3.2 Computing the Nullspace by Elimination: $A = CR$

1 The **nullspace** $\mathbf{N}(A)$ in $\mathbf{R}^n$ contains all solutions $\boldsymbol{x}$ to $A\boldsymbol{x} = \mathbf{0}$. This includes $\boldsymbol{x} = \mathbf{0}$.

2 Elimination from A to R_0 to R does not change the nullspace : $\mathbf{N}(A) = \mathbf{N}(R_0) = \mathbf{N}(R)$.

3 **The reduced row echelon form $\boldsymbol{R_0} = \mathbf{rref}(\boldsymbol{A})$** has $\boldsymbol{I}$ in r columns and $\boldsymbol{F}$ in $n - r$ columns.

4 If column j of R_0 is free (no pivot), there is a "*special solution*" to $A\boldsymbol{x} = \mathbf{0}$ with $x_j = 1$.

5 Every short wide matrix with $m < n$ has nonzero solutions to $A\boldsymbol{x} = \mathbf{0}$ in its nullspace.

This section finds all solutions to $A\boldsymbol{x} = \mathbf{0}$. When A is a square invertible matrix (in this case its rank is $r = n$), the only solution is $\boldsymbol{x} = \mathbf{0}$. Then the nullspace only contains the zero vector : the columns of A are independent. But in general A has r independent columns ($r =$ rank). *The other $n - r$ columns of A are combinations of those independent columns*. We will find $n - r$ vectors in the nullspace—special solutions to $A\boldsymbol{x} = \mathbf{0}$.

With square invertible matrices, Chapter 2 simplified A to an upper triangular U. For matrices of all shapes, elimination will now simplify $A\boldsymbol{x} = \mathbf{0}$ to an "echelon form" $R\boldsymbol{x} = \mathbf{0}$. (Actually $R = I$ when A is invertible, so elimination is now going further than before— as far as it can.) We will start with two examples of R, to show where we are going.

Here is a matrix R of rank $r = 2$. It has $n = 4$ columns so we look for $n - r = 4 - 2 = 2$ independent solutions to $R\boldsymbol{x} = \mathbf{0}$. The nullspace $\mathbf{N}(R)$ will have dimension 2.

Example 1 $\quad R = \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 4 & 6 \end{bmatrix} \qquad R\boldsymbol{x} = \mathbf{0}$ is $\quad \begin{aligned} x_1 + \quad 3x_3 + 5x_4 &= 0 \\ x_2 + 4x_3 + 6x_4 &= 0 \end{aligned}$

Two "special solutions" are easy to find, when x_3 and x_4 are 1 and 0 or 0 and 1.

Set $x_3 = \mathbf{1}$ and $x_4 = \mathbf{0}$. Equation 1 gives $x_1 = \mathbf{-3}$. Equation 2 gives $x_2 = \mathbf{-4}$.

Set $x_3 = \mathbf{0}$ and $x_4 = \mathbf{1}$. Equation 1 gives $x_3 = \mathbf{-5}$. Equation 2 gives $x_2 = \mathbf{-6}$.

These two special solutions $\boldsymbol{s}_1 = (-3, -4, 1, 0)$ and $\boldsymbol{s}_2 = (-5, -6, 0, 1)$ are in the nullspace of R. They give $R\boldsymbol{s}_1 = \mathbf{0}$ and $R\boldsymbol{s}_2 = \mathbf{0}$. Any combination of those two solutions will also be in the nullspace. The matrix R times the vector $\boldsymbol{x} = c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$ produces zero. Soon we will call those vectors $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ a **basis for the nullspace** : the plane of all solutions to $R\boldsymbol{x} = \mathbf{0}$.

In this example, the matrix R was easy to work with. *Its first two columns contained the identity matrix*. It is an example of a matrix in "reduced row echelon form". We will give one more example to show a variation R_0 that is still in reduced row echelon form and still simple. The subscript in R_0 indicates that there is also a row of zeros.

Example 2 $R_0 = \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad R_0\boldsymbol{x} = \mathbf{0}$ is $\begin{aligned} x_1 + 7x_2 + 0x_3 + 8x_4 &= 0 \\ x_3 + 9x_4 &= 0 \\ 0 &= 0 \end{aligned}$

Now the identity matrix is in columns 1 and 3. And row 3 is all zero. This still counts as a reduced row echelon form—elimination can't make it simpler. *The* 1*'s in the identity matrix are still the first nonzeros in their rows*. The word "echelon" refers to the "staircase" of 1's. Any zero rows always come last in R_0.

The special solutions still have 1 and 0 for the "free variables"—which are x_2 and x_4.

Set $\boldsymbol{x_2} = \mathbf{1}$ and $\boldsymbol{x_4} = \mathbf{0}$. Equation 1 gives $\boldsymbol{x_1} = \mathbf{-7}$. Equation 2 gives $\boldsymbol{x_3} = \mathbf{0}$.

Set $\boldsymbol{x_2} = \mathbf{0}$ and $\boldsymbol{x_4} = \mathbf{1}$. Equation 1 gives $\boldsymbol{x_1} = \mathbf{-8}$. Equation 2 gives $\boldsymbol{x_3} = \mathbf{-9}$.

Those special solutions are now $\boldsymbol{s}_1 = (-7, 1, 0, 0)$ and $\boldsymbol{s}_2 = (-8, 0, -9, 1)$. For the free variables x_2 and x_4, we freely choose $1, 0$ and then $0, 1$. Then the equations $R_0\boldsymbol{x} = \mathbf{0}$ tell us x_1 and x_3.

Here is the plan for this section of the book. We start with any m by n matrix A. We apply elimination (to be explained). That changes A into its reduced row echelon form $\boldsymbol{R_0} = \textbf{rref}\,(\boldsymbol{A})$. Our two examples showed the simplest form $R_0 = R$, and then the most general form when R_0 may have zero rows. *Removing all zero rows of* R_0 *leaves* R.

$r, m, n = 2, 2, 4$	**Simplest case**	$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \end{bmatrix}$	as in $\begin{bmatrix} \mathbf{1} & 0 & 3 & 5 \\ 0 & \mathbf{1} & 4 & 6 \end{bmatrix}$
$r, m, n = 2, 3, 4$	**General case**	$\boldsymbol{R_0} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \boldsymbol{P}$	as in $\begin{bmatrix} \mathbf{1} & 7 & 0 & 8 \\ 0 & 0 & \mathbf{1} & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

I and F have r rows. The reduced matrix R_0 and the original A have m rows. So R_0 has $m - r$ rows of zeros. When we remove those zero rows, we have $\boldsymbol{R} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \end{bmatrix} \boldsymbol{P}$.

The identity I has r columns and F has $n - r$ columns. The permutation P is n by n.

$$P = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$ exchanges columns 2 and 3. Then I goes into columns 1 and 3 of R_0 and R.

Example 1 had $P = I$ and we didn't notice P. Columns 1 and 2 were independent in A. Example 2 had column $2 = 7$ (column 1). The first independent columns were 1 and 3. An important job of elimination is to find independent columns. Here is the key to $A = CR$:

$\boldsymbol{A} = \boldsymbol{CR} = \boldsymbol{C}\begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \end{bmatrix}\boldsymbol{P} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{CF} \end{bmatrix}\boldsymbol{P} =$ **[Indep cols Dependent cols] Permute cols**
The dependent columns of A are combinations CF of the independent columns in C.

Chapter 1 described C and R. But we need elimination (to be explained next) to actually find the column matrix C and the row matrix $R = \begin{bmatrix} I & F \end{bmatrix} P$.

Elimination from A to rref(A) : Reduced Row Echelon Form

How does elimination work ? In any order, we may execute these three different steps :

1. Subtract a multiple of one row from another row (above or below **!**)
2. Multiply a row by any nonzero number
3. Exchange any rows.

Let me stay with these two examples, the simplest case and then the general case. Here is a 2 by 4 matrix A that elimination reduces to our 2 by 4 example $R = \begin{bmatrix} I & F \end{bmatrix}$.

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{11} & \mathbf{17} \\ \mathbf{3} & \mathbf{7} & \mathbf{37} & \mathbf{57} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{11} & \mathbf{17} \\ \mathbf{0} & \mathbf{1} & \mathbf{4} & \mathbf{6} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{3} & \mathbf{5} \\ \mathbf{0} & \mathbf{1} & \mathbf{4} & \mathbf{6} \end{bmatrix} = \boldsymbol{R}$$

Elimination starts with column 1. *It subtracts* 3 *times row* 1 *from row* 2. That produces the zero in the middle matrix. Now column 1 is set (the corner pivot was $A_{11} = 1$ which is what we want). Moving to column 2, **we subtract 2 times the new row 2 from row 1**. That produces the second zero in R. Now R starts with the r by r identity matrix I. The rank is $r = 2$ and elimination on this matrix A is complete.

What did elimination actually do ? **It inverted the leading 2 by 2 matrix** $\boldsymbol{W} = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{7} \end{bmatrix}$. W at the start of A became I at the start of R :

$$\textbf{Multiply } \boldsymbol{W^{-1}A} = \boldsymbol{W^{-1}} \begin{bmatrix} \boldsymbol{W} & \boldsymbol{H} \end{bmatrix} \textbf{ to produce } \boldsymbol{R} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{W^{-1}H} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \end{bmatrix}.$$

We always knew that the dependent columns of A (in H) would be some combination of the independent columns (in W). Now we see that $\boldsymbol{H} = \boldsymbol{WF}$. The matrix F is telling us how to combine the independent columns of A to produce the dependent columns. We can understand the echelon form R and the role of F !

$$\begin{array}{l}\textbf{Dependent}\\\textbf{columns}\end{array} \quad \boldsymbol{H} = \begin{bmatrix} 11 & 17 \\ 37 & 57 \end{bmatrix} = \begin{array}{l}\text{Independent}\\\text{columns}\end{array} \quad \boldsymbol{W} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \text{ times } \boldsymbol{F} = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}.$$

This is a key step in showing that **however you compute $\boldsymbol{R}$ from $\boldsymbol{A}$, you always reach the same $\boldsymbol{R}$**. Each piece of R is completely determined by A (even if there are different elimination steps that lead from A to R).

> **1** The first r independent columns of A locate the columns of R containing I
>
> **2** The remaining columns F in R are determined by the equation $H = WF$: (Dependent columns of A) = (Independent columns of A) times F
>
> **3** The last $m - r$ rows of R_0 are rows of zeros.

Example 2 continued Here is a matrix A that leads to our second reduced echelon form R_0. Both A and R_0 are 3 by 4 matrices of rank $r = 2$. Watch each step :

$$\boldsymbol{A} = \begin{bmatrix} 1 & 7 & 3 & 35 \\ 2 & 14 & 6 & 70 \\ 2 & 14 & 9 & 97 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 7 & 3 & 35 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 27 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 27 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{1} & 7 & \mathbf{0} & 8 \\ \mathbf{0} & 0 & \mathbf{1} & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \boldsymbol{R_0}$$

This example shows again the three allowed row operations in elimination from A to R_0 :

> **1)** Subtract a multiple of one row from another row (**below or above**)
>
> **2)** Divide a row like $\begin{bmatrix} 0 & 0 & 3 & 27 \end{bmatrix}$ by its first nonzero entry (*to reach pivot* $= 1$)
>
> **3)** Exchange rows (*to move all zero rows to the bottom of* R_0)

A different series of steps could reach the same R_0. But that result $R_0 = \mathsf{rref}(A)$ can't change. The pieces of R_0 are all fully determined by the original matrix A.

$\boldsymbol{R_0}$ has a zero row because A has rank $r = 2$

$\boldsymbol{I}$ is in columns 1 and 3 of R_0 because those are the first independent columns of A

$\boldsymbol{F}$ in columns 2 and 4 combines columns $1, 3$ of A to give its dependent columns $2, 4$

$$\boldsymbol{C} \;\; \textbf{times} \;\; \boldsymbol{F} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 7 & 35 \\ 14 & 70 \\ 14 & 97 \end{bmatrix} = \begin{array}{l} \textbf{dependent} \\ \textbf{columns} \\ \textbf{2 and 4 of } \boldsymbol{A} \end{array}$$

The Matrix Factorization $A = CR$ and the Nullspace

This is our chance to complete Chapter 1. That chapter introduced the factorization $A = CR$ by small examples: We learned the meaning of independent columns, but we had no systematic way to find them. Now we have a way : *Apply elimination to reduce A to R_0.* ***Then I in R_0 locates the column matrix C in A.*** *And removing any zero rows from R_0 produces the row matrix R.*

$$\boldsymbol{A = CR} \;\; \text{is} \;\; \begin{bmatrix} 1 & 7 & 3 & 35 \\ 2 & 14 & 6 & 70 \\ 2 & 14 & 9 & 97 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \end{bmatrix} \qquad (1)$$

We could never have seen in Chapter 1 that $(35, 70, 97)$ combines columns 1 and 3 of A.

Please remember how the matrix R shows us the nullspace of A. To solve $A\boldsymbol{x} = \mathbf{0}$ we just have to solve $R\boldsymbol{x} = \mathbf{0}$. This is easy because of the identity matrix inside R.

We find two special solutions $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$—one solution for every column of F in R.

$$\boldsymbol{Rs_1}=\mathbf{0} \qquad \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} \mathbf{-7} \\ \mathbf{1} \\ 0 \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} \text{Put } \mathbf{1} \text{ and } \mathbf{0} \\ \text{in positions 2 and 4} \end{matrix}$$

$$\boldsymbol{Rs_2}=\mathbf{0} \qquad \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} \mathbf{-8} \\ \mathbf{0} \\ \mathbf{-9} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} \text{Put } \mathbf{0} \text{ and } \mathbf{1} \\ \text{in positions 2 and 4} \end{matrix}$$

I think $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are easiest to see using the matrix $R = \begin{bmatrix} I & F \end{bmatrix}$ or $\begin{bmatrix} I & F \end{bmatrix} P$.

The special solutions to $\begin{bmatrix} I & F \end{bmatrix} \boldsymbol{x} = \mathbf{0}$ are the columns of	$\begin{bmatrix} -F \\ I \end{bmatrix}$	in Example 1
The special solutions to $\begin{bmatrix} I & F \end{bmatrix} P\boldsymbol{x} = \mathbf{0}$ are the columns of	$P^{\mathrm{T}} \begin{bmatrix} -F \\ I \end{bmatrix}$	in Example 2

The first one is easy because the permutation is $P = I$. The second one is correct because PP^{T} is the identity matrix for any permutation matrix P:

$$\begin{bmatrix} I & F \end{bmatrix} P \ \text{ times } \ P^{\mathrm{T}} \begin{bmatrix} -F \\ I \end{bmatrix} \ \text{ reduces to } \ \begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

Review Suppose the m by n matrix A has rank r. To find the $n - r$ special solutions to $A\boldsymbol{x} = \mathbf{0}$, compute the reduced row echelon form R_0 of A. Remove the $m - r$ zero rows of R_0 to produce $R = \begin{bmatrix} I & F \end{bmatrix} P$ and $A = CR$. Then the special solutions to $A\boldsymbol{x} = \mathbf{0}$ and $R\boldsymbol{x} = \mathbf{0}$ are the $n - r$ columns of $P^{\mathrm{T}} \begin{bmatrix} -F \\ I \end{bmatrix}$.

Example 3 **Elimination on $\boldsymbol{A}$ gives $\boldsymbol{R_0}$ and $\boldsymbol{R}$. Then $\boldsymbol{R}$ reveals the nullspace of $\boldsymbol{A}$.**

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \boldsymbol{R_0} \ \text{ with rank 2}$$

Then $R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and the independent columns of A and R_0 and R are 1 and 3.

To solve $A\boldsymbol{x} = \mathbf{0}$ and $R\boldsymbol{x} = \mathbf{0}$, set $x_2 = 1$. Solve for $x_1 = -2$ and $x_3 = 0$. Special solution $\boldsymbol{s} = (-2, 1, 0)$. All solutions $\boldsymbol{x} = (-2c, c, 0)$. And here is $A = CR$.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 6 & 9 \end{bmatrix} = \boldsymbol{CR} = \begin{bmatrix} 1 & 1 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{(column basis) (row basis)}$$

Can you write each row of A as a combination of the rows of R?

For many matrices, the only solution to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$. The columns of A are independent. The nullspace $\mathbf{N}(A)$ contains only the zero vector: *no special solutions*. The only combination of the columns that produces $A\boldsymbol{x}=\mathbf{0}$ is the zero combination $\boldsymbol{x}=\mathbf{0}$.

This case of a zero nullspace $\mathbf{Z}$ is of the greatest importance. It says that the columns of A are **independent**. No combination of columns gives the zero vector (except $\boldsymbol{x} = \mathbf{0}$). But this can't happen if $n > m$. We can't have n independent columns in $\mathbf{R}^m$.

Important** **Suppose A has more columns than rows. With $n > m$ there is at least one free variable. The system $A\boldsymbol{x} = \mathbf{0}$ has at least one nonzero solution.

Suppose $A\boldsymbol{x} = \mathbf{0}$ has more unknowns than equations ($n > m$). There must be **at least $n - m$ free columns**. **$A\boldsymbol{x} = \mathbf{0}$ has nonzero solutions** in $\mathbf{N}(A)$.

The nullspace is a subspace. Its "dimension" is the number of free variables. This central idea—the ***dimension of a subspace***—is explained in Section 3.5 of this chapter.

Example 4 Find the nullspaces of A, B, M and the two special solutions to $M\boldsymbol{x} = \mathbf{0}$.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix} \quad \boldsymbol{M} = \begin{bmatrix} A & 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}.$$

Solution The equation $A\boldsymbol{x} = \mathbf{0}$ has only the zero solution $\boldsymbol{x} = \mathbf{0}$. *The nullspace is* $\mathbf{Z}$. It contains only the single point $\boldsymbol{x} = \mathbf{0}$ in $\mathbf{R}^2$. This fact comes from elimination:

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R = I \quad \textbf{No free variables}$$

A is invertible. There are no special solutions. Both columns of this matrix have pivots.

The rectangular matrix B has the same nullspace $\mathbf{Z}$. The first two equations in $B\boldsymbol{x} = \mathbf{0}$ again require $\boldsymbol{x} = \mathbf{0}$. The last two equations would also force $\boldsymbol{x} = \mathbf{0}$. When we add extra equations (giving extra rows), the nullspace certainly cannot become larger. Extra rows impose more conditions on the vectors $\boldsymbol{x}$ in the nullspace.

The rectangular matrix M is different. It has extra columns instead of extra rows. The solution vector $\boldsymbol{x}$ has *four* components. Elimination will produce pivots in the first two columns of M. **The last two columns of M are "free". They don't have pivots.**

$$\boldsymbol{M} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \qquad \boldsymbol{R} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & 2 & 0 \\ \mathbf{0} & \mathbf{1} & 0 & 2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \end{bmatrix}$$

For the free variables x_3 and x_4, we make special choices of ones and zeros. First $x_3 = 1$, $x_4 = 0$ and second $x_3 = 0$, $x_4 = 1$. The pivot variables x_1 and x_2 are determined by the equation $R\boldsymbol{x} = \mathbf{0}$. We get two special solutions in the nullspace of M. This is also the nullspace of R: *elimination doesn't change solutions.*

Special solutions to $M\boldsymbol{x} = \mathbf{0}$ $\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ $R\boldsymbol{s}_1 = \mathbf{0} \quad R\boldsymbol{s}_2 = \mathbf{0}$ $\boldsymbol{s_1} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{s_2} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \begin{array}{l} \leftarrow \textbf{ 2 pivot} \\ \leftarrow \textbf{ variables} \\ \leftarrow \textbf{ 2 free} \\ \leftarrow \textbf{ variables} \end{array}$

Elimination in Three Steps

The special value of matrix notation is to show the big picture. So far we have described elimination as it is usually executed, a small step at a time. But if we work with matrices (blocks of the original A), then block elimination can be described in three steps. Start with an m by n matrix A of rank r.

Step 1 Exchange columns of A by P_C and exchange rows of A by P_R to put r independent columns first and r independent rows first in P_RAP_C.

$$P_RAP_C = \begin{bmatrix} \boldsymbol{W} & \boldsymbol{H} \\ \boldsymbol{J} & \boldsymbol{K} \end{bmatrix} \qquad C = \begin{bmatrix} W \\ J \end{bmatrix} \text{ and } B = \begin{bmatrix} W & H \end{bmatrix} \text{ have full rank } r$$

Step 2 Multiply the r top rows by W^{-1} to produce $W^{-1}B = \begin{bmatrix} I & W^{-1}H \end{bmatrix} = \begin{bmatrix} I & F \end{bmatrix}$

Step 3 Subtract $J\begin{bmatrix} I & W^{-1}H \end{bmatrix}$ from the $m-r$ lower rows $\begin{bmatrix} J & K \end{bmatrix}$ to produce $\begin{bmatrix} 0 & 0 \end{bmatrix}$

The result of Steps 1, 2, 3 is the reduced row echelon form $\boldsymbol{R_0}$

$$\boxed{\boldsymbol{P_RAP_C} = \begin{bmatrix} W & H \\ J & K \end{bmatrix} \to \begin{bmatrix} I & W^{-1}H \\ J & K \end{bmatrix} \to \begin{bmatrix} \boldsymbol{I} & \boldsymbol{W^{-1}H} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} = \boldsymbol{R_0}} \qquad (2)$$

There are two facts that need explanation. They led to Step 2 and Step 3 :

1. The $\boldsymbol{r}$ by $\boldsymbol{r}$ matrix $\boldsymbol{W}$ is invertible **2. The blocks satisfy $\boldsymbol{JW^{-1}H = K}$.**

1. For the invertibility of W, we look back to the factorization $A = CR$. Focusing on the r independent rows of A that go into B, this is $B = WR$. Since B and R have rank r and W is r by r, W must have rank r and be invertible.
2. We know that the first r rows $\begin{bmatrix} I & W^{-1}H \end{bmatrix}$ are linearly independent. Since A has rank r, the lower rows $\begin{bmatrix} J & K \end{bmatrix}$ must be combinations of those upper rows. The combinations must be given by J to get the first r columns correct: $JI = J$. Then J times $W^{-1}H$ must equal K to make the last columns correct.

$$\boxed{\textbf{The conclusion is that } \boldsymbol{P_RAP_C} = \begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{J} \end{bmatrix} \boldsymbol{W^{-1}} \begin{bmatrix} \boldsymbol{W} & \boldsymbol{H} \end{bmatrix} = \boldsymbol{CW^{-1}B}}$$

We need that middle factor W^{-1} because the columns C and the rows B both contain W.

To end this important section of the book, here is a note about *computational linear algebra*. Linear equations $A\boldsymbol{x} = \boldsymbol{b}$ are obviously fundamental. In practice, the steps of elimination are reordered for the sake of speed and numerical stability. We can solve systems of order 1000 on a laptop (allowing roundoff errors in single precision or double precision). Supercomputers can solve much larger systems. But there is a limit on the matrix size. What to do beyond that limit ?

The surprising answer is **randomized linear algebra**. We *sample* the columns of A. We accept the errors involved. In practice matrices are not completely random, and the final results are remarkably good. Often the approximation to A is expressed in the 3-factor form $\boldsymbol{A \approx CUR}$. C comes from sampling the columns of A and R comes from the rows of A.

The smaller mixing matrix U is constructed as we go. With high probability, the approximate solution is surprisingly accurate.

Linear algebra is alive. The demands of computation (speed and accuracy) lead to new ideas. The same will be true for eigenvalues and singular values—later in this book.

The Steps from A to $R_0 = \text{rref}(A)$ and R

Finally we describe the individual steps from A to R_0. One important output is a **list L of column numbers** for the first r independent columns of A. That list decides the permutation matrix P. Then $R = \begin{bmatrix} I & F \end{bmatrix} P$ has the columns of I in the right places.

Suppose the 4 by 3 matrix A has the form shown below. The matrix has seven zeros. What steps would a row elimination code take to reach R_0 ?

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & X & x \\ 0 & x & X \\ 0 & x & 0 \end{bmatrix} \quad \begin{array}{l} \text{4 rows and 3 columns, rank 2} \\ \text{Large entries } X, \text{ small entries } x \\ \text{Here are the 9 small steps} \end{array}$$

1 Find the first nonzero column of A. Answer: 2 starts the column list L.

2 Choose the first nonzero or largest nonzero X in column 2 as the pivot.

3 By row exchanges, move that pivot row into row 1.

4 Subtract multiples of row 1 from all other rows so that the rest of column 2 is zero .

5 Divide row 1 by X to change the first pivot of A_2 to 1.

$$A_2 = \begin{bmatrix} 0 & 1 & y \\ 0 & 0 & 0 \\ 0 & 0 & Y \\ 0 & 0 & y \end{bmatrix}$$

6 In the next nonzero column, find the first or largest pivot Y.
7 **The independent column list L is 2, 3**. The rank of A is $r = 2$.
8 By row exchanges, move that pivot row $\begin{bmatrix} 0 & 0 & Y \end{bmatrix}$ into row 2.
9 Divide row 2 by Y to change the second pivot of R_0 to 1.

$$\boldsymbol{R_0} = \begin{bmatrix} 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \boldsymbol{P} \text{ with } F = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Question Do the steps from A to $R_0 = \textbf{rref}(A)$ and to R preserve the column space or row space or nullspace or nullspace of A^{T} (or none of the above)?

Answer Those operations will preserve the **row space of A and the nullspace of A**. The rows themselves are changed (into the rows of R_0).

We will soon say: The rows of R are a **basis** for the row space. This concept emphasizes the importance of **independent rows**. Even better—as Chapters 4 and 7 will show—is to have **basis vectors that are perpendicular**.