2.2 Elimination Matrices and Inverse Matrices

1 Elimination multiplies A by $E_{21}, \ldots, E_{n1}$ then $E_{32}, \ldots, E_{n2}$ as A becomes $\boldsymbol{EA = U}$.

2 In reverse order, the inverses of the E's multiply U to recover $A = E^{-1}U$. **This is** $\boldsymbol{A = LU}$.

3 $\boldsymbol{A^{-1}A = I}$ and $\boldsymbol{(LU)^{-1} = U^{-1}L^{-1}}$. Then $A\boldsymbol{x} = \boldsymbol{b}$ becomes $\boldsymbol{x} = A^{-1}\boldsymbol{b} = \boldsymbol{U^{-1}L^{-1}b}$.

All the steps of elimination can be done with matrices. Those steps can also be *undone* (inverted) with matrices. For a 3 by 3 matrix we can write out each step in detail—almost word for word. But for real applications, matrices are a much better way.

The basic elimination step subtracts a multiple $\boldsymbol{\ell_{ij}}$ of equation $\boldsymbol{j}$ from equation $\boldsymbol{i}$. We always speak about *subtractions* as elimination proceeds. If the first pivot is $a_{11} = 3$ and below it is $a_{21} = -3$, we could just add equation 1 to equation 2. That produces zero. But we stay with subtraction: *subtract* $\ell_{21} = -1$ *times equation* 1 *from equation* 2. Same result. The inverse step is addition. Equation (10) to (11) at the end shows it all.

Here is the matrix that subtracts 2 times row 1 from row 3 : Rows 1 and 2 stay the same.

Elimination matrix $\boldsymbol{E_{ij} = E_{31}}$
Row 3, column 1, multiplier 2

$$E_{31} = \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ \mathbf{-2} & 0 & \mathbf{1} \end{bmatrix}$$

If no row exchanges are needed, then three elimination matrices E_{21} and E_{31} and E_{32} will produce three zeros below the diagonal. This changes A to the triangular U :

$$\boxed{\boldsymbol{E = E_{32}E_{31}E_{21}} \qquad \boldsymbol{EA = U} \textbf{ is upper triangular}} \tag{1}$$

The number ℓ_{32} is affected by the ℓ_{21} and ℓ_{31} that came first. We subtract ℓ_{32} times *row* 2 *of* U (the final second row, not the original second row of A). This is the E_{32} step that produces zero in row 3, column 2 of U. E_{32} gives the last step of 3 by 3 elimination.

Example 1 E_{21} and then E_{31} subtract multiples of row 1 from rows 2 and 3 of A :

$$E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{-2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{1} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{3} & 1 & 0 \\ \mathbf{-3} & 1 & 1 \\ \mathbf{6} & 8 & 4 \end{bmatrix} = \begin{bmatrix} \mathbf{3} & 1 & 0 \\ \mathbf{0} & 2 & 1 \\ \mathbf{0} & 6 & 4 \end{bmatrix} \quad \begin{matrix} \textbf{two new} \\ \textbf{zeros in} \\ \textbf{column 1} \end{matrix} \tag{2}$$

To produce a zero in column 2, E_{32} subtracts $\ell_{32} = 3$ times the **new row 2** from row 3 :

$$(E_{32})\,(E_{31}E_{21}A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mathbf{-3} & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 0 & \mathbf{2} & 1 \\ 0 & \mathbf{6} & 4 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ \mathbf{0} & \mathbf{2} & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} = \boldsymbol{U} \quad \begin{matrix} \boldsymbol{U}\textbf{ has zeros} \\ \textbf{below the} \\ \textbf{main diagonal} \end{matrix} \tag{3}$$

Notice again : E_{32} is subtracting 3 times the row $0, 2, 1$ and not the original row of A. At the end, the pivots $3, 2, 1$ are on the main diagonal of U : zeros below that diagonal.

The **inverse** of each matrix E_{ij} **adds back $\boldsymbol{\ell_{ij}}$(row $\boldsymbol{j}$)** to row i. This leads to the inverse of their product $E = E_{32}E_{31}E_{21}$. That inverse of E is special. *We call it* L.

The Facts About Inverse Matrices

Suppose A is a square matrix. We look for an "***inverse matrix***" A^{-1} of the same size, so that $\boldsymbol{A^{-1}}$ **times** $\boldsymbol{A}$ **equals** $\boldsymbol{I}$. Whatever A does, A^{-1} undoes. Their product is the identity matrix—which does nothing to a vector, so $A^{-1}Ax = x$. *But* A^{-1} *might not exist.*

The n by n matrix A needs n independent columns to be invertible. Then $A^{-1}A = I$.

What a matrix mostly does is to multiply a vector. Multiplying $Ax = b$ by A^{-1} gives $A^{-1}Ax = A^{-1}b$. ***This is*** $x = A^{-1}b$. The product $A^{-1}A$ is like multiplying by a number and then dividing by that number. Numbers have inverses if they are not zero. Matrices are more complicated and interesting. The matrix A^{-1} is called **"$\boldsymbol{A}$ inverse"**.

> **DEFINITION** The matrix A is ***invertible*** if there exists a matrix A^{-1} that "inverts" A:
>
> **Two-sided inverse** $\quad \boldsymbol{A^{-1}A = I}$ and $\boldsymbol{AA^{-1} = I}$. (4)

Not all matrices have inverses. This is the first question we ask about a square matrix: Is A invertible? Its columns must be independent. We don't mean that we actually calculate A^{-1}. In most problems we never compute it! Here are seven "notes" about A^{-1}.

Note 1 ***The inverse exists if and only if elimination produces n pivots*** (row exchanges are allowed). Elimination solves $Ax = b$ without explicitly using the matrix A^{-1}.

Note 2 The matrix A cannot have two different inverses. Suppose $BA = I$ and also $AC = I$. Then $B = C$, according to this "proof by parentheses" = associative law.

$$B(AC) = (BA)C \quad \text{gives} \quad BI = IC \quad \text{or} \quad \boldsymbol{B = C}. \tag{5}$$

This shows that a *left inverse* B (multiplying A from the left) and a *right inverse* C (multiplying A from the right to give $AC = I$) must be the *same matrix*.

Note 3 If A is invertible, the one and only solution to $Ax = b$ is $x = A^{-1}b$:

> ***Multiply*** $Ax = b$ ***by*** $\boldsymbol{A^{-1}}$. ***Then*** $x = A^{-1}Ax = \boldsymbol{A^{-1}b}$.

Note 4 (Important) ***Suppose there is a nonzero vector x such that*** $Ax = \mathbf{0}$. *Then A has dependent columns. It cannot have an inverse.* No matrix can bring $\mathbf{0}$ back to x.

If A is invertible, then $Ax = \mathbf{0}$ only has the zero solution $x = A^{-1}\mathbf{0} = \mathbf{0}$.

Note 5 A square matrix is invertible if and only if its columns are independent.

Note 6 A 2 by 2 matrix is invertible if and only if the number $ad - bc$ is not zero:

$$\textbf{2 by 2 Inverse} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{6}$$

This number $ad - bc$ is the **determinant** of A. A matrix is invertible if its determinant is not zero (Chapter 5). The test for n pivots is usually decided before the determinant appears.

Note 7 A triangular matrix has an inverse provided no diagonal entries d_i are zero:

$$\text{If} \quad A = \begin{bmatrix} \boldsymbol{d_1} & \times & \times & \times \\ 0 & \bullet & \times & \times \\ 0 & 0 & \bullet & \times \\ 0 & 0 & 0 & \boldsymbol{d_n} \end{bmatrix} \quad \text{then} \quad A^{-1} = \begin{bmatrix} \boldsymbol{1/d_1} & \times & \times & \times \\ 0 & \bullet & \times & \times \\ 0 & 0 & \bullet & \times \\ 0 & 0 & 0 & \boldsymbol{1/d_n} \end{bmatrix}$$

Example 2 The 2 by 2 matrix $A = \left[\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right]$ is not invertible. It fails the test in Note 6, because $ad = bc$. It also fails the test in Note 4, because $A\boldsymbol{x} = \boldsymbol{0}$ when $\boldsymbol{x} = (2, -1)$. It fails to have two pivots as required by Note 1. Its columns are clearly dependent.

Elimination turns the second row of this matrix A into a zero row. No pivot.

Example 3 Three of these matrices are invertible, and three are singular. Find the inverse when it exists. Give reasons for noninvertibility (zero determinant, too few pivots, nonzero solution to $A\boldsymbol{x} = \boldsymbol{0}$) for the other three. The matrices are in the order $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{S}, \boldsymbol{T}$:

$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \quad \begin{bmatrix} 4 & 3 \\ 8 & 7 \end{bmatrix} \quad \begin{bmatrix} 6 & 6 \\ 6 & 0 \end{bmatrix} \quad \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution The three matrices with inverses are $\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{S}$:

$$\boldsymbol{B^{-1}} = \frac{1}{4}\begin{bmatrix} 7 & -3 \\ -8 & 4 \end{bmatrix} \quad \boldsymbol{C^{-1}} = \frac{1}{36}\begin{bmatrix} 0 & 6 \\ 6 & -6 \end{bmatrix} \quad \boldsymbol{S^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

A is not invertible because its determinant is $4 \cdot 6 - 3 \cdot 8 = 24 - 24 = 0$. D is not invertible because it has only one pivot; row 2 becomes zero when row 1 is subtracted. T has two equal rows (and the second column minus the first column is zero). In other words $T\boldsymbol{x} = \boldsymbol{0}$ has the nonzero solution $\boldsymbol{x} = (-1, 1, 0)$. *Not invertible.*

The Inverse of a Product AB

For two nonzero numbers a and b, the sum $a + b$ might or might not be invertible. The numbers $a = 3$ and $b = -3$ have inverses $\frac{1}{3}$ and $-\frac{1}{3}$. Their sum $a + b = 0$ has no inverse. But the product $ab = -9$ does have an inverse, which is $\frac{1}{3}$ times $-\frac{1}{3}$.

For matrices A and B, the situation is similar. Their *product* AB has an inverse if and only if A and B are separately invertible (and the same size). The important point is that **A^{-1} and B^{-1} come in reverse order**:

If A and B are invertible (same size) then the inverse of AB is $B^{-1}A^{-1}$.

$$(\boldsymbol{AB})^{-1} = \boldsymbol{B^{-1}A^{-1}} \qquad (\boldsymbol{AB})(\boldsymbol{B^{-1}A^{-1}}) = \boldsymbol{A\,I\,A^{-1}} = \boldsymbol{AA^{-1}} = \boldsymbol{I} \quad (7)$$

We moved parentheses to multiply BB^{-1} first. Similarly $B^{-1}A^{-1}$ times AB equals I.

$B^{-1}A^{-1}$ illustrates a basic rule of mathematics: Inverses come in reverse order. It is also common sense: If you put on socks and then shoes, the first to be taken off are the _____. The same reverse order applies to three or more matrices:

$$\textbf{Reverse order} \qquad (ABC)^{-1} = C^{-1}B^{-1}A^{-1} \tag{8}$$

Example 4 ***Inverse of an elimination matrix***. If E subtracts 5 times row 1 from row 2, then E^{-1} *adds* 5 times row 1 to row 2:

$$\begin{array}{c}\boldsymbol{E}\textbf{ subtracts}\\ \boldsymbol{E^{-1}}\textbf{ adds}\end{array} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-5} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{5} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiply EE^{-1} to get the identity matrix I. Also multiply $E^{-1}E$ to get I. We are adding and subtracting the same 5 times row 1. If $AC = I$ then for square matrices $CA = I$.

For square matrices, an inverse on one side is automatically an inverse on the other side.

Example 5 Suppose F subtracts 4 times row 2 from row 3, and F^{-1} adds it back:

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \quad \text{and} \quad F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}.$$

Now multiply F by the matrix E in Example 4 to find FE. Also multiply E^{-1} times F^{-1} to find $(FE)^{-1}$. Notice the orders FE and $E^{-1}F^{-1}$!

$$FE = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-5} & 1 & 0 \\ \mathbf{20} & \mathbf{-4} & 1 \end{bmatrix} \quad \textbf{is inverted by} \quad E^{-1}F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{5} & 1 & 0 \\ \mathbf{0} & \mathbf{4} & 1 \end{bmatrix}. \tag{9}$$

The result is beautiful and correct. The product FE contains "20" but its inverse doesn't. E subtracts 5 times row 1 from row 2. Then F subtracts 4 times the *new row* 2 (changed by row 1) from row 3. ***In this order FE, row 3 feels an effect of size 20 from row 1.***

In the order $E^{-1}F^{-1}$, that effect does not happen. First F^{-1} adds 4 times row 2 to row 3. After that, E^{-1} adds 5 times row 1 to row 2. There is no 20, because row 3 doesn't change again. ***In this order $\mathbf{E^{-1}F^{-1}}$, row 3 feels no effect from row 1.***

This is why we choose $A = LU$, to go back from the triangular U to the original A. The multipliers fall into place perfectly in the lower triangular L: Equation (11) below.

The elimination order is FE. The inverse order is $\boldsymbol{L} = \boldsymbol{E^{-1}F^{-1}}$.
The multipliers 5 and 4 fall into place below the diagonal of 1's in L.

L is the Inverse of E

E is the product of all the elimination matrices E_{ij}, taking A into its upper triangular form $EA = U$. We are assuming for now that no row exchanges are involved ($P = I$). The difficulty with E is that multiplying all the separate elimination steps E_{ij} does not produce a good formula. But the inverse matrix E^{-1} becomes beautiful when we multiply the inverse steps E_{ij}^{-1}. Remember that those steps come in the *opposite order*.

With $n = 3$, the complication for $E = E_{32}E_{31}E_{21}$ is in the bottom left corner:

$$\boldsymbol{E} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -\ell_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 0 & 1 & \\ -\ell_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -\ell_{21} & 1 & \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -\ell_{21} & 1 & \\ (\boldsymbol{\ell_{32}\ell_{21} - \ell_{31}}) & -\ell_{32} & 1 \end{bmatrix}. \quad (10)$$

Watch how that confusion disappears for $\boldsymbol{E^{-1} = L}$. Reverse order is the good way:

$$\boxed{\boldsymbol{E^{-1}} = \begin{bmatrix} 1 & & \\ \ell_{21} & 1 & \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 0 & 1 & \\ \ell_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & \ell_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ \boldsymbol{\ell_{21}} & 1 & \\ \boldsymbol{\ell_{31}} & \boldsymbol{\ell_{32}} & 1 \end{bmatrix} = \boldsymbol{L}} \quad (11)$$

All the multipliers $\boldsymbol{\ell_{ij}}$ appear in their correct positions in $\boldsymbol{L}$. The next section will show that this remains true for all matrix sizes. **Then $\boldsymbol{EA = U}$ becomes $\boldsymbol{A = LU}$.**

Equation (11) is the key to this chapter: Each $\boldsymbol{\ell_{ij}}$ is in its place for $\boldsymbol{E^{-1} = L}$.

Problem Set 2.2 (more questions than needed)

0 If you exchange columns 1 and 2 of an invertible matrix A, what is the effect on A^{-1}?

Problems 1–11 are about elimination matrices.

1 Write down the 3 by 3 matrices that produce these elimination steps:

(a) E_{21} subtracts 5 times row 1 from row 2.

(b) E_{32} subtracts -7 times row 2 from row 3.

(c) P exchanges rows 1 and 2, then rows 2 and 3.

2 In Problem 1, applying E_{21} and then E_{32} to $\boldsymbol{b} = (1, 0, 0)$ gives $E_{32}E_{21}\boldsymbol{b} =$ _____. Applying E_{32} before E_{21} gives $E_{21}E_{32}\boldsymbol{b} =$ _____. When E_{32} comes first, row _____ feels no effect from row _____.

3 Which three matrices E_{21}, E_{31}, E_{32} put A into triangular form U?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \quad \text{and} \quad E_{32}E_{31}E_{21}A = EA = U.$$

Multiply those E's to get one elimination matrix E. What is $E^{-1} = L$?

4 Include $\boldsymbol{b} = (1, 0, 0)$ as a fourth column in Problem 3 to produce $[\,A \;\; \boldsymbol{b}\,]$. Carry out the elimination steps on this augmented matrix to solve $A\boldsymbol{x} = \boldsymbol{b}$.

5 Suppose $a_{33} = 7$ and the third pivot is 5. If you change a_{33} to 11, the third pivot is ____. If you change a_{33} to ____, there is no third pivot.

6 If every column of A is a multiple of $(1, 1, 1)$, then $A\boldsymbol{x}$ is always a multiple of $(1, 1, 1)$. Do a 3 by 3 example. How many pivots are produced by elimination?

7 Suppose E subtracts 7 times row 1 from row 3.

(a) To *invert* that step you should ____ 7 times row ____ to row ____.

(b) What "inverse matrix" E^{-1} takes that reverse step (so $E^{-1}E = I$)?

(c) If the reverse step is applied first (and then E) show that $EE^{-1} = I$.

8 The ***determinant*** of $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is det $M = ad - bc$. Subtract ℓ times row 1 from row 2 to produce a new M^*. Show that $\det M^* = \det M$ for every ℓ. When $\ell = c/a$, *the product of pivots equals the determinant*: $(a)(d - \ell b)$ equals $ad - bc$.

9 (a) E_{21} subtracts row 1 from row 2 and then P_{23} exchanges rows 2 and 3. What matrix $M = P_{23}E_{21}$ does both steps at once?

(b) P_{23} exchanges rows 2 and 3 and then E_{31} subtracts row 1 from row 3. What matrix $M = E_{31}P_{23}$ does both steps at once? Explain why the M's are the same but the E's are different.

10 (a) What matrix adds row 1 to row 3 and *at the same time* row 3 to row 1 ?

(b) What matrix adds row 1 to row 3 and *then* adds row 3 to row 1 ?

11 Create a matrix that has $a_{11} = a_{22} = a_{33} = 1$ but elimination produces two negative pivots without row exchanges. (The first pivot is 1.)

12 For these "permutation matrices" find P^{-1} by trial and error (with 1's and 0's) :

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

13 Solve for the first column (x, y) and second column (t, z) of A^{-1}. Check AA^{-1}.

$$A = \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \qquad \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix}\begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

14 Find an upper triangular U (not diagonal) with $U^2 = I$. Then $U^{-1} = U$.

15 (a) If A is invertible and $AB = AC$, prove quickly that $B = C$.

(b) If $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, find two different matrices such that $AB = AC$.

16 (Important) If A has row 1 + row 2 = row 3, show that A is not invertible:

(a) Explain why $A\boldsymbol{x} = (0, 0, 1)$ cannot have a solution. Add eqn 1 + eqn 2.

(b) Which right sides (b_1, b_2, b_3) might allow a solution to $A\boldsymbol{x} = \boldsymbol{b}$?

(c) In the elimination process, what happens to equation 3?

17 If A has column 1 + column 2 = column 3, show that A is not invertible:

(a) Find a nonzero solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{0}$. The matrix is 3 by 3.

(b) Elimination keeps columns 1 + 2 = 3. Explain why there is no third pivot.

18 Suppose A is invertible and you exchange its first two rows to reach B. Is the new matrix B invertible? How would you find B^{-1} from A^{-1}?

19 (a) Find invertible matrices A and B such that $A + B$ is not invertible.

(b) Find singular matrices A and B such that $A + B$ is invertible.

20 If the product $C = AB$ is invertible (A and B are square), then A itself is invertible. Find a formula for A^{-1} that involves C^{-1} and B.

21 If the product $M = ABC$ of three square matrices is invertible, then B is invertible. (So are A and C.) Find a formula for B^{-1} that involves M^{-1} and A and C.

22 If you add row 1 of A to row 2 to get B, how do you find B^{-1} from A^{-1}?

23 Prove that a matrix with a column of zeros cannot have an inverse.

24 Multiply $\left[\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right]$ times $\left[\begin{smallmatrix} \mathbf{d} & -\mathbf{b} \\ -\mathbf{c} & \mathbf{a} \end{smallmatrix}\right]$. What is the inverse of each matrix if $ad \neq bc$?

25 (a) What 3 by 3 matrix E has the same effect as these three steps? Subtract row 1 from row 2, subtract row 1 from row 3, then subtract row 2 from row 3.

(b) What single matrix L has the same effect as these three reverse steps? Add row 2 to row 3, add row 1 to row 3, then add row 1 to row 2.

26 If B is the inverse of A^2, show that AB is the inverse of A.

27 Show that $\mathsf{A} = 4 * \mathsf{eye}(4) - \mathsf{ones}(4, 4)$ is *not* invertible: Multiply $\mathsf{A} * \mathsf{ones}(4, 1)$.

28 There are sixteen 2 by 2 matrices whose entries are 1's and 0's. How many of them are invertible?

29 Change I into A^{-1} as elimination reduces A to I (the Gauss-Jordan idea).

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix}$$

30 Could a 4 by 4 matrix A be invertible if every row contains the numbers $0, 1, 2, 3$ in some order? What if every row of B contains $0, 1, 2, -3$ in some order?

31 Find A^{-1} and B^{-1} (*if they exist*) by elimination on $[\,A \quad I\,]$ and $[\,B \quad I\,]$:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

32 **Gauss-Jordan elimination** acts on $[\,\boldsymbol{U} \quad \boldsymbol{I}\,]$ to find the matrix $[\,\boldsymbol{I} \quad U^{-1}\,]$:

$$\text{If } U = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \quad \text{then} \quad U^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}.$$

33 True or false (with a counterexample if false and a reason if true): A is square.

(a) A 4 by 4 matrix with a row of zeros is not invertible.

(b) Every matrix with 1's down the main diagonal is invertible.

(c) If A is invertible then A^{-1} and A^2 are invertible.

34 (Recommended) Prove that A is invertible if $a \neq 0$ and $a \neq b$ (find the pivots or A^{-1}). Then find three numbers c so that C is not invertible:

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix} \qquad C = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}.$$

35 This matrix has a remarkable inverse. Find A^{-1} by elimination on $[\,A \quad I\,]$. Extend to a 5 by 5 "alternating matrix" and guess its inverse; then multiply to confirm.

$$\text{Invert } A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and solve } A\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

36 Suppose the matrices P and Q have the same rows as I but in any order. They are "permutation matrices". Show that $P - Q$ is singular by solving $(P - Q)\,\boldsymbol{x} = \boldsymbol{0}$.

37 Find and check the inverses (assuming they exist) of these **block matrices**:

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}.$$

38 How does elimination from A to U on a 3 by 3 matrix tell you if A is invertible?

39 If $A = I - \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ then $A^{-1} = I + \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}(1 - \boldsymbol{v}^{\mathrm{T}}\boldsymbol{u})^{-1}$. Show that $AA^{-1} = I$ except $A\boldsymbol{u} = \boldsymbol{0}$ when $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = 1$.