LINEAR ALGEBRA

Sixth Edition

MANUAL FOR INSTRUCTORS

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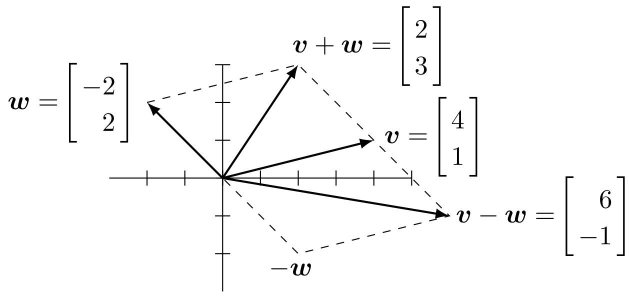
Problem Set 1.1, page 6

1 $c = ma$ and $d = mb$ lead to $ad = amb = bc$. With no zeros, $ad = bc$ is the equation for a 2×2 matrix to have rank 1.

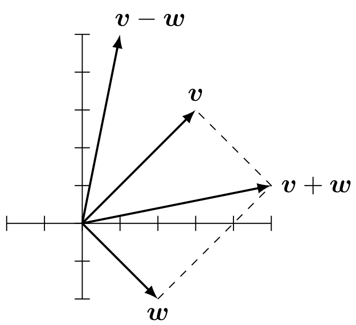
2 The three edges going around the triangle are $\boldsymbol{u} = (5, 0), \boldsymbol{v} = (-5, 12), \boldsymbol{w} = (0, -12)$. Their sum is $u + v + w = (0, 0)$. Their lengths are $||\boldsymbol{u}|| = 5, ||\boldsymbol{v}|| = 13, ||\boldsymbol{w}|| = 12$. This is a $5 - 12 - 13$ right triangle with $5^2 + 12^2 = 25 + 144 = 169 = 13^2$—the best numbers after the $3 - 4 - 5$ right triangle if we don't count $6 - 8 - 10$.

3 The combinations give (a) a line in $\mathbf{R}^3$ (b) a plane in $\mathbf{R}^3$ (c) all of $\mathbf{R}^3$.

4 $\boldsymbol{v} + \boldsymbol{w} = (2, 3)$ and $\boldsymbol{v} - \boldsymbol{w} = (6, -1)$ will be the diagonals of the parallelogram with $\boldsymbol{v}$ and $\boldsymbol{w}$ as two sides going out from $(0, 0)$.



5 This problem gives the diagonals $\boldsymbol{v} + \boldsymbol{w} = (5, 1)$ and $\boldsymbol{v} - \boldsymbol{w} = (1, 5)$ of the parallelogram and asks for the sides $\boldsymbol{v}$ and $\boldsymbol{w}$: The opposite of Problem 4. In this example $\boldsymbol{v} = (3, 3)$ and $\boldsymbol{w} = (2, -2)$. Those come from $\boldsymbol{v} = \frac{1}{2}(\boldsymbol{v} + \boldsymbol{w}) + \frac{1}{2}(\boldsymbol{v} - \boldsymbol{w})$ and $\boldsymbol{w} = \frac{1}{2}(\boldsymbol{v} + \boldsymbol{w}) - \frac{1}{2}(\boldsymbol{v} - \boldsymbol{w})$.

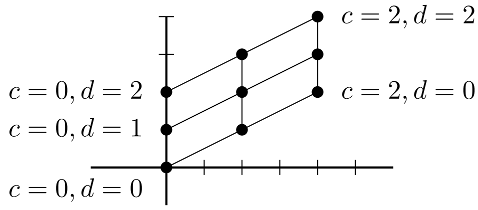


6 $3\boldsymbol{v}+\boldsymbol{w}=(7,5)$ and $c\boldsymbol{v}+d\boldsymbol{w}=(2c+d,c+2d)$.

7 $\boldsymbol{u}+\boldsymbol{v}=(-2,3,1)$ and $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}=(0,0,0)$ and $2\boldsymbol{u}+2\boldsymbol{v}+\boldsymbol{w}=($ add first answers$)=(-2,3,1)$. The vectors $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ are in the same plane because a combination $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}$ gives $(0,0,0)$. Stated another way : $\boldsymbol{u}=-\boldsymbol{v}-\boldsymbol{w}$ is in the plane of $\boldsymbol{v}$ and $\boldsymbol{w}$.

8 The components of every $c\boldsymbol{v}+d\boldsymbol{w}$ add to zero because the components of $\boldsymbol{v}=(1,-2,1)$ and of $\boldsymbol{w}=(0,1,-1)$ add to zero. $c=3$ and $d=9$ give $3\boldsymbol{v}+9\boldsymbol{w}=(3,3,-6)$. There is no solution to $c\boldsymbol{v}+d\boldsymbol{w}=(3,3,6)$ because $3+3+6$ is not zero.

9 The nine combinations $c(2,1)+d(0,1)$ with $c=0,1,2$ and $d=0,1,2$ will lie on a lattice. If we took all whole numbers c and d, the lattice would lie over the whole plane.



10 The question is whether (a,b,c) is a combination $x_1\boldsymbol{u}+x_2\boldsymbol{v}$. Can we solve

$$x_1\begin{bmatrix}1\\1\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}a\\b\\c\end{bmatrix}?$$

Certainly x_1 has to be a. Certainly x_2 has to be c. So the middle components give the **requirement** $\boldsymbol{a+c=b}$.

11 The fourth corner can be $(4,4)$ or $(4,0)$ or $(-2,2)$. Draw 3 possible parallelograms !

12 Four more corners $(1,1,0),(1,0,1),(0,1,1),(1,1,1)$. The center point is $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$. Centers of 6 faces : $\left(\frac{1}{2},\frac{1}{2},0\right),\left(\frac{1}{2},\frac{1}{2},1\right)$ & $\left(0,\frac{1}{2},\frac{1}{2}\right),\left(1,\frac{1}{2},\frac{1}{2}\right)$ & $\left(\frac{1}{2},0,\frac{1}{2}\right),\left(\frac{1}{2},1,\frac{1}{2}\right)$. 12 edges.

13 The combinations of $\boldsymbol{i}=(1,0,0)$ and $\boldsymbol{i}+\boldsymbol{j}=(1,1,0)$ fill the $\boldsymbol{xy}$ **plane** in xyz space.

14 (a) Sum = zero vector. (b) Sum $=-2{:}00$ vector $=8{:}00$ vector.

(c) 2:00 is 30° from horizontal $=(\cos\frac{\pi}{6},\sin\frac{\pi}{6})=(\sqrt{3}/2,1/2)$.

15 Moving the origin to 6:00 adds $\boldsymbol{j} = (0, 1)$ to every vector. So the sum of twelve vectors changes from **0** to $12\boldsymbol{j} = (0, 12)$.

16 First part: $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are all in the same direction.

Second part: Some combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ gives the zero vector but those 3 vectors are not on a line. Then their combinations fill a plane in 3D.

17 The two equations are $c + 3d = 14$ and $2c + d = 8$. The solution is $c = 2$ and $d = 4$.

18 The point $\dfrac{3}{4}\boldsymbol{v} + \dfrac{1}{4}\boldsymbol{w}$ is three-fourths of the way to $\boldsymbol{v}$ starting from $\boldsymbol{w}$. The vector $\dfrac{1}{4}\boldsymbol{v} + \dfrac{1}{4}\boldsymbol{w}$ is halfway to $\boldsymbol{u} = \dfrac{1}{2}\boldsymbol{v} + \dfrac{1}{2}\boldsymbol{w}$. The vector $\boldsymbol{v} + \boldsymbol{w}$ is $2\boldsymbol{u}$ (the far corner of the parallelogram).

19 The combinations $c\boldsymbol{v} + d\boldsymbol{w}$ with $0 \le c \le 1$ and $0 \le d \le 1$ *fill the parallelogram* with sides $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v} = (1, 0)$ and $\boldsymbol{w} = (0, 1)$ then $c\boldsymbol{v} + d\boldsymbol{w}$ fills the unit square. In a special case like $\boldsymbol{v} = (a, 0)$ and $\boldsymbol{w} = (b, 0)$ these combinations only fill a segment of a line.

With $c \ge 0$ and $d \ge 0$ we get the infinite "cone" or "wedge" between $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v} = (1, 0)$ and $\boldsymbol{w} = (0, 1)$, then the cone is the whole first quadrant $x \ge 0$, $y \ge 0$. *Question*: What if $\boldsymbol{w} = -\boldsymbol{v}$? The cone opens to a half-space. But the combinations of $\boldsymbol{v} = (1, 0)$ and $\boldsymbol{w} = (-1, 0)$ only fill a line.

20 (a) $\frac{1}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ is the center of the triangle between $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$; $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{w}$ lies halfway between $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) To fill the triangle keep $c \ge 0$, $d \ge 0$, $e \ge 0$, and $c + d + e = \mathbf{1}$.

21 The sum is $(\boldsymbol{v} - \boldsymbol{u}) + (\boldsymbol{w} - \boldsymbol{v}) + (\boldsymbol{u} - \boldsymbol{w}) =$ **zero vector**. Those three sides of a triangle are in the same plane!

22 The vector $\frac{1}{2}(\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w})$ is *outside* the pyramid because $c + d + e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1$.

23 *All* vectors in 3D are combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ as drawn (not in the same plane). Start by seeing that $c\boldsymbol{u} + d\boldsymbol{v}$ fills a plane, then adding all the vectors $e\boldsymbol{w}$ fills all of $\mathbf{R}^3$. Different answer when $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are in the same plane.

24 A four-dimensional cube has $2^4 = 16$ corners and $2 \cdot 4 = 8$ three-dimensional faces and 24 two-dimensional faces and 32 edges.

25 Fact: For any three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in the plane, some combination $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ is the zero vector (beyond the obvious $c = d = e = 0$). So if there is one combination $C\boldsymbol{u}+D\boldsymbol{v}+E\boldsymbol{w}$ that produces $\boldsymbol{b}$, there will be many more—just add c, d, e or $2c, 2d, 2e$ to the particular solution C, D, E.

The example has $3\boldsymbol{u} - 2\boldsymbol{v} + \boldsymbol{w} = 3(1,3) - 2(2,7) + 1(1,5) = (0,0)$. It also has $-2\boldsymbol{u} + 1\boldsymbol{v} + 0\boldsymbol{w} = \boldsymbol{b} = (0,1)$. Adding gives $\boldsymbol{u} - \boldsymbol{v} + \boldsymbol{w} = (0,1)$. In this case c, d, e equal $3, -2, 1$ and $C, D, E = -2, 1, 0$.

Could another example have $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ that could NOT combine to produce $\boldsymbol{b}$? Yes. The vectors $(1,1), (2,2), (3,3)$ are on a line and no combination produces $\boldsymbol{b}$. We can easily solve $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = 0$ but not $C\boldsymbol{u} + D\boldsymbol{v} + E\boldsymbol{w} = \boldsymbol{b}$.

26 The combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill the plane *unless* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *lie on the same line through* $(0,0)$. Four vectors whose combinations fill 4-dimensional space: one example is the "standard basis" $(1,0,0,0), (0,1,0,0), (0,0,1,0)$, and $(0,0,0,1)$.

27 The equations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = \boldsymbol{b}$ are

$$\begin{array}{rrrl} \mathbf{2}\boldsymbol{c} & -\boldsymbol{d} & & = \mathbf{1} \\ -\boldsymbol{c} & +\mathbf{2}\boldsymbol{d} & -\boldsymbol{e} & = \mathbf{0} \\ & -\boldsymbol{d} & +\mathbf{2}\boldsymbol{e} & = \mathbf{0} \end{array} \qquad \begin{array}{l} \text{So } d = 2e \\ \text{then } c = 3e \\ \text{then } 4e = 1 \end{array} \qquad \begin{array}{l} c = 3/4 \\ d = 2/4 \\ e = 1/4 \end{array}$$