Final Examination in Linear Algebra: 18.06

May 18, 1998 9:00–12:00 Professor Strang

Your name is: ______________________

Grading 1 2 3 4 5 6 7 8

Please circle your recitation:

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2)	M3	2-131	A. Voronov	2-246	3-3299	voronov@math
3)	T10	2-132	A. Edelman	2-380	3-7770	edelman@math
4)	T12	2-132	A. Edelman	2-380	3-7770	edelman@math
5)	T12	2-131	Z. Spasojevic	2-101	3-4470	zoran@math
6)	T1	2-131	Z. Spasojevic	2-101	3-4770	zoran@math
7)	T2	2-132	Y. Ma	2-333	3-7826	yanyuan@math

Answer all 8 questions on these pages. This is a closed book exam. Calculators are not needed in any way and therefore not allowed (to be fair to all). *Grades are known only to your recitation instructor.* Best wishes for the summer and thank you for taking 18.06.

GS

1 If A is a 5 by 4 matrix with linearly independent columns, find each of these **explicitly**:

(a) **(3 points)** The nullspace of A.

(b) **(3 points)** The dimension of the left nullspace $\boldsymbol{N}(A^T)$.

(c) **(3 points)** One particular solution x_p to $Ax_p =$ column 2 of A.

(d) **(3 points)** The general (complete) solution to $Ax =$ column 2 of A.

(e) **(3 points)** The reduced row echelon form R of A.

2 (a) **(5 points)** Find the general (complete) solution to this equation $Ax = b$:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}.$$

(b) **(3 points)** Find a basis for the column space of the 3 by 9 block matrix $[A\ 2A\ A^2]$.

3 (a) **(5 points)** The command $N =$ **null** (A) produces a matrix whose columns are a basis for the nullspace of A. What matrix (describe its properties) is then produced by $B =$ **null** (N')?

(b) **(3 points)** What are the shapes (how many rows and columns) of those matrices N and B, if A is m by n of rank r?

4 Find the determinants of these three matrices:

(a) **(2 points)**

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

(b) **(2 points)**

$$B = \begin{bmatrix} 0 & -A \\ I & -I \end{bmatrix} \qquad \text{(8 by 8, same } A\text{)}$$

(c) **(2 points)**

$$C = \begin{bmatrix} A & -A \\ I & -I \end{bmatrix} \qquad \text{(8 by 8, same } A\text{)}$$

5 **If possible** construct 3 by 3 matrices A, B, C, D with these properties:

(a) **(3 points)** A is a **symmetric** matrix. Its row space is spanned by the vector $(1,1,2)$ and its column space is spanned by the vector $(2,2,4)$.

(b) **(3 points)** All three of these equations have **no solution** but $B \neq 0$:

$$Bx = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad Bx = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad Bx = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(c) **(3 points)** C is a real square matrix but its eigenvalues are not all real and not all pure imaginary.

(d) **(3 points)** The vector $(1,1,1)$ is in the row space of D but the vector $(1,-1,0)$ is not in the nullspace.

6 Suppose u_1, u_2, u_3 is an orthonormal basis for $\mathbf{R}^3$ and v_1, v_2 is an orthonormal basis for $\mathbf{R}^2$.

(a) **(5 points)** What is the **rank**, what are **all vectors** in the **column space**, and what is a **basis for the nullspace** for the matrix $B = u_1(v_1 + v_2)^T$?

(b) **(5 points)** Suppose $A = u_1 v_1^T + u_2 v_2^T$. Multiply AA^T and simplify. Show that this is a projection matrix by checking the required properties.

(c) **(4 points)** Multiply $A^T A$ and simplify. This is the identity matrix! Prove this (for example compute $A^T A v_1$ and then finish the reasoning).

7 (a) **(4 points)** If these three points happen to lie on a line $y = C + Dt$, what system $Ax = b$ of three equations in two unknowns would be solvable?

$$y = 0 \text{ at } t = -1\,, \qquad y = 1 \text{ at } t = 0\,, \qquad y = B \text{ at } t = 1\,.$$

Which value of B puts the vector $b = (0, 1, B)$ into the column space of A?

(b) **(4 points)** For every B find the numbers $\overline{C}$ and $\overline{D}$ that give the best straight line $y = \overline{C} + \overline{D}t$ (closest to the three points in the least squares sense).

(c) **(4 points)** Find the projection of $b = (1, 0, 0)$ onto the column space of A.

(d) **(2 points)** If you apply the Gram-Schmidt procedure to this matrix A, what is the resulting matrix Q that has orthonormal columns?

8 (a) **(5 points)** Find a complete set of eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

(b) **(6 points, 1 each)** Circle all the properties of this matrix A:

A is a projection matrix

A is a positive definite matrix

A is a Markov matrix

A has determinant larger than trace

A has three orthonormal eigenvectors

A can be factored into $A = LU$

(c) **(4 points)** Write the vector $u_0 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ as a combination of eigenvectors of A, and compute the vector $u_{100} = A^{100} u_0$.

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1. (a) zero vector $\{0\}$

   (b) $5 - 4 = 1$

   (c) $x_p = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

   (d) $x = x_p$ because $\boldsymbol{N}(A) = \{0\}$.

   (e) $R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$

2. (a) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} \longrightarrow U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

   The free variable is x_2. The complete solution is

$$x = x_p + x_n = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

   (b) A basis is $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

3. (a) The columns of B are a basis for the row space of A (because the row space is the orthogonal complement of the nullspace).

   (b) N is n by $(n - r)$; B is n by r.

4. (a) $\det A = 6$

   (b) $\det B = 6$

   (c) $\det C = 0$

5. (a) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

   (b) $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

   (c) $C = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

   (d) $D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

   All four matrices are only examples (many other correct answers exist).

6. (a) $\operatorname{rank}(B) = 1$; all multiples of u_1 are in the column space; the vectors $v_1 - v_2$ and v_3 are a basis for the nullspace.

   (b) $AA^T = (u_1v_1^T + u_2v_2^T)(v_1u_1^T + v_2u_2^T) = u_1u_1^T + u_2u_2^T$ since $v_1^Tv_2 = 0$. AA^T is symmetric and it equals $(AA^T)^2$:

$$(u_1u_1^T + u_2u_2^T)(u_1u_1^T + u_2u_2^T) = u_1u_1^T + u_2u_2^T$$

   (The eigenvalues of AA^T are $1, 1, 0$)

   (c) $A^TA = v_1v_1^T + v_2v_2^T$ since $u_1^Tu_2 = 0$.

$$\begin{aligned} A^TAv_1 &= (v_1v_1^T + v_2v_2^T)v_1 = v_1 \\ A^TAv_2 &= (v_1v_1^T + v_2v_2^T)v_2 = v_2\,. \end{aligned}$$

   Since v_1, v_2 are a basis for $\mathbf{R}^2$, $A^TAv = v$ for all v.

7. (a) $Ax = b$ is $\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ B \end{bmatrix}$. This is solvable if $\underline{B = 2}$.

(b) $A^T A\bar{x} = A^T b$ is

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \overline{C} \\ \overline{D} \end{bmatrix} = \begin{bmatrix} 1+B \\ B \end{bmatrix}.$$

Then $\overline{C} = \frac{1+B}{3}$ and $\overline{D} = \frac{B}{2}$

(c) $A^T A\bar{x} = A^T b$ is

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad p = A\bar{x} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1/6 \\ 1/3 \\ 5/6 \end{bmatrix}.$$

(d) $Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$ columns were already orthogonal, now orthonormal

8. (a) $\lambda = 1, 1, 4$. Eigenvectors can be

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(could also be chosen orthonormal because $A = A^T$)

(b) Circle all the properties of this matrix A:

A is a projection matrix

$\boxed{A \text{ is a positive definite matrix}}$

A is a Markov matrix

A has determinant larger than trace

$\boxed{A \text{ has three orthonormal eigenvectors}}$

$\boxed{A \text{ can be factored into } A = LU}$

(c)

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A^{100} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 1^{100} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 4^{100} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4^{100}+1 \\ 4^{100}-1 \\ 4^{100} \end{bmatrix}.$$

18.06 Final Exam December 16, 1999 Closed Book

1 (12 pts.) Let

$$A = \begin{bmatrix} 7 & 0 & 2 & 4 \\ 7 & 1 & 3 & 6 \\ 14 & -1 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 7 & 0 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find bases for the four fundamental subspaces.

(b) Find the conditions on b_1, b_2, and b_3 so that

$$Ax = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

has a solution.

(c) If $Ax = b$ has a solution x_p, describe all of the solutions.

2 (10 pts.) Let A and B be any two matrices so that the product AB is defined.

(a) Explain why every column of AB is in the column space of A.

(b) How does part (a) lead to the conclusion that the rank of AB is less than or equal to the rank of A? State your reasoning in logical steps.

3 (10 pts.) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation satisfying

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -10 \\ 8 \end{bmatrix}.$$

(a) Find $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

(b) What is the matrix A expressing T in terms of the standard basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$? (The same basis is used for the input and the output.)

(c) What is the matrix B expressing T in terms of the basis consisting of eigenvectors of A? (The same basis is used for the input and output.) (There are two possible correct answers, depending on what order you pick the eigenvectors.)

4 (16 pts.) Let V be the subspace of $\mathbf{R}^3$ consisting of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying

$$x + 2y - 5z = 0.$$

(a) Find a 3×2 matrix A whose column space is V.

(b) Find an orthonormal basis for V.

(c) Find the projection matrix P projecting onto the left nullspace (not the column space!) of A.

(d) Find the least squares solution to

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

5 (15 pts.) Suppose

$$Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{has no solution}$$

but

$$Ax = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \text{has infinitely many solutions.}$$

(a) Find all possible information about r, m, and n. (The rank and the shape of A.)

(b) Find an example of such a matrix A with r, m, and n all as small as possible.

(c) How do you know that $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is not in the nullspace of A^{T}?

6 (13 pts.) In each case give all the information you can about the eigenvalues and eigenvectors, when the matrix A has the following property:

(a) The powers A^k approach the zero matrix.

(b) The matrix is symmetric positive definite.

(c) The matrix is not diagonalizable.

(d) The matrix has the form $A = uv^{\mathrm{T}}$, where u and v are vectors in $\mathbf{R}^3$. (You might want to try an example.)

(e) A is similar to a diagonal matrix with diagonal entries 1, 1, and 2.

7 (12 pts.) Define a sequence of numbers in the following way: $G_0 = 0$, $G_1 = 1/2$, and $G_{k+2} = (G_{k+1} + G_k)/2$. (Each number is the average of the two previous numbers.)

(a) Set up a 2×2 matrix A to get from $\begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$ to $\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix}$.

(b) Find an explicit formula for G_k.

(c) What is the limit of G_k as $k \to \infty$?

8 (12 pts.) (a) Suppose A is a 4×4 matrix of rank 3, and let

$$x = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \\ C_{14} \end{bmatrix}$$

be the cofactors of its first row. Explain why $Ax = 0$. (So the cofactors give a formula for a nullspace vector!)

Hint: The first component of Ax and the second component of Ax are determinants of (different) matrices. What are these matrices and why do they have zero determinants? (The 3rd and 4th components of Ax follow similarly, so you can just answer for the 1st and 2nd components.)

(b) Compute the determinant of

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Hint: You might find it convenient to use the fact that the columns are orthogonal.

18.06 Final Exam December 16, 1999 Closed Book

1 (12 pts) Let

$$A = \begin{bmatrix} 7 & 0 & 2 & 4 \\ 7 & 1 & 3 & 6 \\ 14 & -1 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 7 & 0 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find bases for the four fundamental subspaces.

(b) Find the conditions on b_1, b_2, and b_3 so that

$$Ax = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

has a solution.

(c) If $Ax = b$ has a solution x_p, describe all of the solutions.

Solution: Write L and U for the two matrices on the right, so $A = LU$. L is invertible and U is the row reduced form of A.

(a) A basis for $C(A)$ is given by the pivot columns of A, $(7,7,14)$ and $(0,1,-1)$. For $N(A)$: the special solutions, $(-2/7,-1,1,0)$ and $(-4/7,-2,0,1)$. For $C(A^{\mathrm{T}})$: the nonzero rows of U, $(7,0,2,4)$ and $(0,1,1,2)$. For $N(A^{\mathrm{T}})$, row reduce A^{T}. There is one special solution, $(-3,1,1)$.

(b) Row reduce the augmented matrix $[\,A\,|\,b\,]$. There is a solution when $b_3 + b_2 - 3b_1 = 0$.

(c) All solutions are of the form $x_p + c_1(-2/7,-1,1,0) + c_2(-4/7,-2,0,1)$, where c_1 and c_2 are real numbers.

2 (10 pts) Let A and B be any two matrices so that the product AB is defined.

(a) Explain why every column of AB is in the column space of A.

(b) How does part (a) lead to the conclusion that the rank of AB is less than or equal to the rank of A? State your reasoning in logical steps.

Solution:

(a) Write $a_1, \ldots, a_n$ for the columns of A. Then the first column of AB is $b_{11}a_1 + b_{21}a_2 + \cdots + b_{n1}a_n$. This is a linear combination of the columns of A, hence is in $C(A)$. The same reasoning shows that the other columns of AB are in $C(A)$.

(b) From part (a), $C(AB)$ is a subspace of $C(A)$. Therefore $\dim C(AB) \leq \dim C(A)$, or $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$.

3 (10 pts) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation satisfying

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -10 \\ 8 \end{bmatrix}.$$

(a) Find $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

(b) What is the matrix A expressing T in terms of the standard basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$? (The same basis is used for the input and the output.)

(c) What is the matrix B expressing T in terms of the basis consisting of eigenvectors of A? (The same basis is used for the input and output.) (There are two possible correct answers, depending on what order you pick the eigenvectors.)

Solution:

(a) $T\left(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\right) = T\left(\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]\right) - T\left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} -6 \\ 5 \end{smallmatrix}\right]$.

(b) $A = \left[\begin{smallmatrix} -4 & -6 \\ 3 & 5 \end{smallmatrix}\right]$.

(c) The eigenvalues of A are 2 and -1, so $B = \left[\begin{smallmatrix} 2 & 0 \\ 0 & -1 \end{smallmatrix}\right]$.

4 (16 pts) Let V be the subspace of $\mathbf{R}^3$ consisting of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying

$$x + 2y - 5z = 0.$$

(a) Find a 3×2 matrix A whose column space is V.

(b) Find an orthonormal basis for V.

(c) Find the projection matrix P projecting onto the left nullspace (not the column space!) of A.

(d) Find the least squares solution to

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution: Parts (a), (b), and (d) have infinitely many possible correct answers.

(a) The columns of A should be two linearly independent solutions, to $x + 2y - 5z = 0$. For example, $A = \begin{bmatrix} 2 & 5 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) Applying the Gram-Schmidt process to the columns of A yields $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(c) Let Q be the matrix whose columns are the vectors in part (b). The projection matrix onto the column space of A is QQ^{T}. Since the left nullspace is the orthogonal complement of the column space, its projection matrix is $I - QQ^{\mathrm{T}} = \frac{1}{30} \begin{bmatrix} 1 & 2 & -5 \\ 2 & 4 & -10 \\ -5 & -10 & 25 \end{bmatrix}$.

(d) The solution to $A^{\mathrm{T}} A\hat{x} = A^{\mathrm{T}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is $\hat{x} = \begin{bmatrix} -17/15 \\ 2/3 \end{bmatrix}$.

5 (15 pts) Suppose

$$Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{has no solution}$$

but

$$Ax = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \text{has infinitely many solutions.}$$

(a) Find all possible information about r, m, and n. (The rank and the shape of A.)

(b) Find an example of such a matrix A with r, m, and n all as small as possible.

(c) How do you know that $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is not in the nullspace of A^{T}?

Solution:

(a) $m = 3$, $0 < r < 3$, and $r < n$.

(b) With $r = 1$, $m = 3$ and $n = 2$: $A = \begin{bmatrix} 3 & 0 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$.

(c) The column space and the left nullspace are always orthogonal.

6 (13 pts) In each case give all the information you can about the eigenvalues and eigenvectors, when the matrix A has the following property:

(a) The powers A^k approach the zero matrix.

(b) The matrix is symmetric positive definite.

(c) The matrix is not diagonalizable.

(d) The matrix has the form $A = uv^{\mathrm{T}}$, where u and v are vectors in $\mathbf{R}^3$. (You might want to try an example.)

(e) A is similar to a diagonal matrix with diagonal entries 1, 1, and 2.

Solution:

(a) The eigenvalues are between -1 and 1.

(b) The eigenvalues are positive. There are n linearly independent, orthogonal eigenvectors.

(c) There is a repeated eigenvalue and that eigenvalue has fewer linearly independent eigenvectors than its multiplicity as a root of the characteristic polynomial.

(d) The eigenvalues are 0 (the eigenvectors are all vectors orthogonal to v) and $u \cdot v$ (the eigenvectors are the multiples of u).

(e) The eigenvalues are 1 and 2 and there are three linearly independent eigenvectors (two for $\lambda = 1$ and one for $\lambda = 2$).

7 (12 pts) Define a sequence of numbers in the following way: $G_0 = 0$, $G_1 = 1/2$, and $G_{k+2} = (G_{k+1} + G_k)/2$. (Each number is the average of the two previous numbers.)

(a) Set up a 2×2 matrix A to get from $\begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$ to $\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix}$.

(b) Find an explicit formula for G_k.

(c) What is the limit of G_k as $k \to \infty$?

Solution:

(a) $A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$.

(b) A has eigenvalues 1 and $-1/2$ with corresponding eigenvectors $(1, 1)$ and $(1, -2)$. Write the initial state as a linear combination of the eigenvectors: $\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Applying A^k, we get $\begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix} = \frac{1}{3}(1)^k\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{6}(-\frac{1}{2})^k\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, so $G_k = \frac{1}{3}(1 - (-\frac{1}{2})^k)$.

(c) As $k \to \infty$, $G_k \to \frac{1}{3}$.

8 (12 pts) (a) Suppose A is a 4×4 matrix of rank 3, and let

$$x = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \\ C_{14} \end{bmatrix}$$

be the cofactors of its first row. Explain why $Ax = 0$. (So the cofactors give a formula for a nullspace vector!)

Hint: The first component of Ax and the second component of Ax are determinants of (different) matrices. What are these matrices and why do they have zero determinants? (The 3rd and 4th components of Ax follow similarly, so you can just answer for the 1st and 2nd components.)

(b) Compute the determinant of

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Hint: You might find it convenient to use the fact that the columns are orthogonal.

Solution:

(a) The first component of Ax is $\det A$, which is zero because A has rank 3. The second component of Ax is the determinant of the matrix obtained by replacing the first row of A with the second row. This matrix has repeated rows, so its determinant is zero.

(b) $\det B = 16$. You can compute this by brute force. Another way is to note that $B^2 = 4I$, so the eigenvalues of B are ± 2. Since the trace of B is zero, the eigenvalues are $\lambda = 2$, 2, -2, and -2, so $\det B = 16$.

Final Examination in Linear Algebra: 18.06

May 18, 1999 1:30 – 4:30 Professor Strang

Your name is: ______________________________

Secret Code (optional): ______________________

Grading 1 2 3 4 5 6 7 8 9

Please circle your recitation:

1) Mon	2–3	2-131	S. Kleiman		5) Tues	12–1	2-131	S. Kleiman
2) Mon	3–4	2-131	S. Hollander		6) Tues	1–2	2-131	S. Kleiman
3) Tues	11–12	2-132	S. Howson		7) Tues	2–3	2-132	S. Howson
4) Tues	12–1	2-132	S. Howson					

Answer all 9 questions on these pages. This is a closed book exam. Calculators are not needed in any way and therefore not allowed (to be fair to all). *Grades are known only to your recitation instructor* (who is free to post with secret codes and to return exams in person). Solutions will be posted on the web in a few days. Best wishes for the summer and thank you for taking 18.06.

GS

1 Suppose the matrix A is the product

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) **(3 points)** Give a basis for the nullspace of A.

(b) **(3 points)** Give a basis for the row space of A.

(c) **(2 points)** Express row 3 of A as a combination of your basis vectors in your answer to (b).

(d) **(3 points)** What is the dimension of the nullspace of A^T?

2 Suppose A is a 5 by 7 matrix, and $Ax = b$ has a solution for every right side b.

(a) **(3 points)** What do we know about the column space of A?

(b) **(3 points)** What do we know about the rows of A?

(c) **(3 points)** What do we know about the nullspace of A?

(d) **(3 points)** True or false (with reason):

The columns of A are a basis for the column space of A.

Your answers could refer to dimension/basis/linear independence/spanning a space.

3 Assume that A is invertible and permutations are not needed in elimination if possible.

(a) **(2 points)** Are the pivots of A^{-1} equal to $\dfrac{\mathbf{1}}{\text{pivots of } A}$?
If yes, give a reason; if no, give an example.

(b) **(3 points)** Is the product of pivots of A^{-1} equal to $\dfrac{1}{\text{product of pivots of } A}$?
If yes, give a reason; if no, give an example.

(c) **(3 points)** Apply block elimination to the $2n$ by $2n$ matrix

$$M = \begin{bmatrix} A & I \\ -I & 0 \end{bmatrix}.$$

Multiply block row 1 by a suitable matrix and add to block row 2.

What matrix appears in the $(2,2)$ block?

(d) **(3 points)** What is the determinant of M? Explain the final plus or minus sign.

4 Suppose A has eigenvalues $\lambda_1 = 3$, $\lambda_2 = 1$, $\lambda_3 = 0$ with corresponding eigenvectors

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(a) **(3 points)** How do you know that the third column of A contains all zeros?

(b) **(3 points)** Find the matrix A.

(c) **(3 points)** By transposing $S^{-1}AS = \wedge$, find the eigenvectors y_1, y_2, y_3 of A^T. (I am looking for specific vectors like x_1, x_2, x_3 above.)

5 This problem computes the plane $z = Cx + Dy + E$ that is closest (in the least squares sense) to these four measurements:

$$\begin{array}{llll} \text{At} & x = 1, & y = 0 & \text{measurement gives} \quad z = 1 \\ \text{At} & x = 1, & y = 2 & \text{measurement gives} \quad z = 3 \\ \text{At} & x = 0, & y = 1 & \text{measurement gives} \quad z = 5 \\ \text{At} & x = 0, & y = 2 & \text{measurement gives} \quad z = 0 \end{array}$$

(a) **(3 points)** Write down the linear system $Ax = b$ with unknown vector $x = (C, D, E)$ that would give a plane going exactly through the four given points — except that this particular system has no solution.

(b) **(3 points)** Show that this system $Ax = b$ has no solution!

(c) **(3 points)** Find the best least squares solution $\widehat{x} = (\widehat{C}, \widehat{D}, \widehat{E})$.

(d) **(3 points)** The error vector $e = (e_1, e_2, e_3, e_4)$ in the underlying projection problem is perpendicular to which vectors? You don't have to compute e but you do have to say which specific numerical vectors it is perpendicular to.

6 (a) **(3 points)** The vectors $q_1 = \frac{1}{\sqrt{50}} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ and $q_2 = \frac{1}{5} \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$ are orthonormal. Find one more vector to complete an orthonormal basis for $\mathbf{R}^3$.

(b) **(3 points)** In solving part (a), you might start with a vector like $a_3 = (0, 0, 1)$ and find q_3. Which vectors a_3 *would not work* as starting vectors to find q_3 by Gram-Schmidt? How many different real vectors q_3 will give a correct answer to part (a)?

(c) **(3 points)** Project the vector $a_3 = (0, 0, 1)$ onto the plane spanned by q_1 and q_2. Find its projection p.

7 In each part, find the required matrix or explain why such a matrix does not exist.

(a) **(3 points)** The matrices A and A^T and $A + A^T$ have ranks 2 and 1 and 3.

(b) **(3 points)** The solution to $Ax = 0$ is unique, but the solution to $A^Tx = 0$ is not unique.

(c) **(3 points)** The powers A^k do not approach the zero matrix as $k \to \infty$, but the exponential e^{At} does approach the zero matrix as $t \to \infty$.

(d) **(3 points)** The complete solution to

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{is} \quad x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

(e) **(3 points)** The pivots are -1 and -2 but the eigenvalues are $+1$ and $+2$. (Symmetric matrix not required, row exchanges not required.)

8 (a) **(3 points)** The "big formula" for a 6 by 6 determinant has $6! = 720$ terms. How many of those terms are sure to be zero if we know that $a_{15} = 0$?

(b) **(2 points)** If U and V are 3 by 3 orthogonal matrices, is their product UV always orthogonal? Why (give reason) or why not (give example)?

(c) **(2 points)** If A and B are 3 by 3 symmetric matrices, is their product AB always symmetric? Why (give reason) or why not (give example)?

(d) **(3 points)** For which numbers c is the matrix A positive definite? For which numbers c is A^2 positive definite? Why?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & c & 4 \\ 3 & 4 & 9 \end{bmatrix}$$

9 Suppose the 3 by 3 matrix A has the following property Z: Along each of its rows, the entries add up to zero.

(a) **(3 points)** Find a nonzero vector in the nullspace of A.

(b) **(3 points)** Prove that A^2 also has property Z.

(c) **(3 points)** What can you say about the dimension of the nullspace of A^T and why?

(d) **(2 points)** Find an eigenvalue of the matrix $A + 4I$.

Final Examination in Linear Algebra: 18.06

May 18, 1999 — Solutions — Professor Strang

1. (a) $\begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

   (c) $5(\text{row}\,1) + 4(\text{row}\,2)$

   (d) A has rank 2 and A^T is 4 by 3 so its nullspace has dimension $3 - 2 = 1$.

2. (a) $C(A) = \mathbf{R}^5$ since every b is in the column space.

   (b) The rank is 5 so the five rows must be linearly independent.

   (c) The nullspace must have dimension $7 - 5 = 2$.

   (d) This is **false** because the 7 columns cannot be linearly independent.

3. (a) This is generally **false**, as for $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.
   Note that $A = LDU$ gives $A^{-1} = U^{-1}D^{-1}L^{-1}$ (upper times lower!).

   (b) **True** because $\det A^{-1} = 1/(\det A)$.

   (c) Multiply row 1 by A^{-1} and add to row 2 to obtain $\begin{bmatrix} A & I \\ 0 & A^{-1} \end{bmatrix}$.

   (d) The determinant is $+1$. Exchange the first n columns with the last n. This produces a factor $(-1)^n$ and leaves $\begin{bmatrix} I & A \\ 0 & -I \end{bmatrix}$ which is triangular with determinant $(-1)^n$. Then $(-1)^n(-1)^n = +1$.

4. (a) From $Ax_3 = \lambda_3 x_3$ we have $A\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

(b) $A = S \wedge S^{-1} = \begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}3&&\\&1&\\&&0\end{bmatrix}\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix} = \begin{bmatrix}3&0&0\\2&1&0\\2&1&0\end{bmatrix}$.

(c) Transpose $S^{-1}AS = \wedge$ to get $S^T A^T (S^{-1})^T = \wedge$. Then the columns of $(S^{-1})^T$ are the eigenvectors of A^T, and part (b) gives $(S^{-1})^T = \begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}$.

5. (a) $\begin{aligned}1C + 0D + E &= 1\\ 1C + 2D + E &= 3\\ 0C + 1D + E &= 5\\ 0C + 2D + E &= 0\end{aligned}$ is $Ax = b$.

(b) Subtract equation (1) from equation (2):

$$\begin{aligned} 2D &= 2 \quad \text{gives } D = 1\\ D + E &= 5 \quad \text{gives } E = 4\\ 2D + E &= 0 \quad \text{is now false}\end{aligned}$$

(c) Solve $A^T A\hat{x} = A^T b$:

$$\begin{bmatrix}1&1&0&0\\0&2&1&2\\1&1&1&1\end{bmatrix}\begin{bmatrix}1&0&1\\1&2&1\\0&1&1\\0&2&1\end{bmatrix}\begin{bmatrix}\hat{C}\\\hat{D}\\\hat{E}\end{bmatrix} = \begin{bmatrix}1&1&0&0\\0&2&1&2\\1&1&1&1\end{bmatrix}\begin{bmatrix}1\\3\\5\\0\end{bmatrix}$$

$$\begin{bmatrix}2&2&2\\2&9&5\\2&5&4\end{bmatrix}\begin{bmatrix}\hat{C}\\\hat{D}\\\hat{E}\end{bmatrix} = \begin{bmatrix}4\\11\\9\end{bmatrix}$$

$$\begin{bmatrix}2&2&2\\0&7&3\\0&0&\frac{5}{7}\end{bmatrix}\begin{bmatrix}\hat{C}\\\hat{D}\\\hat{E}\end{bmatrix} = \begin{bmatrix}4\\7\\2\end{bmatrix}$$

Back-substitution gives $\hat{E} = \frac{14}{5}$, $\hat{D} = \frac{-1}{5}$, $\hat{C} = \frac{-3}{5}$.

(d) The error vector e is perpendicular to the three columns of A.

6. (a) One way is to solve for x perpendicular to q_1 and q_2:

$$\begin{bmatrix} 3 & 4 & 5 \\ 4 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Another way is Gram-Schmidt and we might as well start with $a_3 = (0, 0, 1)$. Then Gram-Schmidt subtracts off projections:

$$a_3 - (a_3^T q_1)q_1 - (a_3^T q_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{5}{50} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} - 0 = \begin{bmatrix} -.3 \\ -.4 \\ .5 \end{bmatrix}.$$

Normalizing to a unit vector gives

$$q_3 = \frac{1}{\sqrt{50}} \begin{bmatrix} -3 \\ -4 \\ 5 \end{bmatrix}.$$

(b) a_3 will not work if it is in the plane of q_1 and q_2.
The only possible vectors q_3 are $+(\text{our } q_3)$ and $-(\text{our } q_3)$.

(c) The projection is the vector that was subtracted off in part (a):

$$p = \frac{5}{50} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.4 \\ 0.5 \end{bmatrix}.$$

7. (a) Cannot exist because A and A^T have the same rank.

(b) $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or any non-square A with independent columns.

(c) The desired A has an eigenvalue like -2, outside the unit circle and in the left half-plane. In fact, $A = [-2]$ is a 1 by 1 example.

(d) From the two given nullspace vectors we know that $A = \begin{bmatrix} v & v & -v \end{bmatrix}$ for some column v. The particular solution $(1, 1, 1)$ determines v:

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{gives} \quad v + v - v = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{so} \quad v = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

(e) (My favorite this year)

The first pivot must be $a_{11} = -1$. The the trace $1 + 2$ requires $a_{22} = 4$. Then the determinant must be 2, so these matrices will work:

$$A = \begin{bmatrix} -1 & -1 \\ 6 & 4 \end{bmatrix} \quad \text{or any} \quad A = \begin{bmatrix} -1 & -a \\ 6/a & 4 \end{bmatrix}.$$

8. (a) $5! = 120$ terms are sure to be zero.

   (b) **Yes**, $(UV)^T(UV) = V^TU^TUV = V^TV = I$.

   (c) **No**, symmetry would need $AB = (AB)^T = B^TA^T = BA$ and we don't normally have $AB = BA$.

   (d) The 1 by 1, 2 by 2, 3 by 3 determinants are 1, $c - 4$, and -4 (not depending on c!). The last is negative so A is not positive definite. But $\det A = -4$ so A has no zero eigenvalues so A^2 has all three positive eigenvalues.

9. (a) $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has $Ax_0 = 0$.

   (b) $A^2x_0 = A(Ax_0) = 0$

   (c) The dimension of $N(A^T)$ is at least 1 (because A is square and we know that $(1, 1, 1)$ is in $N(A)$).

   (d) A is singular so $\lambda = 0$ is an eigenvalue of A so $\lambda = 4$ is an eigenvalue of $A + 4I$.

Final Examination in Linear Algebra: 18.06

Dec 21, 2000 **9:00 – 12:00** **Professor Strang**

Your name is: ______________________________

Grading 1
2
3
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8

Please circle your recitation:

1)	M2	2-131	Holm	2-181	3-3665	`tsh@math`
2)	M2	2-132	Dumitriu	2-333	3-7826	`dumitriu@math`
3)	M3	2-131	Holm	2-181	3-3665	`tsh@math`
4)	T10	2-132	Ardila	2-333	3-7826	`fardila@math`
5)	T10	2-131	Czyz	2-342	3-7578	`czyz@math`
6)	T11	2-131	Bauer	2-229	3-1589	`bauer@math`
7)	T11	2-132	Ardila	2-333	3-7826	`fardila@math`
8)	T12	2-132	Czyz	2-342	3-7578	`czyz@math`
9)	T12	2-131	Bauer	2-229	3-1589	`bauer@math`
10)	T1	2-132	Ingerman	2-372	3-4344	`ingerman@math`
11)	T1	2-131	Nave	2-251	3-4097	`nave@math`
12)	T2	2-132	Ingerman	2-372	3-4344	`ingerman@math`
13)	T2	1-150	Nave	2-251	3-4097	`nave@math`

Answer all 8 questions on these pages (25 parts, 4 points each). This is a closed book exam. Calculators are not needed in any way and therefore not allowed (to be fair to all). *Grades are known only to your recitation instructor.* Best wishes for the holidays and thank you for taking 18.06. GS

1 (a) Explain why every eigenvector of A is either in the column space $C(A)$ or the nullspace $N(A)$ (or explain why this is false).

(b) From $A = S\Lambda S^{-1}$ find the eigenvalue matrix and the eigenvector matrix for A^{T}. How are the eigenvalues of A and A^{T} related?

(c) Suppose $Ax = 0$ and $A^{\mathrm{T}}y = 2y$. Deduce that x is orthogonal to y. You may prove this directly or use the subspace ideas in (a) or the eigenvector matrices in (b). Write a clear answer.

2 (a) Suppose A is a symmetric matrix. If you first subtract 3 times row 1 from row 3, and after that you subtract 3 times column 1 from column 3, is the resulting matrix B still symmetric? Yes or not necessarily, with a reason.

(b) Create a symmetric positive definite matrix (but not diagonal) with eigenvalues $1, 2, 4$.

(c) Create a nonsymmetric matrix (if possible) with those eigenvalues. Create a rank-one matrix (if possible) with those eigenvalues.

3 Gram-Schmidt is $A = QR$ (start from rectangular A with independent columns, produce Q with orthonormal columns and upper triangular R). The problem is to produce the same Q and R from ordinary (symmetric) elimination on $A^{\mathrm{T}}A$ which gives

$$A^{\mathrm{T}}A = LDL^{\mathrm{T}} = R^{\mathrm{T}}R \qquad \text{(with } R = \sqrt{D}L^{\mathrm{T}}\text{)}.$$

(a) How do you know that the pivots are positive, so $\sqrt{D}$ gives real numbers?

(b) From $A^{\mathrm{T}}A = R^{\mathrm{T}}R$ show that the matrix $Q = AR^{-1}$ has orthonormal columns (what is the test?). Then we have $A = QR$.

(c) Apply Gram-Schmidt to these vectors a_1 and a_2, producing q_1 and q_2. Write your result as QR:

$$a_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad a_2 = \begin{bmatrix} \sin\theta \\ 0 \end{bmatrix}.$$

4 The Fibonacci numbers $F_0, F_1, F_2, F_3, F_4, \ldots$ are $0, 1, 1, 2, 3, \ldots$ and they obey the rule $F_{k+2} = F_{k+1} + F_k$. In matrix form this is

$$\begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \quad \text{or} \quad u_{k+1} = Au_k.$$

The eigenvalues of this particular matrix A will be called a and b.

(a) What quadratic equation connected with A has the solutions (the roots) a and b?

(b) Find a matrix that has the eigenvalues a^2 and b^2. What quadratic equation has the solutions a^2 and b^2?

(c) If you directly compute A^4 you get

$$A^4 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}.$$

Make a guess at the entries of A^k, involving Fibonacci numbers. Then multiply by A to show why your guess is correct. What is the determinant of A^k (not a hard question!)?

5 Suppose A is 3 by 4 and its reduced row echelon form is R:

$$R = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) The four subspaces associated with the original A are $N(A)$, $C(A)$, $N(A^{\mathrm{T}})$, and $C(A^{\mathrm{T}})$. Give the dimension of each subspace and if possible give a basis.

(b) Find the complete solution (when is there a solution?) to the equations

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

(c) Find a matrix A with no zero entries (if possible) whose reduced row echelon form is this same R.

6 Suppose A is a 3 by 3 matrix and you know the three outputs $y_1 = Ax_1$ and $y_2 = Ax_2$ and $y_3 = Ax_3$ from three independent input vectors x_1, x_2, x_3.

(a) Find the matrix A using this hint: Put the vectors x_1, x_2, x_3 into the columns of a matrix X and multiply AX. Why did I require the x's to be independent?

(b) Under what condition on A will the outputs y_1, y_2, y_3 be a basis for R^3? Explain your answer.

(c) If x_1, x_2, x_3 is the input basis and y_1, y_2, y_3 is the output basis, what is the matrix M that represents this same linear transformation (defined by $T(x_1) = y_1,\ \ T(x_2) = y_2,\ \ T(x_3) = y_3$)?

7 (a) Find the eigenvalues of the antidiagonal matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(b) Find as many eigenvectors as possible, with the best possible properties. Are there 4 <u>independent</u> eigenvectors? Are there 4 <u>orthonormal</u> eigenvectors?

(c) What is the rank of $A + 2I$? What is the determinant of $A + 2I$?

8 (a) If $U\Sigma V^{\mathrm{T}}$ is the singular value decomposition of A (m by n) give a formula for the best least squares solution $\bar{x}$ to $Ax = b$. (Simplify your formula as much as possible).

(b) Write down the equations for the straight line $b = C + Dt$ to go through all four of the points (t_1, b_1), (t_2, b_2), (t_3, b_3), (t_4, b_4). Those four points lie on a line provided the vector $b = (b_1, b_2, b_3, b_4)$ lies in ____ .

(c) Suppose S is the subspace spanned by the columns of some m by n matrix A. Give the formula for the projection matrix P that projects each vector in R^m onto the subspace S. <u>Explain where this formula comes from and any condition on A for it to be correct</u>.

(d) Suppose x and y are both in the row space of a matrix A, and $Ax = Ay$. Show that $x - y$ is in the nullspace of A. Then prove that $x = y$.

18.06 Strang, Edelman, Huhtanen Final December 20, 2001

Your name is: ______________________________

Please circle your recitation:

Recitations

	#	Time	Room	Instructor	Office	Phone	Email @math
Lect.	1	MWF 12	4-270	M Huhtanen	2-335	3-7905	huhtanen
Lect.	2	MWF 1	4-370	A Edelman	2-380	3-7770	edelman
Rec.	1	M 2	2-131	D. Sheppard	2-342	3-7578	sheppard
	2	M 2	2-132	M. Huhtanen	2-335	3-7905	huhtanen
	3	M 3	2-131	D. Sheppard	2-342	3-7578	sheppard
	4	T 10	2-132	A. Lachowska	2-180	3-4350	anechka
	5	T 10	2-131	S. Kleiman	2-278	3-4996	kleiman
	6	T 11	2-131	M. Honsen	2-490	3-4094	honsen
	7	T 11	2-132	A. Lachowska	2-180	3-4350	anechka
	8	T 12	2-131	M. Honsen	2-490	3-4094	honsen
	9	T 1	2-132	A. Lachowska	2-180	3-4350	anechka
	10	T 1	2-131	S. Kleiman	2-278	3-4996	kleiman
	11	T 2	2-132	F. Latour	2-090	3-6293	flatour

For full credit, carefully explain your reasoning, as always!

1 (36 pts.) Let

$$a = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \; c = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ and } d = \begin{bmatrix} -7 \\ 2 \\ 2 \end{bmatrix}.$$

(a) Give d as a linear combination of a, b and c.

(b) By using Gram-Schmidt, orthogonalize a, b and c to get ortohonormal vectors q_1, q_2 and q_3.

(c) Give the change of basis matrix from the basis a, b and c to the basis q_1, q_2 and q_3.

2 (36 pts.) Let A be the 3-by-4 matrix defined by

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 + x_3 + x_4 \\ x_1 + 2x_2 - x_4 \\ x_1 + x_2 + 3x_3 - 3x_4 \end{bmatrix}.$$

(a) Give A explicitly.

(b) Find the nullspace matrix of A^T.

(c) What is the dimension of the column space of A?

3 (36 pts.) (a) Let

$$A = \begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix}.$$

Find all diagonal matrices D for which DAD^{-1} is a Jordan matrix.

(b) Find all pairs $a, b \in \mathbf{R}$ such that

$$A = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ a & b \end{bmatrix}$$

is an orthogonal matrix.

(c) Find all orthogonal upper-triangular matrices of size 3-by-3.

4 (32 pts.) (a) Diagonalize the 3-by-3 symmetric matrix A that corresponds to the quadratic form

$$f(x, y, z) = 4xy + 3y^2 + 4z^2.$$

(In your diagonalization $A = S\Lambda S^{-1}$, choose S to be orthogonal.)

(b) Replace the $(1,1)$-entry of A with $a > 0$. For which values of a do you get a positive definite matrix?

5 (32 pts.) Let

$$A = QR = \begin{bmatrix} 1/5 & -2/5 & -4/5 \\ 2/5 & 1/5 & 2/5 \\ 2/5 & -4/5 & 2/5 \\ 4/5 & 2/5 & -1/5 \end{bmatrix} \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & a \end{bmatrix}$$

(Q has orthonormal columns).

(a) Give an orthonormal basis of $C(A)$.

(b) For which values of a the rank of A is 2?

(c) Solve $A^T x = \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}$.

(d) Let $a = 2$. Solve $Ay = b$ in the least squares sense for

$$b = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}.$$

6 (36 pts.) Let A be a symmetric matrix of size 3-by-3 with the 3rd column $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(a) Assume A has the cofactors $C_{1,3} = 0$ and $C_{23} = 1$ and

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

Give A explicitely. Is A invertible?

(b) Compute the SVD of A.

(c) With the help of the SVD, give an orthonormal basis of the column space of A.

18.06 Final Exam, Spring, 2001

Name ________________________ Optional Code ____________________

Recitation Instructor ____________________ Email Address ____________________

Recitation Time ____________________

This final exam is closed book and closed notes. No calculators, laptops, cell phones or other electronic devices may be used during the exam.

There are 6 problems.

Additional paper for your calculations is provided at the back of this booklet.

Good luck.

Problem	Maximum Points	Your Points
1.	15	
2.	15	
3.	15	
4.	20	
5.	20	
6.	15	
Total	100	

1. (15pts.)

(a) Show that the system S:

$$\begin{array}{rcrcrcr} x & + & y & & & = & 3 \\ x & + & y & + & bz & = & 2 \\ ax & + & by & + & (b-a)z & = & 1+3a \end{array}$$

has no solutions if $b = 0$ or $a = b$.

(b) Calculate the solution for $a = 2$, $b = 1$.

(c) Find a general formula for the solution of the system S for $b \neq 0$ and $a \neq b$.

Additional paper for your calculations at the back of this booklet.

2. (15pts.)

(a) Decide whether or not the following vectors form a basis for $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}.$$

(b) Find an orthonormal basis for $\mathrm{Sp}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

Additional paper for your calculations at the back of this booklet.

3. (15pts.) Let

$$A_n = \begin{pmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 \\ 1 & a_2 & -1 & 0 & \cdots & 0 \\ 0 & 1 & a_3 & -1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_{n-1} & -1 \\ 0 & 0 & \cdots & 0 & 1 & a_n \end{pmatrix}.$$

(a) Show for $n \geq 3$ that $\det A_n = a_n \det A_{n-1} + \det A_{n-2}$.

(b) Calculate $\det A_6$ for the cases that (i) $a_j = j$ for all $j = 1, \ldots, 6$, and (ii) $a_j = 6 - j$, for all $j = 1, \ldots, 6$.

Additional paper for your calculations at the back of this booklet.

4. (20pts.) Let U and V be vector spaces.

(a) Define the *kernel* and *image* of a linear transformation $T : U \to V$.

(b) Show that the kernel of T is a subspace of U.

(c) Let T be a linear transformation from U to V and let $\mathbf{u}_1, \dots, \mathbf{u}_k$ form a basis of Ker T. The following steps help you to show that if $\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n$ form a basis of U, then $T\mathbf{u}_{k+1}, \dots, T\mathbf{u}_n$ form a basis of Im T. So assume that $\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1} \dots, \mathbf{u}_n$ are a basis of U.

i. Argue that $T\mathbf{u}_{k+1}, \dots, T\mathbf{u}_n$ are elements of Im T.

ii. Show that any element of Im T can be expressed as a linear combination of $T\mathbf{u}_{k+1}, \dots, T\mathbf{u}_n$.

iii. Show that $T\mathbf{u}_{k+1}, \dots, T\mathbf{u}_n$ are linearly independent.

(d) Deduce a formula relating the dimensions of U, Ker T and Im T.

Additional paper for your calculations at the back of this booklet.

5. (20pts.) Let V be the vector space of polynomials of degree at most 3 with real coefficients. Let T be the map defined by

$$T(f(x)) = \frac{d^2 f}{dx^2} + 2\frac{df}{dx}$$

for all $f(x) \in V$.

(a) Show that T is a linear transformation.

(b) Find the matrices ${}_B(T)_B$ and ${}_C(T)_C$ representing T with respect to the the bases $B = \{1, x, x^2, x^3\}$ and $C = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$, respectively.

(c) Find the matrix ${}_C(I)_B$ representing the change of basis from B to C, and verify that ${}_C(T)_C = {}_C(I)_{B\,B}(T)_{B\,B}(I)_C$.

Additional paper for your calculations at the back of this booklet.

6. (15pts.) Let

$$A = \begin{pmatrix} -3 & 2 & 4 \\ 2 & -6 & 2 \\ 4 & 2 & -3 \end{pmatrix}.$$

(a) Given that one eigenvalue of A is $\lambda_1 = 2$, find the remaining two eigenvalues of A and an eigenvector for each eigenvalue.

(b) Find an orthogonal matrix P such that $P^T AP$ is diagonal.

(c) Find an expression for $\exp(A)$.

Additional paper for your calculations at the back of this booklet.

Your calculations for problem ____.

Your calculations for problem ____.

Your calculations for problem ____.

Your calculations for problem ____.

18.06 Final Exam (Conflict Exam), Spring, 2001

Name ______________________ Optional Code ______________

Recitation Instructor ______________ Email Address ______________

Recitation Time ________________

This final exam is closed book and closed notes. No calculators, laptops, cell phones or other electronic devices may be used during the exam.

There are 6 problems.

Additional paper for your calculations is provided at the back of this booklet.

Good luck.

Problem	Maximum Points	Your Points
1.	15	
2.	15	
3.	15	
4.	20	
5.	20	
6.	15	
Total	100	

1. (15pts.) For which values of a and b does the system of equations

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & ax_3 & + & 2x_4 & = & 1 \\ x_1 & + & & + & 3x_3 & + & 4x_4 & = & b \\ 2x_1 & + & x_2 & + & (a+b)x_3 & + & 7x_4 & = & 2 \end{array}$$

have **no** solutions? Find all solutions in the case that $a = 7$ and $b = 1$.

Additional paper for your calculations at the back of this booklet.

2. (15pts.) Let A_n be the $n \times n$ matrix

$$A_n = \begin{pmatrix} 1 & 2 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 2 & \cdots & 0 & 0 \\ 0 & 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & 0 & \cdots & 2 & 1 \end{pmatrix}.$$

Prove that for $n \geq 3$, $\det(A_n) = \det(A_{n-1}) - 4\det(A_{n-2})$, and evaluate $\det(A_5)$.

Additional paper for your calculations at the back of this booklet.

3. (15pts.) The following are some quick questions. Give only brief reasoning for your answers, no detailed proofs.

(a) Let A, B, C and D be four 3×3 matrices. Let E be the 6×6 matrix

$$E = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Is it necessarily true that $\det(E) = \det(A) \cdot \det C - \det B \cdot \det D$?

(b) Let A be a 3×4 matrix, and B be a 4×3 matrix. Can you say anything about the determinant of their product, BA? How about AB?

(c) Do similar matrices have the same

i. eigenvalues;

ii. eigenvectors;

iii. rank;

iv. column space;

v. determinant?

(d) Does an $n \times n$ matrix with n distinct eigenvalues have an orthogonal set of eigenvectors?

(e) Is the product of two symmetric matrices symmetric?

Additional paper for your calculations at the back of this booklet.

4. (20pts.) Let V be the vector space of polynomials of degree at most 3 with real coefficients. Let T be the map defined by

$$T(f(x)) = f(x) - (1+x)\frac{df}{dx}$$

for all $f(x) \in V$.

(a) Show that T is a linear transformation.

(b) Find the matrices ${}_B(T)_B$ and ${}_C(T)_C$ representing T with respect to the bases $B = \{1, x, x^2, x^3\}$ and $C = \{1+x, x+x^2, x^2+x^3, x^3\}$.

(c) Find the matrix ${}_C(I)_B$ representing the change of basis from B to C, and verify that ${}_C(T)_C = {}_C(I)_B\,{}_B(T)_B\,{}_B(I)_C$.

(d) Find bases for the kernel and image of T.

Additional paper for your calculations at the back of this booklet.

5. (20pts.) Let U and V be vector spaces.

(a) Define the *kernel* and *image* of a linear transformation $T : U \to V$.

(b) Show that the kernel of T is a subspace of U.

(c) Let T be a linear transformation from U to V and let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ form a basis of Ker T. The following steps help you to show that if $\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n$ form a basis of U, then $T\mathbf{u}_{k+1}, \ldots, T\mathbf{u}_n$ form a basis of Im T. So assume that $\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1} \ldots, \mathbf{u}_n$ are a basis of U.

i. Argue that $T\mathbf{u}_{k+1}, \ldots, T\mathbf{u}_n$ are elements of Im T.

ii. Show that any element of Im T can be expressed as a linear combination of $T\mathbf{u}_{k+1}, \ldots, T\mathbf{u}_n$.

iii. Show that $T\mathbf{u}_{k+1}, \ldots, T\mathbf{u}_n$ are linearly independent.

(d) Deduce a formula relating the dimensions of U, Ker T and Im T.

Additional paper for your calculations at the back of this booklet.

6. (15pts.) Find the singular value decomposition of the 3×2 matrix

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \\ 1 & 2 \end{pmatrix}.$$

Additional paper for your calculations at the back of this booklet.

Your calculations for problem ____.

Your calculations for problem ____.

Your calculations for problem ____.

Your calculations for problem ____.

18.06 Professor Strang/Ingerman Final Exam December 17, 2002

Your name is: ______________________________

Please circle your recitation:

1)	M2	2-131	P.-O. Persson	2-088	2-1194	persson
2)	M2	2-132	I. Pavlovsky	2-487	3-4083	igorvp
3)	M3	2-131	I. Pavlovsky	2-487	3-4083	igorvp
4)	T10	2-132	W. Luo	2-492	3-4093	luowei
5)	T10	2-131	C. Boulet	2-333	3-7826	cilanne
6)	T11	2-131	C. Boulet	2-333	3-7826	cilanne
7)	T11	2-132	X. Wang	2-244	8-8164	xwang
8)	T12	2-132	P. Clifford	2-489	3-4086	peter
9)	T1	2-132	X. Wang	2-244	8-8164	xwang
10)	T1	2-131	P. Clifford	2-489	3-4086	peter
11)	T2	2-132	X. Wang	2-244	8-8164	xwang

The ten questions are worth 10 points each.

Thank you for taking 18.06!

1 The 4 by 6 matrix A has all 2's below the diagonal and elsewhere all 1's:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 \end{bmatrix}$$

(a) By elimination factor A into L (4 by 4) times U (4 by 6).

(b) Find the rank of A and a basis for its nullspace (the special solutions would be good).

2 Suppose you know that the 3 by 4 matrix A has the vector $\boldsymbol{s} = (2, 3, 1, 0)$ as a basis for its nullspace.

(a) What is the *rank* of A and the complete solution to $A\boldsymbol{x} = \mathbf{0}$?

(b) What is the exact row reduced echelon form R of A?

3 The following matrix is a *projection matrix*:

$$P = \frac{1}{21}\begin{bmatrix} 1 & 2 & -4 \\ 2 & 4 & -8 \\ -4 & -8 & 16 \end{bmatrix}.$$

(a) What subspace does P project onto?

(b) What is the *distance* from that subspace to $\boldsymbol{b} = (1, 1, 1)$?

(c) What are the three eigenvalues of P? Is P diagonalizable?

4 **(a)** Suppose the product of A and B is the zero matrix: $AB = 0$. Then the $\underline{\quad\textbf{(1)}\quad}$ space of A contains the $\underline{\quad\textbf{(2)}\quad}$ space of B. Also the $\underline{\quad\textbf{(3)}\quad}$ space of B contains the $\underline{\quad\textbf{(4)}\quad}$ space of A. Those blank words are

(1) ____________ **(2)** ____________ **(3)** ____________ **(4)** ____________

(b) Suppose that matrix A is 5 by 7 with rank r, and B is 7 by 9 of rank s. What are the dimensions of spaces **(1)** and **(2)**? From the fact that space **(1)** contains space **(2)**, what do you learn about $r + s$?

5 Suppose the 4 by 2 matrix Q has orthonormal columns.

(a) Find the least squares solution $\widehat{\boldsymbol{x}}$ to $Q\boldsymbol{x} = \boldsymbol{b}$.

(b) Explain why QQ^{T} is not positive definite.

(c) What are the (nonzero) singular values of Q, and why?

6 Let S be the subspace of $\mathbf{R}^3$ spanned by $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}$.

(a) Find an orthonormal basis $\boldsymbol{q}_1, \boldsymbol{q}_2$ for S by Gram-Schmidt.

(b) Write down the 3 by 3 matrix P which projects vectors perpendicularly onto S.

(c) Show how the properties of P (what are they?) lead to the conclusion that $P\boldsymbol{b}$ is orthogonal to $\boldsymbol{b} - P\boldsymbol{b}$.

7 **(a)** If $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a basis for $\mathbf{R}^3$ then the matrix with those three columns is ____.

(b) If $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ span $\mathbf{R}^3$, give all possible ranks for the matrix with those four columns. ____________.

(c) If $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ form an orthonormal basis for $\mathbf{R}^3$, and T is the transformation that projects every vector $\boldsymbol{v}$ onto the plane of $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$, what is the matrix for T in this basis? Explain.

8 Suppose the n by n matrix A_n has 3's along its main diagonal and 2's along the diagonal below and the $(1, n)$ position:

$$A_4 = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}.$$

Find by cofactors of row 1 or otherwise the determinant of A_4 and then the determinant of A_n for $n > 4$.

9 There are six 3 by 3 permutation matrices P.

(a) What numbers can be the *determinant* of P? What numbers can be *pivots*?

(b) What numbers can be the *trace* of P? What *four numbers* can be eigenvalues of P?

10 Suppose A is a 4 by 4 upper triangular matrix with $1, 2, 3, 4$ on its main diagonal. (You could put all 1's above the diagonal.)

(a) For $A - 3I$, which columns have pivots? Which components of the eigenvector $\boldsymbol{x}_3$ (the special solution in the nullspace) are definitely zero?

(b) Using part (a), show that the eigenvector matrix S is also upper triangular.

Course 18.06, Fall 2002: Final Exam, Solutions

1 (a)

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix}.$$

(b) Four pivots $\Rightarrow$ rank of $A = 4$. The row reduced echolon form is

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

The two special solutions $(0,0,0,-1,1,0)$, $(0,0,0,-1,0,1)$ form a basis for the nullspace.

2 (a) The dimension of the nullspace is 1, so the rank of A is $4-1=3$. The complete solution to $A\boldsymbol{x} = 0$ is $\boldsymbol{x} = c \cdot (2,3,1,0)$ for any constant c.

(b) The row reduced echelon form has 3 pivots and the special solution $\boldsymbol{x} = (2,3,1,0)$:

$$R = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3 (a) The projection matrix P projects onto the column space of P which is the line $c \cdot (1,2,-4)$.

(b) The vector from $\boldsymbol{b}$ to the subspace is

$$\boldsymbol{e} = \boldsymbol{b} - P\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{21} \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 22 \\ 23 \\ 17 \end{bmatrix}$$

and the distance is

$$\|\boldsymbol{e}\| = \frac{1}{21}\sqrt{22^2 + 23^2 + 17^2} = \frac{\sqrt{1302}}{21}.$$

(c) Since P projects onto a line, its three eigenvalues are $0, 0, 1$. Since P is symmetric, it has a full set of (orthogonal) eigenvectors, and is then diagonalizable.

4 (a) When $AB = 0$, every column of B is in the nullspace of A. So the **null** space of A contains the **column** space of B. Also the **left null** space of B contains the **row** space of A.

(b) The dimension of the nullspace of A is $n - r = 7 - r$. The dimension of the column space of B is s. Since the first contains the second, $7 - r \geq s$, or $r + s \leq 7$.

5 (a) The least squares solution $\hat{\boldsymbol{x}}$ to $Q\boldsymbol{x} = \boldsymbol{b}$ is

$$\hat{\boldsymbol{x}} = (Q^{\mathrm{T}}Q)^{-1}Q^{\mathrm{T}}\boldsymbol{b} = (I)^{-1}Q^{\mathrm{T}}\boldsymbol{b} = Q^{\mathrm{T}}\boldsymbol{b}.$$

(b) One approach: Q^{T} is 2 by 4 so there are free variables and many solutions to $Q^{\mathrm{T}}\boldsymbol{x} = 0$. Then $QQ^{\mathrm{T}}\boldsymbol{x} = 0$ and QQ^{T} is singular; not positive definite.

Second approach: The rank of Q is 2, so the rank of QQ^{T} must be ≤ 2. But QQ^{T} is a 4 by 4 matrix, so the dimension of its nullspace is 2. This means that it has 2 zero eigenvalues, and QQ^{T} is not positive definite.

(c) The singular values of Q are the square roots of the eigenvalues of $Q^{\mathrm{T}}Q = I$, that is, all 1.

6 (a) Let $\boldsymbol{a} = (1, 2, 2)$ and $\boldsymbol{b} = (5, 4, -2)$. The orthonormal vectors are

$$\boldsymbol{q}_1 = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix},$$

$$\boldsymbol{q}_2 = \frac{\boldsymbol{b} - \frac{\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}}\boldsymbol{q}_1}{\|\cdot\|} = \frac{1}{\sqrt{4^2 + 2^2 + (-4)^2}}\begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

where $\|\cdot\|$ means the norm of the numerator.

(b) Let $Q = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix}$. The projection P is then

$$P = QQ^{\mathrm{T}} = \frac{1}{9}\begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & 2 \\ -2 & 2 & 8 \end{bmatrix}$$

(c) The properties of a projection matrix are $P^2 = P$ and $P^{\mathrm{T}} = P$. This gives

$$(P\boldsymbol{b})^{\mathrm{T}}(\boldsymbol{b} - P\boldsymbol{b}) = \boldsymbol{b}^{\mathrm{T}}P^{\mathrm{T}}(\boldsymbol{b} - P\boldsymbol{b}) = \boldsymbol{b}^{\mathrm{T}}(P\boldsymbol{b} - P^2\boldsymbol{b}) = 0.$$

7 (a) If $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ is a basis for $\mathbf{R}^3$ then the matrix with those three columns is **invertible** (non-singular, full rank).

(b) If $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ span $\mathbf{R}^3$ then the column space is $\mathbf{R}^3$. The only possible rank for the matrix with those four columns is 3.

(c) The transformations of the three basis vectors are

$$\begin{aligned} T(\boldsymbol{q}_1) &= \boldsymbol{q}_1 \\ T(\boldsymbol{q}_2) &= \boldsymbol{q}_2 \\ T(\boldsymbol{q}_3) &= \mathbf{0} \end{aligned}$$

so the transformation matrix T in the basis $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ is

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

8

$$\begin{vmatrix} 3 & 0 & 0 & 2 \\ 2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{vmatrix} = 3\begin{vmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{vmatrix} = 3 \cdot 27 - 2 \cdot 8 = 65.$$

For general $n > 4$, the determinant is $|A_n| = 3^n + (-1)^{n-1}2^n$.

9 (a) The determinant of a permutation matrix P is 1 or -1. The only possible pivot is 1.

(b) For 3 by 3 permutations, the trace of P is 0, 1, or 3. The eigenvalues are 1 and -1 when two rows are exchanged. Otherwise

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 1 = 0 \Rightarrow \lambda = 1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \quad (\text{or } e^{2\pi i/3}, e^{4\pi i/3}),$$

so the four possible eigenvalues are $1, -1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, and $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

10 (a)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \qquad A - 3I = \begin{bmatrix} -2 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Columns 1, 2, and 4 have pivots. Because of the nonzero bottom-right element in $A - 3I$, the fourth component of $\boldsymbol{x}_3$ is definitely zero.

(b) In the same way as above, the special solutions for the matrices $A - 1I$, $A - 2I$, $A - 3I$, and $A - 4I$ must have 3, 2, 1, and 0 zeros as the last components. The eigenvector matrix S is then upper triangular.

Questions from 18.06 Final, Fall 2003

1. Suppose $A = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix}, U = \begin{bmatrix} 5 & 0 & 5 & 1 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) What are the dimensions of the 4 fundamental subspaces associated with A?

(b) Give a basis for each of the 4 fundamental subspaces.

$N(A)$

$R(A)$

$C(A)$

$N(A^T)$

2. Let F be the subspace of R^4 given by

$$F = \{(x, y, z, w) : x - y + 2z + 3w = 0\}.$$

Let P be the projection matrix for projecting onto F. (Many of the subquestions can be answered independently of the others.)

(a) Give an orthonormal basis $\{v_1, \cdots, v_k\}$ for the orthogonal complement to F.

(b) Find an orthonormal basis $\{w_1, \cdots, w_l\}$ for F. Explain how you proceed.

The following questions refer to the projection matrix P for projecting onto F.

(c) What are the eigenvalues of P? Give them with their multiplicities.

(d) What is the projection of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ onto F?

3. (a) Write down the 2×2 rotation matrix, $R(\theta)$, that rotates R^2 in the counterclockwise direction by an angle θ (this matrix is a function of θ).

(b) Compute the eigenvalues of $R(\theta)$. For which value(s) of θ are the eigenvalues real?

(c) What are the eigenvectors of $R(\theta)$.

(d) Write down the singular value decomposition of $R(\theta)$.

4. (a) Give two 3×3 matrices A and B such that AB is not equal to BA.

(b) Suppose A and B are $n \times n$ matrices with the same set of linearly independent eigenvectors $v_1, v_2, \cdots, v_n$. However, the eigenvalues might be different: v_i is the eigenvector for the eigenvalue λ_i of A and the eigenvector for the eigenvalue μ_i of B. Show that $AB = BA$.

5. Consider the differential equation $\begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$.

(a) Solve the differential equation and express $u(t), v(t)$ as functions of $u(0)$ and $v(0)$.

(b) Find a linear transformation $\begin{bmatrix} p \\ q \end{bmatrix} = T \begin{bmatrix} u \\ v \end{bmatrix}$ such that the differential equation simplifies into two independent differential equations in p and in q (one relating $\frac{dp}{dt}$ and p, the other relating $\frac{dq}{dt}$ and q)

(c) Are there initial conditions $u(0)$, $v(0)$ that would make $u(t)$ blow up? If yes, give one such value for $u(0)$ and $v(0)$.

(d) Are there initial conditions $u(0)$, $v(0)$ that would make $u(t)$ go to 0? If yes, give one such value for $u(0)$ and $u'(0)$.

6. True of False. **Circle** the appropriate answer. "True" means "always true", and "false" means "sometimes false". Justify each answer briefly.

   (a) The product of the pivots when performing Gauss-Jordan is equal to the determinant if we do not have to permute rows.
   True or False.

   (b) Let A be an $m \times n$ matrix whose columns are independent. Then AA^T is positive definite.
   True or False.

   (c) If A is a symmetric matrix then the singular values are the absolute values of the nonzero eigenvalues.
   True or False.

   (d) There exists a 5×5 unitary matrix with eigenvalues 1, $1+i$, $1-i$, i and $-i$.
   True or False.

   (e) Suppose V and W are two vector spaces of dimension n. If T is a linear transformation from V to W with only the 0 vector in the kernel, then for any basis of V, there exists an orthonormal basis of W such that the resulting matrix representing T is upper triangular.
   True or False.

7. Consider the following matrix A:

$$A = \begin{bmatrix} 0.5 & 0.4 & 0.2 \\ 0.4 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.6 \end{bmatrix}.$$

(c) Can you immediately tell one of the eigenvalues of A (without computing them)? Explain.

(d) Compute the determinant of A.

(e) Find the eigenvalues of A and the corresponding eigenvectors. (Check your answer.)

(f) Two out of the 3 eigenvectors of A should be orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. How could you have explained this before computing the eigenvectors?

(g) Write an exact expression for A^{100}.

8. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

For each of the following matrices, either complete it (find values for the non-diagonal elements) so that it becomes similar to A, or explain why it is impossible to complete it to a matrix similar to A. Circle whether you are able to complete it or not to a matrix similar to A.

(a)

$$B = \begin{bmatrix} 2 & . & . \\ . & 2 & . \\ . & . & 4 \end{bmatrix}$$

Able to complete it to similar?: Yes No

If yes, give a completion. If not, why not?

(b)

$$C = \begin{bmatrix} 3 & . & . \\ . & 3 & . \\ . & . & 3 \end{bmatrix}.$$

Able to complete it to similar?: Yes No

If yes, give a completion. If not, why not?

18.06 Professor Strang Final Exam December 19, 2005

Your PRINTED name is: ____________________

Grading

1

2

3

4

5

6

7

8

Please circle your recitation:

1)	M 2	2-131	P. Lee	2-087	2-1193	`lee`
2)	M 2	2-132	T. Lawson	4-182	8-6895	`tlawson`
4)	T 10	2-132	P-O. Persson	2-363A	3-4989	`persson`
5)	T 11	2-131	P-O. Persson	2-363A	3-4989	`persson`
6)	T 11	2-132	P. Pylyavskyy	2-333	3-7826	`pasha`
7)	T 12	2-132	T. Lawson	4-182	8-6895	`tlawson`
8)	T 12	2-131	P. Pylyavskyy	2-333	3-7826	`pasha`
9)	T 1	2-132	A. Chan	2-588	3-4110	`alicec`
10)	T 1	2-131	D. Chebikin	2-333	3-7826	`chebikin`
11)	T 2	2-132	A. Chan	2-588	3-4110	`alicec`
12)	T 3	2-132	T. Lawson	4-182	8-6895	`tlawson`

1 (12 pts.) This question is about the matrix $A = I + E$ where E is the all-ones matrix $\texttt{ones}(4,4)$:

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} = I + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

(a) By elimination find the pivots of A.

(b) Factor A into LDL^{T} (if that is possible).

(c) The inverse matrix has the form $A^{-1} = I + cE$. Figure out E^2 and then choose the number c so that $AA^{-1} = I$.

2 (12 pts.) Keep the same matrix A as in Problem 1.

(a) Find the matrix P that projects any vector in $\mathbf{R}^4$ onto the subspace spanned by the first column of A.

(b) Describe the nullspace of $I - P$ and the nullspace of PA.

(c) Find all the eigenvalues of P.

3 (12 pts.) Now suppose $A = I + bE$, with the same $E = \texttt{ones}(4,4)$.

(a) What are the eigenvalues of E?

(b) If $b = 2$, what is the determinant of A?

(c) Suppose you know that $x^{\mathrm{T}}Ax > 0$ for every nonzero vector x. (Same matrix A.) What are the possible values of b?

4 (16 pts.) Suppose A is an 8 by 8 invertible matrix. Throw away any 3 columns of A to get an 8 by 5 matrix B.

(a) You will correctly think that B has rank 5. Give a mathematical reason why this is true.

(b) Tell all you know about the nullspace of B^{T} and the reduced row echelon form `rref`(B).

(c) Give as much information as possible about the eigenvalues and eigenvectors of $B^{\mathrm{T}}B$ and BB^{T} (those are separate questions).

5 (12 pts.) Suppose Q is an m by n matrix with $Q^{\mathrm{T}}Q = I$. Write down the most important facts about

(a) The columns of Q

(b) m and n and the rank of Q

(c) The least squares solution $\widehat{x}$ to $Qx = b$

6 (12 pts.) (a) The eigenvalues of $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ are _________________________.

(b) An orthogonal set of 4 eigenvectors is ______________________________.

(c) CIRCLE every class of matrices to which this matrix A belongs:

diagonalizable	permutation	nonsingular
Jordan matrix	orthogonal	projection
skew-symmetric		

7 (12 pts.) Suppose A is 2 by 3 with this Singular Value Decomposition $U\Sigma V^{\mathrm{T}}$. U and V are orthogonal matrices:

$$A = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^{\mathrm{T}} \\ v_2^{\mathrm{T}} \\ v_3^{\mathrm{T}} \end{bmatrix}.$$

(a) Find a basis for the nullspace of A.

(b) Find all solutions to the equation $Ax = u_1$.

(c) Find the shortest solution to $Ax = u_1$ (minimum length vector) and prove that it is shortest.

8 (12 pts.) Suppose A (3 by 3) has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and independent eigenvectors x_1, x_2, x_3.

(a) What is the general form of the solutions to $u_{k+1} = Au_k$ and $\dfrac{du}{dt} = Au$? (Two questions)

(b) Suppose every solution to $u_{k+1} = Au_k$ approaches a multiple $c\,x_1$ as $k \to \infty$ (c depends on u_0). What does this tell you about $\lambda_1, \lambda_2, \lambda_3$?

(c) For some 3 by 3 matrices, the complete solution to $\dfrac{du}{dt} = Au$ does *not* have the form you gave in part (a). What can go wrong? Give an example of such a matrix A.

1.(a) $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$

(b)$A = \begin{pmatrix} 1 & & & \\ \frac{1}{2} & 1 & & \\ \frac{1}{2} & \frac{1}{3} & 1 & \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} 2 & & & \\ & \frac{3}{2} & & \\ & & \frac{4}{3} & \\ & & & \frac{5}{4} \end{pmatrix} \begin{pmatrix} 1 & & & \\ \frac{1}{2} & 1 & & \\ \frac{1}{2} & \frac{1}{3} & 1 & \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 1 \end{pmatrix}^T$

(c) $c = -1/5$.since $E^2 = 4E$ and $AA^{-1} = (I+E)(I+cE) = I+(c+1+4c)E$ so $5c+1=0$.

2.(a) $P = \frac{aa^T}{a^Ta} = \frac{1}{7}\begin{pmatrix} 4 & 2 & 2 & 2 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix}$, Trick: computation easiest using $a = (2,1,1,1)^T$.

(b) The nullspace of $I-P$ consists of all multiples of a. (One view is that $x = Px$. Another view is that it is the orthogonal complement of the nullspace of P which are all vectors orthogonal to a.)

(c) The matrix has rank 1 and is a projector, so one eigenvalue is 1 and the rest are 0.

3.(a) E is rank one symmetric with trace 4, so the eigenvalues are $0, 0, 0, 4$.

(b) The eigenvalues of A are $1, 1, 1$, and $1 + 2 \cdot 4 = 9$ so the determinant is 9.

(c) We need $1 + 4b > 0$ so $b > -1/4$.

4.(a) The columns of B are independent since the columns of A are. Therefore the span of B is five dimensional. (B has no free columns.)

(b) The nullspace of B^T is a three dimensional subspace of R^8.It is the orthogonal complement of the column space of B in R^8. The rref of B looks like the first five columns of $I_{8.}$

(c) B^TB is a 5×5 matrix with positive eigenvalues that are the squares of the singular values of B. BB^T has the same five positive eigenvalues and three more 0 eigenvalues as well.

5.(a) The columns of Q are n orthonormal vectors in R^m.

(b) $n \leq m$ and the rank of Q is n.

(c) $\hat{x} = Q^T b$.

6.(a) $1, i, -1, -i$

(b) The four columns of the DFT matrix $F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & 1 & -i \end{pmatrix}$

(c) diagonalizable-yes, permutation-yes, nonsingular-yes, Jordan-no, orthogonal-yes, projection-no, skew-symmetric-no

7.(a) The nullspace of A has basis v_2 and v_3.

(b) $x = v_1/4 + c_1 v_2 + c_2 v_3$

(c) Shortest is $v_1/4$. It is the projection of x above onto the span of v_1.

8.(a) $u_k = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + c_3 \lambda_3^k x_3$ and $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + c_3 e^{\lambda_3 t} x_3$

(b)$\lambda_1 = 1$ and $|\lambda_i| < 1$, for $i = 2, 3$.

(c) The matrix may not have a complete set of independent eigenveectors. An example is a three by three Jordan block: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

18.06 Professor Strang Final Exam May 16, 2005

Your PRINTED name is: ____________________

Closed book / 10 wonderful problems.
Thank you for taking this course!

Please circle your recitation:

1)	M 2	2-131	A. Chan	2-588	3-4110	`alicec`
2)	M 3	2-131	A. Chan	2-588	3-4110	`alicec`
3)	M 3	2-132	D. Testa	2-586	3-4102	`damiano`
4)	T 10	2-132	C.I. Kim	2-273	3-4380	`ikim`
5)	T 11	2-132	C.I. Kim	2-273	3-4380	`ikim`
6)	T 12	2-132	W.L. Gan	2-101	3-3299	`wlgan`
7)	T 1	2-131	C.I. Kim	2-273	3-4380	`ikim`
8)	T 1	2-132	W.L. Gan	2-101	3-3299	`wlgan`
9)	T 2	2-132	W.L. Gan	2-101	3-3299	`wlgan`

Grading

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1 (10 pts.) Suppose $P_1, \ldots, P_n$ are points in $\mathbf{R}^n$. The coordinates of P_i are $(a_{i1}, a_{i2}, \ldots, a_{in})$. We want to find a hyperplane $c_1x_1 + \cdots + c_nx_n = 1$ that contains all n points P_i.

(a) What system of equations would you solve to find the c's for that hyperplane?

(b) Give an example in $\mathbf{R}^3$ where no such hyperplane exists (of this form), and an example which allows more than one hyperplane of this form.

(c) Under what conditions on the points or their coordinates is there **not** a unique interpolating hyperplane *with this equation*?

2 (10 pts.) (a) Find a complete set of "special solutions" to $Ax = 0$ by noticing the pivot variables and free variables (those have values 1 or 0).

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) and (c) Prove that those special solutions are a basis for the nullspace $\boldsymbol{N}(A)$. What two facts do you have to prove?? Those are parts (b) and (c) of this problem.

3 (10 pts.) (a) I was looking for an m by n matrix A and vectors b, c such that $Ax = b$ has no solution and $A^{\mathrm{T}}y = c$ has exactly one solution. *Why can I not find* A, b, c?

(b) In $\mathbf{R}^m$, suppose I gave you a vector b and a vector p and n linearly independent vectors $a_1, a_2, \ldots, a_n$. If I claim that p is the projection of b onto the subspace spanned by the a's, what tests would you make to see if this is true?

4 (10 pts.) (a) Find the determinant of

$$B = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

(b) Let A be the 5 by 5 matrix

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}.$$

Find all five eigenvalues of A by noticing that $A - I$ has rank 1 and the trace of A is ______.

(c) ***Find the $(1, 3)$ and $(3, 1)$ entries of A^{-1}.***

5 (10 pts.) (a) Complete the matrix A (fill in the two blank entries) so that A has eigenvectors $x_1 = (3, 1)$ and $x_2 = (2, 1)$:

$$A = \begin{bmatrix} 2 & 6 \\ & \end{bmatrix}$$

(b) Find a different matrix B with those same eigenvectors x_1 and x_2, and with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0$. **What is $\boldsymbol{B^{10}}$?**

6 (10 pts.) We can find the four coefficients of a polynomial $P(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3$ if we know the values y_1, y_2, y_3, y_4 of $P(z)$ at the four points $z = 1, i, i^2, i^3$.

(a) What equations would you solve to find c_0, c_1, c_2, c_3 ?

(b) *Write down one special property of the coefficient matrix.*

(c) Prove that the matrix in those equations is invertible.

7 (10 pts.) Suppose $\mathbf{S}$ is a 4-dimensional subspace of $\mathbf{R}^7$, and P is the projection matrix onto $\mathbf{S}$.

(a) What are the seven eigenvalues of P?

(b) What are all the eigenvectors of P?

(c) If you solve $\frac{du}{dt} = -Pu$ (notice minus sign) starting from $u(0)$, the solution $u(t)$ approaches a steady state as $t \to \infty$. Can you describe that limit vector $u(\infty)$?

8 (10 pts.) Suppose my favorite $-1, 2, -1$ matrix swallowed extra zeros to become

$$A = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}.$$

(a) Find a permutation matrix P so that

$$B = PAP^{\mathrm{T}} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

(b) What are the 4 eigenvalues of B? Is this matrix diagonalizable or not?

(c) How do you know that A has the same eigenvalues as B? Then A is positive definite—what function of u, v, w, z is therefore positive except when $u = v = w = z = 0$?

9 (10 pts.) (a) Describe all vectors that are orthogonal to the nullspace of this singular matrix A. You can do this without computing the nullspace.

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 2 & 6 \\ 2 & 1 & 4 \end{bmatrix}.$$

(b) If you apply Gram-Schmidt to the columns of this A, what orthonormal vectors do you get?

(c) Find a "reduced" LU factorization of A, with only **2** columns in L and **2** rows in U. Can you write A as the sum of two rank 1 matrices?

10 (10 pts.) Suppose the singular value decomposition $A = U\Sigma V^{\mathrm{T}}$ has

$$U = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad V = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of $A^{\mathrm{T}}A$.

(b) Find a basis for the nullspace of A.

(c) Find a basis for the column space of A.

(d) Find a singular value decomposition of $-A^{\mathrm{T}}$.

18.06 - Final Exam, Monday May 16th, 2005

SOLUTIONS

1. (a) We want the coordinates $(a_{i1}, \ldots, a_{in})$ of P_i to satisfy the equation $c_1x_1 + \ldots + c_nx_n = 1$. Thus the system of equations is $Ac = \texttt{ones}$:

$$\begin{aligned} c_1a_{11} + c_2a_{12} + \ldots + c_na_{1n} &= 1 \\ c_1a_{21} + c_2a_{22} + \ldots + c_na_{2n} &= 1 \\ &\cdots \\ c_1a_{n1} + c_2a_{n2} + \ldots + c_na_{nn} &= 1 \end{aligned}$$

(b) There is no plane of the given form, if one of the points P_i is the origin. More than one plane contains the P_i's if the three points are on a line not through the origin.

(c) There is not a unique solution precisely when $\det A = 0$. This means geometrically that the points P_i lie in an $(n-1)-$dimensional subspace of $\mathbf{R}^n$.

2. (a) Subtracting the first row from the second, we find the matrix

$$U = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(In the row reduced echelon form R, the 5 changes to 0.) The pivot variables are the first and the last, while x_2, x_3, x_4 are the free variables. Thus the "special solutions" to $Ax = 0$ are

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

(b) and (c) We need to prove that these three vectors are *linearly independent* and they *span the nullspace*. By considering the second, third and fourth coordinates, a combination of the vectors adding to zero must have zero coefficients. The vectors span the nullspace, since the dimension of the nullspace is three (note that the rank of the matrix A is 2).

3. (a) If $Ax = b$ has no solution, the column space of A must have dimension less than m. The rank is $r < m$. Since $A^{\mathrm{T}}y = c$ has exactly one solution, the columns of A^{T} are independent. This means that the rank of A^{T} is $r = m$. This contradiction proves that we cannot find A, b and c.

(b) We need to check two statements: the vector $b - p$ is orthogonal to the space generated by $a_1, \ldots, a_n$ and the vector p lies in that subspace. The first condition we check by seeing if the scalar products $a_1 \cdot (b-p), \ldots, a_n \cdot (b-p)$ all equal zero. The second condition we check by considering the $(n+1) \times m$ matrix whose first n rows are the coordinates of the a_i's and whose last row consists of the coordinates of p. The vector p is in the span of the a_i's if and only if the last row becomes zero in elimination.

4. (a) To compute the determinant, subtract the second row from all the other rows:

$$\det B = \det \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1\,.$$

(b) $\lambda = 1, 1, 1, 1, 6$. Since $A - I$ has all equal rows, it has rank one. It follows that it has four zero eigenvalues. The eigenvalues of A are the eigenvalues of $A - I$ increased by one, so A has the eigenvalue 1 with multiplicity four. The trace of A equals 10 so $10 - 4 = 6$ is the other eigenvalue.

(c) A is symmetric, and thus so is A^{-1}. The cofactor formula gives:

$$(A^{-1})_{13} = (-1)^{1+3} \frac{\det B}{\det A}\,,$$

and $\det A = 6$ since it equals the product of the eigenvalues of A. We conclude that the $(1,3)$ and the $(3,1)$ entries of A^{-1} are both equal to $-1/6$.

5. (a) The answer is

$$A = \begin{bmatrix} 2 & 6 \\ -1 & 7 \end{bmatrix}.$$

Reason:

$$\begin{bmatrix} 2 & 6 \\ a & b \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 3a+b \end{bmatrix} = \lambda_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

We deduce that $\lambda_1 = 4$ and $3a + b = 4$. Similarly, since x_2 is an eigenvector we have

$$\begin{bmatrix} 2 & 6 \\ a & b \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 2a+b \end{bmatrix} = \lambda_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

We deduce that $\lambda_2 = 5$ and therefore that $2a+b = 5$. We conclude that $\boldsymbol{a = -1}$ **and** $\boldsymbol{b = 7}$**.**

(b) $B = S\Lambda S^{-1}$, where the columns of S are the vectors x_1 and x_2, and Λ is the diagonal matrix with entries 1 and 0:

$$B = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}.$$

Then $\Lambda^{10} = \Lambda$ and therefore $B^{10} = S\Lambda^{10}S^{-1} = S\Lambda S^{-1} = B$.

6. (a) We would solve the equations

$$\begin{aligned} c_0 + c_1 + c_2 + c_3 &= y_1 \\ c_0 + i\,c_1 - c_2 - i\,c_3 &= y_2 \\ c_0 - c_1 + c_2 - c_3 &= y_3 \\ c_0 - i\,c_1 - c_2 + i\,c_3 &= y_4\,. \end{aligned}$$

and the matrix of coefficients is

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1^2 & i^2 & i^4 & i^6 \\ 1^3 & i^3 & i^6 & i^9 \end{bmatrix}.$$

(b) F has *orthogonal columns* and it is *symmetric*. It is also a Vandermonde matrix: each column consists of the first four powers of a number (starting from the zero-th power).

(c) Since the columns of F are orthogonal and non-zero, the matrix is invertible. Its inverse is $\overline{F}/4$. The determinant of this Vandermonde matrix is equal to the product of the differences of $1, i, i^2, i^3$:

$$\det F = (i-1)(-1-1)(-1-i)(-i-1)(-i-i)(-i+1) = -16i\,.$$

7. (a) The seven eigenvalues of P are $1, 1, 1, 1, 0, 0, 0$.

(b) The eigenvectors with eigenvalue 1 are the non-zero vectors in S. The eigenvectors with eigenvalue 0 are the non-zero vectors in the orthogonal complement of S.

(c) The solution $u(t)$ to the differential equation has the form

$$u(t) = v_1 e^{-t} + v_2\,,$$

where v_1 is in S and v_2 is in the orthogonal complement of S. Then $u(\infty) = v_2$, which is the projection of $u(0)$ onto the orthogonal complement of S.

8. (a) The required matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) Since B is block diagonal, its eigenvalues are the eigenvalues of the diagonal blocks. In our case, the two blocks are the same and the eigenvalues of each block are 3 and 1. Thus the eigenvalues of B are $3, 3, 1, 1$.

(c) Since P is a permutation matrix, it is orthogonal and therefore $P^{\mathrm{T}} = P^{-1}$. The matrix B is thus similar to the matrix A and we conclude that A and B have the same eigenvalues.

The function of u, v, w, z which is positive except if $u = v = w = z = 0$ is thus

$$\begin{bmatrix} u & v & w & z \end{bmatrix} A \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = 2\left(u^2 + v^2 + w^2 + z^2 - uw - vz\right).$$

9. (a) The vectors orthogonal to the nullspace of A are the rows of A. Since we know that the matrix A is singular and it is clearly not rank one, it follows that the rank of A is two. The first two rows are independent and therefore the orthogonal complement of the nullspace of A is spanned by the two vectors

$$\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}.$$

(b) We get the vectors

$$\frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(c) The "reduced" LU decomposition, from ignoring the zero row in U, is

Answer $$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 2 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 0 & -4 & -8 \end{bmatrix}.$$

Here are the details: Starting elimination we find

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 2 & 2 & 6 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 7 \\ 0 & -4 & -8 \\ 0 & -5 & -10 \end{bmatrix}, \tag{1}$$

and proceeding further we find

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{5}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 0 & -4 & -8 \\ 0 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 7 \\ 0 & -4 & -8 \\ 0 & 0 & 0 \end{bmatrix}.$$

Collecting all the information together we obtain

$$\begin{bmatrix} 1 & 3 & 7 \\ 2 & 2 & 6 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{5}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 0 & -4 & -8 \\ 0 & 0 & 0 \end{bmatrix},$$

and multiplying the first two matrices on the right-hand side we deduce that

$$\begin{bmatrix} 1 & 3 & 7 \\ 2 & 2 & 6 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & \frac{5}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 0 & -4 & -8 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the last row of the last matrix is all zero, we conclude that

$$\begin{bmatrix} 1 & 3 & 7 \\ 2 & 2 & 6 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 2 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 0 & -4 & -8 \end{bmatrix}.$$

This is the "reduced" LU factorization of A. Multiplying columns of L by rows of U, this is

$$A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \frac{5}{4} \end{bmatrix} \begin{bmatrix} 0 & -4 & -8 \end{bmatrix}.$$

10. (a) The eigenvalues of $A^{\mathrm{T}}A$ are the same as the eigenvalues of $\Sigma^{\mathrm{T}}\Sigma$ which is the 4 by 4 diagonal matrix with entries 1, 16, 0, 0 along the diagonal.

(b) The nullspace $\boldsymbol{N}(A)$ is spanned by the *last two columns* of V.

(c) The column space of A is spanned by the first two columns of U.

(d) A singular value decomposition of $-A^{\mathrm{T}}$ is $-A^{\mathrm{T}} = (-V)\Sigma^{\mathrm{T}}U^{\mathrm{T}}$.

18.06 Professor Strang Final Exam December 18, 2006

Your PRINTED name is: ____________________

Grading

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Please circle your recitation:

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2)	T 10	2-132	A. Barakat	2-172	3-4470	`barakat`
3)	T 11	2-132	A. Barakat	2-172	3-4470	`barakat`
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5)	T 12	2-132	A. Edelman	2-343	3-7770	`edelman`
6)	T 12	2-131	K. Meszaros	2-333	3-7826	`karola`
7)	T 1	2-132	A. Edelman	2-343	3-7770	`edelman`
8)	T 2	2-132	J. Burns	2-333	3-7826	`burns`
9)	T 3	2-132	A. Osorno	2-229	3-1589	`aosorno`

1 (4+7=11 pts.) Suppose A is 3 by 4, and $Ax = 0$ has exactly 2 special solutions:

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(a) Remembering that A is 3 by 4, find its row reduced echelon form R.

(b) Find the dimensions of all four fundamental subspaces $C(A)$, $N(A)$, $C(A^{\mathrm{T}})$, $N(A^{\mathrm{T}})$.

You have enough information to find bases for one or more of these subspaces—find those bases.

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2 (6+3+2=11 pts.) (a) Find the inverse of a 3 by 3 upper triangular matrix U, with **nonzero** entries a, b, c, d, e, f. You could use cofactors and the formula for the inverse. Or possibly Gauss-Jordan elimination.

Find the inverse of $U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$.

(b) Suppose the columns of U are eigenvectors of a matrix A. Show that A is also upper triangular.

(c) Explain why this U **cannot** be the same matrix as the first factor in the Singular Value Decomposition $A = U\Sigma V^{\mathrm{T}}$.

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3 (3+3+5=11 pts.) (a) A and B are any matrices with the same number of rows. What can you say (*and explain why it is true*) about the comparison of

$$\text{rank of } A \qquad\qquad \text{rank of the block matrix } \begin{bmatrix} A & B \end{bmatrix}$$

(b) Suppose $B = A^2$. How do those ranks compare? Explain your reasoning.

(c) If A is m by n of rank r, what are the dimensions of these nullspaces?

$$\text{Nullspace of } A \qquad\qquad \text{Nullspace of } \begin{bmatrix} A & A \end{bmatrix}$$

4 (3+4+5=12 pts.) Suppose A is a 5 by 3 matrix and Ax is never zero (except when x is the zero vector).

(a) What can you say about the columns of A?

(b) Show that $A^{\mathrm{T}}Ax$ is also never zero (except when $x = 0$) by explaining this key step:

If $A^{\mathrm{T}}Ax = 0$ then obviously $x^{\mathrm{T}}A^{\mathrm{T}}Ax = 0$ and then (WHY ?) $Ax = 0$.

(c) We now know that $A^{\mathrm{T}}A$ is invertible. Explain why $B = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ is a one-sided inverse of A (which side of A?). B is NOT a 2-sided inverse of A (*explain why not*).

5 (5+5=10 pts.) If A is 3 by 3 symmetric positive definite, then $Aq_i = \lambda_i q_i$ with positive eigenvalues and orthonormal eigenvectors q_i.

Suppose $x = c_1 q_1 + c_2 q_2 + c_3 q_3$.

(a) Compute $x^{\mathrm{T}} x$ and also $x^{\mathrm{T}} A x$ in terms of the c's and λ's.

(b) Looking at the ratio of $x^{\mathrm{T}} A x$ in part (a) divided by $x^{\mathrm{T}} x$ in part (a), what c's would make that ratio as large as possible? You can assume $\lambda_1 < \lambda_2 < \ldots < \lambda_n$. Conclusion: the ratio $x^{\mathrm{T}} A x / x^{\mathrm{T}} x$ is a maximum when x is ______.

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6 (4+4+4=12 pts.) (a) Find a linear combination w of the linearly independent vectors v and u that is perpendicular to u.

(b) For the 2-column matrix $A = \begin{bmatrix} u & v \end{bmatrix}$, find Q (orthonormal columns) and R (2 by 2 upper triangular) so that $A = QR$.

(c) In terms of Q only, using $A = QR$, find the projection matrix P onto the plane spanned by u and v.

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7 (4+3+4=11 pts.) (a) Find the eigenvalues of

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

(b) Those are both permutation matrices. What are their inverses C^{-1} and $(C^2)^{-1}$?

(c) Find the determinants of C and $C + I$ and $C + 2I$.

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8 (4+3+4=11 pts.) Suppose a rectangular matrix A has independent columns.

(a) How do you find the best least squares solution $\widehat{x}$ to $Ax = b$? By taking those steps, give me a formula (letters not numbers) for $\widehat{x}$ and also for $p = A\widehat{x}$.

(b) The projection p is in which fundamental subspace associated with A? The error vector $e = b - p$ is in which fundamental subspace?

(c) Find by any method the projection matrix P onto the column space of A:

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \\ 0 & -1 \\ 0 & -3 \end{bmatrix}.$$

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9 (3+4+4=11 pts.) This question is about the matrices with 3's on the main diagonal, 2's on the diagonal above, 1's on the diagonal below.

$$A_1 = \begin{bmatrix} 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} \quad A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix} \quad A_n = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & \cdot \\ 0 & 0 & \cdot & \cdot \end{bmatrix}$$

(a) What are the determinants of A_2 and A_3?

(b) The determinant of A_n is D_n. Use cofactors of row 1 and column 1 to find the numbers a and b in the recursive formula for D_n:

$$(*) \qquad D_n = a\,D_{n-1} + b\,D_{n-2}\,.$$

(c) This equation $(*)$ is the same as

$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}.$$

From the eigenvalues of that matrix, how fast do the determinants D_n grow? (If you didn't find a and b, say how you would answer part (c) for any a and b) For 1 point, find D_5.

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18.06 Professor Strang Final Exam December 18, 2006

Your PRINTED name is: <u>SOLUTIONS</u>

Please circle your recitation:

1)	T 10	2-131	K. Meszaros	2-333	3-7826	`karola`
2)	T 10	2-132	A. Barakat	2-172	3-4470	`barakat`
3)	T 11	2-132	A. Barakat	2-172	3-4470	`barakat`
4)	T 11	2-131	A. Osorno	2-229	3-1589	`aosorno`
5)	T 12	2-132	A. Edelman	2-343	3-7770	`edelman`
6)	T 12	2-131	K. Meszaros	2-333	3-7826	`karola`
7)	T 1	2-132	A. Edelman	2-343	3-7770	`edelman`
8)	T 2	2-132	J. Burns	2-333	3-7826	`burns`
9)	T 3	2-132	A. Osorno	2-229	3-1589	`aosorno`

Grading

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1 (4+7=11 pts.) Suppose A is 3 by 4, and $Ax = 0$ has exactly 2 special solutions:

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad x_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(a) Remembering that A is 3 by 4, find its row reduced echelon form R.

(b) Find the dimensions of all four fundamental subspaces $C(A)$, $N(A)$, $C(A^{\mathrm{T}})$, $N(A^{\mathrm{T}})$.

You have enough information to find bases for one or more of these subspaces—find those bases.

Solution.

(a) Each special solution tells us the solution to $Rx = 0$ when we set one free variable $= 1$ and the others $= 0$. Here, the third and fourth variables must be the two free variables, and the other two are the pivots: $R = \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Now multiply out $Rx_1 = 0$ and $Rx_2 = 0$ to find the $*$'s: $R = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(The $*$'s are just the negatives of the special solutions' pivot entries.)

(b) We know the nullspace $N(A)$ has $n - r = 4 - 2 = 2$ dimensions: the special solutions x_1, x_2 form a basis.

The row space $C(A^{\mathrm{T}})$ has $r = 2$ dimensions. It's orthogonal to $N(A)$, so just pick two linearly-independent vectors orthogonal to x_1 and x_2 to form a basis: for example, $x_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, x_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.

(Or: $C(A^{\mathrm{T}}) = C(R^{\mathrm{T}})$ is just the row space of R, so the first two rows are a basis. Same thing!)

The column space $C(A)$ has $r = 2$ dimensions (same as $C(A^{\mathrm{T}})$). We can't write down a basis because we don't know what A is, but we can say that the first two columns of A are a basis.

The left nullspace $N(A^{\mathrm{T}})$ has $m - r = 1$ dimension; it's orthogonal to $C(A)$, so any vector orthogonal to the first two columns of A (whatever they are) will be a basis.

2 (6+3+2=11 pts.) (a) Find the inverse of a 3 by 3 upper triangular matrix U, with **nonzero** entries a, b, c, d, e, f. You could use cofactors and the formula for the inverse. Or possibly Gauss-Jordan elimination.

Find the inverse of $U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$.

(b) Suppose the columns of U are eigenvectors of a matrix A. Show that A is also upper triangular.

(c) Explain why this U **cannot** be the same matrix as the first factor in the Singular Value Decomposition $A = U\Sigma V^{\mathrm{T}}$.

Solution.

(a) *By elimination:* (We keep track of the elimination matrix E on one side, and the product EU on the other. When $EU = I$, then $E = U^{-1}$.)

$$\left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ 0 & d & e & 0 & 1 & 0 \\ 0 & 0 & f & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & b/a & c/a & 1/a & 0 & 0 \\ 0 & 1 & e/d & 0 & 1/d & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/f \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/a & -b/ad & (be-cd)/adf \\ 0 & 1 & 0 & 0 & 1/d & -e/df \\ 0 & 0 & 1 & 0 & 0 & 1/f \end{array}\right] = \begin{bmatrix} I & U^{-1} \end{bmatrix}$$

By cofactors: (Take the minor, then "checkerboard" the signs to get the cofactor matrix, then transpose and divide by $\det(U) = adf$.)

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \rightsquigarrow \begin{bmatrix} df & 0 & 0 \\ bf & af & 0 \\ be-cd & ae & ad \end{bmatrix} \rightsquigarrow \begin{bmatrix} df & 0 & 0 \\ -bf & af & 0 \\ be-cd & -ae & ad \end{bmatrix} \rightsquigarrow \begin{bmatrix} df & -bf & be-cd \\ 0 & af & -ae \\ 0 & 0 & ad \end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix} 1/a & -b/ad & (be-cd)/adf \\ 0 & 1/d & -e/df \\ 0 & 0 & 1/f \end{bmatrix} = U^{-1}$$

(b) We have a complete set of eigenvectors for A, so we can diagonalize: $A = U\Lambda U^{-1}$. We know U is upper-triangular, and so is the diagonal matrix Λ, and we've just shown that U^{-1} is upper-triangular too. So their product A is also upper-triangular.

(c) The columns aren't orthogonal! (For example, the product $u_1^{\mathrm{T}} u_2$ of the first two columns is $ab + 0d + 0 \cdot 0 = ab$, which is nonzero because we're assuming all the entries are nonzero.)

3 (3+3+5=11 pts.) (a) A and B are any matrices with the same number of rows. What can you say (*and explain why it is true*) about the comparison of

$$\text{rank of } A \qquad\qquad \text{rank of the block matrix } \begin{bmatrix} A & B \end{bmatrix}$$

(b) Suppose $B = A^2$. How do those ranks compare? Explain your reasoning.

(c) If A is m by n of rank r, what are the dimensions of these nullspaces?

$$\text{Nullspace of } A \qquad\qquad \text{Nullspace of } \begin{bmatrix} A & A \end{bmatrix}$$

Solution.

(a) All you can say is that rank $A \leq$ rank $[A\,B]$. (A can have any number r of pivot columns, and these will all be pivot columns for $[A\,B]$; but there could be more pivot columns among the columns of B.)

(b) Now rank $A =$ rank $[A\ A^2]$. (Every column of A^2 is a linear combination of columns of A. For instance, if we call A's first column a_1, then Aa_1 is the first column of A^2. So there are no new pivot columns in the A^2 part of $[A\ A^2]$.)

(c) The nullspace $N(A)$ has dimension $n - r$, as always. Since $[A\ A]$ only has r pivot columns — the n columns we added are all duplicates — $[A\ A]$ is an m-by-$2n$ matrix of rank r, and its nullspace $N([A\ A])$ has dimension $2n - r$.

4 (3+4+5=12 pts.) Suppose A is a 5 by 3 matrix and Ax is never zero (except when x is the zero vector).

(a) What can you say about the columns of A?

(b) Show that $A^{\mathrm{T}}Ax$ is also never zero (except when $x = 0$) by explaining this key step:

If $A^{\mathrm{T}}Ax = 0$ then obviously $x^{\mathrm{T}}A^{\mathrm{T}}Ax = 0$ and then (WHY?) $Ax = 0$.

(c) We now know that $A^{\mathrm{T}}A$ is invertible. Explain why $B = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ is a one-sided inverse of A (which side of A?). B is NOT a 2-sided inverse of A (*explain why not*).

Solution.

(a) $N(A) = 0$ so A has full column rank $r = n = 3$: the columns are linearly independent.

(b) $x^{\mathrm{T}}A^{\mathrm{T}}Ax = (Ax)^{\mathrm{T}}Ax$ is the squared length of Ax. The only way it can be zero is if Ax has zero length (meaning $Ax = 0$).

(c) B is a left inverse of A, since $BA = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}A = I$ is the (3-by-3) identity matrix. B is not a right inverse of A, because AB is a 5-by-5 matrix but can only have rank 3. (In fact, $BA = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ is the projection onto the (3-dimensional) column space of A.)

5 (5+5=10 pts.) If A is 3 by 3 symmetric positive definite, then $Aq_i = \lambda_i q_i$ with positive eigenvalues and orthonormal eigenvectors q_i.

Suppose $x = c_1 q_1 + c_2 q_2 + c_3 q_3$.

(a) Compute $x^{\mathrm{T}} x$ and also $x^{\mathrm{T}} A x$ in terms of the c's and λ's.

(b) Looking at the ratio of $x^{\mathrm{T}} A x$ in part (a) divided by $x^{\mathrm{T}} x$ in part (a), what c's would make that ratio as large as possible? You can assume $\lambda_1 < \lambda_2 < \ldots < \lambda_n$. Conclusion: the ratio $x^{\mathrm{T}} A x / x^{\mathrm{T}} x$ is a maximum when x is ______.

Solution.

(a)

$$\begin{aligned}
x^{\mathrm{T}} x &= (c_1 q_1^{\mathrm{T}} + c_2 q_2^{\mathrm{T}} + c_3 q_3^{\mathrm{T}})(c_1 q_1 + c_2 q_2 + c_3 q_3) \\
&= c_1^2 q_1^{\mathrm{T}} q_1 + c_1 c_2 q_1^{\mathrm{T}} q_2 + \cdots + c_3 c_2 q_3^{\mathrm{T}} q_2 + c_3^2 q_3^{\mathrm{T}} q_3 \\
&= c_1^2 + c_2^2 + c_3^2.
\end{aligned}$$

$$\begin{aligned}
x^{\mathrm{T}} A x &= (c_1 q_1^{\mathrm{T}} + c_2 q_2^{\mathrm{T}} + c_3 q_3^{\mathrm{T}})(c_1 A q_1 + c_2 A q_2 + c_3 A q_3) \\
&= (c_1 q_1^{\mathrm{T}} + c_2 q_2^{\mathrm{T}} + c_3 q_3^{\mathrm{T}})(c_1 \lambda_1 q_1 + c_2 \lambda_2 q_2 + c_3 \lambda_3 q_3) \\
&= c_1^2 \lambda_1 q_1^{\mathrm{T}} q_1 + c_1 c_2 \lambda_2 q_1^{\mathrm{T}} q_2 + \cdots + c_3 c_2 \lambda_2 q_3^{\mathrm{T}} q_2 + c_3^2 \lambda_3 q_3^{\mathrm{T}} q_3 \\
&= c_1^2 \lambda_1 + c_2^2 \lambda_2 + c_3^2 \lambda_3.
\end{aligned}$$

(b) We maximize $(c_1^2 \lambda_1 + c_2^2 \lambda_2 + c_3^2 \lambda_3)/(c_1^2 + c_2^2 + c_3^2)$ when $c_1 = c_2 = 0$, so $x = c_3 q_3$ is a multiple of the eigenvector q_3 with the largest eigenvalue λ_3.

(Also notice that the maximum value of this "Rayleigh quotient" $x^{\mathrm{T}} A x / x^{\mathrm{T}} x$ is the largest eigenvalue itself. This is another way of finding eigenvectors: maximize $x^{\mathrm{T}} A x / x^{\mathrm{T}} x$ numerically.)

6 (4+4+4=12 pts.) (a) Find a linear combination w of the linearly independent vectors v and u that is perpendicular to u.

(b) For the 2-column matrix $A = \begin{bmatrix} u & v \end{bmatrix}$, find Q (orthonormal columns) and R (2 by 2 upper triangular) so that $A = QR$.

(c) In terms of Q only, using $A = QR$, find the projection matrix P onto the plane spanned by u and v.

Solution.

(a) You could just write down $w = 0u + 0v = 0$ — that's perpendicular to everything! But a more useful choice is to subtract off just enough u so that $w = v - cu$ is perpendicular to u. That means $0 = w^{\mathrm{T}}u = v^{\mathrm{T}}u - cu^{\mathrm{T}}u$, so $c = (v^{\mathrm{T}}u)/(u^{\mathrm{T}}u)$ and

$$w = v - (\frac{v^{\mathrm{T}}u}{u^{\mathrm{T}}u})u.$$

(b) We already know u and w are orthogonal; just normalize them! Take $q_1 = u/|u|$ and $q_2 = w/|w|$. Then solve for the columns r_1, r_2 of R: $Qr_1 = u$ so $r_1 = \begin{bmatrix} |u| \\ 0 \end{bmatrix}$, and $Qr_2 = v$ so $r_2 = \begin{bmatrix} c|u| \\ |w| \end{bmatrix}$. (Where $c = (v^{\mathrm{T}}u)/(u^{\mathrm{T}}u)$ as before.)

Then $Q = [q_1\ q_2]$ and $R = [r_1\ r_2]$.

(c) $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = (QR)(R^{\mathrm{T}}Q^{\mathrm{T}}QR)^{-1}(R^{\mathrm{T}}Q^{\mathrm{T}}) = (QR)(R^{\mathrm{T}}Q^{\mathrm{T}}) = \underline{QQ^{\mathrm{T}}}$.

7 (4+3+4=11 pts.) (a) Find the eigenvalues of

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

(b) Those are both permutation matrices. What are their inverses C^{-1} and $(C^2)^{-1}$?

(c) Find the determinants of C and $C + I$ and $C + 2I$.

Solution.

(a) Take the determinant of $C - \lambda I$ (I expanded by cofactors): $\lambda^4 - 1 = 0$. The roots of this "characteristic equation" are the eigenvalues: $+1, -1, i, -i$.

The eigenvalues of C^2 are just $\lambda^2 = \pm 1$ (two of each).

(Here's a "guessing" approach. Since $C^4 = I$, all the eigenvalues λ^4 of C^4 are 1: so $\lambda = 1, -1, i, -i$ are the only possibilities. Just check to see which ones work. Then the eigenvalues of C^2 must be ± 1.)

(b) For any permutation matrix, $C^{-1} = C^{\mathrm{T}}$: so

$$C^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and $(C^2)^{-1} = C^2$ is itself.

(c) The determinant of C is the product of its eigenvalues: $1(-1)i(-i) = \underline{-1}$.

Add 1 to every eigenvalue to get the eigenvalues of $C + I$ *(if $C = S\Lambda S^{-1}$, then $C + I = S(\Lambda + I)S^{-1}$)*: $2(0)(1+i)(1-i) = \underline{0}$.

(Or let $\lambda = -1$ in the characteristic equation $\det(C - \lambda I)$.*)*

Add 2 to get the eigenvalues of $C + 2I$ *(or let $\lambda = -2$)*: $3(1)(2+i)(2-i) = \underline{15}$.

8 (4+3+4=11 pts.) Suppose a rectangular matrix A has independent columns.

(a) How do you find the best least squares solution $\widehat{x}$ to $Ax = b$? By taking those steps, give me a formula (letters not numbers) for $\widehat{x}$ and also for $p = A\widehat{x}$.

(b) The projection p is in which fundamental subspace associated with A? The error vector $e = b - p$ is in which fundamental subspace?

(c) Find by any method the projection matrix P onto the column space of A:

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \\ 0 & -1 \\ 0 & -3 \end{bmatrix}.$$

Solution.

(a)

$$\begin{array}{rrcl} & Ax & = & b \\ \text{Least-squares "solution":} & A^{\mathrm{T}}A\hat{x} & = & A^{\mathrm{T}}b \\ A^{\mathrm{T}}A \text{ is invertible:} & \hat{x} & = & (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}b \\ \text{and } p = A\hat{x} \text{ is:} & A\hat{x} & = & A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}b \end{array}$$

(b) $p = A\hat{x}$ is a linear combination of columns of A, so it's in the column space $C(A)$. The error $e = b - p$ is orthogonal to this space, so it's in the left nullspace $N(A^{\mathrm{T}})$.

(c) I used $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. Since $A^{\mathrm{T}}A = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, its inverse is $\begin{bmatrix} 1/10 & 0 \\ 0 & 1/10 \end{bmatrix} = \frac{1}{10}I$, and

$$P = \frac{1}{10}\begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 9 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

9 (3+4+4=11 pts.) This question is about the matrices with 3's on the main diagonal, 2's on the diagonal above, 1's on the diagonal below.

$$A_1 = \begin{bmatrix} 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} \quad A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix} \quad A_n = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & \cdot \\ 0 & 0 & \cdot & \cdot \end{bmatrix}$$

(a) What are the determinants of A_2 and A_3?

(b) The determinant of A_n is D_n. Use cofactors of row 1 and column 1 to find the numbers a and b in the recursive formula for D_n:

$$(*) \qquad D_n = a\,D_{n-1} + b\,D_{n-2}\,.$$

(c) This equation $(*)$ is the same as

$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}.$$

>From the eigenvalues of that matrix, how fast do the determinants D_n grow? (If you didn't find a and b, say how you would answer part (c) for any a and b) For 1 point, find D_5.

Solution.

(a) $\det(A_2) = 3 \cdot 3 - 1 \cdot 2 = 7$ and $\det(A_3) = 3\det(A_2) - 2 \cdot 1 \cdot 3 = 15$.

(b) $D_n = 3D_{n-1} + (-2)D_{n-2}$. *(Show your work.)*

(c) The trace of that matrix A is $a = 3$, and the determinant is $-b = 2$. So the characteristic equation of A is $\lambda^2 - a\lambda - b = 0$, which has roots (the eigenvalues of A)

$$\lambda_\pm = \frac{a \pm \sqrt{a^2 - 4(-b)}}{2} = \frac{3 \pm 1}{2} \quad = 1 \text{ or } 2.$$

D_n grows at the same rate as the largest eigenvalue of A^n, $\lambda_+^n = 2^n$.

The final point: $D_5 = 3D_4 + 2D_3 = 3(3D_3 + 2D_2) + 2D_3 = 11D_3 + 6D_2 = 207$.

18.06 **FINAL EXAM** May 24, 2007

Your PRINTED name is: <u>SOLUTIONS</u>

Please circle your recitation:

(1)	M 2	2-131	A. Osorno
(2)	M 3	2-131	A. Osorno
(3)	M 3	2-132	A. Pissarra Pires
(4)	T 11	2-132	K. Meszaros
(5)	T 12	2-132	K. Meszaros
(6)	T 1	2-132	Jerin Gu
(7)	T 2	2-132	Jerin Gu

Grading

1
2
3
4
5
6
7
8
Total:

Problem 1 *(10 points)*

Let $A = \begin{pmatrix} 3 & 2 & 1 & 1 \\ 6 & 6 & 3 & 3 \\ 3 & 4 & 2 & 2 \end{pmatrix}$.

(a) Calculate the dimensions of the 4 fundamental subspaces associated with A.

(b) Give a basis for each of the 4 fundamental subspaces.

(c) Find the complete solution of the system $A\mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$.

Solution 1

(a) The rank is 2, so the dimensions are:

$C(A) \rightsquigarrow r = 2$

$C(A^T) \rightsquigarrow r = 2$

$N(A) \rightsquigarrow n - r = 4 - 2 = 2$

$N(A^T) \rightsquigarrow m - r = 3 - 2 = 1.$

(b) We can get these by elimination or by inspection:

$C(A) \rightsquigarrow \begin{pmatrix} 3 & 6 & 3 \end{pmatrix}^T, \begin{pmatrix} 2 & 6 & 4 \end{pmatrix}^T$

$C(A^T) \rightsquigarrow \begin{pmatrix} 3 & 2 & 1 & 1 \end{pmatrix}^T, \begin{pmatrix} 6 & 3 & 3 & 1 \end{pmatrix}^T$

$N(A) \rightsquigarrow \begin{pmatrix} 0 & -1/2 & 1 & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & -1/2 & 0 & 1 \end{pmatrix}^T$

$N(A^T) \rightsquigarrow \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}^T$

(c) $x = x_{particular} + x_{nullspace} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 0 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1/2 \\ 0 \\ 1 \end{pmatrix}.$

Problem 2 *(10 points)*

Consider the system of linear equations:

$$\begin{cases} x & + & y & + & z & = & 1 \\ 2x & & & + & z & = & 2 \\ -x & + & y & + & az & = & b \end{cases}$$

In parts (a)–(c) below circle correct answers. Explain your answers.

(a) For $a = 1$, $b = -1$, the system has:

(1) exactly one solution

(2) infinitely many solutions

(3) no solutions

(b) For $a = 0$, $b = 1$, the system has:

(1) exactly one solution

(2) infinitely many solutions

(3) no solutions

(c) For $a = 0$, $b = -1$, the system has:

(1) exactly one solution

(2) infinitely many solutions

(3) no solutions

(d) Solve the system for $a = b = 1$.

Solution 2

If we eliminate the augmented matrix we get $\begin{pmatrix} 1 & 1 & 1 & & 1 \\ 0 & -2 & -1 & & 0 \\ 0 & 0 & a & & b+1 \end{pmatrix}$.

(a) **Exactly one solution:** The matrix is invertible.

(b) **No solutions:** Get a row of zeroes in the matrix with no zero in the augmented column.

(c) **Infinitely many solutions:** Get a row of zeroes with a zero in the augmented column.

(d) Using back substitution we get $x = \begin{pmatrix} 0 & -1 & 2 \end{pmatrix}^T$.

Problem 3 *(10 points)*

Let L be the line in $\mathbb{R}^3$ spanned by the vector $(1,1,1)^T$. Let P be the projection matrix for the projection onto the line L.

(a) What are the eigenvalues of the matrix P? (Indicate their multiplicities.)

(b) Find an *orthonormal* basis of the orthogonal complement $L^\perp$ to the line L.

(c) Calculate the projection of the vector $(1,2,3)^T$ onto the line L.

(d) Calculate the projection of the vector $(1,2,3)^T$ onto the orthogonal complement $L^\perp$.

Solution 3

(a) P is a projection matrix onto a subspace of dimension 1, so the eigenvalues are 1, 0, 0.

(b) $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}.$

(c) $p = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$

(d) The projection onto $L^\perp$ is $b - p = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$

Problem 4 *(10 points)*

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{pmatrix}$.

In parts (a)–(c) below circle correct answers. Explain your answers.

(a) The matrix A is singular: **True** **False**

(b) The matrix $A + 2I$ is singular: **True** **False**

(c) The matrix A is positive definite: **True** **False**

(d) Find all eigenvalues of A and the corresponding eigenvectors.

(e) Find an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^T$.

(f) Solve the system of differential equations $\dfrac{d\mathbf{u}(t)}{dt} = A\,\mathbf{u}(t)$, $\mathbf{u}(0) = (1, 0, 0)^T$.

Solution 4

(a) **True** ($\begin{pmatrix} 1 & -2 & 1 \end{pmatrix}^T$ is in the nullspace).

(b) **True** ($\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T$ is in the nullspace).

(c) **False** The matrix is singular so has 0 as eigenvalue.

(d) A is singular, so 0 is an eigenvalue with eigenvector $\begin{pmatrix} 1 & -2 & 1 \end{pmatrix}^T$.

$A + 2I$ is singular, so -2 is an eigenvalue with eigenvector $\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T$.

We can get the last eigenvalue by looking a the trace: 6. The eigenvector is $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$.

(e)

$$A = \underbrace{\begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{62} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}}_{Q^T}.$$

(f) $u(t) = e^{At}u(0) = Qe^{\Lambda t}Q^T u(0) = \frac{1}{6}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \frac{1}{2}e^{-2t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{3}e^{6t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$

Problem 5 *(10 points)*

Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{pmatrix}$.

(a) What is the rank of A?

(b) Calculate the matrix $A^T A$. Find all its eigenvalues (with multiplicities).

(c) Calculate the matrix AA^T. Find all its eigenvalues (with multiplicities).

(d) Find the matrix Σ in the singular value decomposition $A = U\Sigma V^T$.

Solution 5

(a) 1

(b) $A^T A = \begin{pmatrix} 14 & 14 & 14 & 14 \\ 14 & 14 & 14 & 14 \\ 14 & 14 & 14 & 14 \\ 14 & 14 & 14 & 14 \end{pmatrix}$. It was rank 1 so the eigenvalues are 56, 0, 0, 0.

(c) $AA^T = \begin{pmatrix} 4 & 8 & 12 \\ 8 & 16 & 24 \\ 12 & 24 & 36 \end{pmatrix}$. It was rank 1 so the eigenvalues are 56, 0, 0.

(d) $\Sigma = \begin{pmatrix} \sqrt{56} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Problem 6 *(10 points)*

Let A_n be the tridiagonal $n \times n$-matrix with 2's on the main diagonal, 1's immediately above the main diagonal, 3's immediately below the main diagonal, and 0's everywhere else:

$$A_n = \begin{pmatrix} 2 & 1 & 0 & 0 & \cdots & 0 \\ 3 & 2 & 1 & 0 & \ddots & 0 \\ 0 & 3 & 2 & 1 & \ddots & 0 \\ 0 & 0 & 3 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 \end{pmatrix},$$

(a) Express the determinant $\det(A_n)$ in terms of $\det(A_{n-1})$ and $\det(A_{n-2})$.

(b) Explicitly calculate $\det(A_n)$, for $n = 1, \ldots, 6$.

Solution 6

(a) Using cofactors twice we get $\det(A_n) = 2\det(A_{n-1}) - 3\det(A_{n-2})$.

(b) $\det(A_1) = \det[2] = 2$.

$\det(A_2) = \det \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = 1$.

$\det(A_3) = 2 \cdot 1 - 3 \cdot 2 = -4$.

$\det(A_4) = 2 \cdot (-4) - 3 \cdot 1 = -11$.

$\det(A_5) = 2 \cdot (-11) - 3 \cdot (-4) = -10$.

$\det(A_6) = 2 \cdot (-10) - 3 \cdot (-11) = 13$.

Problem 7 *(10 points)*

Calculate the determinant of the following 6×6-matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Solution 7

This determinant could be computed using cofactors or doing row operations to simplify and then cofactors. it could also be computed as follows.

$$A = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}}_{B}.$$

P is a permutation matrix with determinant -1.

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} - I.$$

The eigenvalues of the matrix with all 1's are 6, 0, 0, 0, 0, 0 so the eigenvalues of B are 5, -1, -1, -1, -1, -1, so the determinant of B is -5.

Thus the determinant of A is 5.

Problem 8 *(10 points)*

(a) Calculate A^{100} for $A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$.

(b) Calculate B^{100} for $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

(c) What will happen with the house (shown below) when we apply the linear transformation $T(v) = Bv$ for $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$? Will it be dilated? Draw the picture of the transformed house.

Solution 8

(a) $A^2 = \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$ so $A^{100} = \begin{pmatrix} 5^{100} & 0 \\ 0 & 5^{100} \end{pmatrix}$.

(b) $B^4 = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$ so $B^{100} = \begin{pmatrix} (-4)^{25} & 0 \\ 0 & (-4)^{25} \end{pmatrix} = \begin{pmatrix} -4^{25} & 0 \\ 0 & -4^{25} \end{pmatrix}$.

(c) From part (b) we see that B^4 is stretching by 4 and rotating by π. Thus B is rotating by $\pi/4$ and stretching by $\sqrt{2}$.

Practice 18.06 Final Questions with Solutions

17th December 2007

Notes on the practice questions

The final exam will be on Thursday, Dec. 20, from 9am to 12noon at the Johnson Track, and will most likely consist of 8–12 questions. The practice problems below mostly concentrate on the material from exams 1–2, since you already have practice problems for exam 3. The real final will have plenty of eigenproblems!

These questions are intended to give you a flavor of how I want you to be able to think about the material, and the flavor of possible questions I might ask. Obviously, these questions do *not* exhaust *all* the material that we covered this term, so you should of course still study your lecture notes and previous exams, and review your homework.

Solutions for these practice problems should be posted on the 18.06 web site by 12/15.

List of potential topics:

Material from exams 1, 2, and 3, and the problem sets (and lectures) up to that point.

Definitely not on final: finite-difference approximations, sparse matrices and iterative methods, non-diagonalizable matrices, generalized eigenvectors, principal components analysis, choosing a basis to convert a linear operator into a matrix, numerical linear algebra and error analysis.

Key ideas:

- The four subspaces of a matrix A and their relationships to one another and the solutions of $A\mathbf{x} = \mathbf{b}$.

- Gaussian elimination $A \to U \to R$ and backsubstitution. Elimination = invertible row operations = multiplying A on the *left* by an invertible matrix. Multiplying on left by an invertible matrix preserves $N(A)$, and hence we can use elimination to find the nullspace. Also, it thus preserves $C(A^H) = N(A)^\perp$. Also, it thus preserves the linear independence of the columns and hence the pivot columns in A are a basis for $C(A)$. Conversely, invertible *column* operations = multiplying A on the *right* by an invertible matrix, hence preserving $N(A^H)$ and $C(A)$.

- Solution of $A\mathbf{x} = \mathbf{b}$ when A is *not* invertible: exactly solvable if $\mathbf{b} \in C(A)$, particular solutions, not unique if $N(A) \neq \{\mathbf{0}\}$. Least-squares solution $A^H A\hat{\mathbf{x}} = A^H\mathbf{b}$ if A full column rank, equivalence to minimizing $\|A\mathbf{x} - \mathbf{b}\|^2$, relationship $A\hat{\mathbf{x}} = P_A\mathbf{b}$ to projection matrix $P_A = A(A^H A)^{-1}A^H$ onto $C(A)$. The fact that $\operatorname{rank}(A^H A) = \operatorname{rank} A = \operatorname{rank} A^H$, hence $A^H A\hat{\mathbf{x}} = A^H\mathbf{b}$ is always solvable for *any* A, and $A^H A$ is always invertible (and positive-definite) if A has full column rank.

- Vector spaces and subspaces. Dot products, transposes/adjoints, orthogonal complements. Linear independence, bases, and orthonormal bases. Gram-Schmidt and QR factorization.

Unitary matrices $Q^H = Q^{-1}$, which preserve dot products $\mathbf{x} \cdot \mathbf{y} = (Q\mathbf{x}) \cdot (Q\mathbf{y})$ and lengths $\|\mathbf{x}\| = \|Q\mathbf{x}\|$.

- Determinants of square matrices: properties, invariance under elimination (row swaps flip sign), = product of eigenvalues. The trace of a matrix = sum of eigenvalues. $\det AB = (\det A)(\det B)$, $\operatorname{trace} AB = \operatorname{trace} BA$, $\operatorname{trace}(A + B) = \operatorname{trace} A + \operatorname{trace} B$.

- For an eigenvector, any complicated matrix or operator acts just like a number λ, and we can do anything we want (inversion, powers, exponentials...) using that number. To act on an arbitrary vector, we expand that vector in the eigenvectors (in the usual case where the eigenvectors form a basis), and then treat each eigenvector individually. Finding eigenvectors and eigenvalues is complicated, though, so we try to infer as much as we can about their properties from the structure of the matrix/operator (Hermitian, Markov, etcetera). SVD as generalization of eigenvectors/eigenvalues.

Problem 1

Suppose that A is some real matrix with full column rank, and we do Gram-Schmidt on it to get an the following orthonormal basis for $C(A)$: $\mathbf{q}_1 = (1, 0, 1, 0, -1)^T/\sqrt{3}$, $\mathbf{q}_2 = (1, 2, -1, 0, 0)^T/\sqrt{6}$, $\mathbf{q}_3 = (-2, 1, 0, 0, 2)^T/\sqrt{9}$.

(a) Suppose we form the matrix B whose columns are $\sqrt{3}\mathbf{q}_1$, $\sqrt{6}\mathbf{q}_2$, and $\sqrt{9}\mathbf{q}_3$. Which of the four subspaces, if any, are guaranteed to be the same for A and B?

(b) Find a basis for the left nullspace $N(A^T)$.

(c) Find a basis for the row space $C(A^T)$.

Solution:

(a) All four subspaces for A and B are the same. Obviously B has the same column space as A, i.e. $C(A) = C(B)$. It follows that they have the same left nullspace, $N(A^T) = N(B^T)$, which are just the orthogonal complement of column space. Since both A and B has full column rank, $N(A) = N(B) = \{0\}$. It follows that the row spaces $C(A^T) = C(B^T)$.

(b) As we have observed above, $N(A^T) = N(B^T)$, where $B = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 2 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{pmatrix}$. To find a basis for B^T, we do Gauss elimination:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & -1 \\ 1 & 2 & -1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 2 & -2 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 2 & -2 & 0 & 1 \\ 0 & 0 & 3 & 0 & -1/2 \end{pmatrix}$$
$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & -5/6 \\ 0 & 2 & 0 & 0 & 2/3 \\ 0 & 0 & 3 & 0 & -1/2 \end{pmatrix},$$

So we may take two vectors $(0, 0, 0, 1, 0)^T$ and $(5, -2, 1, 0, 6)^T$ as a basis of $N(A^T)$.

(c) A is a 5×3 matrix with full column rank, i.e. $\operatorname{rank}(A) = 3$. So the row space have dimension $\dim(C(A^T)) = 3$. However, $C(A^T)$ lies in $\mathbb{R}^3$. So $C(A^T) = \mathbb{R}^3$. We may take the vectors $(1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T$ as a basis.

Problem 2

Perverse physicist Pat proposes a permutation: Pat permutes the columns of some matrix A by some random sequence of column swaps, resulting in a new matrix B.

(a) If you were given B (only), which of the four subspaces of A (if any) could you find? (i.e. which subspaces are preserved by column swaps?)

(b) Suppose $B = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ -1 & 5 & 0 \end{pmatrix}$ and $\mathbf{b} = (0, 1, 2)^T$. Check whether $A\mathbf{x} = \mathbf{b}$ is solvable, and if so whether the solution $\mathbf{x}$ is unique.

(c) Suppose you were given B and it had full column rank. You are also given $\mathbf{b}$. Give an explicit formula (in terms of B and $\mathbf{b}$ only) for the *minimum* possible *value* of $\|A\mathbf{x} - \mathbf{b}\|^2$.

Solution:

(a) Only column swaps are performed, so the column space is not changed. As a consequence, the left nullspace is also not changed. So given B, we can find $C(A)$ and $N(A^T)$.

(b) Since $C(A) = C(B)$, the solvability of $A\mathbf{x} = \mathbf{b}$ is equivalent to the solvability of $B\mathbf{x} = \mathbf{b}$. We do Gauss elimination for the second equation:

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 2 & 1 & 3 & 1 \\ -1 & 5 & 0 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 3 & -1 & 1 \\ 0 & 4 & 2 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 3 & -1 & 1 \\ 0 & 0 & 10/3 & 2/3 \end{pmatrix}.$$

So the equation $B\mathbf{x} = \mathbf{b}$, and thus the equation $A\mathbf{x} = \mathbf{b}$, is solvable. Moreover, the solution $\mathbf{x}$ is unique.

(c) All the vectors $A\mathbf{x}$ are exactly the vectors in the column space $C(A) = C(B)$. Thus the minimum possible value of $\|A\mathbf{x} - \mathbf{b}\|^2$ is exactly the minimum possible value of $\|B\mathbf{x} - \mathbf{b}\|^2$. To find the latter one, we need the vector $B\mathbf{x}$ to be the vector in $C(B)$ which is closest to the vector $\mathbf{b}$. In other words, $B\mathbf{x}$ should be the projection of $\mathbf{b}$ on $C(B)$. Since B is of full column rank, the projection is $B\mathbf{x} = B(B^TB)^{-1}B^T\mathbf{b}$. So the minimum value we want is $\|B(B^TB)^{-1}B^T\mathbf{b} - \mathbf{b}\|^2$

Problem 3

Which of the following sets are vector spaces (under ordinary addition and multiplication by real numbers)?

(a) Given a vector $\mathbf{x} \in \mathbb{R}^n$, the set of all vectors $\mathbf{y} \in \mathbb{R}^n$ with $\mathbf{x} \cdot \mathbf{y} = 3$.

(b) The set of all functions $f(x)$ whose integral $\int_{-\infty}^{\infty} f(x)dx$ is zero.

(c) Given a subspace $V \subseteq \mathbb{R}^n$ and an $m \times n$ matrix A, the set of all vectors $A\mathbf{x}$ for all $\mathbf{x} \in V$.

(d) Given a line $L \subseteq \mathbb{R}^n$ and an $m \times n$ matrix A, the set of all vectors $A\mathbf{x}$ for all $\mathbf{x} \in L$.

(e) The set of $n \times n$ Markov matrices.

(f) The set of eigenvectors with $|\lambda| < \frac{1}{2}$ of an $n \times n$ Markov matrix A.

Solution:

(a) This is not a vector space, since it doesn't contain the zero vector.

(b) This is a vector space.

It is a subset of the vector space consisting all functions. If f and g both has whole integral zero, $af + bg$ also has whole integral zero. So it is a vector subspace.

(c) This is a vector space.

It is just the column space of A, which is a vector subspace of $\mathbb{R}^m$.

(d) This is not a vector space.

A line in $\mathbb{R}^n$ doesn't have to pass the origin, so the set of all vectors $A\mathbf{x}$ need not contain the zero vector.

(e) This is not a vector space.

The zero matrix is not a Markov matrix.

(f) This is not a vector space in general.

If the Markov matrix A has more than one eigenvalue whose absolute value are less than $1/2$, then the corresponding set of eigenvectors is not a vector space, since the summation of two eigenvectors corresponding to different eigenvalues is not a eigenvector.

Problem 4

The *rows* of an $m \times n$ matrix A are linearly independent.

(a) Is $A\mathbf{x} = \mathbf{b}$ necessarily solvable?

(b) If $A\mathbf{x} = \mathbf{b}$ is solvable, is the solution necessarily unique?

(c) What are $N(A^H)$ and $C(A)$?

Solution:

(a) Yes. Since the rows of A are linearly independent, $\text{rank}(A) = m$. So the column space of A is an m-dimensional subspace of $\mathbb{R}^m$, i.e., is $\mathbb{R}^m$ itself. It follows that for any $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ is always solvable.

(b) No, the solution maybe not unique. Since $\text{rank}(A) = m$, the nullspace is $n-m$ dimensional. Thus the solution is not unique if $n > m$, and is unique if $n = m$. (It will never have that $n < m$, otherwise the rows are not linearly independent.)

(c) We have seen in part (a) that $C(A) = \mathbb{R}^m$. So $N(A^T) = \{0\}$. Since $N(A^H)$ consists those points whose conjugate lies in $N(A^T)$, we see $N(A^H) = \{0\}$.

Problem 5

Make up your own problem: give an example of a matrix A and a vector $\mathbf{b}$ such that the solutions of $A\mathbf{x} = \mathbf{b}$ form a line in $\mathbb{R}^3$, $\mathbf{b} \neq \mathbf{0}$, and all the entries of the matrix A are nonzero. Find all solutions $\mathbf{x}$.

Solution:

Such a matrix must be an $m \times 3$ matrix whose nullspace is one dimensional. In other words, the rank is $3 - 1 = 2$. We may take A to be an 2×3 matrix whose rows are linearly independent. As an example, we take

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

To find all solutions, we do elimination

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

so the solutions are given by $\mathbf{x} = (1 - t, t, 0)^T$.

Problem 6

Clever Chris the chemist concocts a conundrum: in ordinary least-squares, you minimize $\|A\mathbf{x}-\mathbf{b}\|^2$ by solving $A^H A\hat{\mathbf{x}} = A^H\mathbf{b}$, but suppose instead that we wanted to minimize $(A\mathbf{x}-\mathbf{b})^H C(A\mathbf{x}-\mathbf{b})$ for some Hermitian positive-definite matrix C?

(a) Suppose $C = B^H B$ for some matrix B. Which of the following (if any) must be properties of B: (i) Hermitian, (ii) Markov, (iii) unitary, (iv) full column rank, (v) full row rank?

(b) In terms of B, A, and $\mathbf{b}$, write down an explicit formula for the $\mathbf{x}$ that minimizes $(A\mathbf{x}-\mathbf{b})^H C(A\mathbf{x}-\mathbf{b})$.

(c) Rewrite your answer from (b) purely in terms of C, A, and $\mathbf{b}$.

(d) Suppose that C was only positive semi-definite. Is there still a minimum value of $(A\mathbf{x}-\mathbf{b})^H C(A\mathbf{x}-\mathbf{b})$? Still a unique solution $\mathbf{x}$?

Solution:

(a) For $C = B^H B$ a Hermitian positive-definite matrix, B must be (iv) full column rank. To show this, we only need to notice that $\mathbf{x}\cdot C\mathbf{x} = \mathbf{x}\cdot B^H B\mathbf{x} = \|Bx\|^2$. Since C is positive definite, B has no nonzero nullspace. It follows that B is of full column rank.

B don't have to be Hermitian or Markov or unitary or full row rank. For example, we may take $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$. Then $C = B^H B = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

(b) We have

$$\begin{aligned}(A\mathbf{x}-\mathbf{b})^H C(A\mathbf{x}-\mathbf{b}) &= (A\mathbf{x}-\mathbf{b})^H B^H B(A\mathbf{x}-\mathbf{b}) = (BA\mathbf{x}-B\mathbf{b})^H(BA\mathbf{x}-B\mathbf{b}) \\ &= \|BA\mathbf{x}-B\mathbf{b}\|^2.\end{aligned}$$

To minimize this, we need to solve the equation $(BA)^H(BA)\hat{x} = (BA)^H B\mathbf{b}$. Since B and A are of full column rank, BA is of full column rank. So $(BA)^H(BA)$ is invertible. The solution is given by $\hat{x} = (A^H B^H BA)^{-1}A^H B^H V\mathbf{b}$.

(c) Since $B^H B = C$, we have $\hat{x} = (A^H CA)^{-1}A^H C\mathbf{b}$.

(d) If C is only positive semi-definite, we still have $C = B^H B$ for some B, but B need not to be of full column rank. As above, we have

$$(A\mathbf{x}-\mathbf{b})^H C(A\mathbf{x}-\mathbf{b}) = \|BA\mathbf{x}-B\mathbf{b}\|^2.$$

So the minimum value of $(A\mathbf{x}-\mathbf{b})^H C(A\mathbf{x}-\mathbf{b})$ still exists. However, since B may be not of full column rank, the matrix need not be invertible. The solution $\mathbf{x}$ need not be unique. For example, if we take C to be the zero matrix, any $\mathbf{x}$ will minimize $(A\mathbf{x}-\mathbf{b})^H C(A\mathbf{x}-\mathbf{b})$.

Problem 7

True or false (explain why if true, give a counter-example if false).

(a) For $n\times n$ real-symmetric matrices A and B, AB and BA always have the same eigenvalues. [Hint: what is $(AB)^T$?]

(b) For $n\times n$ matrices A and B with B invertible, AB and BA always have the same eigenvalues. [Hint: you can write $\det(AB-\lambda I) = \det((A-\lambda B^{-1})B)$. Alternative hint: think about similar matrices.]

(c) Two diagonalizable matrices A and B with the same eigenvalues and eigenvectors must be the same matrix.

(d) Two diagonalizable matrices A and B with the same eigenvalues must be the same matrix.

(e) For $n \times n$ matrices A and B with B invertible, AB and BA always have the same eigenvectors.

Solution:

(a) True.

Since A and B are real-symmetric matrices, $(AB)^T = B^T A^T = BA$. But $(AB)^T$ has the same eigenvalue as AB. So AB and BA have the same eigenvalues.

(b) True.

Since B is invertible, we have $AB = B^{-1}(BA)B$. So BA is similar to AB, and they must have the same eigenvalues.

(c) True.

Let Λ be the diagonal matrix consists of eigenvalues, and S be the matrix of eigenvectors (ordered as the eigenvalue matrix). Then we have $A = S^{-1}\Lambda S$ and also $B = S^{-1}\Lambda S$. So $A = B$.

(d) False.

For example, $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ are both diagonalizable, with the same eigenvalues, but they are different.

(e) False.

For example, let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $BA = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. The eigenvectors of AB are $(0,1)^T$ and $(1,0)^T$, while the eigenvectors or BA are $(0,1)^T$ and $(1,1)^T$.

Problem 8

You are given the matrix

$$A = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

(a) What is the sum of the eigenvalues of A?

(b) What is the product of the eigenvalues of A?

(c) What can you say, without computing them, about the eigenvalues of AA^T?

Solution:

(a) The sum of the eigenvalues of A equals trace of A. So the sum of the eigenvalues of A is $0 + 1 + 1 - 1 = 1$.

(b) The product of the eigenvalues of A equals the determinant of A. We do row transforms

$$\begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & -1/2 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix},$$

where in the first step we add the other rows to the first row. It follows that the product of eigenvalues equals $\det(A) = -2$.

(c) AA^T is a real symmetric matrix, so its eigenvalues are all real and nonnegative. Since A is nonsingular, all eigenvalues of AA^T are positive. The product of all eigenvalues of AA^T are

$\det(AA^T) = \det(A)\det(A^T) = \det(A)^2 = 4$. The sum of all eigenvalues of AA^T is $\operatorname{trace}(AA^T) = \sum_{i,j} a_{ij}^2 = 9$.

Problem 9

You are given the quadratic polynomial $f(x, y, z)$:

$$f(x, y, z) = 2x^2 - 2xy - 4xz + y^2 + 2yz + 3z^2 - 2x + 2z.$$

(a) Write $f(x, y, z)$ in the form $f(x, y, z) = \mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x}$ where $\mathbf{x} = (x, y, z)^T$, A is a real-symmetric matrix, and $\mathbf{b}$ is some constant vector.

(b) Find the point (x, y, z) where $f(x, y, z)$ is at an extremum.

(c) Is this point a minimum, maximum, or a saddle point of some kind?

$\boxed{\text{Solution:}}$

(a) We have

$$A = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 1 & 1 \\ -2 & 1 & 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}.$$

(b) We compute the partial derivatives to find the extremum point:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 4x - 2y - 4z - 2 = 0 \\ \frac{\partial f}{\partial y} &= -2x + 2y + 2z = 0 \\ \frac{\partial f}{\partial x} &= -4x + 2y + 6z + 2 = 0 \end{aligned}$$

The equation is just $2A\mathbf{x} = \mathbf{b}$. The solution to the the equations above is $x = 1, y = 1, z = 0$. So the extreme point is $(1, 1, 0)$.

(c) We look for the pivots of A:

$$\begin{pmatrix} 2 & -1 & -2 \\ -1 & 1 & 1 \\ -2 & 1 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -1 & -2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since A is positive definite, the extreme point is a minimum.

Problem 10

Suppose A is some diagonalizable matrix. Consider the vector $\mathbf{y}(t) = e^{A^2 t}\mathbf{x}$ for some vector $\mathbf{x}$.

(a) If A is 3×3 with eigenvalues λ_1, λ_2, and λ_3 with eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$, and $\mathbf{x} = \mathbf{x}_1 + 3\mathbf{x}_2 + 4\mathbf{x}_3$, what is $\mathbf{y}(t)$?

(b) If $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0}$ for *every* vector $\mathbf{x}$, what does that tell you about the eigenvalues of A and of e^{A^2}?

(c) In terms of A, give a system of differential equations that $\mathbf{y}(t)$ satisfies, and the initial condition.

(d) In terms of A, give a linear recurrence relation that $\mathbf{y}(t)$ satisfies for $t = k\Delta t$ for integers k and some fixed Δt.

Solution:

(a) A^2 has eigenvalues λ_1^2, λ_2^2 and λ_3^2 with eigenvectors $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_3$, so $e^{A^2 t}$ has eigenvalues $e^{\lambda_1^2 t}, e^{\lambda_2^2 t}$ and $e^{\lambda_3^2 t}$ with the same eigenvectors. Thus

$$\mathbf{y}(t) = e^{A^2 t}\mathbf{x} = e^{A^2 t}(\mathbf{x}_1 + 3\mathbf{x}_2 + 4\mathbf{x}_3) = e^{\lambda_1^2 t}\mathbf{x}_1 + 3e^{\lambda_2^2 t}\mathbf{x}_2 + 4e^{\lambda_3^2 t}\mathbf{x}_3.$$

(b) If $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0}$ for every vector $\mathbf{x}$, we must have $\operatorname{Re}\lambda_1^2 < 0, \operatorname{Re}\lambda_2^2 < 0$ and $\operatorname{Re}\lambda_3^2 < 0$. The eigenvalues $e^{\lambda_1^2}, e^{\lambda_2^2}$ and $e^{\lambda_3^2}$ all sastisfies $|e^{\lambda^2}| < 1$.

(c) $\mathbf{y}(t)$ satisfies the system $\mathbf{y}' = A^2\mathbf{y}$. The initial condition is $\mathbf{y}(0) = \mathbf{x}$.

(d) Denote by $\mathbf{y}_k = \mathbf{y}(k\Delta t)$. Then equation $\mathbf{y}(t) = e^{A^2 t}\mathbf{x}$ gives the recurrence relation $\mathbf{y}_{k+1} = e^{A^2\Delta t}\mathbf{y}_k$ with initial condition $\mathbf{y}_0 = \mathbf{x}$.

Problem 11

Suppose A is an $m \times n$ matrix with full *row* rank. Which of the following equations *always* have a solution (possibly non-unique) for any $\mathbf{b}$?

(a) $A\mathbf{x} = \mathbf{b}$

(b) $A^H\mathbf{x} = \mathbf{b}$

(c) $A^H A\mathbf{x} = \mathbf{b}$

(d) $AA^H\mathbf{x} = \mathbf{b}$

(e) $A^H A\mathbf{x} = A^H\mathbf{b}$

(f) $AA^H\mathbf{x} = A\mathbf{b}$

Solution:

(a) The equation $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$, as explained in problem 4.

(b) The equation $A^H\mathbf{x} = \mathbf{b}$ may has no solution, since the column space of A^H is an m-dimensional subspace in the whole space $\mathbb{R}^n$.

(c) The equation $A^H A\mathbf{x} = \mathbf{b}$ may has no solution, since the $C(A^H A) = C(A^H)$ is an m-dimensional subspace in $\mathbb{R}^n$. (That $C(A^H A) = C(A^H)$ comes from the fact $N(A^H A) = N(A)$ which we proved in class.)

(d) The equation $AA^H\mathbf{x} = \mathbf{b}$ will always has a unique solution, since AA^H is an $m \times m$ matrix whose rank is m, i.e. AA^H is invertible.

(e) The equation $A^H A\mathbf{x} = A^H\mathbf{b}$ will always has a solution. In fact, any solution to $A\mathbf{x} = \mathbf{b}$ (from part (a)) is always a solution to $A^H A\mathbf{x} = A^H\mathbf{b}$.

(f) The equation $AA^H\mathbf{x} = A\mathbf{b}$ will always has a unique solution, since AA^H is invertible, as explained in part (d).

Additional Practice Problems

Be sure to look at:

(i) Exams 1, 2, and 3.

(ii) The practice problems for exam 3. (The above problems mostly cover non-eigenvalue stuff.)

(iii) Ideally, also review your homework problems.

18.06 Professor Edelman Final Exam December 18, 2008

Your PRINTED name is: ____________________

Please circle your recitation:

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3)	T 10	2-255	Su Ho Oh	2-333	3-7826	`suho`
4)	T 11	2-131	J. Yu	2-348	4-2597	`jyu`
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12)	T 2	2-147	M. Frankland	2-090	3-6293	`franklan`
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14)	T 3	2-131	J. French	2-489	3-4086	`jfrench`
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Grading

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1 (9 pts.) Given real numbers a, b and c, find x, y and z such that the matrix B below is guaranteed to be singular with real eigenvalues and orthogonal eigenvectors.

$$B = \begin{bmatrix} a & b & a+b \\ b & c & b+c \\ x & y & z \end{bmatrix}.$$

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2 (12 pts.) The matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & p \end{bmatrix}.$$

(a) What are the eigenvalues of A (possibly in terms of p)?

(b) If p is not 0, find an eigenvector that is not in the nullspace.

(c) What are the singular values of A (possibly in terms of p)?

(d) Find a nonzero solution $u(t)$ to $du/dt = (A + 2009I)u$. Check that your answer is correct. (Note that $A + 2009I$ is the matrix above with the upcoming new year added to the diagonal elements.)

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3 (8 pts.) A 4x4 square matrix A has singular values $3, 2, 1$, and 0. Find an eigenvalue of A. Briefly explain your answer.

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4 (9 pts.) The square matrix A has QR decomposition $A = QR$ where Q is orthogonal and R is upper triangular with diagonal elements all equal to 1.

(a) What is the determinant of $A^T A$?

(b) What are all the pivots of $A^T A$?

(c) Are the matrices QR and RQ similar?

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5 (5 pts.) All matrices in this question are $n \times n$. We have that $C = A^{-1}BX$. Propose an X which guarantees that B and C have the same eigenvalues.

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6 (12 pts.) All you are told about a 3×3 matrix A is that five of the nine entries are 1, and the other four are 0. For the ranks below, exhibit a matrix A with this property, or else briefly (but convincingly) argue that it is impossible.

(a) A has rank 0.

(b) A has rank 2.

(c) A has rank 3.

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7 (10 pts.) (Do **two** of the three problems below. Please avoid any confusion to the graders as to which two you chose.)

(a) All you are told about a 100 by 100 matrix is that all of its entries are **even** integers. Must the determinant be odd? Must the determinant be even? Argue your answer convincingly.

(b) Give an example, if possible, of a 100 by 100 matrix with odd integer entries but an even determinant.

(c) All you are told about a 100 by 100 matrix is that all of its entries are **odd** integers. Must the determinant be odd? Must the determinant be even? Argue your answer convincingly.

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8 (15 pts.) The functions of the form

$$f(x) = c_1 + c_2 e^x + c_3 e^{2x},$$

form a three dimensional vector space V.

(a) The transformation d/dx can be written as a 2x3 matrix when the domain is specified to have basis $\{1, e^x, e^{2x}\}$, and the range has basis $\{e^x, e^{2x}\}$. Write down this 2x3 matrix.

(b) On the above three dimensional vector space V, is the evaluation of f at $x = 7$ a linear transformation from that space to R?

(c) On the above three dimensional vector space V, is the transformation that takes $f(x)$ to $\int_0^x f(t)dt$ a linear transformation from that three dimensional space to itself (from V to V) ?

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9 (20 pts.) Suppose an n by n matrix has the property that its nullspace is equal to its column space.

(a) Can the matrix be the zero matrix?

(b) Possibly in terms of n, what is the rank of the matrix?

(c) What are the eigenvalues of this matrix? (Briefly explain your answer. Hint: It might be useful to consider applying A more than once in some way.)

(d) Give an example of a 2 by 2 such matrix.

(e) Perhaps using the previous case twice somehow, give an example of a 4 x 4 such matrix.

Have a great holiday vacation! Thank you for taking linear algebra.

18.06 FINAL SOLUTIONS

Problem 1. (10 points) $B = \begin{pmatrix} a & b & a+b \\ b & c & b+c \\ x & y & z \end{pmatrix}$. We know that symmetric matrices have real eigenvalues and orthogonal eigenvectors. So we set $x = a+b$ and $y = b+c$. This leaves only the singularity of B. For this, we note that setting $z = x+y = a+2b+c$ makes the third column a sum of the first two, thus ensuring singularity.

Problem 2. (4 points each). $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & p \end{pmatrix}$.

a) To find the eigenvalues of A, we compute $det(A-\lambda I) = det \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & p-\lambda \end{pmatrix} = (\lambda^2)(p-\lambda)$ (because A is upper triangular). So the eigenvalues are 0 and p.

b) If $p \neq 0$, the we wish to find $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ so that $A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} pa \\ pb \\ pc \end{pmatrix}$. But $A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ c \\ pc \end{pmatrix}$, so we need $b = pa$ and $c = pb$; and so $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ p \\ p^2 \end{pmatrix}$ works.

c) The singular vaues of A are found by first computing the eigenvalues of $A^TA = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & p \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & p \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & p \\ 0 & 0 & 1+p^2 \end{pmatrix}$. As this is upper triangular, the eigenvalues are 0, 1 and $1+p^2$. So the singular values are 0, 1 and $\sqrt{1+p^2}$.

d) In general, $d\mathbf{u}/dt = B\mathbf{u}$ is solved by $e^{Bt}\mathbf{u}(0)$. For us, $B = \begin{pmatrix} 2009 & 1 & 0 \\ 0 & 2009 & 1 \\ 0 & 0 & 2009+p \end{pmatrix} = A + 2009I$. To compute e^{Bt}, we note that $e^{Bt} = exp(t \begin{pmatrix} 2009 & 0 & 0 \\ 0 & 2009 & 0 \\ 0 & 0 & 2009+p \end{pmatrix} + t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} e^{2009} & 0 & 0 \\ 0 & e^{2009} & 0 \\ 0 & 0 & e^pe^{2009} \end{pmatrix} exp(t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix})$. To compute $exp(t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix})$, we note that $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^3 = 0$. So $exp(t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}) = I + t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + (t^2/2) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$, and finally $e^{Bt} = \begin{pmatrix} e^{2009} & te^{2009} & e^{2009}t^2/2 \\ 0 & e^{2009} & te^{2009} \\ 0 & 0 & e^pe^{2009} \end{pmatrix}$. So our answer is $\begin{pmatrix} e^{2009} & te^{2009} & e^{2009}t^2/2 \\ 0 & e^{2009} & te^{2009} \\ 0 & 0 & e^pe^{2009} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{2009} \\ 0 \\ 0 \end{pmatrix}$.

Problem 3. (8 points) A is 4×4 and has singular values $\{3,2,1,0\}$. As the product of the singular values is (up to sign) the determinant, we get that $det(A) = 0$, so A has nontrivial nullspace, and so 0 is an eigenvalue.

Problem 4. (3 points each). $A = QR$ where Q is orthogonal and R is upper triangular with 1's on the diagonal.

a) $det(A^T A) = det(R^T Q^T QR) = det(R^T R) = det(R)^2 = 1$ since $det(R) = 1$ by the assumption on R.

b) The equation $A^T A = R^T R$ tells us that $(R^{-1})^T (A^T A) = R$; and since $(R^{-1})^T$ is lower triangular, this is exactly the elimination of $A^T A$. So the pivots are all equal to 1.

c) Yes, since $Q^{-1}(QR)Q = RQ$.

Problem 5. (10 points) $C = A^{-1}BX$. We know that similar matrices have the same eigenvalues, so putting $X = A$ forces C and B to have the same eigenvalues.

Problem 6. (4 points each) A is 3×3 and has four 0's and five 1's.

a) A has rank 0 is impossible- it isn't the zero matrix.

b) A has rank 2: $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

c) A has rank 3: $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

Problem 7. (10 points each) A is 100×100.

a) A has all even integers as entries. Therefore each column of A has the form $2\mathbf{c}$ where $\mathbf{c}$ is a vector of integers. So we set $C = (1/2)A$. Then $det(A) = det(2C) = 2^{100} det(C)$; so $det(A)$ is an even integer (note that $det(C)$ really is an integer because all the entries of C are integers and the det can be computed by the big formula).

b) This time, we use the big formula to compute $det(C) = \sum sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$. This is a sum containing 100! terms. Now, 100! is an even number, and each term in the sum is odd (as a product of odd integers). Since the sum of two odd numbers is even, the sum of an even number of odd numbers is even; so this sum is an even integer.

Problem 8. (5 points each) We consider the vector space V of functions of the form $c_1 + c_2 e^x + c_3 e^{2x}$, with basis $\{1, e^x, e^{2x}\}$.

a) d/dx takes V to the space W spanned by $\{e^x, e^{2x}\}$. We have that $\frac{d}{dx}(1) = 0$, $\frac{d}{dx} e^x = e^x$, and $\frac{d}{dx} e^{2x} = 2e^{2x}$. So the linear transformation of d/dx in the given bases is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

b) We conisder the transformation ϕ from V to $\mathbb{R}$ defined by $f \to f(7)$. This is linear: $\phi(f+g) = (f+g)(7) = f(7) + g(7) = \phi(f) + \phi(g)$, and $\phi(cf) \to cf(7) = c\phi(f)$.

c) No. $\int_0^x 1 = x$ is a function not in V.

18.06 Professor Strang Final Exam May 20, 2008

Your PRINTED name is: ____________________

Please circle your recitation:

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13)	T 2	2-132	B. Lehmann	2-089	2-1195	`lehmann`
14)	T 2	26-168	P. McNamara	2-314	4-1459	`petermc`

Grading

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Thank you for taking 18.06.

If you liked it, you might enjoy 18.085 this fall.

Have a great summer. GS

1 (10 pts.) The matrix A and the vector b are

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

(a) The complete solution to $Ax = b$ is $x =$ ________.

(b) $A^{\mathrm{T}}y = c$ can be solved for which column vectors $c = (c_1, c_2, c_3, c_4)$? (Asking for conditions on the c's, not just c in $\boldsymbol{C}(A^{\mathrm{T}})$.)

(c) How do those vectors c relate to the special solutions you found in part (a)?

$\boxed{\text{Solution}}$ (10 points)

a) The complete solution is a particular solution x_p plus any vector in the nullspace x_n. Since the matrix A is already reduced, we can just read the special solutions off: $[-1, 1, 0, 0]^T$ and $[-2, 0, -4, 1]^T$. To find a particular solution to $Ax = b$, we put any numbers (we may as well choose 0) in for the free variables. This yields the two equations $x_1 = 3$ and $x_3 = 1$, so $x_p = [3, 0, 1, 0]^T$. In the end we get

$$x_{comp} = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix} \tag{1}$$

b) You can do this computation by hand by augmenting A^T with the column (c_1, c_2, c_3, c_4) and row reducing. The solution is given by the equations that correspond to 0 rows in the reduced matrix. A quicker way is to note that $A^Ty = c$ has a solution whenever c is in the column space $C(A^T)$, i.e. the row space of A. This is perpendicular to the nullspace. Thus, we can find the equations by taking a basis for the nullspace and using the components as coefficients in our equations. We find equations $-c_1 + c_2 = 0$ and $-2c_1 - 4c_3 + c_4 = 0$.

c) Because these c are in the row space, they are perpendicular to vectors in the nullspace of A, and in particular are perpendicular to the special solutions.

2 (8 pts.) (a) Suppose $q_1 = (1,1,1,1)/2$ is the first column of Q. How could you find three more columns q_2, q_3, q_4 of Q to make an orthonormal basis? (Not necessary to compute them.)

(b) Suppose that column vector q_1 is an eigenvector of A: $Aq_1 = 3q_1$. (The other columns of Q might not be eigenvectors of A.) Define $T = Q^{-1}AQ$ so that $AQ = QT$. Compare the first columns of AQ and QT to discover *what numbers are in the first column of T?*

$\boxed{\text{Solution}}$ (8 points)

a) First, we find additional vectors v_2, v_3 and v_4 that (along with q_1) make up a basis of $\mathbb{R}^4$. Then we run Gram-Schmidt on q_1, v_2, v_3, v_4.

b) Using the column picture of multiplication, we see that the first column of AQ will be $Aq_1 = 3q_1$. Similarly, if we denote the first column of T by (t_1, t_2, t_3, t_4), then the first column of QT will be $t_1q_1 + t_2q_2 + t_3q_3 + t_4q_4$. Since these two are equal, we get an equality of vectors

$$3q_1 = t_1q_1 + t_2q_2 + t_3q_3 + t_4q_4 \tag{2}$$

Since the q_i are linearly independent, we must have $t_1 = 3$ and the other $t_i = 0$, showing that the first column of T is $(3, 0, 0, 0)$.

We can also note that the first column of T is equal to $3Q^Tq_1$, which yields the same answer.

3 (12 pts.) Two eigenvalues of this matrix A are $\lambda_1 = 1$ and $\lambda_2 = 2$. The first two pivots are $d_1 = d_2 = 1$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(a) Find the other eigenvalue λ_3 and the other pivot d_3.

(b) What is the smallest entry a_{33} in the southeast corner that would make A positive semidefinite? What is the smallest c so that $A + cI$ is positive semidefinite?

(c) Starting with one of these vectors $u_0 = (3, 0, 0)$ or $(0, 3, 0)$ or $(0, 0, 3)$, and solving $u_{k+1} = \frac{1}{2} A u_k$, describe the limit behavior of u_k as $k \to \infty$ (with numbers).

$\boxed{\text{Solution}}$ (12 points)

a) The sum of the eigenvalues is the trace, so $1 + 2 + \lambda_3 = 2$. Thus $\lambda_3 = -1$. The product of the pivots is the determinant, which is the product of the eigenvalues as well. So $d_3 = -2$. Note that this means that A is not positive-definite.

b) We can test positive-definiteness using the determinant method. The two top-left determinants of A are both positive, so we just need to check the third one. We obtain the relation:

$$1(c - 1) + 1(-1) \geq 0 \tag{3}$$

so the smallest value of c is 2.

For the second part, we test whether the eigenvalues are non-negative. The eigenvalues of $A + cI$ are just the eigenvalues of A plus c. So when $c = 1$ all the eigenvalues will be non-negative.

c) The matrix $\frac{1}{2}A$ is a Markov matrix. Because it has some 0 entries, we don't automatically know that it has a unique steady state vector. However, since the eigenvalues of $\frac{1}{2}A$ are $1/2$, $-1/2$ and 1, it does have a unique steady state vector (only one eigenvalue has absolute value 1). To find it, we calculate the eigenvector of A with eigenvalue 2 by taking the nullspace of $A - 2I$:

$$A - 2I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \tag{4}$$

$$\rightsquigarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{5}$$

The nullspace is generated by the special solution $(1, 1, 1)$. So, a vector u will have limit $A^{\infty}u$ equal to $c(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, where c is the sum of the components of u. In particular, the vectors $(3, 0, 0)$, etc., all go to $(1, 1, 1)$.

4 (10 pts.) Suppose $Ax = b$ has a solution (maybe many solutions). I want to prove two facts:

A. There is a solution x_{row} in the row space $\boldsymbol{C}(A^{\mathrm{T}})$.

B. There is only *one* solution in the row space.

(a) Suppose $Ax = b$. I can split that x into $x_{\text{row}} + x_{\text{null}}$ with x_{null} in the nullspace. How do I know that $Ax_{\text{row}} = b$? (Easy question)

(b) Suppose x^*_{row} is in the row space and $Ax^*_{\text{row}} = b$. I want to prove that x^*_{row} is the same as x_{row}. Their difference $d = x^*_{\text{row}} - x_{\text{row}}$ is in which subspaces? How to prove $d = 0$?

(c) Compute the solution x_{row} in the row space of this matrix A, by solving for c and d:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \end{bmatrix} x_{\text{row}} = \begin{bmatrix} 14 \\ 9 \end{bmatrix} \quad \text{with} \quad x_{\text{row}} = c \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + d \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

$\boxed{\text{Solution}}$ (10 points)

a) We have $A(x_{row} + x_{null}) = A(x_{row}) + A(x_{null}) = A(x_{row}) + 0$, so $A(x_{row}) = b$.

b) Suppose both $A(x_{row}) = b$ and $A(x^*_{row}) = b$. Then $x^*_{row} - x_{row}$ is in the row space (since it is a linear combination of vectors in the row space) and is in the nullspace (since multiplying by A will give us 0). Because the row space and nullspace are orthogonal complements, the only vector that is in both is the 0 vector: any vector in both will have $|x|^2 = x \cdot x = 0$.

c) Substituting in the given expressions for $Ax_{row} = b$ we find

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 14 \\ 9 \end{bmatrix} \tag{6}$$

or

$$\begin{bmatrix} 14 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 14 \\ 9 \end{bmatrix} \tag{7}$$

We find $(c, d) = (1, 3)$, so $x_{row} = (4, 5, 0)$. Remark: essentially what we are doing here is projecting onto the row space.

5 (10 pts.) The numbers D_n satisfy $D_{n+1} = 2D_n - 2D_{n-1}$. This produces a first-order system for $u_n = (D_{n+1}, D_n)$ with this 2 by 2 matrix A:

$$\begin{bmatrix} D_{n+1} \\ D_n \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} \quad \text{or} \quad u_n = Au_{n-1}\,.$$

(a) Find the eigenvalues λ_1, λ_2 of A. Find the eigenvectors x_1, x_2 with second entry equal to 1 so that $x_1 = (z_1, 1)$ and $x_2 = (z_2, 1)$.

(b) What is the inner product of those eigenvectors? (2 points)

(c) If $u_0 = c_1x_1 + c_2x_2$, give a formula for u_n. For the specific $u_0 = (2, 2)$ find c_1 and c_2 and a formula for D_n.

$\boxed{\text{Solution}}$ (10 points)

a) The eigenvalues of

$$A = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix} \tag{8}$$

satisfy the equation $\lambda^2 - 2\lambda + 2 = 0$, so $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$. We find the eigenvectors by taking the appropriate nullspaces:

$$A - \lambda_1 I = \begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix} \tag{9}$$

has nullspace generated by $x_1 = (1 + i, 1)$, and

$$A - \lambda_2 I = \begin{bmatrix} 1+i & -2 \\ 1 & -1+i \end{bmatrix} \tag{10}$$

has nullspace generated by $x_2 = (1 - i, 1)$. If you pick a different vector in the nullspace, you just rescale so that the bottom entry is 1.

b) The inner product is $x_1^H x_2 = (1-i)^2+1 = 1-2i$, or its conjugate expression $x_2^H x_1 = 1+2i$.

c) If $u_0 = c_1 x_1 + c_2 x_2$, then $u_n = c_1 \lambda_1^n x_1 + c_2 \lambda_2^n x_2$. A matrix always acts on its eigenvectors in a diagonal way. In particular, $(2, 2) = x_1 + x_2$. So we find

$$u_n = (1+i)^n \begin{bmatrix} 1+i \\ 1 \end{bmatrix} + (1-i)^n \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \tag{11}$$

with second entry $D_n = (1+i)^n + (1-i)^n$.

6 (12 pts.) (a) Suppose q_1, q_2, a_3 are linearly independent, and q_1 and q_2 are already orthonormal. Give a formula for a third orthonormal vector q_3 as a linear combination of q_1, q_2, a_3.

(b) Find the vector q_3 in part (a) when

$$q_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad q_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

(c) Find the projection matrix P onto the subspace spanned by the first two vectors q_1 and q_2. You can give a formula for P using q_1 and q_2 or give a numerical answer.

$\boxed{\text{Solution}}$ (12 points)

a) This is the Gram-Schmidt process. We define

$$w_3 = a_3 - (q_1 \cdot a_3)q_1 - (q_2 \cdot a_3)q_2 \tag{12}$$

and then set $q_3 = w_3/\|w_3\|$. Note that we do not need denominators in the expression for w_3 because the q_i are already unit vectors.

b) Substituting in, we find

$$w_3 = a_3 - 5q_1 - (-1)q_2 = (-1, -1, 1, 1) \tag{13}$$

Renormalizing we get $q_3 = \frac{1}{2}(-1, -1, 1, 1)$.

c) The projection matrix P is exactly the expression we used for Gram-Schmidt: $P = q_1 q_1^T + q_2 q_2^T$. There are other more complicated expressions which are also correct. We can start at the most general and simplify to get this one; if A has columns q_1 and q_2 then $P = A(A^T A)^{-1} A^T = A(I)A^T = q_1 q_1^T + q_2 q_2^T$ where we used the column-row picture of multiplication for the last step.

7 (12 pts.) (a) Find the determinant of this N matrix.

$$N = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 2 & 1 & 0 & 3 \\ 3 & 0 & 1 & 2 \\ 4 & 0 & 0 & 1 \end{bmatrix}$$

(b) Using the cofactor formula for N^{-1}, tell me one entry that is zero or tell me that all entries of N^{-1} are nonzero.

(c) What is the rank of $N - I$? Find all four eigenvalues of N.

$\boxed{\text{Solution}}$ (12 points)

a) There are many ways to do this. Perhaps the easiest is cofactors along the top row:

$$\det(N) = 1(1) - 4\det \begin{bmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix} = 1 - 4(4) = -15 \tag{14}$$

Here I found the determinant of the 3 by 3 by swapping the columns to get an upper triangular matrix with diagonal entries $1, 1, 4$.

b) The cofactor formula is $A^{-1} = C^T/\det(A)$ (we know that A is invertible from part a). To check for 0 entries we can ignore the $\det(A)$ part. We just need to find some cofactors that are 0, and we can arrange this by crossing out rows and columns that will give us a smaller matrix with a column of 0s. Some choices are C_{21}, C_{23}, C_{24}, C_{31}, C_{32}, and C_{34}. The corresponding 0 entries of the inverse are the transposes, so we get the entries $(1, 2)$, $(3, 2)$, $(4, 2)$, $(1, 3)$, $(2, 3)$, $(4, 3)$.

c) The matrix $N - I$ has two columns that are all 0s, and the other two columns are clearly independent, so it has rank 2. So $N - I$ has eigenvalue 0 with multiplicity 2. This tells us that N has eigenvalue 1 with multiplicity 2. Calling the other eigenvalues λ_1 and λ_2, we can find them solving the trace and determinant equations:

$$1 + 1 + \lambda_1 + \lambda_2 = 4 \tag{15}$$

$$(1)(1)\lambda_1\lambda_2 = -15 \tag{16}$$

Thus $\lambda_1 = 5$ and $\lambda_2 = -3$.

8 (8 pts.) Every invertible matrix A is the product $A = QH$ of an orthogonal matrix Q and a symmetric positive definite matrix H. I will start the proof:

> A has a singular value decomposition $A = U\Sigma V^{\mathrm{T}}$.
> Then $A = (UV^{\mathrm{T}})(V\Sigma V^{\mathrm{T}})$.

(a) Show that UV^{T} is an orthogonal matrix Q (what is the test for an orthogonal matrix ?).

(b) Show that $V\Sigma V^{\mathrm{T}}$ is a symmetric positive definite matrix. What are its eigenvalues and eigenvectors ? Why did I need to assume that A is invertible ?

$\boxed{\text{Solution}}$ (8 points)

a) To test that $Q = UV^T$ is orthogonal, we must show that $Q^TQ = I$. But $Q^TQ = (UV^T)^TUV^T = VU^TUV^T = V(I)V^T = I$. We used the fact that U and V are orthogonal matrices.

b) The matrix $H = V\Sigma V^T$ is definitely symmetric, as $H^T = V\Sigma^TV^T = V\Sigma V^T$ because Σ is diagonal. Furthermore, note that the expression $H = V\Sigma V^T$ is a diagonalization of H. This means that H has eigenvalues given by the entries of Σ and eigenvectors equal to the columns of V. To show that H is positive-definite, we just need to show that the diagonal entries of Σ are all positive.

Now, we know that they are all non-negative, because the SVD always gives us non-negative singular values. We must also show that none of the singular values are zero. Remember that the singular values are equal to the square roots of the eigenvalues of A^TA. However, because A is invertible, the matrix A^TA is also invertible, and so can't have any eigenvalues equal to 0. So no singular value is 0 either.

9 (7 pts.) (a) Find the inverse L^{-1} of this real triangular matrix L:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & a & 1 \end{bmatrix}$$

You can use formulas or Gauss-Jordan elimination or any other method.

(b) Suppose D is the real diagonal matrix $D = \mathsf{diag}(d, d^2, d^3)$. What are the conditions on a and d so that the matrix $A = LDL^{\mathrm{T}}$ is (*three separate questions*, one point each)

(i) invertible? (ii) symmetric? (iii) positive definite?

$\boxed{\text{Solution}}$ (7 points)

a) I'll do Gauss-Jordan elimination.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ a & 1 & 0 & 0 & 1 & 0 \\ 0 & a & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ 0 & a & 1 & 0 & 0 & 1 \end{array}\right] \tag{17}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ 0 & 0 & 1 & a^2 & -a & 1 \end{array}\right] \tag{18}$$

b) Note that L is invertible no matter what a is, and D is invertible so long as $d \neq 0$. So $A = LDL^T$ will be invertible whenever $d \neq 0$. If $d = 0$, then of course A can't be invertible. The matrix A is always symmetric, since $A^T = (LDL^T)^T = LD^TL^T = LDL^T$.

Because A is always symmetric, to check positive-definiteness we just need to check that the pivots are all positive. But $A = LDL^T$ is the "pivot" decomposition for A. So the pivots of A are d, d^2, d^3, and we need $d > 0$.

10 (11 pts.) This problem uses least squares to find the *plane* $C + Dx + Ey = b$ that best fits these 4 points:

$$\begin{array}{lll} x=0 & y=0 & b=2 \\ x=1 & y=1 & b=1 \\ x=1 & y=-1 & b=0 \\ x=-2 & y=0 & b=1 \end{array}$$

(a) Write down 4 equations $Ax = b$ with unknown $x = (C, D, E)$ that would hold if the plane went through the 4 points. Then write down the equations to solve for the best (least squares) solution $\widehat{x} = (\widehat{C}, \widehat{D}, \widehat{E})$.

(b) Find the best $\widehat{x}$ and the error vector e (*is the vector* e *in* $\mathbf{R}^3$ *or* $\mathbf{R}^4$?).

(c) If you change this $b = (2, 1, 0, 1)$ to the vector $p = A\widehat{x}$, what will be the best plane to fit these four new points (p_1, p_2, p_3, p_4) ? What will be the new error vector ?

$\boxed{\text{Solution}}$ (11 points)

a) The equations are of the form $C + 0D + 0E = 2$, etc., or in matrix form

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tag{19}$$

Of course this system is not solvable. The best solution is given by $A^T A\widehat{x} = A^T b$.

b) We have

$$A^T A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{20}$$

and

$$A^T b = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} \tag{21}$$

It is a diagonal system, so we immediately find $(C, D, E) = (1, -1/6, 1/2)$. The error vector is the difference of the real b and the approximate values we get for our plane: $e = b - A\widehat{x}$. Since $A\widehat{x} = [1, 4/3, 1/3, 4/3]^T$, we get $e = (1, -1/3, -1/3, -1/3)$.

c) We know $p = A\widehat{x}$ is the projection of b onto the column space of A. So the system $Ax = p$ is solvable exactly; we don't need any approximations. The best fit plane will be the same plane as in part b: $1 - x/6 + y/2 = b$ (we changed the b-coordinates of the points so that they lie on this plane, so of course it is the best fit). The error vector will become 0 because it is an exact fit.

18.06 Final Solution

Hold on Tuesday, 19 May 2009 at 9am in Walker Gym.
Total: 100 points.

Problem 1:

A sequence of numbers $f_0, f_1, f_2, \ldots$ is defined by the recurrence

$$f_{k+2} = 3f_{k+1} - f_k,$$

with starting values $f_0 = 1$, $f_1 = 1$. (Thus, the first few terms in the sequence are $1, 1, 2, 5, 13, 34, 89, \ldots$.)

(a) Defining $\mathbf{u}_k = \begin{pmatrix} f_{k+1} \\ f_k \end{pmatrix}$, re-express the above recurrence as $\mathbf{u}_{k+1} = A\mathbf{u}_k$, and give the matrix A.

(b) Find the eigenvalues of A, and use these to predict what the ratio f_{k+1}/f_k of successive terms in the sequence will approach for large k.

(c) The sequence above starts with $f_0 = f_1 = 1$, and $|f_k|$ grows rapidly with k. Keep $f_0 = 1$, but give a *different* value of f_1 that will make the sequence (with the *same recurrence* $f_{k+2} = 3f_{k+1} - f_k$) approach *zero* ($f_k \to 0$) for large k.

$\boxed{\text{Solution}}$ (18 points = 6+6+6)

(a) We have

$$\begin{pmatrix} f_{k+2} \\ f_{k+1} \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{k+1} \\ f_k \end{pmatrix} \quad \Rightarrow \quad A = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}.$$

(b) Eigenvalues of A are roots of $\det(A - \lambda I) = \lambda^2 - 3\lambda + 1 = 0$. They are $\lambda_1 = \dfrac{3+\sqrt{5}}{2}$ and $\lambda_2 = \dfrac{3-\sqrt{5}}{2}$. Note that $\lambda_1 > \lambda_2$, so the ratio f_{k+1}/f_k will approach $\lambda_1 = \dfrac{3+\sqrt{5}}{2}$ for large k.

(c) Let $\mathbf{v}_1, \mathbf{v}_2$ be the eigenvectors with eigenvalues λ_1 and λ_2, respectively. So, we can write $\mathbf{u}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ and then $\mathbf{u}_k = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2$. If we need $f_k \to 0$, we have to make $c_1 = 0$. In other words, $\mathbf{u}_0$ must be proportional to the eigenvector $\mathbf{v}_2$.

$$A - \lambda_2 I = \begin{pmatrix} \dfrac{3+\sqrt{5}}{2} & -1 \\ 1 & -\dfrac{3-\sqrt{5}}{2} \end{pmatrix} \quad \Rightarrow \quad \mathbf{v}_2 = \begin{pmatrix} \dfrac{3-\sqrt{5}}{2} \\ 1 \end{pmatrix}.$$

Hence, we need to take $f_1 = \dfrac{3-\sqrt{5}}{2}$ so that f_k will approach zero for large k.

Problem 2: For the matrix $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ with rank 2, consider the system of equations $A\mathbf{x} = \mathbf{b}$.

(i) $A\mathbf{x} = \mathbf{b}$ has a solution whenever $\mathbf{b}$ is orthogonal to some nonzero vector $\mathbf{c}$. Explicitly compute such a vector $\mathbf{c}$. Your answer can be multiplied by any overall constant, because $\mathbf{c}$ is any basis for the ____________________ space of A.

(ii) Find the orthogonal projection $\mathbf{p}$ of the vector $\mathbf{b} = \begin{pmatrix} 9 \\ 9 \\ 9 \end{pmatrix}$ onto $C(A)$. (*Note*: The matrix $A^{\mathrm{T}}A$ is singular, so you *cannot* use your formula $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ to obtain the projection matrix P onto the column space of A. But I have repeatedly discouraged you from computing P explicitly, so you don't need to be reminded anyway, right?)

(iii) If $\mathbf{p}$ is your answer from (ii), then a solution $\mathbf{y}$ of $A\mathbf{y} = \mathbf{p}$ minimizes what? [You need not answer (ii) or compute $\mathbf{y}$ for this part.]

$\boxed{\text{Solution}}$ (18 points = 7+7+4)

(i) The system of equations $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ lies in the column space of A, which is orthogonal to the left nullspace of A. We solve for a (nonzero) vector $\mathbf{c}$ in the left nullspace using Gaussian elimination, as follows.

$$A^{\mathrm{T}} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{c} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

The answer can by any nonzero multiple of $\mathbf{c}$, which will be a basis for the *left nullspace* of A.

(ii) <u>Method 1:</u> Since $\mathbf{c}$ is a basis of the orthogonal complement of the column space $C(A)$, the projection of $\mathbf{b}$ onto $C(A)$ can be computed as

$$\mathbf{p} = \mathbf{b} - \frac{\mathbf{c}^T\mathbf{b}\mathbf{c}}{\|\mathbf{c}\|^2} = \begin{pmatrix} 9 \\ 9 \\ 9 \end{pmatrix} - \frac{-9}{3}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix}.$$

Method 2: (not recommended) We know that $\mathbf{p}$ is the best linear approximation of $\mathbf{b}$. So we solve

$$A^{\mathrm{T}}A\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix} = A^{\mathrm{T}}\begin{pmatrix}9\\9\\9\end{pmatrix},$$

$$\begin{pmatrix}6&3&0\\3&2&1\\0&1&2\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix} = \begin{pmatrix}36\\18\\0\end{pmatrix}$$

We can get a particular solution $\mathbf{y} = (6,0,0)^{\mathrm{T}}$. (There are other solutions too.) Hence,

$$\mathbf{p} = A\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix} = \begin{pmatrix}1&0&-1\\1&1&1\\2&1&0\end{pmatrix}\begin{pmatrix}6\\0\\0\end{pmatrix} = \begin{pmatrix}6\\6\\12\end{pmatrix}.$$

(iii) Since $\mathbf{p}$ is the orthogonal projection of $\mathbf{b}$ onto $C(A)$, A solution $\mathbf{y}$ of $A\mathbf{y} = \mathbf{p}$ minimizes the distance $\|A\mathbf{y} - \mathbf{b}\|$.

Problem 3: True or false. Give a counter-example if *false*. (You need not provide a reason if true.)

(a) If Q is an orthogonal matrix, then $\det Q = 1$.

(b) If A is a Markov matrix, then $d\mathbf{u}/dt = A\mathbf{u}$ approaches some finite constant vector (a "steady state") for any initial condition $\mathbf{u}(0)$.

(c) If S and T are subspaces of $\mathbb{R}^2$, then their intersection (points in *both* S *and* T) is also a subspace.

(d) If S and T are subspaces of $\mathbb{R}^2$, then their union (points in *either* S *or* T) is also a subspace.

(e) The rank of AB is less than or equal to the ranks of A and B for any A and B.

(f) The rank of $A + B$ is less than or equal to the ranks of A and B for any A and B.

$\boxed{\text{Solution}}$ (12 points = 2+2+2+2+2+2)

(a) False. For example, $Q = (-1)$ is an orthogonal matrix: $Q^{\mathrm{T}}Q = (-1)(-1) = (1)$.

REMARK: In general, for a real orthogonal matrix Q, $\det Q = \pm 1$. This is because $\det(Q^{\mathrm{T}}Q) = \det(I) = 1 \;\Rightarrow\; \det(Q)^2 = \det(Q^{\mathrm{T}})\det(Q) = 1$.

(b) False. Be careful here that we are discussing differential equations but not the powers of A. For example, $A = (1)$, the differential equation has solution $\mathbf{u} = ce^t$ for some constant c, which does not approach to any finite constant vector.

REMARK: It is true that for the Markov process $\mathbf{u}_{k+1} = A\mathbf{u}_k$, $\mathbf{u}_k$ approaches some finite constant vector (a "steady state") for any initial condition $\mathbf{u}_0$.

(c) True. Intersections of subspaces are always subspaces.

(d) False. For example, S and T are the x- and y-axes. Then $(1, 1) = (1, 0) + (0, 1)$ is a linear combination of points in the union of S and T, but does not lie in the union itself. So the union of S and T is not a subspace.

(e) Ture. One may see this by arguing as follows. Since the column space of AB is a subspace of the column space of A, the rank of AB is less than or equal to the rank of A. Similarly, since the row space of AB is a subspace of the row space of B, the rank of AB is less than or equal to the rank of B.

(f) False. $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ both have rank 1. But $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has rank 2.

REMARK: It is true that $\text{rank}(A + B) \leq \text{rank}A + \text{rank}B$.

Problem 4: Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix}$$

(a) Find an orthonormal basis for $C(A)$ using Gram-Schmidt, forming the columns of a matrix Q.

(b) Write each step of your Gram-Schmidt process from (a) as a multiplication of A on the __________ (*left* or *right*) by some invertible matrix. Explain how the product of these invertible matrices relates to the matrix R from the QR factorization $A = QR$ of A.

(c) Gram-Schmidt on another matrix B (of the same size as A) gives the *same* orthonormal basis (the same Q) as in part (a). Which of the four subspaces, if any, must be the same for the matrices AA^{T} and BB^{T}? [*You can do this part without doing (a) or (b).*]

$\boxed{\text{Solution}}$ (18 points = 6+6+6)

(a) From $\mathbf{u}_1 = (1,1,1,1)^{\mathrm{T}}$, we get $\mathbf{q}_1 = \mathbf{u}_1/\|\mathbf{u}_1\| = \frac{1}{2}(1,1,1,1)^{\mathrm{T}}$.

$$\begin{aligned}
\mathbf{v}_2 &= (1,-1,0,0)^{\mathrm{T}}, \\
\mathbf{u}_2 &= \mathbf{v}_2 - \mathbf{q}_1^{\mathrm{T}}\mathbf{v}_2\mathbf{q}_1 = \mathbf{v}_2 = (1,-1,0,0)^{\mathrm{T}}, \\
\mathbf{q}_2 &= \mathbf{v}_2/\|\mathbf{v}_2\| = \frac{1}{\sqrt{2}}(1,-1,0,0)^{\mathrm{T}}; \\
\mathbf{v}_3 &= (1,-1,-3,-1)^{\mathrm{T}}, \\
\mathbf{u}_3 &= \mathbf{v}_3 - \mathbf{q}_1^{\mathrm{T}}\mathbf{v}_3\mathbf{q}_1 - \mathbf{q}_1^{\mathrm{T}}\mathbf{v}_3\mathbf{q}_1 = \mathbf{v}_3 + \mathbf{u}_1 - \mathbf{u}_2 = (1,1,-2,0)^{\mathrm{T}}, \\
\mathbf{q}_3 &= \mathbf{v}_3/\|\mathbf{v}_3\| = \frac{1}{\sqrt{6}}(1,1,-2,0)^{\mathrm{T}}.
\end{aligned}$$

Hence, we have

$$Q = \begin{pmatrix} 1/2 & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/2 & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/2 & 0 & -2/\sqrt{6} \\ 1/2 & 0 & 0 \end{pmatrix}.$$

(b) Each step of the Gram Schmidt process from (a) is a multiplication of A on the *right* as follows.

$$
\begin{aligned}
A \rightsquigarrow A \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &\rightsquigarrow A \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&\rightsquigarrow A \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \\
&\rightsquigarrow A \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\sqrt{6} \end{pmatrix} = Q.
\end{aligned}
$$

The product of these invertible 3×3 matrices is exactly R^{-1}.

(c) Since the Gram-Schmidt of A and B gives the same outcome, the column space of A and B are the same. We know that A and AA^{T} have the same column space, and B and BB^{T} have the same column space. Hence AA^{T} and BB^{T} have the same column space. Moreover, since left nullspace is always orthogonal to the column space, AA^{T} and BB^{T} have the same left nullspace too. Also, notice that AA^{T} and BB^{T} are symmetric matrices, their row spaces are the same as the column spaces, and their nullspaces are the same as the left nullspaces. Therefore, all four subspaces of AA^{T} are the same as BB^{T}.

Problem 5: The complete solution to $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

for any arbitrary constants c and d.

(i) If A is an $m \times n$ matrix with rank r, give as much true information as possible about the integers m, n, and r.

(ii) Construct an explicit example of a possible matrix A and a possible right-hand side $\mathbf{b}$ with the solution $\mathbf{x}$ above. (There are many acceptable answers; you only have to provide one.)

$\boxed{\text{Solution}}$ (16 points = 8+8)

(i) Since we can multiply A with $\mathbf{x}$, $n = 3$. Also, since the nullspace of A is 2-dimensional, $r = n - 2 = 1$. There is no restriction on m except that $m \geq r = 1$.

(ii) We construct a minimal one, namely, $A = (a_1\ a_2\ a_2)$ is 1×3. For this, we need $A \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$ and $A \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 0$. That is

$$\begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A special solution is $A = (1\ -1\ 2)$. In this case, $\mathbf{b} = A\mathbf{x} = (1\ -1\ 2) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = (-1)$.

So, an example is

$$\begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \mathbf{x} = (-1).$$

Problem 6: Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

(i) A has one eigenvalue $\lambda = -1$, and the other eigenvalue is a double root of $\det(A - \lambda I)$. What is the other eigenvalue? (Very little calculation required.)

(ii) Is A defective? Why or why not?

(iii) Using the above A, suppose we want to solve the equation

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u} + c\mathbf{u}$$

where c is some real number, for some initial condition $\mathbf{u}(0)$.

(a) For what values of c will the solutions $\mathbf{u}(t)$ always to go zero as $t \to \infty$?

(b) For what values of c will the solutions $\mathbf{u}(t)$ typically diverge ($\|\mathbf{u}(t)\| \to \infty$) as $t \to \infty$?

(c) For what values of c will the solutions $\mathbf{u}(t)$ approach a constant vector (possibly zero) as $t \to \infty$?

$\boxed{\text{Solution}}$ (18 points = 6+6+6 (2+2+2))

(i) Let $\lambda_1 = -1$ and let $\lambda_2 = \lambda_3$ denote the double roots. Then from the trace of A, we have $\lambda_1 + 2\lambda_2 = \text{trace}(A) = 3$. Hence, $\lambda_2 = 2$.

(ii) A is not defective. There are two ways to see it. For one way, since A is symmetric, it is always non-defective; for another way, we compute $A - \lambda_2 I = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$, which has rank 1 and hence its nullspace is 2-dimentional.

(iii) The key point here is that $A + cI$ would have eigenvalues $\lambda_1 + c$ and $\lambda_2 + c$ (with multiplicity 2). An alternative point of view is as follows. If we write the initial condition $\mathbf{u}(t) = c_1(t)\mathbf{v}_1 + c_2(t)\mathbf{v}_2 + c_3(t)\mathbf{v}_3$, then the differential equation becomes

$$\frac{dc_1(t)}{dt}\mathbf{v}_1 + \frac{dc_2(t)}{dt}\mathbf{v}_2 + \frac{dc_3(t)}{dt}\mathbf{v}_3 = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3 + cc_1\mathbf{v}_1 + cc_2\mathbf{v}_2 + cc_3\mathbf{v}_3.$$

We have

$$\begin{cases} \dfrac{dc_1(t)}{dt} = c_1\lambda_1 + cc_1, & \Rightarrow \quad c_1 = e^{(\lambda_1+c)t}; \\ \dfrac{dc_2(t)}{dt} = c_2\lambda_2 + cc_2, & \Rightarrow \quad c_2 = e^{(\lambda_2+c)t}; \\ \dfrac{dc_3(t)}{dt} = c_3\lambda_3 + cc_3, & \Rightarrow \quad c_3 = e^{(\lambda_3+c)t}; \end{cases}$$

(a) If we require $\mathbf{u}(t)$ always go zero as $t \to \infty$, $\lambda_1 + c < 0$, $\lambda_2 + c = \lambda_3 + c < 0$. Hence, we require $c < -2$.

(b) If the solution $\mathbf{u}(t)$ typically diverge, we need either $\lambda_1 + c > 0$ or $\lambda_2 + c = \lambda_3 + c > 0$. Hence, we require $c > -2$.

(c) If we allow the solution to approach to some constant vector, we allow to have the extreme case of (a), that is to say $c \leq -2$.