18.06 Professor Edelman Quiz 3 December 1, 2010

Grading

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Your PRINTED name is: ____________________

Please circle your recitation:

R01	T 9	2-132	S. Kleiman	2-278	3-4996	`kleiman`
R02	T 10	2-132	S. Kleiman	2-278	3-4996	`kleiman`
R03	T 11	2-132	S. Sam	2-487	3-7826	`ssam`
R04	T 12	2-132	Y. Zhang	2-487	3-7826	`yanzhang`
R05	T 1	2-132	V. Vertesi	2-233	3-2689	`18.06`
R06	T 2	2-131	V. Vertesi	2-233	3-2689	`18.06`

1 (30 pts.)

In the following six problems produce a real 2×2 matrix with the desired properties, or argue concisely, simply, and convincingly that no example can exist.

(a) (5 pts.) A 2×2 symmetric, positive definite, Markov Matrix.

(b) (5 pts.) A 2×2 symmetric, negative definite (i.e., negative eigenvalues), Markov Matrix.

(c) (5 pts.) A 2×2 symmetric, Markov Matrix with one positive and one negative eigenvalue.

(d) (5 pts.) A 2×2 matrix $\neq 3I$ whose only eigenvalue is the double eigenvalue 3.

(e) (5 pts.) A 2×2 **symmetric** matrix $\neq 3I$ whose only eigenvalue is the double eigenvalue 3.
(Note the word "symmetric" in problem (e).)

(f) (5 pts.) A 2×2 non-symmetric matrix with eigenvalues 1 and -1.

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2 (35 pts.)

Let

$$A = -\begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

(Note the minus sign in the definition of A.)

(a) (15 pts.) Write down a valid SVD for A. (No partial credit for this one so be careful.)

(b) (20 pts.) The 4×4 matrix $e^{At} = I + f(t)A$. Find the scalar function $f(t)$ in simplest possible form. (Hint: the power series is one way; eigendecomposition is another.)

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3 (35 pts.)

(a) (15 pts.) The matrix A has independent columns. The matrix C is square, diagonal, and has positive entries. Why is the matrix $K = A^T C A$ positive definite? You can use any of the basic tests for positive definiteness.

(b) (20 pts.) If a diagonalizable matrix A has orthonormal eigenvectors and real eigenvalues must it be symmetric? (Briefly why or give a counterexample)

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18.06 Fall 2010 Exam 3 Solutions

(1) (a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) No example exists. There are many ways to see this.
First way: Negative definite means that all upper left submatrices have negative determinant. In particular, the $(1,1)$ entry needs to be negative, but this violates the definition of Markov.
Second way: Since the trace is the sum of the eigenvalues, all negative eigenvalues implies that the trace is negative. But the diagonal entries are nonnegative by the Markov property, which is another violation.
Third way: A Markov matrix always has 1 as an eigenvalue. All eigenvalues of negative definite are negative.

(c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Really, any matrix of the form $\begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}$ where $1/2 > a \geq 0$ would do: then the trace would be < 1 and since the trace is the sum of the eigenvalues and we know that one of the eigenvalues is 1, this means the other one has to be negative.

(d) Take a Jordan block: $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$.

(e) The answer we were looking for: all symmetric real matrices are diagonalizable. So if A is symmetric and has 3 as both of its eigenvalues, then its Jordan normal form is $3I$, and A is similar to $3I$. But the only matrix that is similar to $3I$ is $3I$ itself, which means A would have to be $3I$. Hence no example exists.

Another approach people tried but usually didn't fully justify: write the matrix as $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. The trace is the sum of the eigenvalues and hence 6, and the determinant is the product of the eigenvalues and hence 9. So $a + c = 6$ and $ac - b^2 = 9$. Since $b^2 \geq 0$, we get $ac \geq 9$. The way to maximize ac subject to $a + c = 6$ is to set $a = c = 3$. This can be justified with elementary calculus or citing the arithmetic-mean/geometric-mean inequality (AM-GM) but not many people really explained clearly why this was true. Anyway, this means that $a = c = 3$ and $b = 0$, but then $A = 3I$ again.

(f) $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

(2) (a) A is clearly rank 1, so a reduced SVD like we saw in class is the easiest:

$$A = U\Sigma V^T = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} (1) \begin{pmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{pmatrix}^T$$

(b) **Answer 1**: Notice that $A^2 = -A$, so $e^{At} = I + A(t - t^2/2 + t^3/3! - t^4/4! + \cdots)$. Thus $f(t) = 1 - e^{-t}$.
Answer 2: The eigenvalues of the symmetric rank 1 matrix A are $-1, 0, 0, 0$. In matrix language $\Lambda = \begin{pmatrix} -1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$ and $e^{\Lambda t} - I = \begin{pmatrix} e^{-t} - 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} = (1 - e^{-t})\Lambda$.
If $A = Q\Lambda Q^T$, then $e^{At} - I = Q(e^{\Lambda t} - I)Q^T = (1 - e^{-t})Q\Lambda Q^T = (1 - e^{-t})A$ giving the same answer $f(t) = 1 - e^{-t}$.

(3) (a) Again, there are many ways to do this.
First way: rememaber that a symmetric matrix is positive definite if and only if it can be written as $R^T R$ where R has linearly independent columns. In our case, let $c_1, \ldots, c_n$ be the diagonal entries of C and let B be the diagonal matrix with diagonal entries $\sqrt{c_1}, \ldots, \sqrt{c_n}$ (take the positive square roots). Then $C = B^T B$ and so $K = A^T B^T B A = (BA)^T (BA)$ so we

take $R = BA$. Since A has linearly independent columns and B is invertible (because the c_i are nonzero numbers), we conclude that BA also has linearly independent columns.

Second way: Use the energy definition. Let x be a nonzero vector. We have to show that $x^T Kx > 0$. First, since A has linearly independent columns, this means that its null space is 0, so $Ax \neq 0$. Set $y = Ax$. Since C is diagonal and has positive diagonal entries, it is positive definite (this follows from the eigenvalue definition, or the submatrices definition, for example). So $y^T Cy > 0$, but $y^T Cy = x^T Kx$, so we're done.

(b) True, since A is diagonalizable with real eigenvalues, we can write $A = S\Lambda S^{-1}$ where Λ has real entries and the columns of S are some eigenvectors. Since we also know that A has orthonormal eigenvectors, we may choose these for S, and hence $S^{-1} = S^T$. But then $A = S\Lambda S^T$ and $A^T = (S^T)^T \Lambda^T S^T$. But $(S^T)^T = S$ for any matrix S, and $\Lambda^T = \Lambda$ because it is diagonal and square. So $A = A^T$ and A is symmetric.

Remark: Some people said that Hermitian matrices (topic not covered) are diagonalizable with orthonormal eigenvectors and have real eigenvalues but are not symmetric in general. This is true, and technically did not violate the directions of the problem since it did not specify that A has to have real entries, so received full credit.

18.06 Quiz 3 Solutions May 8, 2010 Professor Strang

Your PRINTED name is: ____________________ **1.**

Your recitation number is ____________________ **2.**

3.

1. **(40 points)** Suppose u is a unit vector in R^n, so $u^T u = 1$. This problem is about the n by n symmetric matrix $H = I - 2u\,u^T$.

   (a) Show directly that $H^2 = I$. Since $H = H^T$, we now know that H is not only

   symmetric but also ____________________.

   $\boxed{\text{Solution}}$ Explicitly, we find $H^2 = (I - 2uu^T)^2 = I^2 - 4uu^T + 4uuu^Tuu^T$ (2 points): since $u^Tu = 1$, $H^2 = I$ (3 points). Since $H = H^T$, we also have $H^TH = 1$, implying that H is an orthogonal (or unitary) matrix.

   (b) One eigenvector of H is u itself. Find the corresponding eigenvalue.

   $\boxed{\text{Solution}}$ Since $Hu = (I - 2uu^T)u = u - 2uu^Tu = u - 2u = -u$, $\lambda = -1$.

   (c) If v is any vector perpendicular to u, show that v is an eigenvector of H and **find the eigenvalue.** With all these eigenvectors v, that eigenvalue must be repeated how many times? Is H **diagonalizable**? Why or why not?

   $\boxed{\text{Solution}}$ For any vector v orthogonal to u (i.e. $u^Tv = 0$), we have $Hv = (I-2uu^T)v = v - 2uu^Tv = v$, so the associated λ is 1. The orthogonal complement to the space spanned by u has dimension $n-1$, so there is a basis of $(n-1)$ orthonormal eigenvectors with this eigenvalue. Adding in the eigenvector u, we find that H is diagonalizable.

   (d) Find the diagonal entries H_{11} and H_{ii} in terms of $u_1, \ldots, u_n$. Add up $H_{11} + \ldots + H_{nn}$ and separately add up the eigenvalues of H.

   $\boxed{\text{Solution}}$ Since ith diagonal entry of uu^T is u_i^2, the i diagonal entry of H is $H_{ii} = 1-2u_i^2$ (3 points). Summing these together gives $\sum_{i=1}^n H_{ii} = n - 2\sum_{i=1}^n u_i^2 = n - 2$ (3 points). Adding up the eigenvalues of H also gives $\sum \lambda_i = (1) - 1 + (n-1)(1) = n - 2$ (4 points).

2. **(30 points)** Suppose A is a positive definite symmetric n by n matrix.

(a) How do you know that A^{-1} is also positive definite? (We know A^{-1} is symmetric. I just had an e-mail from the International Monetary Fund with this question.)

$\boxed{\text{Solution}}$ Since a matrix is positive-definite if and only if all its eigenvalues are positive (5 points), and since the eigenvalues of A^{-1} are simply the inverses of the eigenvalues of A, A^{-1} is also positive definite (the inverse of a positive number is positive) (5 points).

(b) Suppose Q is any **orthogonal** n by n matrix. How do you know that $Q\,A\,Q^T = Q\,A\,Q^{-1}$ is positive definite? Write down which test you are using.

$\boxed{\text{Solution}}$ Using the energy text ($x^TAx > 0$ for nonzero x), we find that $x^TQAQ^Tx = (Q^Tx)^TA(Q^Tx) > 0$ for all nonzero x as well (since Q is invertible). Using the positive eigenvalue test, since A is similar to QAQ^{-1} and similar matrices have the same eigenvalues, QAQ^{-1} also has all positive eigenvalues. (5 points for test, 5 points for application)

(c) Show that the block matrix

$$B = \begin{bmatrix} A & A \\ A & A \end{bmatrix}$$

is positive **semidefinite**. How do you know B is not positive definite?

$\boxed{\text{Solution}}$ First, since B is singular, it cannot be positive definite (it has eigenvalues of 0). However, the pivots of B are the pivots of A in the first n rows followed by 0s in the remaining rows, so by the pivot test, B is still semi-definite. Similarly, the first n upper-left determinants of B are the same as those of A, while the remaining ones are 0s, giving another proof. Finally, given a nonzero vector

$$u = \begin{bmatrix} x \\ y \end{bmatrix}$$

where x and y are vectors in $\mathbf{R}^n$, one has $u^TBu = (x+y)^TA(x+y)$ which is nonnegative (and zero when $x + y = 0$).

3. **(30 points)** This question is about the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}$$

.

(a) Find its eigenvalues and eigenvectors.

Write the vector $u(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ as a combination of those eigenvectors.

$\boxed{\text{Solution}}$ Since $\det(A - \lambda I) = \lambda^2 + 4$, the eigenvalues are $2i, -2i$ (4 points). Two associated eigenvectors are $[1 \quad -2i]^T, [1 \quad 2i]^T$, though there are many other choices (4 points). $u(0)$ is just the sum of these two vectors (2 points).

(b) Solve the equation $\frac{du}{dt} = Au$ starting with the same vector $u(0)$ at time $t = 0$.

In other words: the solution $u(t)$ is what combination of the eigenvectors of A?

$\boxed{\text{Solution}}$ One simply adds in factors of $e^{\lambda t}$ to each term, giving

$$u(t) = e^{2it} \begin{bmatrix} 1 \\ -2i \end{bmatrix} + e^{-2it} \begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

(c) Find the 3 matrices in the Singular Value Decomposition $A = U\Sigma V^T$ in two steps.

–First, compute V and Σ using the matrix A^TA.

–Second, find the (orthonormal) columns of U.

$\boxed{\text{Solution}}$ Note that $A^TA = V\Sigma^TU^TU\Sigma V^T = V\Sigma^2V^T$, so the diagonal entries of Σ are simply the positive roots of the eigenvalues of

$$A^TA = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e. $\sigma_1 = 4, \sigma_2 = 1$. Since A^TA is already diagonal, V is the identity matrix. The columns of U should satisfy $Au_1 = \sigma_1 v_1, Au_2 = \sigma_2 v_2$: by inspection, one obtains

$$u_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

18.06 Professor Edelman Quiz 3 December 5, 2011

Grading

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Your PRINTED name is: ____________________

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Please circle your recitation:

1	T 9	2-132	Kestutis Cesnavicius	2-089	2-1195	`kestutis`
2	T 10	2-132	Niels Moeller	2-588	3-4110	`moller`
3	T 10	2-146	Kestutis Cesnavicius	2-089	2-1195	`kestutis`
4	T 11	2-132	Niels Moeller	2-588	3-4110	`moller`
5	T 12	2-132	Yan Zhang	2-487	3-4083	`yanzhang`
6	T 1	2-132	Taedong Yun	2-342	3-7578	`tedyun`

1 (24 pts.)

Let $A = \begin{pmatrix} .5 & 0 & 0 \\ .5 & .9 & 0 \\ 0 & .1 & 1 \end{pmatrix}$.

1. (4 pts) True or False: The matrix A is Markov.

   True. Markov matrices have columns that sum to 1 and have non-negative entries. The answer of false applies to what is known as "Positive Markov Matrices."

2. (6 pts) Find a vector $x \neq 0$ and a scalar λ such that $A^T x = \lambda x$.

   The obvious choice is (1,1,1) with λ =1 as this is the column sum property. Also easy to see is (1,0,0) with $\lambda = 0.5$.

3. (4 pts) True or False: The matrix A is diagonalizable. (Explain briefly.)

True. The three eigenvalues, on the diagonal, are distinct.

4. (4 pts) True or False: One singular value of A is $\sigma = 0$. (Explain briefly.)

False. The matrix is nonsingular, since it has no zero eigenvalues. Nonsingular square matrices have all n singular values positive.

5. (6 pts) Find the three diagonal entries of e^{At} as functions of t.

They are $e^t, e^{0.5t}, e^{0.9t}$.

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2 (30 pts.)

1. (5 pts) An orthogonal matrix Q satisfies $Q^TQ = QQ^T = I$. What are the n singular values of Q?

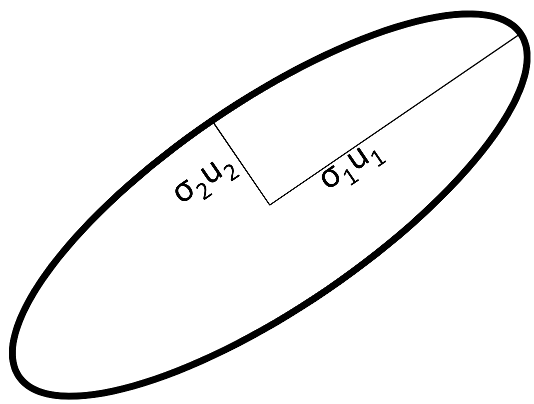
> They are all 1. The singular values are the positive square roots of the eigenvalues of $QQ^T = Q^TQ = I$.

2. (10 pts) Let $A = \begin{pmatrix} 1 & & \\ & -2 & \\ & & 3 \end{pmatrix}$. Find an SVD, meaning $A = U\Sigma V^T$, where U and V are orthogonal, and $\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix}$ is diagonal with $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0$. (Be sure that the factorization is correct and satisifies all stated requirements.)

> $A = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} \begin{pmatrix} 3 & & \\ & 2 & \\ & & 1 \end{pmatrix} \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$. The singular values are in decreasing order and are positive. One can compute AA^T and A^TA, but easier to rig the permutation matrices and correct the sign.

3. (15 pts) The 2×2 matrix $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$, where $\sigma_1 > \sigma_2 > 0$ and both u_1, u_2 and v_1, v_2 are orthonormal bases for R^2.

The set of all vectors x with $\|x\| = 1$ describes a circle in the plane. What shape best describes the set of all vectors Ax with $\|x\| = 1$? Draw a general picture of that set labeling all the relevant quantities σ_1, σ_2, u_1, u_2 and v_1, v_2. (Hint: Why are u_1, u_2 relevant and v_1, v_2 not relevant?)



The svd rotates (or reflects) the circle with V^T, scales to an ellipse with axes in the coordinate directions through Σ, and then a rotated ellipse with axes in the direction $u_1$and u_2 after U is applied. The Σ scales the x and y axes by $\sigma_1$and σ_2 respectively, and σ_1 is the longer of the two.

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3 (16 pts.)

1. (6 pts) Let $x \neq 0$ be a vector in R^3. How many eigenvalues of $A = xx^T$ are positive? zero? negative? (Explain your answer.) (Hint: What is the rank?)

A is symmetric pos semidefinite and rank 1, so there are 1 positive, 2 zero, and no negative eigenvalues.

2. (6 pts) a) What are the possible eigenvalues of a projection matrix?

0 and 1 (Since $P^2 = P$, $\lambda^2 = \lambda$.)

   b) True or False: every projection matrix is diagonalizable.

True, every projection matrix is symmetric, hence diagonalizable.

3. (4 pts) True or False: If every eigenvalue of A is 0, then A is similar to the zero matrix.

False. A Jordan block with zero eigenvalues is not similar to the zero matrix for $n > 1$.

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4 (30 pts.)

Consider the matrix $A = \begin{pmatrix} x & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ with parameter x in the (1,1) position.

1. (10 pts) Specify all numbers x, if any, for which A is positive definite. (Explain briefly.)

No x, the matrix is clearly singular with two equal rows and two equal columns.

2. (10 pts) Specify all numbers x, if any, for which e^A is positive definite. (Explain briefly.)

The eigenvalues of e^A are the exponentials of the eigenvalues of the matrix A. Since A is symmetric the eigenvalues are real, and thus exponentials are positive. A symmetric matrix with positive eigenvalues is positive definite.

3. (10 pts) Find an x, if any, for which $4I - A$ is positive definite. (Explain briefly.)

> One can take any $x < 3$. The easiest choice is $x = 1$. With this guess the matrix has two eigenvalues 0 and one eigenvalue 3 both less than 4, so $4 - \lambda > 0$ for all three eigenvalues. Systematically, one can consider the three upper left determinants of $4I - A$ which are $4 - x$, $11 - 3x$, and $24 - 8x$. They are all positive if and only if $x < 3$.

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18.06 Quiz 3 Professor Strang May 6, 2011

Your PRINTED name is ____________ 1.

Your Recitation Instructor (and time) is ____________ 2.

Instructors: (Hezari)(Pires)(Sheridan)(Yoo) 3.

Please show enough work so we can see your method and give due credit.

1. (a) Find two eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}.$$

$p(\lambda) = \det(A - \lambda I) = \lambda^2 - 7\lambda + 10 = (\lambda-2)(\lambda-5) = 0 \Rightarrow \lambda_1 = 2$, $\lambda_2 = 5$

$N(A - 2I) = N\left(\begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$. $\to x_1$

$N(A - 5I) = N\left(\begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$. $\to x_2$

(b) Express any vector $u_0 = \begin{bmatrix} a \\ b \end{bmatrix}$ as a combination of the eigenvectors.

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S^{-1} u_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a-b \\ b \end{bmatrix}$

So $u_0 = c_1 x_1 + c_2 x_2 = (a-b)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(c) What is the solution $u(t)$ to $\frac{du}{dt} = Au$ starting from $u(0) = u_0$?

$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = (a-b)e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

(d) Find a formula $u_k =$ ____________ for the solution to $u_{k+1} = Au_k$ which starts from that vector u_0. **Set $k = -1$ to find $A^{-1}u_0$.**

$u_k = A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 = (a-b)2^k\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b5^k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$u_{-1} = (a-b)2^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b5^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{a-b}{2} + \frac{b}{5} \\ \frac{b}{5} \end{bmatrix}$

2. This problem is about the matrix

$$A = \begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix}.$$

(a) Find all eigenvectors of A. Exactly why is it impossible to diagonalize A in the form $A = S\Lambda S^{-1}$?

$p(\lambda) = (\lambda - \sqrt{2})(\lambda - \sqrt{2}) = 0 \Rightarrow \lambda_1 = \lambda_2 = \sqrt{2}$ repeated eigenvalues

$N(A - \sqrt{2} I) = N\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

There are not enough independent eigenvectors to form an invertible matrix S with eigenvectors as its columns.

(b) Find the matrices U, Σ, V^T in the Singular Value Decomposition $A = U\,\Sigma\,V^T$.

Tell me *two orthogonal vectors* v_1, v_2 in the plane so that Av_1 and Av_2 are also orthogonal.

$B = A^TA = \begin{bmatrix} \sqrt{2} & 0 \\ 1 & \sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}$

eigenvalues of B $\rightarrow P_B(\lambda) = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4) = 0 \Rightarrow \lambda_1 = 4,\ \lambda_2 = 1$ for B.

$\Rightarrow \sigma_1 = \sqrt{\lambda_1} = 2,\ \sigma_2 = \sqrt{\lambda_2} = 1. \Rightarrow \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$

eigenvectors of B $\rightarrow \begin{cases} N(B - 4I) = N\begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}\right\} \xRightarrow[\text{normalize}]{} v_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \\ N(B - I) = N\begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}\right\} \Rightarrow v_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}. \end{cases}$

$u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}.$

$u_2 = \frac{Av_2}{\sigma_2} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}.$

$A = [u_1 | u_2]\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}[v_1 | v_2]^T.$

$\boxed{v_1 \perp v_2 \text{ and } Av_1 \perp Av_2 \text{ because } u_1 \perp u_2.}$

(c) Find a matrix B that is similar to A (but different from A).

Show that A and B meet the requirement to be similar *(what is it?)*.

we say $B \underset{\text{similar}}{\sim} A$ if $B = MAM^{-1}$ for some invertible M.

choose for example $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. (You can choose any M you like!)

Then $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \cdots = \begin{bmatrix} \sqrt{2} & 0 \\ 1 & \sqrt{2} \end{bmatrix}$

and $B \neq A$ but $B \sim A$!

3. Suppose A is a real m by n matrix.

(a) Prove that the symmetric matrix A^TA has the property $x^T(A^TA)x \geq 0$ for every vector x in R^n. Explain each step in your reason.

$x^T(A^TA)x = (x^TA^T)Ax = (Ax)^TAx = (Ax)\cdot(Ax) \geq 0.$

(b) According to part (a), the matrix A^TA is positive semidefinite at least — and possibly positive definite. Under what condition on A is A^TA positive definite?

we want to see under what condition

$x^T(A^TA)x = 0$ implies $x = 0$.

So let $x^TA^TAx = 0$. By (a) we get $(Ax)\cdot(Ax) = 0$.

So $Ax = 0$. Hence to get $x = 0$ from $Ax = 0$,

we need $N(A) = \{0\}$ or A must have independent columns.

(c) If $m < n$ prove that A^TA is *not* positive definite.

we use (b) and show that if $m < n$ then $N(A) \neq \{0\}$.

OK! we know that $\dim N(A) = n - r$ where $r = \text{rank}(A)$.

But $r \leq m$. So

$\dim N(A) = n - r \geq n - m$. Since $m < n$, $n - m > 0$

therefore: $N(A) \neq \{0\}$.

18.06 Professor Edelman Quiz 3 December 3, 2012

Grading

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Your PRINTED name is: ________________________

Please circle your recitation:

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2	T 10	2-132	Rosalie Belanger-Rioux	2-331	3-5029	`robr`
3	T 10	2-146	Andrey Grinshpun	2-349	3-7578	`agrinshp`
4	T 11	2-132	Rosalie Belanger-Rioux	2-331	3-5029	`robr`
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6	T 1	2-132	Tiankai Liu	2-491	3-4091	`tiankai`
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1 (16 pts.)

a) (4 pts.) Suppose C is $n \times n$ and positive definite. If A is $n \times m$ and $M = A^T CA$ is not positive definite, find the smallest eigenvalue of M. (Explain briefly.)

Solution. The smallest eigenvalue of M is 0.

The problem only asks for brief explanations, but to help students understand the material better, I will give lengthy ones.

First of all, note that $M^T = A^T C^T A = A^T CA = M$, so M is symmetric. That implies that all the eigenvalues of M are real. (Otherwise, the question wouldn't even make sense; what would the "smallest" of a set of complex numbers mean?)

Since we are assuming that M is *not* positive definite, at least one of its eigenvalues must be nonpositive. So, to solve the problem, we just have to explain why M cannot have any negative eigenvalues. The explanation is that M is **positive semidefinite**. That's the buzzword we were looking for.

Why is M positive semidefinite? Well, note that, since C is positive definite, we know that for every vector y in $\mathbb{R}^n$

$$y^T Cy \geqslant 0,$$

with equality if and only if y is the zero vector. Then, for any vector x in $\mathbb{R}^m$, we may set $y = Ax$, and see that

$$x^T Mx = x^T A^T CAx = (Ax)^T C(Ax) \geqslant 0. \tag{*}$$

Since M is symmetric, the fact that $x^T Mx$ is always non-negative means that M is positive semidefinite. Such a matrix never has negative eigenvalues. Why? Well, if M did have a negative eigenvalue, say $\lambda < 0$, with a corresponding eigenvector $v \neq 0$, then

$$v^T Mv = v^T(\lambda v) = \lambda v^T v = \lambda \|v\|^2 < 0,$$

which would contradict $(*)$ above, which is supposed to hold for *every* x in $\mathbb{R}^m$. $\square$

Remark: Some students wrote that M is similar to C, but this is totally false. In the given problem, if $m \neq n$, then M and C don't even have the same dimensions, so they cannot possibly be similar. (Remember that two $n \times n$ matrices A and B are similar if there is an invertible $n \times n$ matrix M such that $A = M^{-1}BM$, which isn't usually the same thing as M^TBM, unless M is an orthogonal matrix.)

b) (12 pts.) If A is symmetric, which of these four matrices are necessarily positive definite? A^3, $(A^2+I)^{-1}$, $A+I$, e^A. (Explain briefly.)

Solution. The answer is that $(A^2+I)^{-1}$ and e^A have to be positive definite, but A^3 and $A+I$ don't.

The key is to use **the $Q\Lambda Q^{-1}$ factorization**. Let me remind you what that is. Since A is symmetric, there is an *orthonormal basis* of $\mathbb{R}^n$ (if A is an $n \times n$ matrix) consisting of eigenvectors $q_1, q_2, \ldots, q_n$ of A, and the corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all *real*. Form an $n \times n$ matrix Q whose columns are these n eigenvectors $q_1, q_2, \ldots, q_n$, and let Λ be a diagonal $n \times n$ matrix whose diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$, so that $A = Q\Lambda Q^{-1}$. (In case you're wondering, it would also be correct to write $A = Q\Lambda Q^T$. Since Q is an orthogonal matrix, $Q^{-1} = Q^T$.)

Note that all four matrices we are asked to discuss are symmetric. So the question of positive definiteness is just a question about the positivity of their eigenvalues.

- $A^3 = (Q\Lambda Q^{-1})^3 = Q\Lambda^3Q^{-1}$, so A^3 **is *similar* to** Λ^3, and these two matrices have the same eigenvalues. But Λ^3 is just the diagonal matrix whose diagonal entries are ${\lambda_1}^3, {\lambda_2}^3, \ldots, {\lambda_n}^3$. Do these numbers all have to be positive? Of course not. For example, we could have $A = \Lambda = 0$, the zero matrix. Then $A^3 = \Lambda^3 = 0$, which isn't positive definite.

- Before we discuss $(A^2+I)^{-1}$, let's check that this actually makes sense, i.e., that A^2+I is really invertible. Well,

$$A^2 + I = (Q\Lambda Q^{-1})^2 + I = Q(\Lambda^2 + I)Q^{-1}.$$

  Now $\Lambda^2 + I$ is a diagonal matrix whose diagonal entries ${\lambda_1}^2 + 1$, ${\lambda_2}^2 + 1$, $\ldots$, ${\lambda_n}^2 + 1$ are all nonzero, so $\Lambda^2 + I$ really is invertible. Then $A^2 + I$, which is similar to $\Lambda^2 + I$, must also be invertible, and in fact we can write down its inverse:

$$(A^2 + I)^{-1} = Q(\Lambda^2 + 1)^{-1}Q^{-1}.$$

  Now $(A^2 + I)^{-1}$ **is *similar* to** $(\Lambda^2 + 1)^{-1}$, and these two matrices have the same eigenvalues, namely $({\lambda_1}^2 + 1)^{-1}$, $({\lambda_2}^2 + 1)^{-1}, \ldots$, $({\lambda_n}^2 + 1)^{-1}$. These eigenvalues are all positive, because $(\lambda^2 + 1)^{-1} > 0$ for any real number λ. So $(A^2 + I)^{-1}$ is positive definite.

- $A+I = Q(\Lambda+I)Q^{-1}$, so $A+I$ **is *similar* to** $\Lambda+I$, and these two matrices have the same eigenvalues, namely $\lambda_1 + 1$, $\lambda_2 + 1, \ldots$, $\lambda_n + 1$. Do these numbers all have to be positive? Of course not. For example, we could have $A = -I$. Then $A + I = 0$, which isn't positive definite.

- Finally, we have e^A. Note that

$$e^A = e^{Q\Lambda Q^{-1}} = \sum_{k=0}^{\infty} \frac{1}{k!}(Q\Lambda Q^{-1})^k = \sum_{k=0}^{\infty} \frac{1}{k!} Q\Lambda^k Q^{-1} = Q\left(\sum_{k=0}^{\infty} \frac{1}{k!}\Lambda^k\right) Q^{-1} = Qe^{\Lambda}Q^{-1},$$

  so e^A **is *similar* to** e^{Λ}. But e^{Λ} is just the diagonal matrix with diagonal entries $e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}$, which are all positive, because $e^{\lambda} > 0$ for all real λ. So the eigenvalues of e^A are all positive, and e^A must be positive definite.

You see, diagonalization allows us to reduce a problem about matrices to a problem about real numbers. The general philosophy is this: If A is similar to a diagonal matrix to Λ, then often some expression[1] in A is similar to the same expression in Λ, and the expression in

[1] Here I mean a polynomial (e.g., A^3 or $A + I$; think of I as being akin to the constant 1), a rational function (e.g., $(A^2 + I)^{-1}$), or a convergent power series (e.g., e^A) in the variable A alone. We do *not* allow expressions involving A^T in addition to A, or anything more complicated than that.

Λ can be computed just by plugging in the diagonal entries one by one. So the question basically comes to this: which of the functions λ^3, $(\lambda^2+1)^{-1}$, $\lambda+1$, e^λ is everywhere positive (i.e., positive for all real λ)? Of course, your solution should explain *why* it comes to this. $\square$

Remarks: (i) Some students thought that A must itself be positive definite. Some even wrote a "proof" that all symmetric matrices are positive definite! Please disabuse yourself of this notion. Positive definite matrices (at least the ones with real entries) are required to be symmetric, but there are lots of symmetric matrices that aren't positive definite: for example, 0 and $-I$. (ii) Some students discussed only the matrices that *are* necessarily positive definite, and didn't write anything at all about A^3 and $I+A$. A complete solution should convince people that it is correct. And in order to convince people that "$(A^2+I)^{-1}$ and e^A" is the correct answer, one should explain *both* why these two matrices are necessarily positive definite, *and* why the other two aren't.

2 (30 pts.)

Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

a) (6 pts.) What are the eigenvalues of A ? (Explain briefly.)

This matrix is upper triangular. For such a matrix, the determinant is the product of the diagonal entries. Using this observation, if we try to compute $|A - xI|$, we find $-x^3$. This implies that the only eigenvalue is 0 with multiplicity 3.

b) (6 pts.) What is the rank of A?

It is clear that the last two columns of A are pivot columns. Therefore, the rank is 2.

c) (6 pts.) What are the singular values of A?

The singular values of A are the square roots of the eigenvalues of $A^T A$.

$$A^T A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

We have $|xI - A^T A| = x((x-1)(x-2) - 1) = x(x^2 - 3x + 1)$

The roots are 0, $\frac{3+\sqrt{5}}{2}$, $\frac{3-\sqrt{5}}{2}$. Therefore, the singular values of A are 0, $\sqrt{\frac{3+\sqrt{5}}{2}}$ and $\sqrt{\frac{3-\sqrt{5}}{2}}$.

d) (6 pts.) What is the Jordan form of A? (Explain briefly.)

In general, the Jordan form has zeroes everywhere except on the diagonal where you put the eigenvalues on the second diagonal where you have 1 and 0. Note that the matrix A as it is is not in Jordan normal form because you have a 1 in the upper right corner. There are 3 possibilities for what the Jordan form can be. One with two ones over the diagonal and two with one one and one zero. To determine which is the actual Jordan form, you can look at the rank. We know that A has rank 2 and the Jordan form is similar to A so it must have rank 2 as well. Therefore, the only possibility is :

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

e) (6 pts.) Compute in simplest form e^{tA}.

We can use the series expression for e^{tA}. In general this is an infinite sum which is unpleasant but in this particular case, the powers of A quickly become zero. Indeed, we have :

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = 0$$

Therefore, we have :

$$e^{tA} = I + tA + \frac{t^2}{2}A^2 = \begin{pmatrix} 1 & t & t + t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

3 (28 pts.)

We are told that A is 2×2, symmetric, and Markov and one of the real eigenvalues is y with $-1 < y < 1$.

a) (7 pts.) What is this matrix A in terms of y?

We have a symmetric matrix, hence $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. Also, it is Markov, so we want $a+b=1$ and $b+c=1$, with all entries non-negative. So $a=c$ and we have $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$.

Now, we want the eigenvalues of this matrix to be y and 1 (recall that Markow matrices ALWAYS have 1 as an eigenvalue, with the all-ones vector as the corresponding eigenvector). But we know the eigenvalues of A satisfy $det(A-\lambda I)=0$, or $(a-\lambda)^2-b^2=0$ or $a-\lambda=\pm b$. So $\lambda_1 = a+b = 1$ and $\lambda_2 = a-b = y$ (since $b \geq 0$). Using $a+b=1$ into $a-b=y$ we get $2a-1=y$ or $a=(y+1)/2$, and then $b=1-a=(1-y)/2$. So we have found our symmetric Markov matrix with eigenvalues 1 and y: $A = \begin{pmatrix} (1+y)/2 & (1-y)/2 \\ (1-y)/2 & (1+y)/2 \end{pmatrix}$.

b) (7 pts.) Compute the eigenvectors of A.

An easy way to find the eigenvector corresponding to the eigenvalue 1 is to recall we have a symmetric Markov matrix, so columns add to 1 but rows too, hence the constant vector will be an eigenvector. So for $\lambda_1 = 1$ we have $v_1 = (1/2\ 1/2)^T$. And for $\lambda_2 = y$, we find a vector in the nullspace of $A - yI = \begin{pmatrix} (1-y)/2 & (1-y)/2 \\ (1-y)/2 & (1-y)/2 \end{pmatrix}$. This is easy, we find $v_2 = (1/2\ \ -1/2)^T$.

c) (7 pts.) What is A^{2012} in simplest form?

We have now diagonalized A: $A = S\Lambda S^{-1}$, where columns of S are the eigenvectors and Λ is a diagonal matrix with 1 and y. So we have

$$A^{2012} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}^{2012} \frac{1}{-1/2} \begin{pmatrix} -1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

so that

$$A^{2012} = \begin{pmatrix} (1+y^{2012})/2 & (1-y^{2012})/2 \\ (1-y^{2012})/2 & (1+y^{2012})/2 \end{pmatrix}$$

d) (7 pts.) What is $\lim_{n\to\infty} A^n$ in simplest form? (Explain Briefly.)

From the above, and the fact that $-1 < y < 1$, we can see clearly that

$$\lim_{n\to\infty} A^n = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

Another way to reason: we know the steady-state is the eigenvector of the dominating eigenvalue, in this case $\lambda_1 = 1$ and so $v_1 = (1/2\ 1/2)^T$. But we are asking for the matrix which will give us this steady-state, no matter what probability vector we start with. And so its column space has to be along the line of v_1, and no bigger. But there is only one vector proportional to v_1 which could also be a column of a Markov matrix, i.e. whose entries sum to 1. So both columns of the answer have to be v_1. (You could also use the fact that the answer should be symmetric too.)

4 (26 pts.)

a) (5 pts.) P is a three by three permutation matrix. List all the possible values of a singular value. (Explain briefly.)

A permutation matrix satisfies $P^T P = I$ which has all ones as eigenvalues, so all the singular values of P are $\sqrt{1} = 1$.

b) (9 pts.) P is a three by three permutation matrix. List all the possible values of an eigenvalue. (Explain briefly.)

We will do part (c) first.

c) (12 pts.) There are six 3×3 permutation matrices. Which are similar to each other? (Explain briefly.)

Let's list the six matrices. There is the identity matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

There are the three transposition matrices:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

There are the two three-cycles:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

If two matrices have different traces, then they must have different eigenvalues and so are not similar. The trace of the identity is 3, the trace of the transpositions is 1, and the trace of the three cycles is 0.

We first show that all of the transpositions are similar to each other. Every permutation matrix P satisfies

$$P\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

so they all have eigenvalue $\lambda_1 = 1$. Note that each of the transposition matrices has a fixed point and so has a standard basis vector as an eigenvector with eigenvalue $\lambda_2 = 1$. For example,

$$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Since they all have trace 1, their final eigenvalue λ_3 must be -1 so that $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Thus we have shown that the transposition matrices all have the eigenvalue 1 repeated twice with two linearly independent eigenvectors as well as the eigenvalue -1. Therefore, they are similar as each of their Jordan canonical forms must be

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}.$$

Finally, we show the two three-cycles are similar to each other. As before, they have eigenvalue $\lambda_1 = 1$ corresponding to the all ones vector. Their other two eigenvalues must satisfy $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Then $\lambda_2 + \lambda_3 = -1$. However, we must have that $|\lambda_2| = |\lambda_3| = 1$ since the permutation matrices are orthonormal. Note that if λ_2, λ_3 were real then they must each be 1 or -1 and it is impossible to have $\lambda_2 + \lambda_3 = -1$. Therefore, they are complex and must satisfy $\lambda_2 = \overline{\lambda_3}$. Then their real parts are the same and must add to -1, so they each have real part $-1/2$. Using that $|\lambda_2| = |\lambda_3| = 1$, we get that one of λ_2, λ_3 must be $1/2+i\sqrt{3}/2$ and the other must be $1/2 - i\sqrt{3}/2$. Therefore, the three cycles both have the same eigenvalues, namely the three different cubed roots of 1 in the complex plane, and so are similar.

Returning to part (b) of the problem, we have shown that the possible eigenvalues are the square roots of 1 and the cubed roots of 1.

18.06 Professor Strang Quiz 3 – Solutions May 7th, 2012

Grading

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Your PRINTED name is: ____________________

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r05	T 1	4-153	Dimiter Ostrev	ostrev
r06	T 1	4-159	Uhi Rinn Suh	ursuh
r07	T 1	66-144	Ailsa Keating	ailsa
r08	T 2	66-144	Niels Martin Moller	moller
r09	T 2	4-153	Dimiter Ostrev	ostrev
r10	ESG		Gabrielle Stoy	gstoy

1 (33 pts.)

Suppose an $n \times n$ matrix A has n independent eigenvectors $x_1, \ldots, x_n$. Then you could write the solution to $\frac{du}{dt} = Au$ in three ways:

$$u(t) = e^{At}u(0), \quad \text{or}$$

$$u(t) = Se^{\Lambda t}S^{-1}u(0), \quad \text{or}$$

$$u(t) = c_1 e^{\lambda_1 t}x_1 + \ldots + c_n e^{\lambda_n t}x_n.$$

Here, $S = [x_1 \mid x_2 \mid \ \ldots \ \mid x_n]$.

(a) From the definition of the exponential of a matrix, show why e^{At} is the same as $Se^{\Lambda t}S^{-1}$.

Solution. Recall that $A = S\Lambda S^{-1}$, and $A^k t^k = S\Lambda^k t^k S^{-1}$. Then, definition of the exponential:

$$\boxed{\exp(At) = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = S\left(\sum_{k=0}^{\infty} \frac{\Lambda^k t^k}{k!}\right)S^{-1} = Se^{\Lambda t}S^{-1}.}$$

□

(b) How do you find $c_1, \ldots, c_n$ from $u(0)$ and S?

Solution. Since $e^0 = 1$, we see that

$$u(0) = c_1 x_1 + \ldots + c_n x_n = S\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},$$

where we used the definition of the matrix product. Thus the answer is:

$$\boxed{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = S^{-1}u(0).}$$

□

(c) For this specific equation, write $u(t)$ in any one of the (added: latter two of the) three forms, using *numbers* not symbols: You can choose which form.

$$\frac{du}{dt} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} u, \quad \text{starting from} \quad u(0) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

Solution. We diagonalize A and get:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

Thus $c = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so for the second form

$$\boxed{u(t) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix},}$$

while in the third form:

$$\boxed{u(t) = e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.}$$

□

2 (30 pts.)

This question is about the real matrix

$$A = \begin{bmatrix} 1 & c \\ 1 & -1 \end{bmatrix}, \quad \text{for} \quad c \in \mathbb{R}.$$

(a) - Find the eigenvalues of A, depending on c.

- For which values of c does A have real eigenvalues?

Solution. Since $0 = \mathrm{tr} A = \lambda_1 + \lambda_2$, we see that $\lambda_2 = -\lambda_1$.

Also, $-1 - c = \det A = -\lambda_1^2$. Thus,

$$\boxed{\lambda = \pm\sqrt{1+c}.}$$

Therefore,

$$\boxed{\text{the eigenvalues are real precisely when } c \geq -1.}$$

$\square$

(b) - For one particular value of c, convince me that A is similar to both the matrix

$$B = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix},$$

and to the matrix

$$C = \begin{bmatrix} 2 & 2 \\ 0 & -2 \end{bmatrix}.$$

- Don't forget to say which value c this happens for.

Solution. If two matrices are similar, then they do have the same eigenvalues (those are $2, -2$ for both B and C). Here we must therefore have $0 = \mathrm{tr}A$ and $-1 - c = \det A = -4$. We see that this happens precisely when $c = 3$, where we check that indeed the eigenvalues are $2, -2$. However, this does not guarantee that they are similar - and hence is not convincing.

$\boxed{\text{Convincing: The eigenvalues } 2, -2 \text{ are different, so both } A, B \text{ and } C \text{ are diagonalizable,}}$

with the same diagonal matrix (for example to $\Lambda = B$!). Therefore A, B and C are all similar when $c = 3$. $\square$

(c) For one particular value of c, convince me that A cannot be diagonalized. It is not similar to a diagonal matrix Λ, when c has that value.

- Which value c?

- Why not?

Solution. As we saw above, $\text{tr}A = 0$, so regardless of c the eigenvalues come in pairs $\lambda_2 = -\lambda_1$. This means that whenever $\lambda_1 \neq 0$, we have two different eigenvalues, and hence A is diagonalizable (not what we're after).

Thus we need $\lambda_1 = \lambda_2 = 0$, a repeated eigenvalue, which happens when $c = -1$ (so $\det A = 0$) as the only suspect – does it work?

$$\boxed{\text{Convincing: For } c = -1, \text{ we have } N(A - 0 \cdot I) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}}$$

With only a 1-dimensional space of eigenvectors for the matrix, we are convinced that A is not diagonalizable for $c = -1$. $\square$

3 (37 pts.)

(a) Suppose A is an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

- What is the largest number real number c that can be subtracted from the diagonal entries of A, so that $A - cI$ is positive semidefinite?

- Why?

Solution. - We first realize that: If A is symmetric, then $A - cI$ is also symmetric, since in general $(A + B)^T = A^T + B^T$ (simple, but very important to check!).

- Then we realize that the eigenvalues of $A - cI$ are $\lambda_1 - c \leq \lambda_2 - c \leq \ldots \leq \lambda_n - c$. Therefore:

$$\boxed{c = \lambda_1 \text{ is the largest that can ensure positive semidefiniteness (and it does).}}$$

$\square$

(b) Suppose B is a matrix with independent columns.

- What is the nullspace $N(B)$?

- Show that $A = B^T B$ is positive definite. Start by saying what that means about $x^T Ax$.

Solution. - Then $Bx = 0$ only has the zero solution, so $N(B) = \{0\}$.

- Again, we start by observing that $A^T = A$ is symmetric. Then we recall what positive definite means (the "energy" test):

$$\boxed{x^T Ax > 0 \quad \text{whenever} \quad x \neq 0.}$$

Thus, we see here (by definition the inner product property of the transpose of a matrix):

$$\boxed{x^T Ax = x^T(B^T(Bx)) = (Bx)^T(Bx) = \|Bx\|^2 \geq 0.}$$

So $A = B^T B$ is positive semidefinite. But finally, the equality $\|Bx\|^2 = 0$, only happens when $Bx = 0$ which by $N(B) = \{0\}$ means $x = 0$. □

(c) This matrix A has rank $r = 1$:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

- Find its largest singular value σ from $A^T A$.

- From its column space and row space, respectively, find unit vectors u and v so that

$$Av = \sigma u, \quad \text{and} \quad A = u\sigma v^T.$$

- From the nullspaces of A and A^T put numbers into the full SVD (Singular Value Decomposition) of A:

$$A = \begin{bmatrix} | & | \\ u & \dots \\ | & | \end{bmatrix} \begin{bmatrix} \sigma & 0 \\ & \\ 0 & \dots \end{bmatrix} \begin{bmatrix} | & | \\ v & \dots \\ | & | \end{bmatrix}^T.$$

Solution. We compute:

$$A^T A = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}.$$

Thus the two eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 10$, and $\sigma = \sqrt{10}$. For v, we find a vector in $N(A^T A - 10I)$, and normalize to unit length:

$$v = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

Then we find u using

$$u = \frac{Av}{\sigma} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}.$$

Since we have the orthogonal sums of subspaces $\mathbb{R}^2 = \mathbb{R}^m = c(A) \oplus N(A^T)$ and also $\mathbb{R}^2 = \mathbb{R}^n = c(A^T) \oplus N(A)$, we need to find one unit vector from each of $N(A)$ and $N(A^T)$ and augment to v and u, respectively:

$$v_2 = \begin{bmatrix} 1\sqrt{2} \\ -1\sqrt{2} \end{bmatrix} \in N(A),$$

$$u_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \in N(A^T),$$

Thus, we finally see the full SVD:

$$\boxed{A = U\Sigma V^T = \begin{bmatrix} 1/\sqrt{5} & -2\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T.}$$

We remember, as a final check, to verify that the square matrices U and V both contain orthonormal bases of $\mathbb{R}^2$ as they should:

$$UU^T = I_2,$$

$$VV^T = I_2.$$

□

18.06 Professor Edelman Quiz 3 December 4, 2013

Grading

1

2

3

Your PRINTED name is: ____________________

Please circle your recitation:

1	T 9	Dan Harris	E17-401G	3-7775	`dmh`
2	T 10	Dan Harris	E17-401G	3-7775	`dmh`
3	T 10	Tanya Khovanova	E18-420	4-1459	`tanya`
4	T 11	Tanya Khovanova	E18-420	4-1459	`tanya`
5	T 12	Saul Glasman	E18-301H	3-4091	`sglasman`
6	T 1	Alex Dubbs	32-G580	3-6770	`dubbs`
7	T 2	Alex Dubbs	32-G580	3-6770	`dubbs`

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1 (32 pts.) (2 points each)

There are sixteen 2×2 matrices whose entries are either 0 or 1. For each of the sixteen, write down the two singular values. Time saving hint: if you really understand singular values, then there is really no need to compute AA^T or A^TA, but it is okay if you must.

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2 (30 pts.) (3 points each: Please circle true or false, and either way, explain briefly.)

a) If A and B are invertible, then so is $(A+B)/2$. True? False? (Explain briefly).

b) If A and B are Markov, then so is $(A+B)/2$. True? False? (Explain briefly).

c) If A and B are positive definite, then so is $(A+B)/2$. True? False? (Explain briefly).

d) If A and B are diagonalizable, then so is $(A+B)/2$. True? False? (Explain briefly).

e) If A and B are rank 1, then so is $(A+B)/2$. True? False? (Explain briefly).

f) If A is symmetric then so is e^A. True? False? (Explain briefly).

g) If A is Markov then so is e^A. True? False? (Explain briefly).

h) If A is symmetric, then e^A is positive definite. True? False? (Explain briefly).

i) If A is singular, then so is e^A. True? False? (Explain briefly).

j) If A is orthogonal, then so is e^A. True? False? (Explain briefly).

3 (38 pts.)

Let $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$.

a) (10 pts.) Find a nonzero solution $y(t)$ in R^2 to $dy/dt = Ay$ that is independent of t, in other words, $y(t)$ is a constant vector in R^2. (Hint: why would a vector in the nullspace of A have this property?)

b) (10 pts.) Show that e^{At} is Markov for every value of $t \geq 0$.

c) (10 pts.) What is the limit of e^{At} as $t \to \infty$?

d) (8 pts.) What is the steady state vector of the Markov matrix e^{A} ?

18.06 F13 EXAM 3 SOLUTIONS

Q1. Useful facts:

- Row/column swaps don't alter singular values
- Sum of squares of SVs = sum of squares of entries of matrix
- For symmetric matrix, SVs = |eigenvalues|

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ SVs 0, 0

$\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]$ All related to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ by row/column swaps. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ symmetric, evals $(1, 0)$ $\Rightarrow$ SVs 1, 0

$\left[\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right]$ All same as $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

Rank 1 $\therefore$ one SV is 0. If σ is the other,

$$0^2 + \sigma^2 = 1^2 + 1^2$$

so $\sigma = \sqrt{2}$.

$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$ Related by col swap. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has evals 1, 1, so SVs 1, 1.

Q1 cont.

$$\left[\begin{pmatrix}1&1\\1&0\end{pmatrix}, \begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix}, \begin{pmatrix}0&1\\1&1\end{pmatrix}\right]$$

All same as $\begin{pmatrix}1&1\\1&0\end{pmatrix}$, which has evals

$$\frac{1+\sqrt{5}}{2}, \quad \frac{1-\sqrt{5}}{2}.$$

$\frac{1-\sqrt{5}}{2}$ is negative, so SVs are $\frac{\sqrt{5}+1}{2}$, $\frac{\sqrt{5}-1}{2}$.

$\left[\begin{pmatrix}1&1\\1&1\end{pmatrix}\right]$ Symmetric, evals $(0,2)$

$\therefore$ SVs $0, 2$.

Q2. a) No. e.g. $A=I$, $B=-I$, $\frac{A+B}{2}=0$ not invertible.

b) True.

kth column sum of $\frac{A+B}{2}$

$$= \frac{(\text{kth column sum of } A) + (\text{kth column sum of } B)}{2}$$

$$= \frac{1+1}{2} = 1$$

Q2. cont.

c). True.

$$x^T \left(\frac{A+B}{2}\right) x$$

$$= \frac{x^T A x + x^T B x}{2} > 0 \qquad \text{if } x^T A x,\ x^T B x > 0$$

d) False. $A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$ are both diagonalizable,

but $\frac{A+B}{2} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not.

e) False. $A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ are both rank 1,

but $\frac{A+B}{2} = I$ is not.

f) True. In the sum

$$e^A = I + A + \frac{A^2}{2!} + \cdots$$

every term is symmetric, so the sum is symmetric.

g) False. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is Markov but $e^A = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$ is not.

Q2 cont.

h) True. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A, then $e^{\lambda_1}, \dots, e^{\lambda_n}$ are the eigenvalues of e^A, and e^{λ_i} is positive since λ_i is real.

i) False. $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is singular but $e^A = I$ is not.

j) False. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is orthogonal but $e^A = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$ is not.

Q3. a) $y(t)$ is constant $\Rightarrow \frac{dy}{dt} = 0$.

So $Ay = \frac{dy}{dt} \iff Ay = 0$

$\therefore y \in \text{null } A$.

Can take $y = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

b) We calculate e^{At}. A has evals $-2, 0$ with corresponding evecs $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So

$$A = S \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} S^{-1}$$

$$At = S \begin{pmatrix} -2t & 0 \\ 0 & 0 \end{pmatrix} S^{-1} \quad \text{where} \quad S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$e^{At} = S \begin{pmatrix} e^{-2t} & 0 \\ 0 & 1 \end{pmatrix} S^{-1}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t} & -e^{-2t} \\ 1 & 1 \end{pmatrix}$$

Q3. cont

$$= \frac{1}{2}\begin{pmatrix} e^{-2t}+1 & -e^{-2t}+1 \\ -e^{-2t}+1 & e^{-2t}+1 \end{pmatrix}$$

$\frac{1}{2}((e^{-2t}+1)+(-e^{-2t}+1))=1$

$\frac{1}{2}((-e^{-2t}+1)+(e^{-2t}+1))=1$

so Markov.

c) As $t\to\infty$, $e^{-2t}\to 0$, so

$$e^{At} \to \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

d) Steady state of e^A

$$= \lim_{n\to\infty} e^{nA} v$$ where v is any probability vector e.g. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= \lim_{t\to\infty} e^{tA} v$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

18.06 Quiz 3 — Professor Strang — May 10, 2013

Please PRINT your name ____________________ **1.**

2.

Please Circle Your Recitation: **3.**

r1	T	10	36-156	Russell Hewett	r7	T	1	36-144	Vinoth Nandakumar
r2	T	11	36-153	Russell Hewett	r8	T	1	24-307	Aaron Potechin
r3	T	11	24-407	John Lesieutre	r9	T	2	24-307	Aaron Potechin
r4	T	12	36-153	Stephen Curran	r10	T	2	36-144	Vinoth Nandakumar
r5	T	12	24-407	John Lesieutre	r11	T	3	36-144	Jennifer Park
r6	T	1	36-153	Stephen Curran					

(1) **(40 pts)**

In all of this problem, the 3 by 3 matrix A has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with independent eigenvectors x_1, x_2, x_3.

(a) What are the trace of A and the determinant of A?

(b) Suppose: $\lambda_1 = \lambda_2$. Choose the true statement from 1, 2, 3:

1. A can be diagonalized. Why?
2. A can not be diagonalized. Why?
3. I need more information to decide. Why?

(c) From the eigenvalues and eigenvectors, how could you find the matrix A? Give a formula for A and explain each part carefully.

(d) Suppose $\lambda_1 = 2$ and $\lambda_2 = 5$ and $x_1 = (1, 1, 1)$ and $x_2 = (1, \text{-}2, 1)$. Choose λ_3 and x_3 so that A is symmetric positive semidefinite but not positive definite.

(2) **(30 pts.)**

Suppose A has eigenvalues $1, \frac{1}{3}, \frac{1}{2}$ and its eigenvectors are the columns of S:

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{with} \quad S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

(a) What are the eigenvalues and eigenvectors of A^{-1}?

(b) What is the general solution (with 3 arbitrary constants c_1, c_2, c_3) to the differential equation $du/dt = Au$? Not enough to write e^{At}. Use the c's.

(c) Start with the vector $u = (1, 4, 3)$ from adding up the three eigenvectors: $u = x_1 + x_2 + x_3$. Think about the vector $v = A^k u$ for VERY large powers k. What is the limit of v as $k \to \infty$?

(3) **(30 pts.)**

(a) For a really large number N, will this matrix be positive definite? Show why or why not.

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 4 & N & 1 \\ 3 & 1 & 4 \end{bmatrix}.$$

(b) Suppose: A is positive definite symmetric

Q is orthogonal (same size as A)

B is $Q^T AQ = Q^{-1}AQ$

Show that: 1. B is also symmetric.

2. B is also positive definite.

(c) If the SVD of A is $U\Sigma V^T$, how do you find the orthogonal V and the diagonal Σ from the matrix A?

Math 18.06, Spring 2013
Problem Set #Exam 3
May 14, 2013

Problem 1. *In all of this problem, the 3 by 3 matrix A has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with independent eigenvectors x_1, x_2, x_3.*

a) *What are the trace of A and the determinant of A?*

The trace of A is $\lambda_1 + \lambda_2 + \lambda_3$ and the determinant is $\lambda_1\lambda_2\lambda_3$.

b) *Suppose: $\lambda_1 = \lambda_2$. Choose the true statement from 1, 2, 3:*

1. *A can be diagonalized.*
2. *A can not be diagonalized.*
3. *I need more information to decide.*

(1) is the correct option, because we know that there exists a full set of independent eigenvectors.

c) *From the eigenvalues and eigenvectors, how could you find the matrix A? Give a formula for A and explain each part carefully.*

We can recover A using $A = S\Lambda S^{-1}$, where S is a matrix whose columns are x_1, x_2, x_3, and Λ is a diagonal matrix whose diagonal entries are $\lambda_1, \lambda_2, \lambda_3$.

d) *Suppose $\lambda_1 = 2$ and $\lambda_2 = 5$ and $x_1 = (1, 1, 1)$ and $x_2 = (1, -2, 1)$. Choose λ_3 and x_3 so that A is symmetric positive semidefinite but not positive definite.*

If we want A to be symmetric, the third eigenvector x_3 had better be orthogonal to the other two. The quick way to find a vector orthogonal to two given ones in $\mathbb{R}^3$ is via cross product: $x_3 = x_1 \times x_2 = (3, 0, -3)$.

Alternately, you can use elimination: x_3 should be in the nullspace of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

You could also just notice the first and last entries match and guess the answer from that. Either way, x_3 should be a multiple of $(1, 0, -1)$.

As for the eigenvalue, to get a matrix that's positive semidefinite but not positive definite, we need to use $\lambda_3 = 0$.

It doesn't actually ask you to compute A, but here's one that works:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 3 & -2 & 3 \\ -2 & 8 & -2 \\ 3 & -2 & 3 \end{bmatrix}$$

Problem 2. *Suppose A has eigenvalues 1, 1/3, 1/2 and its eigenvectors are the columns of S:*

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \textit{with} \quad S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

a) *What are the eigenvalues and eigenvectors of A^{-1}?*

The eigenvectors of A^{-1} are the same as those of A. Its eigenvalues are the inverses of those of A: 1, 3, and 2.

b) *What is the general solution (with 3 arbitrary constants c_1, c_2, c_3) to the differential equation $du/dt = Au$? Not enough to write e^{At}. Use the c's.*

The general solution is

$$\begin{aligned} u(t) &= c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + c_3 e^{\lambda_3 t} x_3 \\ &= c_1 e^{t} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + c_2 e^{t/3} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{t/2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

c) *Start with the vector $u = (1, 4, 3)$ from adding up the three eigenvectors: $u = x_1 + x_2 + x_3$. Think about the vector $v = A^k u$ for VERY large powers k. What is the limit of v as $k \to \infty$?*

We have

$$A^k u = A^k(x_1 + x_2 + x_3) = \lambda_1^k x_1 + \lambda_2^k x_2 + \lambda_3^k x_3 = x_1 + \left(\frac{1}{3}\right)^k x_2 + \left(\frac{1}{2}\right)^k x_3.$$

When k is very large, the two rightmost terms both go to 0, while the first one is an unchanging x_1. The limit v is therefore equal to x_1.

Problem 3. a) *For a really large number N, will this matrix be positive definite? Show why or why not.*

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 4 & N & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

The easiest test to use here is going to be to check whether the upper-left determinants are positive.

1×1: This is 2, which is always greater than 0.

2×2: This is $2N - 16$, which is greater than 0 if N is really large (in particular if $N > 8$).

3×3: Use the method of your choice to compute the determinant of A, in terms of N. By the (not so) big formula, it's

$$\det A = 8N + 12 + 12 - 2 - 64 - 9N = -42 - N.$$

This is going to be very negative if N is really large. So the matrix will not be positive definite.

b)

Suppose: *A is positive definite symmetric*
Q is orthogonal (same size as A)
B is $Q^T AQ = Q^{-1}AQ$.

Show that: *B is also symmetric.*
B is also positive definite.

First we show that B is symmetric. This means we need to check $B^T = B$. Using what we're told,

$$B^T = (Q^T AQ)^T = Q^T A^T (Q^T)^T = Q^T AQ = B,$$

Note that in the next-to-last step we used the fact that A itself is symmetric ($A^T = A$).

For positive definiteness, one way is to use the energy test. If x is any nonzero vector, then

$$x^T Bx = x^T (Q^T AQ)x = (Qx)^T A(Qx) = y^T Ay,$$

where $y = Qx$. We know that y is nonzero, because Q is orthogonal and therefore has no nullspace.

Another approach is via eigenvalues. We know that $B = Q^{-1}AQ$, so B is similar to A. That means that they have the same eigenvalues. Since A is positive definite, its eigenvalues are all positive, so those of B are as well.

A third approach: A is positive definite, so $A = R^T R$ for some R with independent columns. Then $B = Q^T R^T RQ = (RQ)^T (RQ)$. RQ is a matrix with independent columns, since Q is orthogonal. So B is positive definite.

c) *If the SVD of A is $U\Sigma V^T$, how do you find the orthogonal V and the diagonal Σ from the matrix A?*

First compute the matrix $A^T A$. Find the eigenvalues and eigenvectors. The first r columns of V should be length 1 eigenvectors of $A^T A$, corresponding to nonzero eigenvalues, arranged in order of decreasing eigenvalue. (A small complication: if there is a repeated eigenvalue, make sure to pick orthogonal eigenvectors for that eigenvalue). The diagonal entries of Σ should be square roots of the eigenvalues of $A^T A$, again in decreasing order.

The remaining columns of V should be an orthonormal basis for the nullspace of $A^T A$ (which is the same thing as the nullspace of A). This will give enough columns for V to be a square matrix. The other diagonal entries of Σ should be 0.

18.06 **EXAM 3** December 3, 2014

Your PRINTED name is: ________________________

Please circle your recitation:

R01	T 9	E17-136	Darij Grinberg
R02	T 10	E17-136	Darij Grinberg
R03	T 10	24-307	Carlos Sauer
R04	T 11	24-307	Carlos Sauer
R05	T 12	E17-136	Tanya Khovanova
R06	T 1	E17-139	Michael Andrews
R07	T 2	E17-139	Tanya Khovanova

Grading	
1	
2	
3	
4	
Total:	

Each problem is 25 points, and each of its five parts (a)–(e) is 5 points.

In all problems, write all details of your solutions. Just giving an answer is not enough to get a full credit. Explain how you obtained the answer.

Problem 1. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(a) Find the eigenvalues λ_1 and λ_2 of A.

(b) Solve the initial value problem $d\mathbf{u}/dt = A\,\mathbf{u}\,,\ \ \mathbf{u}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(c) Find a diagonal matrix which is similar to the matrix A.

(d) Find the singular values σ_1 and σ_2 of A.

(e) Is the matrix A positive definite?

Problem 2. True or false? If your answer is "true", explain why. If your answer is "false", give a counterexample.

(a) Every positive definite matrix is nonsingular.

(b) If A is an $n \times n$ matrix with real eigenvalues and with n linearly independent eigenvectors which are orthogonal to each other, then A is symmetric.

(c) If a matrix B is similar to A, then B has the same *eigenvectors* as A.

(d) Any symmetric matrix is similar to a diagonal matrix.

(e) Any matrix which is similar to a diagonal matrix is symmetric.

Problem 3. (a–c) Consider the matrix $A = \begin{pmatrix} 2 & t & 0 \\ t & 2 & t \\ 0 & t & 2 \end{pmatrix}$ that depends on a parameter t.

(a) Find all values of t, for which the matrix A has 3 nonzero eigenvalues.

(b) Find all values of t, for which the matrix A has 3 positive eigenvalues.

(c) Find all values of t, for which the matrix A has 3 negative eigenvalues.

(d) Find a singular value decomposition $B = U\Sigma V^T$ for the matrix $B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

(e) Find orthonormal bases of the 4 fundamental subspaces of the matrix B from part (d).

Problem 4. (a–d) Consider the following operations on the space of quadratic polynomials $f(x) = ax^2 + bx + c$. Which of them are linear transformations?

If they are linear transformations, find their matrices in the basis $1, x, x^2$.

If they are not linear transformations, explain it using the definition of linear transformation.

(a) $T_1(f) = f(x) - f(1)$.

(b) $T_2(f) = f(x) - 1$.

(c) $T_3(f) = x - f(1)$.

(d) $T_4(f) = x^2 f(1/x)$.

(e) The linear transformation $R : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection with respect to the line $x+y = 0$. Find the matrix of R in the standard basis of $\mathbb{R}^2$.

If needed, you can use this extra sheet for your calculations.

If needed, you can use this extra sheet for your calculations.

Exam Solutions

Problem 1

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(a) Find the eigenvalues λ_1 and λ_2 of A.

(b) Solve the initial value problem $d\mathbf{u}/dt = A\mathbf{u}$, $\mathbf{u}(0) = (1, -1)^T$.

(c) Find a diagonal matrix which is similar to the matrix A.

(d) Find the singular values σ_1 and σ_2 of A.

(e) Is the matrix A positive definite?

Solutions:

(a) $\det(xI - A) = x^2 - 1$ has roots $\lambda_1 = 1$ and $\lambda_2 = -1$.

(b) $\mathbf{u}(t) = (e^{-t}, -e^{-t})^T$.

(c) Let $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} / \sqrt{2}$. Then Q is orthogonal and $A = Q\Lambda Q^{-1}$, where $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(d) Let $U = Q\Lambda$ and $V = Q$. U and V are orthogonal, $A = UIV^T$ is a singular value decomposition of A, and we see that the singular values are $\sigma_1 = |\lambda_1| = 1$ and $\sigma_2 = |\lambda_2| = 1$.

(e) No, since $\lambda_2 = -1 < 0$.

Problem 2

True or false? If your answer is "true", explain why. If your answer is "false", give a counterexample.

(a) Every positive definite matrix is nonsingular.

(b) If A is an $n \times n$ matrix with real eigenvalues and with n linearly independent eigenvectors which are orthogonal to each other, then A is symmetric.

(c) If a matrix B is similar to A, then B has the same eigenvectors as A.

(d) Any symmetric matrix is similar to a diagonal matrix.

(e) Any matrix which is similar to a diagonal matrix is symmetric.

Solutions:

(a) True: if A is singular, then there exists a nonzero x with $Ax = 0$; thus $x^T Ax = 0$, and A is not positive definite.

(b) True: if the given eigenvectors are $v_1, \ldots, v_n$, and their eigenvalues are $\lambda_1, \ldots, \lambda_n$, respectively, then $A = Q\Lambda Q^T$, where

$$Q = \left(v_1/|v_1| \,\middle|\, \cdots \,\middle|\, v_n/|v_n| \right) \text{ and } \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n);$$

thus, $A^T = (Q\Lambda Q^T)^T = (Q^T)^T \Lambda^T Q^T = Q\Lambda Q^T = A$.

(c) False: we saw in 1) that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has eigenvectors $(1,1)^T$ and $(1,-1)^T$; we also saw that it is similar to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which has eigenvectors $(1,0)^T$ and $(0,1)^T$.

(d) True: this is the $Q\Lambda Q^T = Q\Lambda Q^{-1}$ decomposition.

(e) False: $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ has distinct eigenvalues 0 and 1, and so it is similar to a diagonal matrix, but it is not symmetric.

Problem 3

Let $A = \begin{pmatrix} 2 & t & 0 \\ t & 2 & t \\ 0 & t & 2 \end{pmatrix}$

(a) Find all values of t, for which the matrix A has 3 nonzero eigenvalues.

(b) Find all values of t, for which the matrix A has 3 positive eigenvalues.

(c) Find all values of t, for which the matrix A has 3 negative eigenvalues.

(d) Find a singular value decomposition $B = U\Sigma V^T$ for the matrix $B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

(e) Find orthonormal bases of the 4 fundamental subspaces of the matrix B from part (d).

Solutions:

(a) A has 3 nonzero eigenvalues whenever $\det A = 8 - 4t^2$ is nonzero, i.e. whenever $t \neq \pm\sqrt{2}$.

(b) Since $2 > 0$, A has 3 positive eigenvalues whenever $\det A = 8 - 4t^2$ and $\det \begin{pmatrix} 2 & t \\ t & 2 \end{pmatrix} = 4 - t^2$ are positive, i.e. whenever $-\sqrt{2} < t < \sqrt{2}$.

(c) Since $\mathrm{tr} A = 6 > 0$, there are no values of t for which A has 3 negative eigenvalues.

(d) Use `http://web.mit.edu/18.06/www/Fall14/Recitation10_Michael.pdf` algorithm.

   (a) $BB^T = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ has orthonormal eigenvectors $u_1 = (1,1)^T/\sqrt{2}$ and $u_2 = (1,-1)/\sqrt{2}$ with eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 0$.

   (b) $\sigma_1 = \sqrt{\lambda_1} = 2$ and $v_1 = B^T u_1/2 = (1,0,1)^T/\sqrt{2}$.

   (c) We can choose $v_2 = (1,0,-1)/\sqrt{2}$ and $v_3 = (0,1,0)$, since they give an orthonormal basis for $N(B^T B) = N(B)$.

   (d) This gives $B = U\Sigma V^T$ where

$$U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} /\sqrt{2},\ \Sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \text{and } V = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -1 & 0 \end{pmatrix} /\sqrt{2}.$$

(e) $C(B) = \left\langle (1,1)^T/\sqrt{2} \right\rangle$, $N(B) = \left\langle (1,0,-1)^T/\sqrt{2},\ (0,1,0)^T \right\rangle$,
$C(B^T) = \left\langle (1,0,1)^T/\sqrt{2} \right\rangle$, $N(B^T) = \left\langle (1,-1)^T/\sqrt{2} \right\rangle$.

Problem 4

Consider the following operations on the space of quadratic polynomials $f(x) = ax^2 + bx + c$. Which of them are linear transformations? If they are linear transformations, find their matrices in the basis $1, x, x^2$. If they are not linear transformations, explain it using the definition of linear transformation.

(a) $T_1(f) = f(x) - f(1)$.

(b) $T_2(f) = f(x) - 1$.

(c) $T_3(f) = x - f(1)$.

(d) $T_4(f) = x^2 f(1/x)$.

(e) The linear transformation $R : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the reflection with respect to the line $x + y = 0$. Find the matrix of R in the standard basis of $\mathbb{R}^2$.

Solutions:

(a) T_1 is linear. Its matrix is $\begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(b) T_2 is not linear because $T_2(0) = -1 \neq 0$.

(c) T_3 is not linear because $T_3(0) = x \neq 0$.

(d) T_4 is linear. Its matrix is $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

(e) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

18.06 Exam III Professor Strang May 7, 2014

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R05	T	11	E17-136	Oren Mangoubi
R06	T	12	36-144	Benjamin Iriarte Giraldo
R07	T	12	4-149	Goncalo Tabuada
R08	T	12	36-112	Adrian Vladu
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R10	T	1	36-153	Benjamin Iriarte Giraldo
R11	T	1	36-155	Tanya Khovanova
R12	T	2	36-144	Jui-En (Ryan) Chang
R13	T	2	36-155	Tanya Khovanova
R14	T	3	36-144	Xuwen Zhu
ESG	T	3		G. Stoy

Grading
1:

2:

3:

4:

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1. (28 points) This question is about the differential equation

$$\frac{dy}{dt} = Ay = \begin{bmatrix} 5 & 2 \\ 8 & 5 \end{bmatrix} y \quad \text{with } y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a) Find an eigenvalue matrix Λ and an eigenvector matrix S so that $A = S\Lambda S^{-1}$. Compute the matrix exponential e^{tA} by using $e^{t\Lambda}$.

(b) Find $y(t)$ as a **combination of the eigenvectors** of A that has the correct value $y(0)$ at $t = 0$.

2. **(a)** (24 points) Suppose a symmetric n by n matrix S has eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_n$ and orthonormal eigenvectors $q_1, \ldots, q_n$.

If $x = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n$ show that $x^T x = c_1^2 + \cdots + c_n^2$ and $x^T S x = \lambda_1 c_1^2 + \cdots + \lambda_n c_n^2$.

(b) What is the largest possible value of $R(x) = \frac{x^T S x}{x^T x}$ for nonzero x ?

Describe a vector x that gives this maximum value for this ratio $R(x)$?

3. (24 points)

(a) Show that the matrix $S = A^T A$ is positive semidefinite, for any matrix A. Which test will you use and how will you show it is passed?

(b) If A is 3 by 4, show that $A^T A$ is **not** positive definite.

4. (24 points)

(a) Show that none of the singular values of A are larger than 3.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Why does $B = AQ$ have the same singular values as A? (Q is an orthogonal matrix.)

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18.06 Exam III Professor Strang May 7, 2014

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R05	T	11	E17-136	Oren Mangoubi
R06	T	12	36-144	Benjamin Iriarte Giraldo
R07	T	12	4-149	Goncalo Tabuada
R08	T	12	36-112	Adrian Vladu
R09	T	1	36-144	Jui-En (Ryan) Chang
R10	T	1	36-153	Benjamin Iriarte Giraldo
R11	T	1	36-155	Tanya Khovanova
R12	T	2	36-144	Jui-En (Ryan) Chang
R13	T	2	36-155	Tanya Khovanova
R14	T	3	36-144	Xuwen Zhu
ESG	T	3		Gabrielle Stoy

Grading 1:

2:

3:

4:

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1. (28 points) This question is about the differential equation

$$\frac{dy}{dt} = Ay = \begin{bmatrix} 5 & 2 \\ 8 & 5 \end{bmatrix} y \quad \text{with } y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a) Find an eigenvalue matrix Λ and an eigenvector matrix S so that $A = S\Lambda S^{-1}$. Compute the matrix exponential e^{tA} by using $e^{t\Lambda}$.

(b) Find $y(t)$ as a **combination of the eigenvectors** of A that has the correct value $y(0)$ at $t = 0$.

Solutions:

(a) $\det(A - \lambda I) = 0 \Leftrightarrow \lambda^2 - 10\lambda + 9 = 0$. Eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 9$. The eigenvector associated to λ_1 is $v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and the eigenvector associated to λ_2 is $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The matrix $S = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}$ and $S^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}$. Finally, $e^{tA} = Se^{t\Lambda}S^{-1} = \begin{pmatrix} \frac{1}{2}e^t + \frac{1}{2}e^{9t} & -\frac{1}{4}e^t + \frac{1}{4}e^{9t} \\ -e^t + e^{9t} & -\frac{1}{2}e^t + \frac{1}{2}e^{9t} \end{pmatrix}$.

(b) $y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ -2 \end{pmatrix} + b\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. This implies that $a = \frac{1}{2}$ and $b = \frac{1}{2}$. Hence, $y(t) = ae^{\lambda_1 t}v_1 + be^{\lambda_2 t}v_2 = \frac{1}{2}e^t\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \frac{1}{2}e^{9t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

2. (a) (24 points) Suppose a symmetric n by n matrix S has eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_n$ and orthonormal eigenvectors $q_1, \ldots, q_n$.

If $x = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n$ show that $x^T x = c_1^2 + \cdots + c_n^2$ and $x^T S x = \lambda_1 c_1^2 + \cdots + \lambda_n c_n^2$.

(b) What is the largest possible value of $R(x) = \frac{x^T S x}{x^T x}$ for nonzero x ?

Describe a vector x that gives this maximum value for this ratio $R(x)$?

Solutions:

(a) Since the eigenvectors are orthonormal, one has $x^T x = (c_1 q_1 + \cdots + c_n q_n)^T (c_1 q_1 + \cdots + c_n q_n) = c_1^2 q_1^T q_1 + \cdots + c_n^2 q_n^T q_n = c_1^2 + \cdots + c_n^2$.
On the other hand, $x^T S x = (c_1 q_1 + \cdots + c_n q_n)^T S (c_1 q_1 + \cdots + c_n q_n) = (c_1 q_1 + \cdots + c_n q_n)^T (\lambda_1 c_1 q_1 + \cdots + \lambda_n c_n q_n) = \lambda_1 c_1^2 q_1^T q_1 + \cdots + \lambda_n c_n^2 q_n^T q_n = \lambda_1 c_1^2 + \cdots + \lambda_n c_n^2$.

(b) Using (a), $R(X) = \frac{\lambda_1 c_1^2 + \cdots + \lambda_n c_n^2}{c^1 + \cdots + c_n}$. Since $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, $R(X)$ is maximal when $c_2 = \cdots = c_n = 0$ and $c_1 \neq 0$. In this case the largest value of $R(x)$ is λ_1 and the associated vector x is any non-zero multiple of q_1.

3. (24 points)

(a) Show that the matrix $S = A^T A$ is positive semidefinite, for any matrix A. Which test will you use and how will you show it is passed?

(b) If A is 3 by 4, show that $A^T A$ is **not** positive definite.

Solutions:

(a) Energy test. For every vector x we have $x^T S x = x^T A^T A x = (Ax)^T (Ax) = ||Ax||^2 \geq 0$. Hence, S is positive semidefinite.

(b) Since A is 3×4, one has $\dim(C(A)) \leq 3$ and $\dim(N(A)) \geq 1$. Hence, there exists a non-zero vector v such that $Av = 0$. As a consequence, $A^T A$ is **not** positive definite.

4. (24 points)

(a) Show that none of the singular values of A are larger than 3.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Why does $B = AQ$ have the same singular values as A? (Q is an orthogonal matrix.)

Solutions:

(a) $A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$. Hence, $\mathrm{tr}(A^T A) = 6$. However, $A^T A$ is positive semidefinite, therefore all the eigenvalues are nonnegative. This implies that $0 \leq \lambda_i \leq 6$ and hence that $\sigma_i \leq \sqrt{6} \leq 3$.

(b) Since $B^T B = Q^T A^T A Q$, the matrixes $B^T B$ and $A^T A$ are similar. This implies that they have the same eingenvalues and therefore that B and A have the same singular values $\sigma_i = \sqrt{\lambda_i}$.

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18.06 Exam III Professor Strang May 4, 2015

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R10	T1pm	36-144	Tanya Khovanova
R11	T1pm	26-322	Jay Shah
R12	T2pm	36-144	Tanya Khovanova
R13	T2pm	26-322	Jay Shah
R14	T3pm	26-322	Carlos Sauer
ESG			Gabrielle Stoy

Grading 1:

2:

3:

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1. (33 points)

(a) Suppose A has the eigenvalues $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -1$ with eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ in the columns of this $S = \left[\begin{array}{c|c|c} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{array}\right]$:

$$S = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

What are the eigenvalues and eigenvectors of the matrix $B = A^9 + I$?

(b) How could you find that matrix $B = A^9 + I$ using the eigenvectors in S and the eigenvalues $1, 0, -1$?

(c) Give a reason why the matrix B does have or doesn't have each of these properties:

i. B is invertible

ii. B is symmetric

iii. trace $= B_{11} + B_{22} + B_{33} = 3$.

2. (33 points)

(a) Show that $\lambda_1 = 0$ is an eigenvalue of A and find an eigenvector $\mathbf{x}_1$ with that zero eigenvalue:

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

(b) Find the other eigenvalues λ_2 and λ_3 of this symmetric matrix. Does A have two more independent eigenvectors $\mathbf{x}_2$ and $\mathbf{x}_3$? Give a reason why or why not. (Not required to find $\mathbf{x}_2$ and $\mathbf{x}_3$.)

(c) Suppose $\dfrac{d\mathbf{u}}{dt} = A\mathbf{u}$ starts from $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Explain why this $\mathbf{u}(t)$ approaches a steady state $\mathbf{u}(\infty)$ as $t \to \infty$. You can use the general formula $\mathbf{u}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + c_3 e^{\lambda_3 t}\mathbf{x}_3$ or $e^{At} = Se^{\Lambda t}S^{-1}$ without putting in all eigenvectors. **Find** that steady state $\mathbf{u}(\infty)$.

3. (34 points)

(a) If C is any symmetric matrix, show that e^C is a positive definite matrix. We can see that e^C is symmetric — which test will you use to show that e^C is positive definite?

(b) A is a 3 by 3 matrix. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are orthonormal eigenvectors (with eigenvalues $1, 2, 3$) of the symmetric matrix $A^T A$. Show that $A\mathbf{v}_1, A\mathbf{v}_2$, $A\mathbf{v}_3$ are orthogonal by rewriting and simplifying $(A\mathbf{v}_i)^T(A\mathbf{v}_j)$.

(c) For the 3 by 3 matrix A in part (b), find three matrices U, Σ, V that go into the Singular Value Decomposition $A = U\Sigma V^T$.

(d) TRUE or FALSE: If A is any symmetric 4 by 4 matrix and M is any invertible 4 by 4 matrix, then $B = M^{-1}AM$ is also symmetric. Give a reason for true or false.

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Exam 3 Solutions

Question 1

(a) $A = S\text{diag}(1,0,-1)S^{-1}$. Thus $B = A^9 + I = S(\text{diag}(1,0,-1)^9 + I)S^{-1} = S\text{diag}(2,1,0)S^{-1}$. So B has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = 0$ and the same eigenvectors as A.

(b) Write $A = S\text{diag}(1,0,-1)S^{-1}$ and multiply to give $B = S\text{diag}(2,1,0)S^{-1}$.

(c) (i) B has 0 as an eigenvalue and so cannot be invertible.

(ii) B has distinct eigenvalues, with eigenvectors which are not orthogonal, and so it cannot be symmetric. (The point about distinct eigenvalues is not needed for full credit.)

(iii) True: the trace of B is the sum of the eigenvalues, $2 + 1 + 0 = 3$.

Question 2

(a) $A(1,1,1)^T = 0$ and so $x_1 = (1,1,1)^T$ is an eigenvector with eigenvalue $\lambda_1 = 0$.

(b) Each of the columns of $A + 3I$ is $(1,1,1)^T$ and so it is rank 1. In particular, the null space of $A + 3I$ has dimension 2 and so the other eigenvalues are $\lambda_2 = -3 = \lambda_3$.

(c) $u(t) = c_1 e^{0t} x_1 + c_2 e^{-3t} x_2 + c_3 e^{-3t} x_3 \longrightarrow u(\infty) = c_1 x_1$ as $t \longrightarrow \infty$. We just need to find c_1. But $u(0) = c_1 x_1 + c_2 x_2 + c_3 x_3 = (1,2,3)^T$. Since x_1 is orthogonal to x_2 and x_3 we see, by dotting with x_1, that $c_1 x_1 \cdot x_1 = (1,2,3)^T \cdot x_1$. Remembering that $x_1 = (1,1,1)^T$ we obtain $3c_1 = 6$ so that $c_1 = 2$ and $u(\infty) = (2,2,2)^T$.

Question 3

(a) We can write $C = Q\Lambda Q^T$ for some orthogonal matrix Q and some diagonal matrix Λ. Then $e^C = Qe^{\Lambda}Q^T$, which immediately shows that e^C is symmetric. If $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, then $e^{\Lambda} = \text{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})$ so that each eigenvalue of e^C is of the form e^{λ}. In particular, it is positive, so that e^C is positive definite.

(b) $(Av_i) \cdot (Av_j) = (Av_i)^T(Av_j) = v_i^T(A^TA)v_j = jv_i^Tv_j = jv_i \cdot v_j$. Since v_1, v_2 and v_3 are orthogonal, we see that $(Av_i) \cdot (Av_j) = 0$ when $i \neq j$, i.e. Av_1, Av_2 and Av_3 are orthogonal.

(c) $V = (v_1|v_2|v_3)$, $\Sigma = \text{diag}(1, \sqrt{2}, \sqrt{3})$, and $U = (Av_1|\frac{Av_2}{\sqrt{2}}|\frac{Av_3}{\sqrt{3}})$.

(d) False. Take $A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then $M^{-1}AM = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

In fact, any diagonal A with distinct eigenvalues, together with any M with nonorthogonal columns, will provide a counterexample.

Almost full credit for correctly saying false, e.g. just a rewording that says less about M. An unsymmetric B can be similar to a symmetric (diagonal) Λ as in question 1.