2.2 Elimination Matrices and Inverse Matrices

1. Elimination multiplies $A$ by $E_{21}, \ldots, E_{n1}$ then $E_{32}, \ldots, E_{n2}$ as $A$ becomes $EA = U$.

2. In reverse order, the inverses of the $E$’s multiply $U$ to recover $A = E^{-1}U$. This is $A = LU$.

3. $A^{-1}A = I$ and $(LU)^{-1} = U^{-1}L^{-1}$. Then $Ax = b$ becomes $x = A^{-1}b = U^{-1}L^{-1}b$.

All the steps of elimination can be done with matrices. Those steps can also be undone (inverted) with matrices. For a 3 by 3 matrix we can write out each step in detail—almost word for word. But for real applications, matrices are a much better way.

The basic elimination step subtracts a multiple $\ell_{ij}$ of equation $j$ from equation $i$. We always speak about subtractions as elimination proceeds. If the first pivot is $a_{11} = 3$ and below it is $a_{21} = -3$, we could just add equation 1 to equation 2. That produces zero. But we stay with subtraction: subtract $\ell_{21} = -1$ times equation 1 from equation 2. Same result. The inverse step is addition. Equation (10) to (11) at the end shows it all.

Here is the matrix that subtracts 2 times row 1 from row 3: Rows 1 and 2 stay the same.

$$\text{Elimination matrix } E_{ij} = E_{31} \quad \text{Row 3, column 1, multiplier 2}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

If no row exchanges are needed, then three elimination matrices $E_{21}$ and $E_{31}$ and $E_{32}$ will produce three zeros below the diagonal. This changes $A$ to the triangular $U$:

$$E = E_{32}E_{31}E_{21} \quad EA = U \text{ is upper triangular}$$

(1)

The number $\ell_{32}$ is affected by the $\ell_{21}$ and $\ell_{31}$ that came first. We subtract $\ell_{32}$ times row 2 of $U$ (the final second row, not the original second row of $A$). This is the $E_{32}$ step that produces zero in row 3, column 2 of $U$. $E_{32}$ gives the last step of 3 by 3 elimination.

Example 1 $E_{21}$ and then $E_{31}$ subtract multiples of row 1 from rows 2 and 3 of $A$:

$$E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ -3 & 1 & 1 \\ 6 & 8 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 6 & 4 \end{bmatrix}$$

(2)

$$\text{two new zeros in column 1}$$

To produce a zero in column 2, $E_{32}$ subtracts $\ell_{32} = 3$ times the new row 2 from row 3:

$$(E_{32})(E_{31}E_{21}A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

(3)

$U$ has zeros below the main diagonal

Notice again: $E_{32}$ is subtracting 3 times the row 0, 2, 1 and not the original row of $A$. At the end, the pivots 3, 2, 1 are on the main diagonal of $U$: zeros below that diagonal.

The inverse of each matrix $E_{ij}$ adds back $\ell_{ij}$ to row $j$ to row $i$. This leads to the inverse of their product $E = E_{32}E_{31}E_{21}$. That inverse of $E$ is special. We call it $L$. 
The Facts About Inverse Matrices

Suppose $A$ is a square matrix. We look for an “inverse matrix” $A^{-1}$ of the same size, so that $A^{-1}$ times $A$ equals $I$. Whatever $A$ does, $A^{-1}$ undoes. Their product is the identity matrix—which does nothing to a vector, so $A^{-1}Ax = x$. But $A^{-1}$ might not exist.

The $n$ by $n$ matrix $A$ needs $n$ independent columns to be invertible. Then $A^{-1}A = I$.

What a matrix mostly does is to multiply a vector. Multiplying $Ax = b$ by $A^{-1}$ gives $A^{-1}Ax = A^{-1}b$. This is $x = A^{-1}b$. The product $A^{-1}A$ is like multiplying by a number and then dividing by that number. Numbers have inverses if they are not zero. Matrices are more complicated and interesting. The matrix $A^{-1}$ is called “$A$ inverse”.

**DEFINITION** The matrix $A$ is **invertible** if there exists a matrix $A^{-1}$ that “inverts” $A$:

\[ A^{-1}A = I \quad \text{and} \quad AA^{-1} = I. \quad (4) \]

**Not all matrices have inverses.** This is the first question we ask about a square matrix: Is $A$ invertible? Its columns must be independent. We don’t mean that we actually calculate $A^{-1}$. In most problems we never compute it! Here are seven “notes” about $A^{-1}$.

**Note 1** The inverse exists if and only if elimination produces $n$ pivots (row exchanges are allowed). Elimination solves $Ax = b$ without explicitly using the matrix $A^{-1}$.

**Note 2** The matrix $A$ cannot have two different inverses. Suppose $BA = I$ and also $AC = I$. Then $B = C$, according to this “proof by parentheses” = associative law.

\[ B(AC) = (BA)C \quad \text{gives} \quad BI = IC \quad \text{or} \quad B = C. \quad (5) \]

This shows that a left inverse $B$ (multiplying $A$ from the left) and a right inverse $C$ (multiplying $A$ from the right to give $AC = I$) must be the same matrix.

**Note 3** If $A$ is invertible, the one and only solution to $Ax = b$ is $x = A^{-1}b$:

Multiply $Ax = b$ by $A^{-1}$. Then $x = A^{-1}Ax = A^{-1}b$.

**Note 4** (Important) Suppose there is a nonzero vector $x$ such that $Ax = 0$. Then $A$ has dependent columns. It cannot have an inverse. No matrix can bring 0 back to $x$.

If $A$ is invertible, then $Ax = 0$ only has the zero solution $x = A^{-1}0 = 0$.

**Note 5** A square matrix is invertible if and only if its columns are independent.

**Note 6** A 2 by 2 matrix is invertible if and only if the number $ad - bc$ is not zero:

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.
\]

This number $ad - bc$ is the determinant of $A$. A matrix is invertible if its determinant is not zero (Chapter 5). The test for $n$ pivots is usually decided before the determinant appears.
2.2. Elimination Matrices and Inverse Matrices

Note 7  A triangular matrix has an inverse provided no diagonal entries $d_i$ are zero:

\[
\begin{bmatrix}
d_1 & \times & \times & \times \\
0 & \times & \times & \\
0 & 0 & \times & \\
0 & 0 & 0 & d_n
\end{bmatrix}
\]

then

\[
\begin{bmatrix}
1/d_1 & \times & \times & \times \\
0 & \times & \times & \\
0 & 0 & \times & \\
0 & 0 & 0 & 1/d_n
\end{bmatrix}
\]

Example 2  The 2 by 2 matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ is not invertible. It fails the test in Note 6, because $ad = bc$. It also fails the test in Note 4, because $Ax = 0$ when $x = (2, -1)$. It fails to have two pivots as required by Note 1. Its columns are clearly dependent. Elimination turns the second row of this matrix $A$ into a zero row. No pivot.

Example 3  Three of these matrices are invertible, and three are singular. Find the inverse when it exists. Give reasons for noninvertibility (zero determinant, too few pivots, nonzero solution to $Ax = 0$) for the other three. The matrices are in the order $A, B, C, D, S, T$:

\[
\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 8 & 7 \end{bmatrix}, \begin{bmatrix} 6 & 6 \\ 6 & 0 \end{bmatrix}, \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}
\]

Solution  The three matrices with inverses are $B, C, S$:

\[
B^{-1} = \frac{1}{4} \begin{bmatrix} 7 & -3 \\ -8 & 4 \end{bmatrix}, \quad C^{-1} = \frac{1}{36} \begin{bmatrix} 0 & 6 \\ 6 & -6 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}
\]

$A$ is not invertible because its determinant is $4 \cdot 6 - 3 \cdot 8 = 24 - 24 = 0$. $D$ is not invertible because it has only one pivot; row 2 becomes zero when row 1 is subtracted. $T$ has two equal rows (and the second column minus the first column is zero). In other words $Tx = 0$ has the nonzero solution $x = (-1, 1, 0)$. Not invertible.

The Inverse of a Product $AB$

For two nonzero numbers $a$ and $b$, the sum $a + b$ might or might not be invertible. The numbers $a = 3$ and $b = -3$ have inverses $\frac{1}{3}$ and $-\frac{1}{3}$. Their sum $a + b = 0$ has no inverse. But the product $ab = -9$ does have an inverse, which is $\frac{1}{3}$ times $-\frac{1}{3}$.

For matrices $A$ and $B$, the situation is similar. Their product $AB$ has an inverse if and only if $A$ and $B$ are separately invertible (and the same size). The important point is that $A^{-1}$ and $B^{-1}$ come in reverse order:

\[
(AB)^{-1} = B^{-1}A^{-1}
\]

\[
(AB)(B^{-1}A^{-1}) = AIA^{-1} = AA^{-1} = I
\]

(7)
We moved parentheses to multiply $BB^{-1}$ first. Similarly $B^{-1}A^{-1}$ times $AB$ equals $I$.

$B^{-1}A^{-1}$ illustrates a basic rule of mathematics: Inverses come in reverse order. It is also common sense: If you put on socks and then shoes, the first to be taken off are the _____. The same reverse order applies to three or more matrices:

Reverse order 

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$ (8)

**Example 4  Inverse of an elimination matrix.** If $E$ subtracts 5 times row 1 from row 2, then $E^{-1}$ adds 5 times row 1 to row 2:

\[
E \text{ subtracts } \quad E = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Multiply $EE^{-1}$ to get the identity matrix $I$. Also multiply $E^{-1}E$ to get $I$. We are adding and subtracting the same 5 times row 1. If $AC = I$ then for square matrices $CA = I$.

For square matrices, an inverse on one side is automatically an inverse on the other side.

**Example 5**  Suppose $F$ subtracts 4 times row 2 from row 3, and $F^{-1}$ adds it back:

\[
F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \quad \text{and} \quad F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}.
\]

Now multiply $F$ by the matrix $E$ in Example 4 to find $FE$. Also multiply $E^{-1}$ times $F^{-1}$ to find $(FE)^{-1}$. Notice the orders $FE$ and $E^{-1}F^{-1}$!

\[
FE = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 20 & -4 & 1 \end{bmatrix} \quad \text{is inverted by} \quad E^{-1}F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}.
\] (9)

The result is beautiful and correct. The product $FE$ contains “20” but its inverse doesn’t. $E$ subtracts 5 times row 1 from row 2. Then $F$ subtracts 4 times the new row 2 (changed by row 1) from row 3. In this order $FE$, row 3 feels an effect of size 20 from row 1.

In the order $E^{-1}F^{-1}$, that effect does not happen. First $F^{-1}$ adds 4 times row 2 to row 3. After that, $E^{-1}$ adds 5 times row 1 to row 2. There is no 20, because row 3 doesn’t change again. In this order $E^{-1}F^{-1}$, row 3 feels no effect from row 1.

This is why we choose $A = LU$, to go back from the triangular $U$ to the original $A$. The multipliers fall into place perfectly in the lower triangular $L$: Equation (11) below.

The elimination order is $FE$. The inverse order is $L = E^{-1}F^{-1}$.

*The multipliers 5 and 4 fall into place below the diagonal of 1’s in $L$.*
2.2. Elimination Matrices and Inverse Matrices

$L$ is the Inverse of $E$

$E$ is the product of all the elimination matrices $E_{ij}$, taking $A$ into its upper triangular form $EA = U$. We are assuming for now that no row exchanges are involved ($P = I$). The difficulty with $E$ is that multiplying all the separate elimination steps $E_{ij}$ does not produce a good formula. But the inverse matrix $E^{-1}$ becomes beautiful when we multiply the inverse steps $E_{ij}^{-1}$. Remember that those steps come in the opposite order.

With $n = 3$, the complication for $E = E_{32} E_{31} E_{21}$ is in the bottom left corner:

$$
E = \begin{bmatrix}
1 & 0 & 1 \\
0 & -\ell_{32} & 1 \\
0 & -\ell_{31} & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
-\ell_{21} & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & -\ell_{21} & 1 \\
-\ell_{31} & (\ell_{32} \ell_{21} - \ell_{31}) & -\ell_{32} & 1
\end{bmatrix}.
$$

(10)

Watch how that confusion disappears for $E^{-1} = L$. Reverse order is the good way:

$$
E^{-1} = \begin{bmatrix}
1 & \ell_{21} & 1 \\
\ell_{31} & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & \ell_{32} & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
\ell_{21} & 1 & 1 \\
\ell_{31} & \ell_{32} & 1
\end{bmatrix} = L
$$

(11)

All the multipliers $\ell_{ij}$ appear in their correct positions in $L$. The next section will show that this remains true for all matrix sizes. Then $EA = U$ becomes $A = LU$.

Equation (11) is the key to this chapter: Each $\ell_{ij}$ is in its place for $E^{-1} = L$.

Problem Set 2.2  

(more questions than needed)

0. If you exchange columns 1 and 2 of an invertible matrix $A$, what is the effect on $A^{-1}$?

Problems 1–11 are about elimination matrices.

1. Write down the 3 by 3 matrices that produce these elimination steps:

   (a) $E_{21}$ subtracts 5 times row 1 from row 2.

   (b) $E_{32}$ subtracts $-7$ times row 2 from row 3.

   (c) $P$ exchanges rows 1 and 2, then rows 2 and 3.

2. In Problem 1, applying $E_{21}$ and then $E_{32}$ to $b = (1, 0, 0)$ gives $E_{32} E_{21} b = \text{____}$. Applying $E_{32}$ before $E_{21}$ gives $E_{21} E_{32} b = \text{____}$. When $E_{32}$ comes first, row ____ feels no effect from row ____.

3. Which three matrices $E_{21}, E_{31}, E_{32}$ put $A$ into triangular form $U$?

   $$
   A = \begin{bmatrix}
   1 & 1 & 0 \\
   4 & 6 & 1 \\
   -2 & 2 & 0
   \end{bmatrix}
   \quad \text{and} \quad
   E_{32} E_{31} E_{21} A = EA = U.
   $$

   Multiply those $E$’s to get one elimination matrix $E$. What is $E^{-1} = L$?
Include \( b = (1, 0, 0) \) as a fourth column in Problem 3 to produce \([A \ b]\). Carry out the elimination steps on this augmented matrix to solve \( Ax = b \).

Suppose \( a_{33} = 7 \) and the third pivot is 5. If you change \( a_{33} \) to 11, the third pivot is ______. If you change \( a_{33} \) to _____, there is no third pivot.

If every column of \( A \) is a multiple of \((1, 1, 1)\), then \( Ax \) is always a multiple of \((1, 1, 1)\). Do a 3 by 3 example. How many pivots are produced by elimination?

Suppose \( E \) subtracts 7 times row 1 from row 3.

(a) To invert that step you should _____ 7 times row _____ to row _____.

(b) What “inverse matrix” \( E^{-1} \) takes that reverse step (so \( E^{-1} E = I \))?

(c) If the reverse step is applied first (and then \( E \)) show that \( EE^{-1} = I \).

The determinant of \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is \( \det M = ad - bc \). Subtract \( \ell \) times row 1 from row 2 to produce a new \( M^* \). Show that \( \det M^* = \det M \) for every \( \ell \). When \( \ell = c/a \), the product of pivots equals the determinant: \( \text{(a)}(d - \ell b) \) equals \( ad - bc \).

(a) \( E_{21} \) subtracts row 1 from row 2 and then \( P_{23} \) exchanges rows 2 and 3. What matrix \( M = P_{23}E_{21} \) does both steps at once?

(b) \( P_{23} \) exchanges rows 2 and 3 and then \( E_{31} \) subtracts row 1 from row 3. What matrix \( M = E_{31}P_{23} \) does both steps at once? Explain why the \( M \)’s are the same but the \( E \)’s are different.

(a) What matrix adds row 1 to row 3 and at the same time row 3 to row 1 ?

(b) What matrix adds row 1 to row 3 and then adds row 3 to row 1 ?

Create a matrix that has \( a_{11} = a_{22} = a_{33} = 1 \) but elimination produces two negative pivots without row exchanges. (The first pivot is 1.)

For these “permutation matrices” find \( P^{-1} \) by trial and error (with 1’s and 0’s):

\[
P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.
\]

Solve for the first column \((x, y)\) and second column \((t, z)\) of \( A^{-1} \). Check \( AA^{-1} \).

\[
A = \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \quad \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

Find an upper triangular \( U \) (not diagonal) with \( U^2 = I \). Then \( U^{-1} = U \).

(a) If \( A \) is invertible and \( AB = AC \), prove quickly that \( B = C \).

(b) If \( A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \), find two different matrices such that \( AB = AC \).
(Important) If $A$ has row 1 + row 2 = row 3, show that $A$ is not invertible:

(a) Explain why $Ax = (0, 0, 1)$ cannot have a solution. Add eqn 1 + eqn 2.
(b) Which right sides $(b_1, b_2, b_3)$ might allow a solution to $Ax = b$?
(c) In the elimination process, what happens to equation 3?

If $A$ has column 1 + column 2 = column 3, show that $A$ is not invertible:

(a) Find a nonzero solution $x$ to $Ax = 0$. The matrix is 3 by 3.
(b) Elimination keeps columns 1 + 2 = 3. Explain why there is no third pivot.

Suppose $A$ is invertible and you exchange its first two rows to reach $B$. Is the new matrix $B$ invertible? How would you find $B^{-1}$ from $A^{-1}$?

Find invertible matrices $A$ and $B$ such that $A + B$ is not invertible.

Find singular matrices $A$ and $B$ such that $A + B$ is invertible.

If the product $C = AB$ is invertible ($A$ and $B$ are square), then $A$ itself is invertible. Find a formula for $A^{-1}$ that involves $A^{-1}$ and $B$.

If the product $M = ABC$ of three square matrices is invertible, then $B$ is invertible. (So are $A$ and $C$.) Find a formula for $B^{-1}$ that involves $M^{-1}$ and $A$ and $C$.

If you add row 1 of $A$ to row 2 to get $B$, how do you find $B^{-1}$ from $A^{-1}$?

Prove that a matrix with a column of zeros cannot have an inverse.

Multiply $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ times $\begin{bmatrix} -d & -b \\ -c & -a \end{bmatrix}$. What is the inverse of each matrix if $ad \neq bc$?

(a) What 3 by 3 matrix $E$ has the same effect as these three steps? Subtract row 1 from row 2, subtract row 1 from row 3, then subtract row 2 from row 3.
(b) What single matrix $L$ has the same effect as these three reverse steps? Add row 2 to row 3, add row 1 to row 3, then add row 1 to row 2.

If $B$ is the inverse of $A^2$, show that $AB$ is the inverse of $A$.

Show that $A = 4 \ast \text{eye}(4) - \text{ones}(4, 4)$ is not invertible: Multiply $A \ast \text{ones}(4, 1)$.

There are sixteen 2 by 2 matrices whose entries are 1’s and 0’s. How many of them are invertible?

Change $I$ into $A^{-1}$ as elimination reduces $A$ to $I$ (the Gauss-Jordan idea).

$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix}$

Could a 4 by 4 matrix $A$ be invertible if every row contains the numbers 0, 1, 2, 3 in some order? What if every row of $B$ contains 0, 1, 2, $-3$ in some order?
31 Find $A^{-1}$ and $B^{-1}$ (if they exist) by elimination on $[A \ I]$ and $[B \ I]$:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

32 **Gauss-Jordan elimination** acts on $[U \ I]$ to find the matrix $[I \ U^{-1}]$:

If $U = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ then $U^{-1} = \begin{bmatrix} \phantom{0} & \phantom{0} & \phantom{0} \\ \phantom{0} & \phantom{0} & \phantom{0} \\ \phantom{0} & \phantom{0} & \phantom{0} \end{bmatrix}$.

33 True or false (with a counterexample if false and a reason if true): $A$ is square.

(a) A 4 by 4 matrix with a row of zeros is not invertible.
(b) Every matrix with 1’s down the main diagonal is invertible.
(c) If $A$ is invertible then $A^{-1}$ and $A^2$ are invertible.

34 (Recommended) Prove that $A$ is invertible if $a \neq 0$ and $a \neq b$ (find the pivots or $A^{-1}$). Then find three numbers $c$ so that $C$ is not invertible:

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}.$$

35 This matrix has a remarkable inverse. Find $A^{-1}$ by elimination on $[A \ I]$. Extend to a 5 by 5 “alternating matrix” and guess its inverse; then multiply to confirm.

Invert $A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and solve $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

36 Suppose the matrices $P$ and $Q$ have the same rows as $I$ but in any order. They are “permutation matrices”. Show that $P - Q$ is singular by solving $(P - Q)x = 0$.

37 Find and check the inverses (assuming they exist) of these block matrices:

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}.$$

38 How does elimination from $A$ to $U$ on a 3 by 3 matrix tell you if $A$ is invertible?

39 If $A = I - uv^T$ then $A^{-1} = I + uv^T(1 - v^Tu)^{-1}$. Show that $AA^{-1} = I$ except $Au = 0$ when $v^Tu = 1$. 
