A1 The Ranks of AB and $A+B$

From Chapters 1 to 3, we know that rank of A = rank of A^{T}. This page establishes more key facts about ranks: **When we multiply matrices, the rank cannot increase**. You will see this by looking at column spaces and row spaces. **3** shows one special situation $B = A^{\mathrm{T}}$ when the rank stays the same. Then you know the rank of $A^{\mathrm{T}}A$. Statement 4 becomes important when data science factors A into $U\Sigma V^{\mathrm{T}}$ or CR.
Here are five key facts in one place. The most important fact is **rank of $\boldsymbol{A}$ = rank of $\boldsymbol{A}^{\mathrm{T}}$**.

1 **Rank of $AB \le$ rank of A**   **Rank of $AB \le$ rank of B**
2 **Rank of $A + B \le$ (rank of A) + (rank of B)**
3 **Rank of $A^{\mathrm{T}}A$ = rank of AA^{T} = rank of A = rank of A^{T}**
4 **If A is m by r and B is r by n**—both with rank r—**then AB also has rank r**

Statement 1 involves the column space of AB and the row space of AB:

$\mathbf{C}(AB)$ is contained in $\mathbf{C}(A)$ so the dimension of $\mathbf{C}(AB)$ cannot be larger

Every column of AB is a combination of the columns of A (*matrix multiplication*)
Every row of AB is a combination of the rows of B (*matrix multiplication*)

Remember from Section 1.4 that **row rank = column rank**. We can use rows or columns. *The rank cannot grow when we multiply AB.* Statement **1** in the box is frequently used.

Statement 2 Each column of $A + B$ is the sum of (column of A) + (column of B).

rank $(A+B) \le$ rank (A) + rank (B) is **true**. Bases for $\mathbf{C}(A)$ and $\mathbf{C}(B)$ span $\mathbf{C}(A+B)$.

rank $(A+B)$ = rank (A) + rank (B) is **not** always true. It is certainly false if $A = B = I$.

Statement 3 A and $A^{\mathrm{T}}A$ both have n columns. **They also have the same nullspace**. (See Problem 4.19.) So $n - r$ is the same for both, and *the rank r is the same for both*. Then rank$(A^{\mathrm{T}}) \ge$ rank$(A^{\mathrm{T}}A)$ = rank(A). Exchange A and A^{T} to show **equal ranks**.

Statement 4 We are told that A and B have rank r. By Statement **3**, $A^{\mathrm{T}}A$ and BB^{T} have rank r. Those are r by r matrices so they are invertible. So is their product $A^{\mathrm{T}}ABB^{\mathrm{T}}$. Then

$$\boldsymbol{r} = \textbf{rank of } (\boldsymbol{A^{\mathrm{T}}ABB^{\mathrm{T}}}) \le \textbf{rank of } (\boldsymbol{AB}) \le \textbf{rank of } \boldsymbol{A} = \boldsymbol{r}. \text{ So } AB \text{ has rank } r.$$

Note This does not mean that every product of rank r matrices will have rank r. Statement 4 assumes that **A has exactly r columns** and **B has r rows**. BA can easily fail.

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-3} \end{bmatrix} \qquad \boldsymbol{AB} \textbf{ has rank 1} \qquad \textbf{But } \boldsymbol{BA} \textbf{ is zero!}$$