\mathrm{T}} CA\boldsymbol{x} = (A\boldsymbol{x})^{\mathrm{T}} C(A\boldsymbol{x})$. Since C is assumed positive definite, this energy can drop to zero only when $A\boldsymbol{x} = \mathbf{0}$. Sine A is assumed to have independent columns, $A\boldsymbol{x} = \mathbf{0}$ only happens when $\boldsymbol{x} = \mathbf{0}$. Thus $A^{\mathrm{T}} CA$ has positive energy and is positive definite.

My textbooks *Computational Science and Engineering* and *Introduction to Applied Mathematics* start with many examples of $A^{\mathrm{T}} CA$ in a wide range of applications. I believe this is a unifying concept from linear algebra.

36 (a) The eigenvectors of $\lambda_1 I - S$ are $\lambda_1 - \lambda_1, \lambda_1 - \lambda_2, \ldots, \lambda_1 - \lambda_n$. Those are ≥ 0; $\lambda_1 I - S$ is semidefinite.

(b) Semidefinite matrices have energy $\boldsymbol{x}^{\mathrm{T}} (\lambda_1 I - S)\, \boldsymbol{x}_2 \geq 0$. Then $\lambda_1 \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \geq \boldsymbol{x}^{\mathrm{T}} S\boldsymbol{x}$.

(c) Part (b) says $\boldsymbol{x}^{\mathrm{T}} S\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \leq \lambda_1$ for all $\boldsymbol{x}$. Equality at the eigenvector with $S\boldsymbol{x} = \lambda_1\boldsymbol{x}$.

37 Energy $\boldsymbol{x}^{\mathrm{T}} S\boldsymbol{x} = a\,(x_1+x_2+x_3)^2 + c\,(x_2-x_3)^2 \geq 0$ if $a \geq 0$ and $c \geq 0$: semidefinite.

S has rank ≤ 2 and determinant $= 0$; cannot be positive definite for any a and c.

Problem Set 6.6, page 360

1 $B = GCG^{-1} = GF^{-1}AFG^{-1}$ so $M = FG^{-1}$. C similar to A and $B \Rightarrow A$ similar to B.

2 $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = M^{-1}AM$ with $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

3 $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = M^{-1}AM$;

$B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$;

$B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

4 A has no repeated λ so it can be diagonalized: $S^{-1}AS = \Lambda$ makes A similar to Λ.

5 $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ are similar (they all have eigenvalues 1 and 0).

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is by itself and also $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is by itself with eigenvalues 1 and -1.

6 *Eight families* of similar matrices: six matrices have $\lambda = 0, 1$ (one family); three matrices have $\lambda = 1, 1$ and three have $\lambda = 0, 0$ (two families each!); one has $\lambda = 1, -1$; one has $\lambda = 2, 0$; two matrices have $\lambda = \frac{1}{2}(1 \pm \sqrt{5})$ (they are in one family).

7 (a) $(M^{-1}AM)(M^{-1}\boldsymbol{x}) = M^{-1}(A\boldsymbol{x}) = M^{-1}\mathbf{0} = \mathbf{0}$ (b) The nullspaces of A and of $M^{-1}AM$ have the same *dimension*. Different vectors and different bases.

8 Same Λ / Same S But $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ have the same line of eigenvectors and the same eigenvalues $\lambda = 0, 0$.

9 $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, every $A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$. $A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

10 $J^2 = \begin{bmatrix} c^2 & 2c \\ 0 & c^2 \end{bmatrix}$ and $J^k = \begin{bmatrix} c^k & kc^{k-1} \\ 0 & c^k \end{bmatrix}$; $J^0 = I$ and $J^{-1} = \begin{bmatrix} c^{-1} & -c^{-2} \\ 0 & c^{-1} \end{bmatrix}$.

11 $\boldsymbol{u}(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} v(0) \\ w(0) \end{bmatrix}$. The equation $\dfrac{d\boldsymbol{u}}{dt} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \boldsymbol{u}$ has $\dfrac{dv}{dt} = \lambda v + w$ and $\dfrac{dw}{dt} = \lambda w$. Then $w(t) = 2e^{\lambda t}$ and $v(t)$ must include $2te^{\lambda t}$ (this comes from the repeated λ). To match $v(0) = 5$, the solution is $v(t) = 2te^{\lambda t} + 5e^{\lambda t}$.

12 If $M^{-1}JM=K$ then $JM=\begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & \mathbf{0} & \mathbf{0} & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix} = MK = \begin{bmatrix} \mathbf{0} & m_{12} & m_{13} & \mathbf{0} \\ 0 & m_{22} & m_{23} & 0 \\ 0 & m_{32} & m_{33} & 0 \\ 0 & m_{42} & m_{43} & 0 \end{bmatrix}$.

That means $m_{21} = m_{22} = m_{23} = m_{24} = 0$. M is not invertible, J not similar to K.

13 The five 4 by 4 Jordan forms with $\lambda = 0, 0, 0, 0$ are J_1 = zero matrix and

$$J_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad J_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad J_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 12 showed that J_3 and J_4 are *not similar*, even with the same rank. Every matrix with all $\lambda = 0$ is "*nilpotent*" (its nth power is A^n = zero matrix). You see $J^4 = 0$ for these matrices. How many possible Jordan forms for $n = 5$ and all $\lambda = 0$?

14 (1) Choose M_i = reverse diagonal matrix to get $M_i^{-1} J_i M_i = M_i^{\mathrm{T}}$ in each block (2) M_0 has those diagonal blocks M_i to get $M_0^{-1} J M_0 = J^{\mathrm{T}}$. (3) $A^{\mathrm{T}} = (M^{-1})^{\mathrm{T}} J^{\mathrm{T}} M^{\mathrm{T}}$ equals $(M^{-1})^{\mathrm{T}} M_0^{-1} J M_0 M^{\mathrm{T}} = (M M_0 M^{\mathrm{T}})^{-1} A (M M_0 M^{\mathrm{T}})$, and A^{T} *is similar to* A.

15 $\det(M^{-1}AM - \lambda I) = \det(M^{-1}AM - M^{-1}\lambda I M)$. This is $\det(M^{-1}(A - \lambda I)M)$. By the product rule, the determinants of M and M^{-1} cancel to leave $\det(A - \lambda I)$.

16 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is similar to $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$; $\begin{bmatrix} b & a \\ d & c \end{bmatrix}$ is similar to $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$. So two pairs of similar matrices but $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not similar to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$: different eigenvalues!

17 (a) *False*: Diagonalize a nonsymmetric $A = S\Lambda S^{-1}$. Then Λ is symmetric and similar (b) *True*: A singular matrix has $\lambda = 0$. (c) *False*: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are similar

(they have $\lambda = \pm 1$) (d) *True*: Adding I increases all eigenvalues by 1

18 $AB = B^{-1}(BA)B$ so AB is similar to BA. *If* $AB\boldsymbol{x} = \lambda\boldsymbol{x}$ *then* $BA(B\boldsymbol{x}) = \lambda(B\boldsymbol{x})$.

19 Diagonal blocks 6 by 6, 4 by 4; AB has the same eigenvalues as BA plus $6 - 4$ zeros.

20 (a) $A = M^{-1}BM \Rightarrow A^2 = (M^{-1}BM)(M^{-1}BM) = M^{-1}B^2M$. So A^2 is similar to B^2. (b) A^2 equals $(-A)^2$ but A may not be similar to $B = -A$ (it could be!). (c) $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$ is diagonalizableto $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ because$\lambda_1 \neq \lambda_2$, sothesematrices are similar. (d) $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ has only one eigenvector, so not diagonalizable (e) PAP^{T} is similar to A.

21 J^2 has three 1's down the *second* superdiagonal, and *two* independent eigenvectors for $\lambda = 0$. Its 5 by 5 Jordan form is $\begin{bmatrix} J_3 & \\ & J_2 \end{bmatrix}$ with $J_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Note to professors: An interesting question: *Which matrices A have (complex) square roots* $R^2 = A$? If A is invertible, no problem. But any Jordan blocks for $\lambda = 0$ must have sizes $n_1 \geq n_2 \geq \ldots \geq n_k \geq n_{k+1} = 0$ that come in pairs like 3 and 2 in this example: $n_1 = (n_2$ or $n_2{+}1)$ and $n_3 = (n_4$ or $n_4{+}1)$ and so on.

A list of all 3 by 3 and 4 by 4 Jordan forms could be $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, $\begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$,

$\begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$ (for any numbers a, b, c) with $3, 2, 1$ eigenvectors; $\text{diag}(a, b, c, d)$ and $\begin{bmatrix} a & 1 & & \\ & a & & \\ & & b & \\ & & & c \end{bmatrix}$,

$\begin{bmatrix} a & 1 & & \\ & a & & \\ & & b & 1 \\ & & & b \end{bmatrix}$, $\begin{bmatrix} a & 1 & & \\ & a & 1 & \\ & & a & \\ & & & b \end{bmatrix}$, $\begin{bmatrix} a & 1 & & \\ & a & 1 & \\ & & a & 1 \\ & & & a \end{bmatrix}$ with $4, 3, 2, 1$ eigenvectors.

22 If all roots are $\lambda = 0$, this means that $\det(A - \lambda I)$ must be just λ^n. The Cayley-Hamilton Theorem in Problem 6.2.32 immediately says that $A^n =$ zero matrix. The key example is a single n by n Jordan block (with $n - 1$ ones above the diagonal): Check directly that $J^n =$ zero matrix.

23 Certainly $Q_1 R_1$ is similar to $R_1 Q_1 = Q_1^{-1}(Q_1 R_1)Q_1$. Then $A_1 = Q_1 R_1 - cs^2 I$ is similar to $A_2 = R_1 Q_1 - cs^2 I$.

24 A could have eigenvalues $\lambda = 2$ and $\lambda = \frac{1}{2}$ (A could be diagonal). Then A^{-1} has the same two eigenvalues (and is similar to A).

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1 $A = U\Sigma V^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}^{\mathrm{T}} = \dfrac{\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}}{\sqrt{10}} \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix} \dfrac{\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}{\sqrt{5}}$

2 This $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is a 2 by 2 matrix of rank 1. Its row space has basis $\boldsymbol{v}_1$, its nullspace has basis $\boldsymbol{v}_2$, its column space has basis $\boldsymbol{u}_1$, its left nullspace has basis $\boldsymbol{u}_2$:

$$\text{Row space} \quad \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ Nullspace} \quad \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\text{Column space} \quad \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \ \boldsymbol{N}(A^{\mathrm{T}}) \quad \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

3 If A has rank 1 then so does $A^{\mathrm{T}}A$. The only nonzero eigenvalue of $A^{\mathrm{T}}A$ is its trace, which is the sum of all a_{ij}^2. (Each diagonal entry of $A^{\mathrm{T}}A$ is the sum of a_{ij}^2 down one column, so the trace is the sum down all columns.) Then $\sigma_1 =$ square root of this sum, and $\sigma_1^2 =$ this sum of all a_{ij}^2.

4 $A^{\mathrm{T}}A = AA^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenvalues $\sigma_1^2 = \dfrac{3+\sqrt{5}}{2}, \sigma_2^2 = \dfrac{3-\sqrt{5}}{2}$. But A is indefinite
$\sigma_1 = (1+\sqrt{5})/2 = \lambda_1(A), \ \sigma_2 = (\sqrt{5}-1)/2 = -\lambda_2(A)$; $\boldsymbol{u}_1 = \boldsymbol{v}_1$ but $\boldsymbol{u}_2 = -\boldsymbol{v}_2$.

5 A proof that ***eigshow*** finds the SVD. When $\boldsymbol{V}_1 = (1,0), \boldsymbol{V}_2 = (0,1)$ the demo finds $A\boldsymbol{V}_1$ and $A\boldsymbol{V}_2$ at some angle θ. A 90° turn by the mouse to $\boldsymbol{V}_2, -\boldsymbol{V}_1$ finds $A\boldsymbol{V}_2$ and $-A\boldsymbol{V}_1$ at the angle $\pi - \theta$. Somewhere between, the constantly orthogonal $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ must produce $A\boldsymbol{v}_1$ and $A\boldsymbol{v}_2$ *at angle* $\pi/2$. Those orthogonal directions give $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

6 $AA^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $\sigma_1^2 = 3$ with $\boldsymbol{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\sigma_2^2 = 1$ with $\boldsymbol{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

$A^{\mathrm{T}}A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ has $\sigma_1^2 = 3$ with $\boldsymbol{v}_1 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$, $\sigma_2^2 = 1$ with $\boldsymbol{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$;

and $\boldsymbol{v}_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$. Then $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = [\,\boldsymbol{u}_1 \ \ \boldsymbol{u}_2\,] \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} [\,\boldsymbol{v}_1 \ \ \boldsymbol{v}_2 \ \ \boldsymbol{v}_3\,]^{\mathrm{T}}$.

7 The matrix A in Problem 6 had $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$ in Σ. The smallest change to rank 1 is **to make $\boldsymbol{\sigma}_2 = \mathbf{0}$**. In the factorization

$$A = U\Sigma V^{\mathrm{T}} = \boldsymbol{u}_1\sigma_1\boldsymbol{v}_1^{\mathrm{T}} + \boldsymbol{u}_2\sigma_2\boldsymbol{v}_2^{\mathrm{T}}$$

this change $\sigma_2 \to 0$ will leave the closest rank–1 matrix as $\boldsymbol{u}_1\sigma_1\boldsymbol{v}_1^{\mathrm{T}}$. See Problem 14 for the general case of this problem.

8 The number $\sigma_{\max}(A^{-1})\sigma_{\max}(A)$ is the same as $\sigma_{\max}(A)/\sigma_{\min}(A)$. This is certainly ≥ 1. It equals 1 if all σ's are equal, and $A = U\Sigma V^{\mathrm{T}}$ is a multiple of an orthogonal matrix. The ratio $\sigma_{\max}/\sigma_{\min}$ is the important **condition number** of A studied in Section 9.2.

9 $A = UV^{\mathrm{T}}$ since all $\sigma_j = 1$, which means that $\Sigma = I$.

10 A rank–1 matrix with $A\boldsymbol{v} = 12\boldsymbol{u}$ would have $\boldsymbol{u}$ in its column space, so $A = \boldsymbol{u}\boldsymbol{w}^{\mathrm{T}}$ for some vector $\boldsymbol{w}$. I intended (but didn't say) that $\boldsymbol{w}$ is a multiple of the unit vector $\boldsymbol{v} = \frac{1}{2}(1,1,1,1)$ in the problem. Then $A = 12\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ to get $A\boldsymbol{v} = 12\boldsymbol{u}$ when $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v} = 1$.

11 If A has orthogonal columns $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ of lengths $\sigma_1, \ldots, \sigma_n$, then $A^{\mathrm{T}}A$ will be diagonal with entries $\sigma_1^2, \ldots, \sigma_n^2$. So the σ's are definitely the singular values of A (as expected). The eigenvalues of that diagonal matrix $A^{\mathrm{T}}A$ are the columns of I, so $V = I$ in the SVD. Then the $\boldsymbol{u}_i$ are $A\boldsymbol{v}_i/\sigma_i$ which is the unit vector $\boldsymbol{w}_i/\sigma_i$.

The SVD of this A with orthogonal columns is $A = U\Sigma V^{\mathrm{T}} = (A\Sigma^{-1})(\Sigma)(I)$.

12 Since $A^{\mathrm{T}} = A$ we have $\sigma_1^2 = \lambda_1^2$ and $\sigma_2^2 = \lambda_2^2$. But λ_2 is negative, so $\sigma_1 = 3$ and $\sigma_2 = 2$. The unit eigenvectors of A are the same $\boldsymbol{u}_1 = \boldsymbol{v}_1$ as for $A^{\mathrm{T}}A = AA^{\mathrm{T}}$ and $\boldsymbol{u}_2 = -\boldsymbol{v}_2$ (notice the sign change because $\sigma_2 = -\lambda_2$, as in Problem 4).

13 Suppose the SVD of R is $R = U\Sigma V^{\mathrm{T}}$. Then multiply by Q to get $A = QR$. So the SVD of this A is $(QU)\Sigma V^{\mathrm{T}}$. (Orthogonal Q times orthogonal U = orthogonal QU.)

14 The smallest change in A is to set its smallest singular value σ_2 to zero. See # **7**.

15 The singular values of $A + I$ are not $\sigma_j + 1$. They come from eigenvalues of $(A+I)^{\mathrm{T}}(A+I)$.

16 This simulates the random walk used by *Google* on billions of sites to solve $A\boldsymbol{p} = \boldsymbol{p}$. It is like the power method of Section 9.3 except that it follows the links in one "walk" where the vector $p_k = A^k p_0$ averages over all walks.

17 $A = U\Sigma V^{\mathrm{T}} =$ [cosines including $\boldsymbol{u}_4$] **diag**(**sqrt**$(2-\sqrt{2}, 2, 2+\sqrt{2})$) [sine matrix]$^{\mathrm{T}}$. $AV = U\Sigma$ says that differences of sines in V are cosines in U times σ's.

The SVD of the *derivative* on $[0,\pi]$ with $f(0) = 0$ has $\boldsymbol{u} = \sin nx$, $\sigma = n$, $\boldsymbol{v} = \cos nx$!