3.5 Dimensions of the Four Subspaces

1 The column space $\mathbf{C}(A)$ and the row space $\mathbf{C}(A^{\mathrm{T}})$ both have *dimension* r (the rank of A).

2 The nullspace $\mathbf{N}(A)$ has *dimension* $n-r$. The left nullspace $\mathbf{N}(A^{\mathrm{T}})$ has *dimension* $m-r$.

3 Elimination from A to R_0 changes $\mathbf{C}(A)$ and $\mathbf{N}(A^{\mathrm{T}})$ (but their dimensions don't change).

The main theorem in this chapter connects ***rank*** and ***dimension***. The ***rank*** of a matrix counts independent columns. The ***dimension*** of a subspace is the number of vectors in a basis. We can count pivots or basis vectors. *The rank of A reveals the dimensions of all four fundamental subspaces.* Here are the subspaces, including the new one.

Two subspaces come directly from A, and the other two come from A^{T}.

	Four Fundamental Subspaces	**Dimensions**
1.	The ***row space*** is $\mathbf{C}(A^{\mathrm{T}})$, a subspace of $\mathbf{R}^n$.	$\boldsymbol{r}$
2.	The ***column space*** is $\mathbf{C}(A)$, a subspace of $\mathbf{R}^m$.	$\boldsymbol{r}$
3.	The ***nullspace*** is $\mathbf{N}(A)$, a subspace of $\mathbf{R}^n$.	$\boldsymbol{n-r}$
4.	The ***left nullspace*** is $\mathbf{N}(A^{\mathrm{T}})$, a subspace of $\mathbf{R}^m$.	$\boldsymbol{m-r}$

We know $\mathbf{C}(A)$ and $\mathbf{N}(A)$ pretty well. Now $\mathbf{C}(A^{\mathrm{T}})$ and $\mathbf{N}(A^{\mathrm{T}})$ come forward. The row space contains all combinations of the rows. *This row space is the column space of* A^{T}.

For the left nullspace we solve $A^{\mathrm{T}}\boldsymbol{y}=\mathbf{0}$—that system is n by m. In Example 2 this produces one of the great equations of applied mathematics—**Kirchhoff's Current Law**. The currents flow around a network, and they can't pile up at the nodes. The matrix A is the **incidence matrix of a graph**. Its four subspaces come from nodes and edges and loops and trees. Those subspaces are connected in an absolutely beautiful way.

Part 1 of the Fundamental Theorem finds the dimensions of the four subspaces. One fact stands out: ***The row space and column space have the same dimension*** r. This number r is the rank of A (Chapter 1). The other important fact involves the two nullspaces:

$\mathbf{N}(\boldsymbol{A})$ ***and*** $\mathbf{N}(\boldsymbol{A}^{\mathrm{T}})$ ***have dimensions*** $n-r$ ***and*** $m-r$***, to make up the full*** n ***and*** m***.***

Part 2 of the Fundamental Theorem will describe how the four subspaces fit together: Nullspace perpendicular to row space, and $\mathbf{N}(A^{\mathrm{T}})$ perpendicular to $\mathbf{C}(A)$. That completes the "right way" to understand $A\boldsymbol{x}=\boldsymbol{b}$. Stay with it—you are doing real mathematics.

The Four Subspaces for R_0

Suppose A is reduced to its row echelon form R_0. For that special form, the four subspaces are easy to identify. We will find a basis for each subspace and check its dimension. Then we watch how the subspaces change (two of them don't change!) as we look back at A. The main point is that ***the four dimensions are the same for A and R_0***.

For A and R, one of the four subspaces can have different dimensions—because zero rows are removed in R, which changes m.

As a specific 3 by 5 example, look at the four subspaces for this echelon matrix R_0 :

$$\begin{matrix} \boldsymbol{m=3} \\ \boldsymbol{n=5} \\ \boldsymbol{r=2} \end{matrix} \qquad R_0 = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{5} & 0 & \mathbf{7} \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} \textbf{pivot rows 1 and 2} \\ \\ \textbf{pivot columns 1 and 4} \end{matrix}$$

The rank of this matrix is $\boldsymbol{r = 2}$ (*two pivots*). Take the four subspaces in order.

1. The ***row space*** has dimension 2, matching the rank.

Reason : The first two rows are a basis. The row space contains combinations of all three rows, but the third row (the zero row) adds nothing to the row space.

The pivot rows 1 and 2 are independent. That is obvious for this example, and it is always true. If we look only at the pivot columns, we see the r by r identity matrix. There is no way to combine its rows to give the zero row (except by the combination with all coefficients zero). So the r pivot rows (the rows of R) are a basis for the row space.

The dimension of the row space is the rank r. The nonzero rows of R_0 form a basis.

2. The ***column space*** of R_0 also has dimension $r = 2$.

Reason : The pivot columns 1 and 4 form a basis. They are independent because they contain the r by r identity matrix. No combination of those pivot columns can give the zero column (except the combination with all coefficients zero). And they also span the column space. Every other (free) column is a combination of the pivot columns. Actually the combinations we need are the three special solutions !

Column 2 is 3 (column 1). The special solution is $(-3, 1, 0, 0, 0)$.

Column 3 is 5 (column 1). The special solution is $(-5, 0, 1, 0, 0,)$.

Column 5 is 7 (column 1) $+$ 2 (column 4). That solution is $(-7, 0, 0, -2, 1)$.

The pivot columns are independent, and they span $\mathbf{C}(R_0)$, so they are a basis for $\mathbf{C}(R_0)$.

The dimension of the column space is the rank r. The pivot columns form a basis.

> **3.** The ***nullspace*** of R_0 has dimension $\boldsymbol{n-r=5-2}$. The 3 free variables give **3 special solutions** to $R_0\boldsymbol{x}=\mathbf{0}$. Set the free variables to 1 and 0 and 0.

$$\boldsymbol{s}_2=\begin{bmatrix}\mathbf{-3}\\\mathbf{1}\\\mathbf{0}\\0\\\mathbf{0}\end{bmatrix}\quad \boldsymbol{s}_3=\begin{bmatrix}\mathbf{-5}\\\mathbf{0}\\\mathbf{1}\\0\\\mathbf{0}\end{bmatrix}\quad \boldsymbol{s}_5=\begin{bmatrix}\mathbf{-7}\\\mathbf{0}\\\mathbf{0}\\\mathbf{-2}\\\mathbf{1}\end{bmatrix}$$

$R_0\boldsymbol{x}=\mathbf{0}$ has the complete solution $\boldsymbol{x}=x_2\boldsymbol{s}_2+x_3\boldsymbol{s}_3+x_5\boldsymbol{s}_5$ The nullspace has dimension 3.

Reason : There is a special solution for each free variable. With n variables and r pivots, that leaves $n-r$ free variables and special solutions. The special solutions are independent, because you can see the identity matrix in rows 2, 3, 5.

The nullspace $\mathbf{N}(A)$ ***has dimension*** $\boldsymbol{n-r}$***. The special solutions form a basis.***

> **4.** The ***nullspace of*** $\boldsymbol{R_0^{\mathrm{T}}}$ ***(left nullspace of*** $\boldsymbol{R_0}$***) has dimension*** $\boldsymbol{m-r=3-2}$.

Reason : R_0 has r independent rows and $\boldsymbol{m-r}$ **zero rows**. Then R_0^{T} has r independent columns and $\boldsymbol{m-r}$ **zero columns**. So $\boldsymbol{y}$ in the nullspace of R_0^{T} can have nonzeros in its last $m-r$ entries. The example has $m-r=1$ zero column in R_0^{T} and 1 nonzero in $\boldsymbol{y}$.

$$\boldsymbol{R_0^{\mathrm{T}}y}=\begin{bmatrix}\mathbf{1}&0&0\\\mathbf{3}&0&0\\\mathbf{5}&0&0\\0&\mathbf{1}&0\\\mathbf{7}&\mathbf{2}&0\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\quad\text{is solved by } \boldsymbol{y}=\begin{bmatrix}0\\0\\y_3\end{bmatrix}. \tag{1}$$

Because of zero rows in R_0 and zero columns in R_0^{T}, it is easy to see the dimension (and even a basis) for this fourth fundamental subspace :

If $\boldsymbol{R_0}$ **has** $\boldsymbol{m-r}$ **zero rows, its left nullspace has dimension** $\boldsymbol{m-r}$**.**

Why is this a "*left* nullspace"? Because we can transpose $R_0^{\mathrm{T}}\boldsymbol{y}=\mathbf{0}$ to $\boldsymbol{y}^{\mathrm{T}}R_0=\mathbf{0}^{\mathrm{T}}$. Now $\boldsymbol{y}^{\mathrm{T}}$ is a row vector to the *left* of R. This subspace came fourth, and some linear algebra books omit it—but that misses the beauty of the whole subject.

> ***In*** $\mathbf{R}^n$ ***the row space and nullspace have dimensions*** $\boldsymbol{r}$ and $\boldsymbol{n-r}$ (adding to n).
>
> ***In*** $\mathbf{R}^m$ ***the column space and left nullspace have dimensions*** $\boldsymbol{r}$ and $\boldsymbol{m-r}$ (total m).

We have a job still to do. ***The four subspace dimensions for*** $\boldsymbol{A}$ ***are the same as for*** $\boldsymbol{R_0}$**.** The job is to explain why. A is now any matrix that reduces to $R_0=\mathrm{rref}(A)$.

This A reduces to R_0 $$A=\begin{bmatrix}1&3&5&0&7\\0&0&0&1&2\\\mathbf{1}&\mathbf{3}&\mathbf{5}&\mathbf{1}&\mathbf{9}\end{bmatrix}$$ **Same row space as R_0**
Different column space
But same dimension !

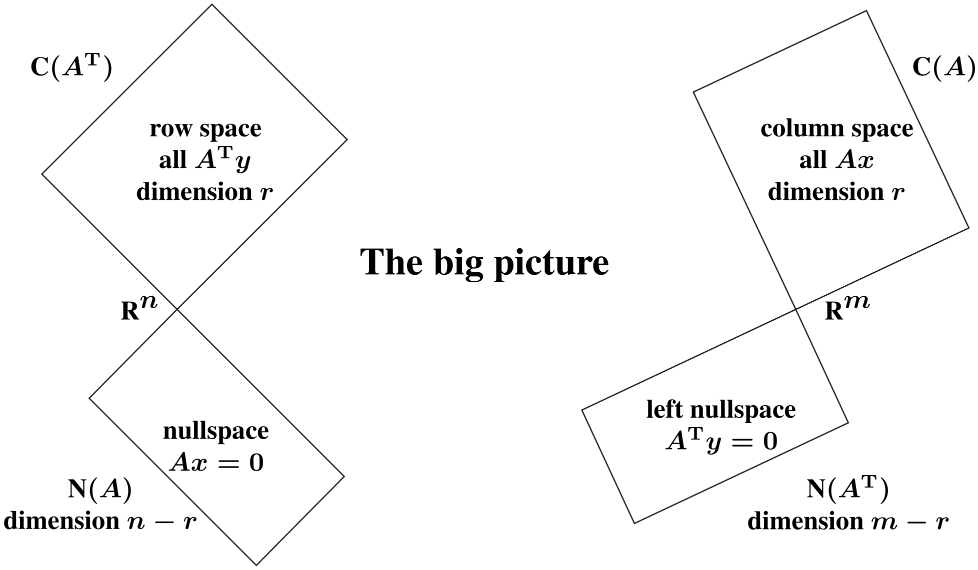


Figure 3.3: The dimensions of the Four Fundamental Subspaces (for R_0 and for A).

The Four Subspaces for A

1 ***A has the same row space as R_0 and R.* Same dimension r and same basis**.

Reason: Every row of A is a combination of the rows of R_0. Also every row of R_0 is a combination of the rows of A. **Elimination changes rows, but not row spaces**.

Since A has the same row space as R_0, the first r rows of R_0 are still a basis. Or we could choose r suitable rows of the original A. They might not always be the *first* r rows of A, because those could be dependent. The good r rows of A are the ones that end up as pivot rows in R_0 and R.

2 ***The column space of A has dimension r. The column rank equals the row rank.***

The number of independent columns = the number of independent rows.

Wrong reason : "A and R_0 have the same column space." This is false. *The columns of R_0 often end in zeros*. The columns of A don't often end in zeros. Then $\mathbf{C}(A)$ is not $\mathbf{C}(R_0)$.

Right reason : The ***same combinations*** of the columns are zero (or not) for A and R_0. Dependent in $A \Leftrightarrow$ dependent in R_0. Say that another way: $A\boldsymbol{x} = \mathbf{0}$ *exactly when* $R_0\boldsymbol{x} = \mathbf{0}$. The column spaces are different, but their *dimensions* are the same—equal to the rank r.

Conclusion The r pivot columns of A are a basis for *its* column space $\mathbf{C}(A)$.

3 ***A has the same nullspace as R_0.*** **Same dimension $n - r$ and same basis.**

Reason: Elimination doesn't change the solutions to $A\boldsymbol{x} = \boldsymbol{0}$. The special solutions are a basis for this nullspace (as we always knew). There are $n-r$ free variables, so the nullspace dimension is $n - r$. This is the **Counting Theorem** : $\boldsymbol{r + (n - r)}$ **equals** $\boldsymbol{n}$.

(**dimension of column space**)$+$(**dimension of nullspace**) $=$ **dimension of** $\mathbf{R}^{\boldsymbol{n}}$.

4 ***The left nullspace of A*** (the nullspace of A^{T}) ***has dimension*** $\boldsymbol{m - r}$.

Reason: A^{T} is just as good a matrix as A. When we know the dimensions for every A, we also know them for A^{T}. Its column space was proved to have dimension r. Since A^{T} is n by m, the "whole space" is now $\mathbf{R}^m$. The counting rule for A was $r+(n-r) = n$. **The counting rule for $\boldsymbol{A^{\mathrm{T}}}$ is $\boldsymbol{r + (m - r) = m}$.** We have all details of a big theorem:

***Fundamental Theorem of Linear Algebra*, Part 1**

The column space and row space both have dimension $\boldsymbol{r}$.

The nullspaces have dimensions $\boldsymbol{n - r}$ and $\boldsymbol{m - r}$.

By concentrating on *spaces* of vectors, not on individual numbers or vectors, we get these clean rules. You will soon take them for granted—eventually they begin to look obvious. But if you write down an 11 by 17 matrix with 187 nonzero entries, I don't think most people would see why these facts are true:

Two key facts

dimension of $\mathbf{C}(A) =$ dimension of $\mathbf{C}(A^{\mathrm{T}}) =$ rank of A
dimension of $\mathbf{C}(A) +$ dimension of $\mathbf{N}(A) = 17$.

Every vector $A\boldsymbol{x} = \boldsymbol{b}$ in the column space comes from exactly one $\boldsymbol{x}$ in the row space! (If we also have $A\boldsymbol{y} = \boldsymbol{b}$ then $A(\boldsymbol{x} - \boldsymbol{y}) = \boldsymbol{b} - \boldsymbol{b} = \boldsymbol{0}$. So $\boldsymbol{x} - \boldsymbol{y}$ is in the nullspace as well as the row space, which forces $\boldsymbol{x} = \boldsymbol{y}$.) From its row space to its column space, *A is like an r by r invertible matrix.*

It is the nullspaces that force us to define a **"pseudoinverse of $\boldsymbol{A}$"** in Section 4.5.

Example 1 $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ has $m = 2$ with $n = 3$. The rank is $r = 1$.

The row space is the line through $(1, 2, 3)$. The nullspace is the plane $x_1 + 2x_2 + 3x_3 = 0$. The line and plane dimensions still add to $1 + 2 = 3$. The column space and left nullspace are **perpendicular lines in $\mathbf{R}^2$**. Dimensions $1 + 1 = 2$.

$$\text{Column space} = \text{line through } \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{Left nullspace} = \text{line through } \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

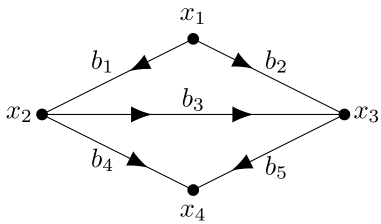
Final point: The $\boldsymbol{y}$'s in the left nullspace combine the rows of A to give the zero row.

Example 2 You have nearly finished three chapters with made-up equations, and this can't continue forever. Here is a better example of five equations (one equation for every edge in Figure 3.4). The five equations have four unknowns (one for every node). **The important matrix in $\boldsymbol{Ax} = \boldsymbol{b}$ is an incidence matrix**. It has 1 and -1 on every row.

Differences $\boldsymbol{Ax} = \boldsymbol{b}$
across edges $1, 2, 3, 4, 5$
between nodes $1, 2, 3, 4$
$\boldsymbol{m} = 5$ and $\boldsymbol{n} = 4$

$$\begin{array}{|rrrrl|} -x_1 & +x_2 & & & = b_1 \\ -x_1 & & +x_3 & & = b_2 \\ & -x_2 & +x_3 & & = b_3 \\ & -x_2 & & +x_4 & = b_4 \\ & & -x_3 & +x_4 & = b_5 \end{array} \qquad (2)$$

If you understand the four fundamental subspaces for this matrix (*the column spaces and the nullspaces for A and A^{T}*) you have captured a central idea of linear algebra.



$$\begin{array}{c} \text{nodes} \quad x_1 \quad x_2 \quad x_3 \quad x_4 \qquad \text{edges} \end{array}$$

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{-1} & \mathbf{1} & & \\ \mathbf{-1} & & \mathbf{1} & \\ & \mathbf{-1} & \mathbf{1} & \\ & \mathbf{-1} & & \mathbf{1} \\ & & \mathbf{-1} & \mathbf{1} \end{bmatrix} \quad \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

Figure 3.4: A **"graph"** with 5 edges and 4 nodes. **$\boldsymbol{A}$ is its 5 by 4 incidence matrix**.

The nullspace $\mathbf{N}(A)$ To find the nullspace we set $\boldsymbol{b} = \mathbf{0}$. Then the first equation says $x_1 = x_2$. The second equation is $x_3 = x_1$. Equation 4 is $x_2 = x_4$. *All four unknowns x_1, x_2, x_3, x_4 have the same value c.* The vectors $\boldsymbol{x} = (c, c, c, c)$ fill the nullspace of A.

That nullspace is a line in $\mathbf{R}^4$. The special solution $\boldsymbol{x} = (1, 1, 1, 1)$ is a basis for $\mathbf{N}(A)$. The dimension of $\mathbf{N}(A)$ is 1 (one vector in the basis). *The rank of A must be* 3, *since* $n - r = 4 - 3 = 1$. We now know the dimensions of all four subspaces.

The column space $\mathbf{C}(A)$ There must be $r = 3$ independent columns. The fast way is to look at the first 3 columns of A. The systematic way is to find $R_0 = \mathsf{rref}(A)$.

$$\begin{array}{c} \textbf{Columns} \\ \mathbf{1, 2, 3} \\ \textbf{of } \boldsymbol{A} \end{array} \quad \begin{array}{rrr} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array} \qquad \boldsymbol{R_0} = \begin{array}{c} \textbf{reduced row} \\ \textbf{echelon form} \\ \textbf{of } \boldsymbol{A} \end{array} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From R_0 we see again the special solution $\boldsymbol{x} = (1, 1, 1, 1)$. The first 3 columns are basic, the fourth column is free. To produce a basis for $\mathbf{C}(A)$ and not $\mathbf{C}(R_0)$, we must go back to **columns $\mathbf{1, 2, 3}$ of $\boldsymbol{A}$**. The column space has dimension $r = 3$.

The row space C(A^{T}) The dimension must again be $r = 3$. But the first 3 rows of A are *not independent*: row $3 =$ row $2 -$ row 1. So row 3 became zero in elimination, and row 3 was exchanged with row 4. *The first three independent rows are rows* $1, 2, 4$. Those three rows are a basis (one possible basis) for the row space.

Edges $1, 2, 3$ form a **loop** in the graph :	Dependent rows $1, 2, 3$.
Edges $1, 2, 4$ form a **tree**. **Trees have no loops**!	Independent rows $1, 2, 4$.

The left nullspace N(A^{T}) Now we solve $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. Combinations of the rows give zero. We already noticed that row $3 =$ row $2 -$ row 1, so one solution is $\boldsymbol{y} = (1, -1, 1, 0, 0)$. I would say : *That* $\boldsymbol{y}$ *comes from following the upper loop in the graph*. Another $\boldsymbol{y}$ comes from going around the lower loop and it is $\boldsymbol{y} = (0, 0, \mathbf{-1}, \mathbf{1}, \mathbf{-1})$: *row* $3 =$ *row* $4 -$ *row* 5. Those two $\boldsymbol{y}$'s are independent, they solve $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$, and the dimension of $\mathbf{N}(A^{\mathrm{T}})$ is $\boldsymbol{m} - \boldsymbol{r} = 5 - 3 = \mathbf{2}$. So we have a basis for the left nullspace.

You may ask how "loops" and "trees" got into this problem. That didn't have to happen. We could have used elimination to solve $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. The 4 by 5 matrix A^{T} would have three pivot columns $1, 2, 4$ and two free columns $3, 5$. There are two special solutions and the nullspace of A^{T} has dimension two: $m - r = 5 - 3 = 2$. But **loops** and **trees** identify *dependent rows* and *independent rows* in a beautiful way for every incidence matrix.

The equations $A\boldsymbol{x} = \boldsymbol{b}$ give "voltages" x_1, x_2, x_3, x_4 at the four nodes. The equations $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ give "currents" y_1, y_2, y_3, y_4, y_5 on the five edges. These two equations are **Kirchhoff's Voltage Law** and **Kirchhoff's Current Law**. Those laws apply to an electrical network. But the ideas behind the words apply all over engineering and science and economics and business. Linear algebra connects the laws to the four subspaces.

Graphs are *the most important model in discrete applied mathematics*. You see graphs everywhere: roads, pipelines, blood flow, the brain, the Web, the economy of a country or the world. We can understand their matrices A and A^{T}. Here is a summary.

The incidence matrix $\boldsymbol{A}$ comes from a connected graph with n nodes and m edges. The row space and column space have dimensions $r = n - 1$. The nullspaces of A and A^{T} have dimensions 1 and $m - n + 1$:

$\mathbf{N}(A)$	The constant vectors $(c, c, \ldots, c)$ make up the nullspace of A : $\dim = 1$.
$\mathbf{C}(A^{\mathrm{T}})$	The edges of any tree give r independent rows of A : $r = n - 1$.
$\mathbf{C}(A)$	***Voltage Law***: The components of $A\boldsymbol{x}$ add to zero around all loops: $\dim = n - 1$.
$\mathbf{N}(A^{\mathrm{T}})$	***Current Law***: $A^{\mathrm{T}}\boldsymbol{y} =$ (**flow in**)$-$(**flow out**) $= \mathbf{0}$ is solved by loop currents.
	There are $\boldsymbol{m - r = m - n + 1}$ ***independent small loops in the graph.***

For every graph in a plane, linear algebra yields ***Euler's formula*** : Theorem 1 in topology !

$$(\textit{\textbf{nodes}}) - (\textit{\textbf{edges}}) + (\textit{\textbf{small loops}}) = (\boldsymbol{n}) - (\boldsymbol{m}) + (\boldsymbol{m} - \boldsymbol{n} + \mathbf{1}) = \mathbf{1}$$

Rank Two Matrices = Rank One plus Rank One

Rank one matrices have the form $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$. Here is a matrix A of rank $r = 2$. We can't see r immediately from A. So we reduce the matrix by row operations to R_0. **R_0 has the same row space as A**. Throw away its zero row to find R—also with the same row space.

$$\textbf{Rank two} \qquad \boldsymbol{A} = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 7 \\ 4 & 2 & 20 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \\ \mathbf{4} & \mathbf{2} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{3} \\ \mathbf{0} & \mathbf{1} & \mathbf{4} \end{bmatrix} = \boldsymbol{CR} \qquad (3)$$

Now look at columns. The pivot columns of R are clearly $(1, 0)$ and $(0, 1)$. Then the pivot columns of A are also in columns 1 and 2: $\boldsymbol{u}_1 = (1, 1, 4)$ and $\boldsymbol{u}_2 = (0, 1, 2)$. Notice that C has those same first two columns! That was guaranteed since multiplying by two columns of the identity matrix (in R) won't change the pivot columns $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

When you put in letters for the columns and rows, you see **rank 2 = rank 1 + rank 1**.

$$\begin{array}{l}\textbf{Matrix } \boldsymbol{A} \\ \textbf{Rank two}\end{array} \qquad A = \begin{bmatrix} & & \\ \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \\ & & \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1^{\mathrm{T}} \\ \boldsymbol{v}_2^{\mathrm{T}} \\ \textbf{zero row} \end{bmatrix} = \boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}}$$

Columns of $\boldsymbol{C}$ times rows of $\boldsymbol{R}$. *Every rank r matrix is a sum of r rank one matrices*

■ WORKED EXAMPLES ■

3.5 A **Put four 1's into a 5 by 6 matrix of zeros**, keeping the dimension of its *row space* as small as possible. Describe all the ways to make the dimension of its *column space* as small as possible. Then describe all the ways to make the dimension of its *nullspace* as small as possible. How to make the *sum of the dimensions of all four subspaces small*?

Solution The rank is 1 if the four 1's go into the same row, or into the same column. They can also go into *two rows and two columns* (so $a_{ii} = a_{ij} = a_{ji} = a_{jj} = 1$). Since the column space and row space always have the same dimensions, this answers the first two questions: Dimension 1.

The nullspace has its smallest possible dimension $6 - 4 = 2$ when the rank is $r = 4$. To achieve rank 4, the 1's must go into four different rows and four different columns.

You can't do anything about the sum $r + (n - r) + r + (m - r) = \boldsymbol{n} + \boldsymbol{m}$. It will be $6 + 5 = 11$ no matter how the 1's are placed. The sum is 11 even if there aren't any 1's...

If all the other entries of A are 2's instead of 0's, how do these answers change?

3.5 B All the rows of AB are combinations of the rows of B. So the row space of AB is contained in (possibly equal to) the row space of B. **Rank $(AB) \leq$ rank (B).**

All columns of AB are combinations of the columns of A. So the column space of AB is contained in (possibly equal to) the column space of A. **Rank $(AB) \leq$ rank (A).**

If we multiply A by an *invertible* matrix B, the rank will not change. The rank can't drop, because when we multiply by the inverse matrix the rank can't jump back up.

Appendix 1 collects the key facts about the ranks of matrices.

Problem Set 3.5

1 (a) If a 7 by 9 matrix has rank 5, what are the dimensions of the four subspaces? What is the sum of all four dimensions?

(b) If a 3 by 4 matrix has rank 3, what are its column space and left nullspace?

2 Find bases and dimensions for the four subspaces associated with A and B:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}.$$

3 Find a basis for each of the four subspaces associated with A:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4 Construct a matrix with the required property or explain why this is impossible:

(a) Column space contains $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, row space contains $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

(b) Column space has basis $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, nullspace has basis $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

(c) Dimension of nullspace $= 1 +$ dimension of left nullspace.

(d) Nullspace contains $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, column space contains $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(e) Row space $=$ column space, nullspace $\neq$ left nullspace.

5 If $\mathbf{V}$ is the subspace spanned by $(1, 1, 1)$ and $(2, 1, 0)$, find a matrix A that has $\mathbf{V}$ as its row space. Find a matrix B that has $\mathbf{V}$ as its nullspace. Multiply AB.

6 Without using elimination, find dimensions and bases for the four subspaces for

$$A = \begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}.$$

7 Suppose the 3 by 3 matrix A is invertible. Write down bases for the four subspaces for A, and also for the 3 by 6 matrix $B = [\,A \quad A\,]$. (*The basis for $\boldsymbol{Z}$ is empty.*)

8 What are the dimensions of the four subspaces for A, B, and C, if I is the 3 by 3 identity matrix and 0 is the 3 by 2 zero matrix?

$$A = \begin{bmatrix} I & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} I & I \\ 0^{\mathrm{T}} & 0^{\mathrm{T}} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 \end{bmatrix}.$$

9 Which subspaces are the same for these matrices of different sizes?

$$\text{(a)} \ \begin{bmatrix} A \end{bmatrix} \text{ and } \begin{bmatrix} A \\ A \end{bmatrix} \qquad \text{(b)} \ \begin{bmatrix} A \\ A \end{bmatrix} \text{ and } \begin{bmatrix} A & A \\ A & A \end{bmatrix}.$$

Prove that all three of those matrices have the *same rank* r.

10 If the entries of a 3 by 3 matrix are chosen randomly between 0 and 1, what are the most likely dimensions of the four subspaces? What if the random matrix is 3 by 5?

11 (Important) A is an m by n matrix of rank r. Suppose there are right sides $\boldsymbol{b}$ for which $A\boldsymbol{x} = \boldsymbol{b}$ has *no solution*.

(a) What are all inequalities ($<$ or $\leq$) that must be true between m, n, and r?

(b) How do you know that $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has solutions other than $\boldsymbol{y} = \mathbf{0}$?

12 Construct a matrix with $(1, 0, 1)$ and $(1, 2, 0)$ as a basis for its row space and its column space. Why can't this be a basis for the row space and nullspace?

13 True or false (with a reason or a counterexample):

(a) If $m = n$ then the row space of A equals the column space.

(b) The matrices A and $-A$ share the same four subspaces.

(c) If A and B share the same four subspaces then A is a multiple of B.

14 Without computing A, find bases for its four fundamental subspaces:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

15 If you exchange the first two rows of A, which of the four subspaces stay the same? If $\boldsymbol{v} = (1, 2, 3, 4)$ is in the left nullspace of A, write down a vector in the left nullspace of the new matrix after the row exchange.

16 *Explain why* $\boldsymbol{v} = (1, 0, -1)$ *cannot be a row of* A *and also in the nullspace of* A.

17 Describe the four subspaces of $\mathbf{R}^3$ associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad I + A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

18 Can tic-tac-toe be completed (5 ones and 4 zeros in A) so that rank $(A) = 2$ but neither side passed up a winning move?

19 (Left nullspace) Add the extra column $\boldsymbol{b}$ and reduce A to echelon form:

$$\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}.$$

A combination of the rows of A has produced the zero row. What combination is it? (Look at $b_3 - 2b_2 + b_1$ on the right side.) Which vectors are in the nullspace of A^{T} and which vectors are in the nullspace of A?

20 (**Patience needed**) Describe the row operations that reduce a matrix A to its echelon form R_0.

21 Suppose A is the sum of two matrices of rank one: $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$.

(a) Which vectors span the column space of A ?

(b) Which vectors span the row space of A?

(c) The rank is less than 2 if _____ or if _____.

(d) Compute A and its rank if $\boldsymbol{u} = \boldsymbol{z} = (1,0,0)$ and $\boldsymbol{v} = \boldsymbol{w} = (0,0,1)$.

22 Construct $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ whose column space has basis $(1,2,4), (2,2,1)$ and whose row space has basis $(1,0), (1,1)$. Write A as (3 by 2) times (2 by 2).

23 Without multiplying matrices, find bases for the row and column spaces of A:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

How do you know from these shapes that A cannot be invertible?

24 (Important) $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{d}$ is solvable when $\boldsymbol{d}$ is in which of the four subspaces? The solution $\boldsymbol{y}$ is unique when the _____ contains only the zero vector.

25 True or false (with a reason or a counterexample):

(a) A and A^{T} have the same number of pivots.

(b) A and A^{T} have the same left nullspace.

(c) If the row space equals the column space then $A^{\mathrm{T}} = A$.

(d) If $A^{\mathrm{T}} = -A$ then the row space of A equals the column space.

26 If a, b, c are given with $a \neq 0$, how would you choose d so that $\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix}$ has rank 1? Find a basis for the row space and nullspace. Show they are perpendicular!

Challenge Problems

27 Find the ranks of the 8 by 8 checkerboard matrix B and the chess matrix C:

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} r & n & b & q & k & b & n & r \\ p & p & p & p & p & p & p & p \\ & & & \text{four zero rows} & & & & \\ p & p & p & p & p & p & p & p \\ r & n & b & q & k & b & n & r \end{bmatrix}$$

The numbers r, n, b, q, k, p are all different. Find bases for the row space and left nullspace of B and C. Find a basis for the nullspace of C.

28 If $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ is a 2 by 2 matrix of rank 1, redraw Figure 3.5 to show clearly the Four Fundamental Subspaces. If B produces those same four subspaces, what is the exact relation of B to A?

29 $\mathbf{M}$ is the space of 3 by 3 matrices. Multiply every matrix X in $\mathbf{M}$ by A:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}. \quad \text{Notice: } A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Which matrices X lead to $AX =$ zero matrix?

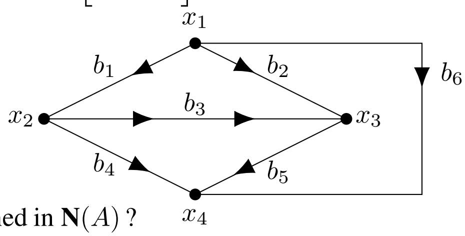
(b) Which matrices have the form AX for some matrix X?

(a) finds the "nullspace" of that operation AX and (b) finds the "column space". What are the dimensions of those two subspaces of $\mathbf{M}$? Why do they add to 9?

30 Suppose the m by n matrices A and B have *the same four subspaces*. If they are both in row reduced echelon form, is it true that ***F* must equal *G***?

$$A = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} I & G \\ 0 & 0 \end{bmatrix}.$$

31 Find the **incidence matrix** and its rank and one vector in each subspace for this complete graph—all six edges included.



32 (Review) (a) Is $\mathbf{N}(AB)$ or $\mathbf{N}(BA)$ contained in $\mathbf{N}(A)$?

(b) Is $\mathbf{C}(AB)$ or $\mathbf{C}(BA)$ contained in $\mathbf{C}(A)$?

33 Suppose A is m by n and B is M by n and $T = \begin{bmatrix} A \\ B \end{bmatrix}$.

(a) What are the relations between the nullspaces of A and B and T?

(b) What are the relations between the row spaces of A and B and T?

34 Suppose A is m by n. What can you say about each of the four fundamental subspaces for the matrices A and $W = \begin{bmatrix} A & A \end{bmatrix}$?

35 If A and B are n by n, is it always true that rank $\begin{bmatrix} A \\ B \end{bmatrix} =$ rank $\begin{bmatrix} A & B \end{bmatrix}$?

Thoughts on Chapter 3 : The Big Picture of Elimination

This page explains elimination at the vector level and subspace level, when A is reduced to R. You know the steps and I won't repeat them. Elimination starts with the first pivot. It moves a column at a time (left to right) and a row at a time (top to bottom) for $\boldsymbol{U}$. Continuing elimination **upward** produces $\boldsymbol{R_0}$ and $\boldsymbol{R}$. Elimination answers two questions:

Question 1 Is this column a combination of previous columns?

If the column contains a pivot, the answer is no. Pivot columns are "independent" of previous columns. If column 4 has no pivot, it is a combination of columns $1, 2, 3$.

Question 2 Is this row a combination of previous rows?

If the row contains a pivot, the answer is no. Pivot rows are independent of previous rows, and their first nonzero is 1 from I. Rows that are all zero in R_0 were not and are not independent. Those zero rows disappear in R. That matrix is r by n.

It is amazing to me that one pass through the matrix answers both questions 1 and 2. Elimination acts on the rows but the result tells us about the columns ! The identity matrix in R locates the first r independent columns in A. Then the free columns F in R tell us *the combinations of those independent columns that produce the dependent columns in A.* This is easy to miss without seeing the factorization $A = CR$.

$\boldsymbol{R}$ **tells us the special solutions to** $\boldsymbol{Ax = 0}$. We could reach R from A by different row exchanges and elimination steps, but it will always be the same R. (This is because the special solutions are fully decided by A. The formula comes before Problem Set 3.2.) In the language coming soon, R reveals a "basis" for three of the fundamental subspaces:

The **column space** of A—choose the $\boldsymbol{r}$ **columns of** $\boldsymbol{A}$ that produce pivots in R.

The **row space** of A—choose the $\boldsymbol{r}$ **rows of** $\boldsymbol{R}$ as a basis.

The **nullspace** of A—choose the $\boldsymbol{n - r}$ **special solutions** to $R\boldsymbol{x} = \mathbf{0}$ (and $A\boldsymbol{x} = \mathbf{0}$).

For the **left nullspace** $\mathbf{N}(A^{\mathrm{T}})$, we look at the elimination step $EA = R_0$. The last $m - r$ rows of R_0 are zero. The **last** $\boldsymbol{m - r}$ **rows of** $\boldsymbol{E}$ are a basis for the left nullspace ! In reducing the extended matrix $[A \;\; I]$ to $[R_0 \;\; E]$, the matrix E keeps a record of elimination that is otherwise lost.

Suppose we fix C and B (m by r and r by n, both rank r). Choose any invertible r by r mixing matrix M. **All the matrices** $\boldsymbol{CMB}$ (and only those) have the **same four fundamental subspaces**.

Note This is the first textbook to express the result of elimination in its matrix form $\boldsymbol{A} = \boldsymbol{CR} = \boldsymbol{C}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{F}\end{bmatrix}\boldsymbol{P}$. Elimination reveals $\boldsymbol{C}$ and $\boldsymbol{F}$ and $\boldsymbol{P}$ and $\boldsymbol{A} = \begin{bmatrix}\boldsymbol{C} & \boldsymbol{CF}\end{bmatrix}\boldsymbol{P}$.

$$\boldsymbol{A} = \Big[\textbf{Independent columns in } \boldsymbol{C} \quad \textbf{Dependent columns in } \boldsymbol{CF}\Big]\ \textbf{Permute columns}$$

Row Operations on an m by n Matrix A : Review

(i)	Subtract a multiple of one row from another row
(ii)	Exchange two rows
(iii)	Multiply a row by any nonzero constant

The important point is : **Those row operations are reversible (invertible)**.

(i) Add back the multiple of one row to the other row

(ii) Exchange the rows again

(iii) Divide the row by that nonzero constant

Total effect of those row operations : **An m by m invertible matrix E multiplies A**.

The nullspace is not changed by E : $A\boldsymbol{x} = \mathbf{0} \Rightarrow EA\boldsymbol{x} = \mathbf{0} \Rightarrow A\boldsymbol{x} = \mathbf{0}$.

A and EA have different rows but **the same row space and nullspace**.

Reduced Row Echelon Form : $\boldsymbol{E}$ can produce $EA = R_0 = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} P = \textbf{rref}(A)$

The identity I is r by r, F is r by $n-r$, P puts the n columns in correct order.

Factorization : $A = CR =$ [First r independent columns] $\begin{bmatrix} I & F \end{bmatrix} P$

$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{CF} \end{bmatrix} \boldsymbol{P} = \begin{bmatrix} \textbf{Independent cols} & \textbf{Dependent cols} \end{bmatrix}$ **Reorder columns**

Nullspace of $\boldsymbol{A}$: Each column of F leads to one of the $n-r$ "special solutions" to $A\boldsymbol{x} = \mathbf{0}$:

Special solution
page 88, Example I

$$\boldsymbol{s}_k = P^{\mathrm{T}} \begin{bmatrix} -\text{ column } k \text{ of } F(r \text{ by } n-r) \\ +\text{ column } k \text{ of } I(n-r \text{ by } n-r) \end{bmatrix}$$

That permutation P^{T} puts the n components of the solution $\boldsymbol{s}_k$ in the right order.

Example Special solution $\boldsymbol{s}_1$ to $A\boldsymbol{x} = \mathbf{0}$ and $R\boldsymbol{x} = \mathbf{0}$ with $P = I$ and rank $r = 3$

$$R = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad \boldsymbol{s}_1 = \begin{bmatrix} -3 \\ -4 \\ -5 \\ 1 \end{bmatrix} \qquad R\boldsymbol{s}_1 = \mathbf{0} \qquad A\boldsymbol{s}_1 = CR\boldsymbol{s}_1 = \mathbf{0}$$