18.06 Professor Strang Quiz 3 May 9, 1997

1.(a) (20) Compute $u_{20} = A^{20}u_0$ starting from

$$A = \begin{bmatrix} .7 & .5 \\ .3 & .5 \end{bmatrix} \quad \text{and} \quad u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(b) (10) For the same A, which real numbers c have the property that $(A - cI)^n\, u_0$ approaches zero as $n \to \infty$?

(c) (10) Find the eigenvalues and eigenvectors of $A^{-1} + A^{20}$ without computing A^{-1} (for the same A as above).

2.(a) (10) If you transpose $S^{-1}AS = \Lambda$ you learn that

The eigenvalues of A^T are ________________________

The eigenvectors of A^T are ________________________

(b) (10) Complete the last row so that B is a singular matrix, with real eigenvalues and orthogonal eigenvectors:

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ - & - & - \end{bmatrix}.$$

(c) (10) C is a 3×3 matrix. I add row 1 to row 2 to get K:

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} C.$$

This probably changes the eigenvalues. What should I do to the *columns* of K (answer in words) to get back to the original eigenvalues of C?

3.(a) (10) For which numbers c is this matrix positive definite?

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & c & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

What function $F(x_1, x_2, x_3)$ has A as its matrix of second derivatives?

(b) (12) What is the 2×2 matrix P that projects every vector onto the "θ-line" containing all multiples of $\mathbf{a} = (\cos\theta, \sin\theta)$? What are the *eigenvalues* of P?

(c) (8) That projection is a linear transformation. Suppose we choose the basis vectors $v_1 = (\cos\theta, \sin\theta)$ along the θ-line and $v_2 = (-\sin\theta, \cos\theta)$ perpendicular to the θ-line. What matrix represents P with respect to this basis?

18.06 **Solutions** **Quiz 3** **May 9, 1997**

1.(a) A is a Markov matrix. So $\lambda = 1$ is an eigenvalue. Then $\lambda = .2$ is the other eigenvalue because trace = 1.2

Eigenvectors

$$\lambda = 1: \quad A - I = \begin{bmatrix} -.3 & .5 \\ .3 & -.5 \end{bmatrix} \to x = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\lambda = .2: \quad A - .2I = \begin{bmatrix} .5 & .5 \\ .3 & .3 \end{bmatrix} \to x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$S = \begin{bmatrix} 5 & 1 \\ 3 & -1 \end{bmatrix}$ has inverse $\frac{1}{8}\begin{bmatrix} 1 & 1 \\ 3 & -5 \end{bmatrix} = S^{-1}$ so that $S^{-1}u_0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}\frac{1}{8}$

We want

$$A^{20}u_0 = S \wedge^{20} S^{-1}u_0 = \begin{bmatrix} 5 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & \\ & .2^{20} \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}\frac{1}{8}$$

$$= \begin{bmatrix} 5 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 3x \quad .2^{20} \end{bmatrix}\frac{1}{8}$$

$$= \begin{bmatrix} 5 + 3 \quad (.2^{20}) \\ 3 - 3 \quad (.2^{20}) \end{bmatrix}\frac{1}{8}$$

Check: Change 20th power to 0th and we get $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = u_0$

(b) A has $\lambda = .2$ and 1

$A - cI$ has $\lambda = .2 - c$ and $1 - c$

Need $|.2 - c| < 1$ (need $c < 1.2$) and $|1 - c| < 1$ (need $c > 0$)

Therefore the condition for $|\lambda| < 1$ and stability is $0 < c < 1.2$

(c) Same eigenvectors for $A^{-1} + A^{20}$ as in part (a) for A itself

$$\text{Eigenvalues} \qquad \frac{1}{1} + 1^{20} = 2$$

$$\frac{1}{.2} + (.2)^{20}$$

2.(a) If you transpose $S^{-1}AS = \Lambda$ you learn that $\quad S^TA^T(S^{-1})^T = \Lambda^T = \Lambda$

The eigenvalues of A^T are **the same as the eigenvalues of** A

The eigenvectors of A^T are **the columns of** $(S^{-1})^T$

(b)

$$B = \begin{array}{ccc} \boxed{1} & 2 & \boxed{3} \\ \boxed{2} & 3 & \boxed{4} \\ \boxed{3} & 4 & \boxed{5} \end{array}$$

for symmetry   to make B singular

(c) Multiply on the right by $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So subtract the 2nd column of K from the first.

Then the result $\begin{bmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{bmatrix} C \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{bmatrix}^{-1}$ is **similar** to C and has the same eigenvalues

---

3.(a) (8pts) Check determinants

$$\begin{array}{rcll} 1 & > & 0 & \quad 1 \times 1 \\ c-1 & > & 0 & \quad 2 \times 2 \\ c-2 & > & 0 & \quad 3 \times 3 \end{array}$$
$$\boxed{\text{Need } c > 2 \text{ for positive definiteness}}$$

(2 pts)

$$\begin{aligned} F &= \frac{1}{2}\left(x_1^2 - 2x_1x_2 + cx_2^2 - 2x_2x_3 + x_3^2\right) \\ &= \frac{1}{2}x^T A\, x\,. \end{aligned}$$

Then $\dfrac{\partial^2 F}{\partial x_i \partial x_j} = A_{ij} =$ second derivative matrix.

(b)

$$P = \frac{aa^T}{a^Ta} = \frac{\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} [\cos\theta \ \sin\theta]}{\cos^2\theta + \sin^2\theta} = \begin{bmatrix} \cos^2\theta & \cos\theta\ \sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \quad \boxed{\lambda = 1, 0}$$

(c) Since $Pv_1 = v_1$ and $Pv_2 = 0$ the projection matrix in this basis is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

3 December 1997 **Profs. S. Lee and A. Kirillov**

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Hour Exam III for Course 18.06: Linear Algebra

Recitation Instructor: Your Name: **SOLUTIONS**

Recitation Time: Lecturer:

Grading

1. **30**
2. **22**
3. **16**
4. **32**

TOTAL: 100

Show all your work on these pages.
No calculators or notes.
Please work carefully, and check your intermediate results.
Point values (total of 100) are marked on the left margin.

1. Let $A = \begin{bmatrix} 4 & 1 \\ -1 & 4 \end{bmatrix}$.

[10] **1a.** Find the eigenvalues of A.

$$\boxed{\lambda_1 = 4 + i\,,\ \ \lambda_2 = 4 - i\,.}$$

`Find roots of quadratic equation:`
$\det(A - \lambda I) = (4-\lambda)(4-\lambda) - (-1) = \lambda^2 - 8\lambda + 17 = 0\,.$

[10] **1b.** Find an eigenvector for each eigenvalue of A.

$$\boxed{x_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix},\ \ x_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}.}$$

$$\underbrace{\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}}_{(A-\lambda_1 I)} \underbrace{\begin{bmatrix} -i \\ 1 \end{bmatrix}}_{x_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad \underbrace{\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}}_{(A-\lambda_2 I)} \underbrace{\begin{bmatrix} i \\ 1 \end{bmatrix}}_{x_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

[10] **1c.** Compute $x_1^H x_2$.
(Note: x_1 and x_2 are the complex eigenvectors that you obtained in **1b**.)

$\boxed{x_1^H x_2 = 0.}$

2. Let $A = \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$.

[12] **2a.** Find an invertible matrix S that makes $S^{-1}AS$ a diagonal matrix.

$$\boxed{S = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}.}$$

`The diagonal entries of` A `are its eigenvalues.`
`The columns of` S `are the eigenvectors of` A `for` $\lambda_1 = -2$, $\lambda_2 = 0$.

[10] **2b.** For the differential equation $\frac{du}{dt} = Au$, give a nonzero initial vector $u(0)$ such that $u(t) \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $t \to \infty$.

$$\boxed{u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \texttt{ or any nonzero multiple of it.}}$$

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = c_1 e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}.$$

`Choose` $c_2 = 0$ `and` $c_1 \neq 0$ `for initial vector to approach` $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ `as` $t \to \infty$.

[16] **3.** Fill in the matrix $A = \begin{bmatrix} 0.5 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ so that A is a positive Markov matrix with the steady state vector $x_1 = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}$.

(Recall that the limit of $A^k u_0$ is always a multiple of x_1.)

$$\boxed{A = \begin{bmatrix} 1/2 & 1/6 \\ 1/2 & 5/6 \end{bmatrix}.}$$

`The first column of A adds to 1 when` $a_{21} = 1/2$.

`Next, we solve` $A = \begin{bmatrix} 1/2 & a_{12} \\ 1/2 & a_{22} \end{bmatrix} \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$ `for the second column of` A.

`Along the first row:` $(1/8) + (3/4)a_{12} = 1/4$, `which gives` $a_{12} = 1/6$.

`The second column of A adds to 1 for` $a_{22} = 5/6$.

`Observe that the steady state vector satisfies` $Ax_1 = (1)x_1$.

4. Each independent question refers to the matrix $A = \begin{bmatrix} 4 & 1 \\ d & -4 \end{bmatrix}$.

In each case, find the value of d that makes the statement true (and show your work!).

[10] **4a.** Give a value for d such that $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is an eigenvector of A.

$\boxed{d = 41/25.}$

$$\begin{bmatrix} 4 & 1 \\ d & -4 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 21 \\ 5d-4 \end{bmatrix} = (21/5) \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

`In the second component, solve` $5d - 4 = (21/5)$ `for` d.

[10] **4b.** Give a value for d such that 2 is one of the eigenvalues of A.

$\boxed{d = -12.}$

`When` 2 `is an eigenvalue of` A,

$A - 2I = \begin{bmatrix} 2 & 1 \\ d & -6 \end{bmatrix}$ `must have linearly dependent columns.`

[12] **4c.** Give a value for d such that A is a nondiagonalizable matrix.

Recall that $A = \begin{bmatrix} 4 & 1 \\ d & -4 \end{bmatrix}$.

$\boxed{d = -16.}$

The issue of nondiagonalizability only comes up for a matrix that has some repeated eigenvalues.
In this case, 0 is a twice repeated eigenvalue of A when $d = -16$.
The eigenvalue is repeated twice, but we only find one linearly independent eigenvector (via the special solution to $(A - 0I)x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$).

18.06 Professor Edelman Quiz 3 November 20, 1998

Your name is ____.

Please circle your recitation:

1)	M2	2-131	Darren Crowdy	crowdy@math	2-335	3-7905
2)	M2	2-132	Yue Lei	yuelei@math	2-586	3-4102
3)	M3	2-131	Darren Crowdy	crowdy@math	2-335	3-7905
4)	T10	2-131	Sergiu Moroianu	bebe@math	2-491	3-4091
5)	T10	2-132	Gabrielle Stoy	stoy@math	2-235	3-4984
6)	T11	2-131	Sergiu Moroianu	bebe@math	2-491	3-4091
7)	T11	2-132	Gabrielle Stoy	stoy@math	2-235	3-4984
8)	T12	2-132	Anda Degeratu	anda@math	2-229	3-1589
9)	T12	2-131	Edward Goldstein	egold@math	2-092	3-6228
10)	T1	2-131	Anda Degeratu	anda@math	2-229	3-1589
11)	T2	2-132	Yue Lei	yuelei@math	2-586	3-4102

1. (a.) (10 pts) Find ALL the eigenvalues and ONE eigenvector of each of the matrices below:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 5 & 0 \\ -2 & 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ -3 & 0 & -2 \end{bmatrix}$$

1. (b.) (10 pts) Find ONLY one eigenvalue of each of the matrices below: (This can be done with no arithmetic.)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

2. (20 pts) Let A have eigenvalues $\lambda_1, \ldots, \lambda_n$ (all nonzero) and corresponding eigenvectors $x_1, \ldots, x_n$ forming a basis for R^n. Let C be its cofactor matrix. (The answers to the questions below should be in terms of the λ_i.)

(a) (5 pts) What is trace(A^{-1})? det(A^{-1})?

(b) (15 pts) What is trace(C)? What is det(C)? (Hint: $A^{-1} = \frac{C^T}{\det A}$)

3. (30 pts.) Suppose A is symmetric $(n \times n)$ with rank $r = 1$ and one eigenvalue equal to 7. Let the general solution to

$$\frac{du}{dt} = -Au$$

be written as $u(t) = M(t)u(0)$. (Note the minus sign!)

(a) (5 pts.) Write down an expression for $M(t)$ in terms of A and t.
(b) (15 pts.) Is it true that for all t, $\text{trace}(M(t)) \geq \det(M(t))$? Explain your answer by finding all the eigenvalues of $M(t)$.
(c) (5 pts.) Can $u(t)$ blow up when $t \to \infty$? Explain.
(d) (5 pts.) Can $u(t)$ approach 0 when $t \to \infty$? Explain.

4. (30pts.) (a). If B is invertible prove that AB has the same eigenvalues as BA. (Hint: Find a matrix M such that $ABM = MBA$.)

(b). Find a diagonalizable matrix $A \neq 0$ that is similar to $-A$. Also find a nondiagonalizable matrix A that is similar to $-A$.

18.06 Professor Edelman Quiz 3 November 20, 1998

Your name is ______________________________.

Please circle your recitation:

1)	M2	2-131	Darren Crowdy	crowdy@math	2-335	3-7905
2)	M2	2-132	Yue Lei	yuelei@math	2-586	3-4102
3)	M3	2-131	Darren Crowdy	crowdy@math	2-335	3-7905
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10)	T1	2-131	Anda Degeratu	anda@math	2-229	3-1589
11)	T2	2-132	Yue Lei	yuelei@math	2-586	3-4102

1. (a.) (10 pts) Find ALL the eigenvalues and ONE eigenvector of each of the matrices below:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 5 & 0 \\ -2 & 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ -3 & 0 & -2 \end{bmatrix}$$

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector of A and B.

$\det(A - \lambda I) = (5-\lambda)\begin{vmatrix} -\lambda & -1 \\ -2 & 3-\lambda \end{vmatrix} = (5-\lambda)(\lambda^2 - 3\lambda + 2) \Rightarrow \lambda = -1, -2, 5$

B is lower triangular. The eigenvalues are on the diagonal: 1, 5, -2.

1. (b.) (10 pts) Find ONLY one eigenvalue of each of the matrices below: (This can be done with no arithmetic.)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

A is singular since column 1 + column 3 = 2 x column 2 . So A has an eigenvalue 0.

$$B \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$$

so 5 is an eigenvalue of B.

2. (20 pts) Let A have eigenvalues $\lambda_1, \ldots, \lambda_n$ (all nonzero) and corresponding eigenvectors $x_1, \ldots, x_n$ forming a basis for R^n. Let C be its cofactor matrix. (The answers to the questions below should be in terms of the λ_i.)

(a) (5 pts) What is trace(A^{-1})? det(A^{-1})?

(b) (15 pts) What is trace(C)? What is det(C)? (Hint: $A^{-1} = \frac{C^T}{\det A}$)

(a) A^{-1} has eigenvalues $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda n}$.
trace$(A^{-1}) = \frac{1}{\lambda_1} + \ldots + \frac{1}{\lambda n}$
det$(A^{-1}) = \frac{1}{\lambda_1}\frac{1}{\lambda_2} \cdots \frac{1}{\lambda n}$.
(b) The eigenvalues of C^T are the same as that of C or det$(A)\times$ those of A^{-1}.
Thus they are $\mu_i = \frac{\lambda_1 \lambda_2 \ldots \lambda_n}{\lambda_i}$
trace$(C) = \lambda_1\lambda_2 \ldots \lambda_n(\frac{1}{\lambda_1} + \ldots + \frac{1}{\lambda_n})$
det$(C) = (\lambda_1 \ldots \lambda_n)^{n-1}$

3. (30 pts.) Suppose A is symmetric $(n \times n)$ with rank $r = 1$ and one eigenvalue equal to 7. Let the general solution to

$$\frac{du}{dt} = -Au$$

be written as $u(t) = M(t)u(0)$. (Note the minus sign!)

(a) (5 pts.) Write down an expression for $M(t)$ in terms of A and t.
(b) (15 pts.) Is it true that for all t, $\text{trace}(M(t)) \geq \det(M(t))$? Explain your answer by finding all the eigenvalues of $M(t)$.
(c) (5 pts.) Can $u(t)$ blow up when $t \to \infty$? Explain.
(d) (5 pts.) Can $u(t)$ approach 0 when $t \to \infty$? Explain.

(a) $M(t) = e^{-At}$
(b) $M(t)$ has one eigenvalue e^{-7t} and the rest are 1.
(c) No blow up. All eigs are ≤ 1.
(d) If $u(0)$ is the eigenvector corresponding to to e^{7t} then $u(t)$ approaches 0.

4. (30pts.) (a). If B is invertible prove that AB has the same eigenvalues as BA. (Hint: Find a matrix M such that $ABM = MBA$.)

$M = B^{-1}$ so $AB = MBAM^{1}$ is similar to BA.

(b). Find a diagonalizable matrix $A \neq 0$ that is similar to $-A$. Also find a nondiagonalizable matrix A that is similar to $-A$.

$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

18.06 Professor Strang Exam 3 May 6, 1998

Your name is: ______________________________

Grading 1
2
3
4

Please circle your recitation:

1)	M2	2-132	M. Nevins	2-588	3-4110	monica@math
2)	M3	2-131	A. Voronov	2-246	3-3299	voronov@math
3)	T10	2-132	A. Edelman	2-380	3-7770	edelman@math
4)	T12	2-132	A. Edelman	2-380	3-7770	edelman@math
5)	T12	2-131	Z. Spasojevic	2-101	3-4470	zoran@math
6)	T1	2-131	Z. Spasojevic	2-101	3-4770	zoran@math
7)	T2	2-132	Y. Ma	2-333	3-7826	yanyuan@math

1 Find the eigenvalues and eigenvectors of these matrices:

(a) **(10)** Projection $P = \dfrac{aa^T}{a^Ta}$ with $a = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(b) **(10)** Rotation $Q = \begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}$

(c) **(8)** Reflection $R = 2P - I$

2 (a) **(10)** Find the eigenvalues λ_1, λ_2, λ_3 (NOT the eigenvectors x_1, x_2, x_3) of this Markov matrix:

$$A = \begin{bmatrix} .6 & .6 & 0 \\ .2 & .2 & .2 \\ .2 & .2 & .8 \end{bmatrix}$$

(b) **(10)** Suppose u_0 is the sum $x_1 + x_2 + x_3$ of the three eigenvectors that you didn't compute. What is $A^n u_0$?

(c) **(4)** As $n \to \infty$ what is the limit of $A^n u_0$?

3 (a) **(2 each)** Suppose M is any invertible matrix. Circle all the properties of a matrix A that remain the same for $M^{-1}AM$:

same rank

same nullspace

same determinant

real eigenvalues

orthonormal eigenvectors

symmetric positive definiteness

(b) **(2 each)** This is a similar question but now Q is an orthonormal matrix. Circle the properties of A that remain the same for $Q^{-1}AQ$:

same column space

A^k approaches zero as k increases

orthonormal eigenvectors

symmetric positive definiteness

projection matrix

4 (a) **(3 each)** Suppose the 5 by 4 matrix A has independent columns. What is the most information you can give about

the eigenvalues of A^TA: ______________________________

the eigenvectors of A^TA: ______________________________

the determinant of A^TA: ______________________________

(b) **(9)** Find the singular value decomposition (SVD) for this matrix:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix} .$$

(c) **(8)** When the input basis is $v_1, \ldots, v_n$ and the output basis is $w_1, \ldots, w_n$ and the matrix of the linear transformation T using these bases is the identity matrix, what is $T(v_1 + v_2)$?

18.06 Professor Strang Exam 3 Solutions May 6, 1998

1. (a)

$$\begin{aligned} \lambda_1 &= 1 & \lambda_2 &= 0 \\ x_1 &= a = \begin{bmatrix} 3 \\ 4 \end{bmatrix} & x_2 &= \begin{bmatrix} 4 \\ -3 \end{bmatrix} \end{aligned}$$

(or multiples of x_1 and x_2)

(b)

$$\begin{aligned} \lambda_1 &= .6 + .8i & \lambda_2 &= .6 - .8i \\ x_1 &= \begin{bmatrix} 1 \\ -i \end{bmatrix} & x_2 &= \begin{bmatrix} 1 \\ i \end{bmatrix} \end{aligned}$$

(c)

$$\begin{aligned} \lambda_1 &= 2(1) - 1 = 1 \\ \lambda_2 &= 2(0) - 1 = -1 \end{aligned}$$

R has the same eigenvectors as P

2. (a)

$$\begin{aligned} \lambda_1 &= 1 && \text{(for any Markov matrix)} \\ \lambda_2 &= 0 && \text{(since } A \text{ is singular)} \\ \lambda_3 &= .6 && \text{(since the trace of } A \text{ is 1.6)} \end{aligned}$$

(1, 2, 3 can be permuted)

(b)

$$\begin{aligned} n \geq 1: \quad A^n u_0 &= 1^n x_1 + 0^n x_2 + (.6)^n x_3 \\ &= x_1 + (.6)^n x_3 \end{aligned}$$

(c)

$A^n u_0$ approaches x_1.

3. (a) Suppose M is any invertible matrix. Circle all the properties of a matrix A that remain the same for $M^{-1}AM$:

   $\boxed{\text{same rank}}$

   same nullspace

   $\boxed{\text{same determinant}}$

   $\boxed{\text{real eigenvalues}}$

   orthonormal eigenvectors

   symmetric positive definiteness

   (b) This is a similar question but now Q is an orthonormal matrix. Circle the properties of A that remain the same for $Q^{-1}AQ$:

$$= Q^T AQ\ !$$

   same column space

   $\boxed{A^k \text{ approaches zero as } k \text{ increases}}$

   $\boxed{\text{orthonormal eigenvectors}}$

   $\boxed{\text{symmetric positive definiteness}}$

   $\boxed{\text{projection matrix}}$

4. (a) Suppose the 5 by 4 matrix A has independent columns. What is the most information you can give about

   the eigenvalues of A^TA: **They are real and positive**

   the eigenvectors of A^TA: **They are orthogonal**

   the determinant of A^TA: **The determinant is positive**

   (b)

$$A = \underset{U}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

   (c)

$$\begin{aligned} T(v_1 + v_2) &= T(v_1) + T(v_2) \\ &= Iw_1 + Iw_2 = w_1 + w_2 \\ &\quad \text{(intermediate steps may be ommitted)} \end{aligned}$$

18.06 Professor Strang Quiz 3 May 5, 1999

Your name is: ______________________ **Grading** 1 2 3 4

Please circle your recitation:

1)	Mon	2–3	2-131	S. Kleiman
2)	Mon	3–4	2-131	S. Hollander
3)	Tues	11–12	2-132	S. Howson
4)	Tues	12–1	2-132	S. Howson
5)	Tues	12–1	2-131	S. Kleiman
6)	Tues	1–2	2-131	S. Kleiman
7)	Tues	2–3	2-132	S. Howson

1 (27 pts.) Suppose $A = \begin{bmatrix} .7 & .4 \\ .3 & .6 \end{bmatrix}$.

(a) Find the matrices $\wedge$ and S in the diagonalization formula $S^{-1}AS = \wedge$.

(b) Find the matrix A^k (all four entries of the k^{th} power of A).

(c) Find the limit as $k \to \infty$ of $u_k = A^k u_0$ if $u_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2 (25 pts.) Suppose a 3 by 3 real symmetric matrix A has eigenvalues $4, 4, 0$.

(a) Find the determinant from the eigenvalues of $(2A - I)^{-1}$.

(b) True or false or not enough information:

This matrix A has 3 independent eigenvectors and can be diagonalized.

(c) True or false or not enough information:

The function $x^T Ax$ is never negative for any vector x.

(d) True or false or not enough information:

The matrix $\frac{1}{4}A$ is a Markov matrix.

(e) If A has orthonormal eigenvectors q_1, q_2, q_3 with $\lambda = 4, 4, 0$, find a formula for A in terms of q_1, q_2, q_3 using diagonalization.

3 (24 pts.) Suppose that A is an invertible 3 by 3 matrix.

(a) Show me how to prove that $x^T A^T Ax$ is *always positive* if x is not the zero vector. Why will this fail if A is not invertible?

(b) Show me how to prove that $A^T A$ is *similar* to AA^T. Does it follow that these matrices have the same eigenvalues and eigenvectors?

(c) If the SVD is written in the usual form $A = U\Sigma V^T$, what is the matrix $A^T A$ (reduced to the simplest form)?

4 (24 pts.) (a) Find the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then find the eigenvalues of $A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$ and $R = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix}$. The numbers a and b are real.

(b) Under what condition on "a" do all solutions of $du/dt = Au$ approach zero as $t \to \infty$?

(c) Under what conditions on "b" do all solutions of $dv/dt = Rv$ approach zero as $t \to \infty$?

(d) Under what condition on "a" is the matrix A positive definite?

18.06 Quiz 3 December 3, 1999 Closed Book

1 (36 pts.) The differential equation is

$$\frac{du}{dt} = Au \qquad \text{with} \qquad A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \qquad \text{and} \qquad u(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors and diagonalize to $A = S\Lambda S^{-1}$.

A is not invertible, hence one eigenvalue is 0.

$Tr(A) = -5$, so the other eigenvalue of A must be -5.

An eigenvector of A with eigenvalue 0 is $(3, 2)$.

An eigenvector of A with eigenvalue -5 is $(1, -1)$.

$$A = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 1/5 & 1/5 \\ 2/5 & -3/5 \end{bmatrix}.$$

(b) Solve for $u(t)$ starting from the given $u(0)$.

General solution is $u(t) = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 \, e^{-5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The condition $u(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ is satisfied when $c_1 = 1$, $c_2 = 2$.

(c) Compute the matrix e^{At} using S and Λ.

$A = S\Lambda S^{-1} \Rightarrow e^{At} = Se^{\Lambda t}S^{-1}$. So

$$e^{At} = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-5} \end{bmatrix} \begin{bmatrix} 1/5 & 1/5 \\ 2/5 & -3/5 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2e^{-5t} + 3 & -3e^{-5t} + 3 \\ -2e^{-5t} + 2 & 3e^{-5t} + 2 \end{bmatrix}$$

(d) As t approaches infinity, find the limits of $u(t)$ and e^{At}.

$$\text{As} \quad t \to \infty, \quad e^{-5t} \to 0, \quad u(t) \to \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \text{and} \quad e^{At} \to \frac{1}{5} \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$$

2 (40 pts.) The matrix A has 3's on the diagonal and 2's everywhere else:

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

(a) Decide if A is positive definite. What is the minimum value of $x^T A x$ for **all** vectors x in $\mathbb{R}^3$?

A is positive definite since:

1. A is symmetric,
2. The upper left determinants are $3, 5, 7$.

A is positive definite $\Rightarrow x^T A x \geq 0$.

When $x = 0$, $x^T A x = 0$. So 0 is the minimum.

(b) All entries of $B = A - I$ are 2's. From its rank find all the eigenvalues of B and then all the eigenvalues of A.

All three columns of B are the same

$\Rightarrow Rank(B) = 1$

$\Rightarrow N(B)$ has dimension 2

$\Rightarrow B$ has two independent eigenvectors with eigenvalue 0

$\Rightarrow \lambda_1 = \lambda_2 = 0$, and $\lambda_3 = tr(B) = 6$.

The eigenvalues of A are $1, 1, 7$.

(c) Write down any one specific symmetric matrix C that is similar to A. Write down if possible any one nonsymmetric matrix N that is similar to A. Write down a matrix J with the same eigenvalues as A that is **not** similar to A. (Give the 9 numbers in C, N, J.)

$$C = \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \qquad N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{bmatrix}, \qquad J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

Explanation: A is symmetric(so you could have let $C = A$) and hence can be diagonalized to Λ. Consider

$$D = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 7 \end{bmatrix}$$

D has eigenvalues 1, 1, 7.

If D is similar to A, then D can also be diagonalized to Λ and hence must have two eigenvectors with eigenvalue 1. This is possible iff $x = 0$. Our choice of C is obtained by letting $y = z = 0$. Our choice of N is obtained by letting $y = 0$, $z = 1$.

When $x \neq 0$, D is not similar to A. Our choice of J is obtained by letting $x = 1$.

Note: There are choices for C, N, and J which are not upper triangular.

(d) For the 6 by 6 matrix A_6 with 3's on the diagonal and 2's everywhere else use the same method (with $A_6 - I$) to find the six eigenvalues. If you make a good choice of eigenvectors, in what form can you factor A?

The matrix $B_6 = A_6 - I$ has rank 1, so it has 5 independent eigenvectors with eigenvalue 0. It follows that
the eigenvalues of B_6 are $0, 0, 0, 0, 0, 12 = tr(B)$ and
the eigenvalues of A_6 are $1, 1, 1, 1, 1, 13$.

We already know that $A = S\Lambda S^{-1}$ for some S whose columns are independent eigenvectors of A. But A is symmetric, so we can choose its eigenvectors to be orthonormal and have $A = Q\Lambda Q^{-1}$, where Q is orthogonal.

3 (24 pts.) Suppose $A = U\Sigma V^T =$ (orthogonal 2×2) (diagonal) (orthogonal 3×3)

$$U = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

(a) What are the eigenvalues and eigenvectors of A^TA?

$A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma^T\Sigma V^T$.

$$\Sigma^T\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of A^TA are $16, 1, 0$.

v_1 is an eigenvector of A^TA with eigenvalue 16.

v_2 is an eigenvector of A^TA with eigenvalue 1.

v_3 is an eigenvector of A^TA with eigenvalue 0.

(b) What is the nullspace of A? (Describe the whole nullspace.)

The nullspace of A is the linear span of v_3.

(c) What is the row space of A? (Describe the whole row space.)

The row space of A is the linear span of v_1 , v_2.

18.06 Professor Strang Quiz 3 Solutions May 5, 1999

1 (a) $\lambda_1 = 1$ (because A is a Markov matrix) and $\lambda_2 = .3$ (from the trace).

$$S^{-1}AS = \begin{bmatrix} 4 & 1 \\ 3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} .7 & .4 \\ .3 & .6 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & .3 \end{bmatrix}$$

(b) $$A^k = S \wedge^k S^{-1} = \begin{bmatrix} 4 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ & (.3)^k \end{bmatrix} \frac{1}{7} \begin{bmatrix} 1 & .1 \\ .3 & -4 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 4+3(.3)^k & 4-4(.3)^k \\ 3-3(.3)^k & 3+4(.3)^k \end{bmatrix}$$

(c) The limit is $\frac{1}{7}\begin{bmatrix} 8 \\ 6 \end{bmatrix} = \frac{2}{7}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, a multiple of the eigenvector x_1.

2 (a) $(2A - I)^{-1}$ has eigenvalues $\frac{1}{7}, \frac{1}{7}, -1$ so the determinant is $-\frac{1}{49}$.

(b) **True** (A is given as symmetric)

(c) **True** (no negative eigenvalues; A is positive semidefinite)

(d) **Not enough information**: $\frac{1}{4}A = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ is not Markov, $\frac{1}{4}A = \begin{bmatrix} 1 & & \\ & .5 & .5 \\ & .5 & .5 \end{bmatrix}$ is Markov.

(e) $$A = Q \wedge Q^T = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} 4 & & \\ & 4 & \\ & & 0 \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ q_3^T \end{bmatrix} = 4q_1q_1^T + 4q_2q_2^T.$$

3 (a) $x^TA^TAx = (Ax)^T(Ax)$ is positive unless $Ax = 0$. (Then it is zero — so if x is in the nullspace of A we do have $x^TA^TAx = 0$.)

(b) $A(A^TA)A^{-1} = (AA^T)$ so the matrices in parentheses are similar.

(c) $A^TA = (V\Sigma^TU^T)(U\Sigma V^T) = V\Sigma^T\Sigma V^T$.

4 (a) A has eigenvalues 1 and -1 (then $a+1$ and $a-1$).
R has eigenvalues i and $-i$ (then $b+i$ and $b-i$).

(b) We need $a+1 < 0$ for stability so the condition is a $a < -1$.

(c) We need the real parts of $b+i$ and $b-i$ to be negative, so the condition is $b < 0$.

(d) We need $a+1$ and $a-1$ to be positive, so the condition is $a > 1$.

18.06 Professor Strang Quiz 3 December 6, 2000

Your name is: ______________________________

Please circle your recitation:

1)	M2	2-131	Holm	2-181	3-3665	tsh@math
2)	M2	2-132	Dumitriu	2-333	3-7826	dumitriu@math
3)	M3	2-131	Holm	2-181	3-3665	tsh@math
4)	T10	2-132	Ardila	2-333	3-7826	fardila@math
5)	T10	2-131	Czyz	2-342	3-7578	czyz@math
6)	T11	2-131	Bauer	2-229	3-1589	bauer@math
7)	T11	2-132	Ardila	2-333	3-7826	fardila@math
8)	T12	2-132	Czyz	2-342	3-7578	czyz@math
9)	T12	2-131	Bauer	2-229	3-1589	bauer@math
10)	T1	2-132	Ingerman	2-372	3-4344	ingerman@math
11)	T1	2-131	Nave	2-251	3-4097	nave@math
12)	T2	2-132	Ingerman	2-372	3-4344	ingerman@math
13)	T2	1-150	Nave	2-251	3-4097	nave@math

1 (30 pts.) (a) Find the diagonalization $A = S\Lambda S^{-1}$ of

$$A = \begin{bmatrix} 0.5 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) What is the limit of A^k as $k \to \infty$?

(c) Suppose B^k approaches I (the 2 by 2 identity) as $k \to \infty$. How do you know that $B = I$? Explain using eigenvalues and Jordan forms like

$$J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

2 (40 pts.) (a) Suppose the diagonalization $A = S\Lambda S^{-1}$ is exactly the same as the singular value decomposition $A = U\Sigma V^{\mathrm{T}}$ (so $S = U = V$ and $\Lambda = \Sigma$). What information does this give about A? Can it be singular?

(b) What are the eigenvalues of a 3 by 3 Markov projection matrix that has trace 2? Create one matrix that has these properties.

(c) Here is a matrix with orthogonal columns. Find its SVD $A = U\Sigma V^{\mathrm{T}}$.

$$A = \begin{bmatrix} 3 & 0 \\ 4 & 0 \\ 0 & 7 \end{bmatrix}$$

(d) Suppose A is similar to a 3 by 3 matrix B that has eigenvalues 1, 1, 2. What can you say about

1. the eigenvalues of A
2. diagonalizability of A
3. symmetry of A
4. positive definiteness of A

In each of (2) (3) and (4) decide if A can't have or might have or must have this property.

3 (30 pts.) (a) Find the eigenvalues of the matrix (and fill in the blanks)

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

These eigenvalues are all __________ because this matrix A is __________

(b) If the eigenvectors are x_1, x_2, x_3 (not required to compute them) describe the general solution to the differential equation $\frac{du}{dt} = Au$.

(c) At what time T is the solution $u(T)$ guaranteed to equal its initial value $u(0)$?

Math 18.06 Quiz 3 Solutions

1 (30 pts.) (a)

$$A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 6 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -6 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$A^{\infty} = \begin{bmatrix} 1 & 6 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -6 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) The eigenvalues of B must both be 1. Suppose B has the Jordan form $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, with $B = MJM^{-1}$. Then $B^n = MJ^nM^{-1}$ and $J^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, which cannot converge. So B can NOT have Jordan Form J. The only alternative is that B has Jordan form I, in which case $B = MIM^{-1} = I$

2 (40 pts.)

(a) $S^{-1} = S^T$, so $A = S\Lambda S^T$ is symmetric. Singular values are always nonnegative, so from $\Lambda = \Sigma$ the eigenvalues of A are nonnegative, so A is symmetric positive semidefinite. It can be singular (the all zeros matrix is an example).

(b) The eigenvalues of a projection matrix are either 0 or 1, and their sum is 2, so they must be 1, 1, 0. For example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

(c) $A^T A = \begin{bmatrix} 25 & 0 \\ 0 & 49 \end{bmatrix}$ so the singular values are 7 and 5. So

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{and } A = \begin{bmatrix} 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(d) 1. The eigenvalues of A are 1,1,2 - the same as the eigenvalues of B.
2. A might or might not be diagonalizable
3. A might or might not be symmetric
4. A definitely (!) has positive eigenvalues. However it might not be symmetric, so A might or might not be positive definite.

3 (30 pts.)

(a) The eigenvalues are $0, \sqrt{2}i, -\sqrt{2}i$. They are all pure imaginary (including zero!) because A is skew symmetric

(b) The general solution is $\vec{u}(T) = c_1\vec{x}_1 + c_2 e^{\sqrt{2}iT}\vec{x}_2 + e^{-\sqrt{2}iT}\vec{x}_3$

(c) $e^{i\theta} = \cos\theta + i\sin\theta$. This function has a period of 2π, so when $\sqrt{2}T = 2n\pi$, we have $\vec{u}(T) = \vec{u}(0)$. In particular, T can be $\sqrt{2}\pi$.

18.06 Exam 3 May 3, 2000 Closed Book

Your name is: ____________________

Please circle your recitation:

1)	M 2	2–131	P. Clifford	2)	M 3	2–131	P. Clifford
3)	T 11	2–132	T. de Piro	4)	T 12	2–132	T. de Piro
5)	T 1	2–131	T. Bohman	6)	T 1	2–132	T. Pietraho
7)	T 2	2–132	T. Pietraho	8)	T 2	2–131	T. Bohman

Note: Make sure your exam has 4 problems.

Problem		**Points possible**
1	______	25
2	______	25
3	______	25
4	______	25
Total	______	100

1 (25 pts) Let

$$A = \begin{bmatrix} 5 & 3 & 0 \\ 3 & 5 & 0 \end{bmatrix}, \quad \text{so} \quad A^{\mathrm{T}}A = \begin{bmatrix} 34 & 30 & 0 \\ 30 & 34 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The following are eigenvectors of $A^{\mathrm{T}}A$:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(a) What are the eigenvalues of $A^{\mathrm{T}}A$?

(b) What are the singular values of A?

(c) Give the singular value decomposition of A.

Note: You must show your work to receive credit for this problem.

2 (25 pts) True (give a reason) or False (give a counterexample):

(a) If A is a symmetric matrix, any two eigenvectors of A are perpendicular.

(b) If A is $n \times n$ and has n orthonormal eigenvectors, then A is symmetric.

(c) Any eigenvector matrix S of a symmetric matrix is symmetric.

Note: You must show your work to receive credit for this problem.

3 (25 pts) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

For what (if any) values of d does A have all positive eigenvalues? (Hint: Do not try to compute the eigenvalues of A.)

Note: You must show your work to receive credit for this problem.

4 (25 pts) Suppose A is a 3×3 matrix with eigenvalues $\lambda = 1$ and $\lambda = 2$. Suppose also that $A - I$ has rank one.

(a) Which eigenvalue of A is repeated? **Explain why.**

(b) Write down a specific matrix which is similar to A and symmetric. **Explain why** they are similar.

(c) Write down a specific matrix which is similar to A and not symmetric. **Explain why** they are similar.

(d) Write down a specific matrix which has the same eigenvalues as A but is not similar to A. **Explain why** they are not similar.

18.06 Exam 3 Solutions May 3, 2000

Problem 1 (a) Find the eigenvalues by multiplying each eigenvector by $A^{\mathrm{T}}A$: 64, 4, and 0.
(b) The singular values are the square roots of the nonzero eigenvalues of $A^{\mathrm{T}}A$: 8 and 2.
(c) The SVD is

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 2 (a) False. The 2×2 identity matrix is symmetric, but it has plenty of non-perpendicular eigenvectors, e.g., $(1,0)$ and $(1,1)$.
(b) True, because A is diagonalizable using an orthogonal matrix: $A = Q\Lambda Q^{\mathrm{T}}$. Such a matrix is symmetric: $(Q\Lambda Q^{\mathrm{T}})^{\mathrm{T}} = Q\Lambda Q^{\mathrm{T}}$.
(c) False. Same example as in part (a).

Problem 3 There is no such value of $d > 0$. To have positive eigenvalues means that A is positive definite. The upper left determinants are 1, $d-4$, and $12-4d$. These are never all positive.

Problem 4 (a) $\lambda = 1$ is repeated. The number of independent $\lambda = 1$ eigenvectors is given by the dimension of $N(A-I)$, which is two. So A has two independent $\lambda = 1$ eigenvectors, so $\lambda = 1$ must be repeated.
(b) A has three independent eigenvectors, so it is diagonalizable, i.e., similar to

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(c) The matrix

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

has the same eigenvalues and number of independent eigenvalues as A, so is similar to A.
(d) The matrix

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

has the same eigenvalues as A but is missing an eigenvector: the rank of $C - I$ is two, so C has only one independent $\lambda = 1$ eigenvector.

18.06 Strang, Edelman, Huhtanen Quiz 3 December 5, 2001

Your name is: ______________________________

Please circle your recitation:

Recitations

#	Time	Room	Instructor	Office	Phone	Email @math
Lect. 1	MWF 12	4-270	M Huhtanen	2-335	3-7905	huhtanen
Lect. 2	MWF 1	4-370	A Edelman	2-380	3-7770	edelman
Rec. 1	M 2	2-131	D. Sheppard	2-342	3-7578	sheppard
2	M 2	2-132	M. Huhtanen	2-335	3-7905	huhtanen
3	M 3	2-131	D. Sheppard	2-342	3-7578	sheppard
4	T 10	2-132	A. Lachowska	2-180	3-4350	anechka
5	T 10	2-131	S. Kleiman	2-278	3-4996	kleiman
6	T 11	2-131	M. Honsen	2-490	3-4094	honsen
7	T 11	2-132	A. Lachowska	2-180	3-4350	anechka
8	T 12	2-131	M. Honsen	2-490	3-4094	honsen
9	T 1	2-132	A. Lachowska	2-180	3-4350	anechka
10	T 1	2-131	S. Kleiman	2-278	3-4996	kleiman
11	T 2	2-132	F. Latour	2-090	3-6293	flatour

For full credit, carefully explain your reasoning, as always!

1 (36 pts.) Let A be the square matrix

$$A = \begin{bmatrix} 2 & 1 \\ x & y \end{bmatrix}.$$

(a) With $x = 2$ and $y = 1$ diagonalize A. That is, compute $A = S\Lambda S^{-1}$, where Λ is a diagonal matrix. (12p)

(b) With $y = 2$ pick x so that S can be orthogonal in a diagonalization of A. Compute then one such S. (12p)

(c) If $y = 2$, can you find $x > 0$ such that A and $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$ are similar? (Hint: look at the eigenvalues.) (12p)

2 (32 pts.) (a) Choose x and y so that

$$M = \begin{bmatrix} 1/2 & x \\ y & 1/4 \end{bmatrix}$$

is a Markov matrix. (4p)

Compute the steady state eigenvector x_1 of unit length. (That is, $\|x_1\| = 1$). (8p)

(b) Is

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

positive definite? (4p)

Find the singular value decomposition of A. (16p)

3 (32 pts.) Let

$$A = \begin{bmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

and

$$X = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \text{ so that } X^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 0 \end{bmatrix}$$

(a) Compute $M = X^{-1}AX$. (4p)

What are the eigenvalues of A? (4p)

How many linearly independent eigenvectors does A have? (4p)

Is A diagonalizable? (4p)

(b) Let

$$B = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Compute e^{Bt} explicitly. (12p)

Compute $\lim_{t\to\infty} e^{Bt}x$. (4p)

18.06 Midterm Exam 3, Spring, 2001

Name ______________________ Optional Code ______________

Recitation Instructor ______________ Email Address ______________

Recitation Time ________________

This midterm is closed book and closed notes. No calculators, laptops, cell phones or other electronic devices may be used during the exam.

There are 3 problems. Good luck.

1. (40pts.) Consider the matrix

$$A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{pmatrix}.$$

(a) Given that one eigenvalue of A is $\lambda = 6$, find the remaining eigenvalues.

(b) Find three linearly independent eigenvectors of A.

(c) Find an *orthogonal* matrix Q and a diagonal matrix Λ, so that $A = Q\Lambda Q^T$.

2. (20pts.) Consider the system of first order linear ODEs

$$\frac{dx}{dt} = -7x + 2y \quad \frac{dy}{dt} = -6x.$$

Find two independent real-valued solutions $\begin{pmatrix} x^{(1)} \\ y^{(1)} \end{pmatrix}$ and $\begin{pmatrix} x^{(2)} \\ y^{(2)} \end{pmatrix}$ of this system and hence find the solution $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ which satisfies the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

3. (40pts.) Let A_n be the $n \times n$ tridiagonal matrix

$$A_n = \begin{pmatrix} 1 & -a & 0 & 0 & \cdots & 0 \\ -a & 1 & -a & 0 & \cdots & 0 \\ 0 & -a & 1 & -a & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -a & 1 & -a \\ 0 & 0 & \cdots & 0 & -a & 1 \end{pmatrix}.$$

(a) Show for $n \geq 3$ that

$$\det(A_n) = \det(A_{n-1}) - a^2 \cdot \det(A_{n-2}). \tag{1}$$

(b) Show that eq.(1) can equivalently be written as $\mathbf{x}_n = B\,\mathbf{x}_{n-1}$, where

$$\mathbf{x}_n = \begin{pmatrix} \det(A_n) \\ \det(A_{n-1}) \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -a^2 \\ 1 & 0 \end{pmatrix}.$$

(c) For $a^2 = \frac{3}{16}$, find an expression for $\det(A_n)$ for any n. (*Hint:* One method starts by writing B in the form $B = S\Lambda S^{-1}$, where Λ is a diagonal matrix.)

18.06 Solutions to Midterm Exam 3, Spring, 2001

1. (40pts.) Consider the matrix

$$A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{pmatrix}.$$

(a) Given that one eigenvalue of A is $\lambda = 6$, find the remaining eigenvalues.

•

$$\begin{aligned} \det \begin{pmatrix} 4-\lambda & -1 & 1 \\ -1 & 4-\lambda & -1 \\ 1 & -1 & 4-\lambda \end{pmatrix} &= 0 \\ \Leftrightarrow -\lambda^3 + 12\lambda^2 - 45\lambda + 54 &= 0 \\ \Leftrightarrow \lambda_1 = 6,\ \lambda_2 = \lambda_3 = 3 & \end{aligned}$$

(b) Find three linearly independent eigenvectors of A.

• For $\lambda_1 = 6$, we have $\begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix} \mathbf{v}_1 = \mathbf{0}$, and so $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = \lambda_3 = 3$, we have $\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \mathbf{v} = \mathbf{0}$, and two linearly independent solutions are $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

(c) Find an *orthogonal* matrix Q and a diagonal matrix Λ, so that $A = Q\Lambda Q^T$.

• To obtain the first column of Q, let $\mathbf{q}_1 = \mathbf{v}_1/\|\mathbf{v}_1\| = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$. The second column of Q is given by $\mathbf{q}_2 = \mathbf{v}_1/\|\mathbf{v}_2\| = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$. The vector $\mathbf{v}_3$ is not orthogonal to $\mathbf{q}_2$, so we need to use Gram-Schmidt to make it so:

$$\tilde{\mathbf{q}}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ -1 \end{pmatrix}.$$

Normalising gives $\mathbf{q}_3 = \tilde{\mathbf{q}}_3/\|\tilde{\mathbf{q}}_3\| = \begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix}$. Hence,

$$Q = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

2. (20pts.) Consider the system of first order linear ODEs

$$\frac{dx}{dt} = -7x + 2y \quad \frac{dy}{dt} = -6x.$$

Find two independent real-valued solutions $\begin{pmatrix} x^{(1)} \\ y^{(1)} \end{pmatrix}$ and $\begin{pmatrix} x^{(2)} \\ y^{(2)} \end{pmatrix}$ of this system and hence find the solution $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ which satisfies the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

- We can write this system as $\frac{d}{dt}\mathbf{x} = \begin{pmatrix} -7 & 2 \\ -6 & 0 \end{pmatrix}\mathbf{x}$. The eigenvalues of this matrix are given by

$$\begin{aligned} \det \begin{pmatrix} -7-\lambda & 2 \\ -6 & -\lambda \end{pmatrix} &= 0 \\ \Leftrightarrow \lambda^2 + 7\lambda + 12 &= 0 \\ \Leftrightarrow \lambda_1 = -4,\ \lambda_2 = -3. \end{aligned}$$

The corresponding eigenvectors are: for $\lambda_1 = -4$, $\begin{pmatrix} -3 & 2 \\ -6 & 4 \end{pmatrix}\mathbf{v}_1 = \mathbf{0}$, so $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$; and for $\lambda_2 = -3$, $\begin{pmatrix} -4 & 2 \\ -6 & 3 \end{pmatrix}\mathbf{v}_2 = \mathbf{0}$, so $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Hence the general solution to this system of equations is

$$\mathbf{x}(t) = C_1 e^{-4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 e^{-3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

To satisfy the initial condition, we need to solve

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which gives $C_1 = 1$ and $C_2 = -1$.

3. (40pts.) Let A_n be the $n \times n$ tridiagonal matrix

$$A_n = \begin{pmatrix} 1 & -a & 0 & 0 & \cdots & 0 \\ -a & 1 & -a & 0 & \cdots & 0 \\ 0 & -a & 1 & -a & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -a & 1 & -a \\ 0 & 0 & \cdots & 0 & -a & 1 \end{pmatrix}.$$

(a) Show for $n \geq 3$ that

$$\det(A_n) = \det(A_{n-1}) - a^2 \cdot \det(A_{n-2}). \qquad (1)$$

- Let us expand the determinant of A_n along the first column,

$$\begin{aligned} \det(A_n) &= 1 \cdot \det \begin{pmatrix} 1 & -a & 0 & \cdots & 0 \\ -a & 1 & -a & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -a & 1 & -a \\ 0 & \cdots & 0 & -a & 1 \end{pmatrix} + a \det \begin{pmatrix} -a & 0 & 0 & \cdots & 0 \\ -a & 1 & -a & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -a & 1 & -a \\ 0 & \cdots & 0 & -a & 1 \end{pmatrix} \\ &= \det(A_{n-1}) - a^2 \det \begin{pmatrix} 1 & -a & 0 & \cdots \\ -a & 1 & -a & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & -a & 1 \end{pmatrix} \\ &= \det(A_{n-1}) - a^2 \det(A_{n-2}). \end{aligned}$$

Here, we have expanded the determinant in the second term of line 1 along the first row.

(b) Show that eq.(1) can equivalently be written as $\mathbf{x}_n = B\,\mathbf{x}_{n-1}$, where

$$\mathbf{x}_n = \begin{pmatrix} \det(A_n) \\ \det(A_{n-1}) \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -a^2 \\ 1 & 0 \end{pmatrix}.$$

-

$$\begin{aligned} \mathbf{x}_n = \begin{pmatrix} \det(A_n) \\ \det(A_{n-1}) \end{pmatrix} &= \begin{pmatrix} \det(A_{n-1}) - a^2 \det(A_{n-2}) \\ \det(A_{n-1}) \end{pmatrix} \\ &= \begin{pmatrix} 1 & -a^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \det(A_{n-1}) \\ \det(A_{n-2}) \end{pmatrix} = \begin{pmatrix} 1 & -a^2 \\ 1 & 0 \end{pmatrix} \mathbf{x}_{n-1}. \end{aligned}$$

(c) For $a^2 = \frac{3}{16}$, find an expression for $\det(A_n)$ for any n. (*Hint:* One method starts by writing B in the form $B = S\Lambda S^{-1}$, where Λ is a diagonal matrix.)

• The answer is given by $\mathbf{x}_n = B^{n-2}\mathbf{x}_2$. To determine $\mathbf{x}_2$, we need $\det(A_1) = 1$, and $\det(A_2) = \det\begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix} = 1 - a^2 = 1 - \frac{3}{16} = \frac{13}{16}$. To find B^{n-2}, we first need to diagonalise B. Eigenvalues are given by

$$\begin{aligned} \det\begin{pmatrix} 1-\lambda & -\frac{3}{16} \\ 1 & -\lambda \end{pmatrix} &= 0 \\ \Leftrightarrow \lambda^2 - \lambda + \frac{3}{16} &= 0 \\ \Leftrightarrow \lambda_1 = \frac{3}{4},\ \lambda_2 = \frac{1}{4}. \end{aligned}$$

The eigenvector corresponding to $\lambda_1 = \frac{3}{4}$ is given by $\begin{pmatrix} 1/4 & -3/16 \\ 1 & -3/4 \end{pmatrix}\mathbf{v}_1 = \mathbf{0}$, so that $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. The eigenvector corresponding to $\lambda_2 = \frac{1}{4}$ is given by $\begin{pmatrix} 3/4 & -3/16 \\ 1 & -1/4 \end{pmatrix}\mathbf{v}_2 = \mathbf{0}$, and hence $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. So $B = S\Lambda S^{-1}$, where

$$S = \begin{pmatrix} 3 & 1 \\ 4 & 4 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}, \quad S^{-1} = \frac{1}{8}\begin{pmatrix} 4 & -1 \\ -4 & 3 \end{pmatrix}.$$

Hence,

$$\mathbf{x}_n = S\Lambda^{n-2}S^{-1}\mathbf{x}_2 = \frac{1}{8}\begin{pmatrix} \frac{27}{4}\left(\frac{3}{4}\right)^{n-2} - \frac{1}{4}\left(\frac{1}{4}\right)^{n-2} \\ 9\left(\frac{3}{4}\right)^{n-2} - \left(\frac{1}{4}\right)^{n-2} \end{pmatrix}$$

and

$$\det(A_n) = \frac{2}{4^{n+1}}\left(3^{n+1} - 1\right).$$

18.06 Professor Strang/Ingerman Quiz 3 December 6, 2002

Your name is: ______________________________

Please circle your recitation:

1)	M2	2-131	P.-O. Persson	2-088	2-1194	persson
2)	M2	2-132	I. Pavlovsky	2-487	3-4083	igorvp
3)	M3	2-131	I. Pavlovsky	2-487	3-4083	igorvp
4)	T10	2-132	W. Luo	2-492	3-4093	luowei
5)	T10	2-131	C. Boulet	2-333	3-7826	cilanne
6)	T11	2-131	C. Boulet	2-333	3-7826	cilanne
7)	T11	2-132	X. Wang	2-244	8-8164	xwang
8)	T12	2-132	P. Clifford	2-489	3-4086	peter
9)	T1	2-132	X. Wang	2-244	8-8164	xwang
10)	T1	2-131	P. Clifford	2-489	3-4086	peter
11)	T2	2-132	X. Wang	2-244	8-8164	xwang

1 (36 pts.) (a) What are the eigenvalues of the 5 by 5 matrix $A = \boldsymbol{ones}(5)$ with all entries $a_{ij} = 1$? Please look at A, not at $\det(A - \lambda I)$.

(b) Solve this differential equation to find $\boldsymbol{u}(t)$:

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} \qquad \text{starting from } \boldsymbol{u}(0) = (0, 1, 1, 1, 2)\,.$$

First split $\boldsymbol{u}(0)$ into two eigenvectors of A.

(c) Using part (a), what are the *eigenvalues* and *trace* and *determinant* of the matrix $B =$ same as A except zeros on the diagonal.

2 (20 pts.) (a) If A is similar to B show that e^A is similar to e^B. *First define "similar" and* e^A!!

(b) If A has 3 eigenvalues $\lambda = 0, 2, 4$, *find the eigenvalues of* e^A.

Using part (a) explain this connection with determinants:

$$\text{determinant of } e^A = e^{\text{trace of } A}$$

3 (22 pts.) Suppose the SVD $A = U\Sigma V^{\mathrm{T}}$ is

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

(a) For which angles θ and α (0 to $\frac{\pi}{2}$) is A a positive definite symmetric matrix? No computing needed.

(b) What are the eigenvalues and eigenvectors of $A^{\mathrm{T}}A$? No computing!

4 (22 pts.) Multinational companies in the US, Asia, and Europe have assets of \$ 12 trillion. At the start, \$ 6 trillion are in the US, \$ 6 trillion in Europe. Each year half the US money stays home, $\frac{1}{4}$ each goes to Asia and Europe. For Asia and Europe, half stays home and half is sent to the US.

$$\begin{bmatrix} \text{US} \\ \text{Asia} \\ \text{Europe} \end{bmatrix}_{\text{year } k+1} = \begin{bmatrix} .5 & .5 & .5 \\ .25 & .5 & 0 \\ .25 & 0 & .5 \end{bmatrix} \begin{bmatrix} \text{US} \\ \text{Asia} \\ \text{Europe} \end{bmatrix}_{\text{year } k}$$

(a) The eigenvalues and eigenvectors of this *singular* matrix A are

(b) The limiting distribution of the \$ 12 trillion as the world ends is

$$\begin{aligned} \text{US} &= \\ \text{Asia} &= \\ \text{Europe} &= \end{aligned}$$

Course 18.06, Fall 2002: Quiz 3, Solutions

1 (a) One eigenvalue of $A = \mathsf{ones}(5)$ is $\lambda_1 = 5$, corresponding to the eigenvector $\boldsymbol{x}_1 = (1,1,1,1,1)$. Since the rank of A is 1, all the other eigenvalues $\lambda_2, \ldots, \lambda_5$ are zero. Check: The trace of A is 5.

(b) The initial condition $\boldsymbol{u}(0)$ can be written as a sum of the two eigenvectors $\boldsymbol{x}_1 = (1,1,1,1,1)$ and $\boldsymbol{x}_2 = (-1,0,0,0,1)$, corresponding to the eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 0$:

$$\boldsymbol{u}(0) = (0,1,1,1,2) = (1,1,1,1,1) + (-1,0,0,0,1) = \boldsymbol{x}_1 + \boldsymbol{x}_2.$$

The solution to $\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$ is then

$$\boldsymbol{u}(t) = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + c_2 e^{\lambda_2 t}\boldsymbol{x}_2 = (1,1,1,1,1)e^{5t} + (-1,0,0,0,1).$$

(c) The eigenvectors of $B = A - I$ are the same as for A, and the eigenvalues are smaller by 1:

$$B\boldsymbol{x} = (A - I)\boldsymbol{x} = A\boldsymbol{x} - \boldsymbol{x} = \lambda\boldsymbol{x} - \boldsymbol{x} = (\lambda - 1)\boldsymbol{x},$$

where $\boldsymbol{x}, \lambda$ are an eigenvector and an eigenvalue of A. The eigenvalues of B are then $4, -1, -1, -1, -1$, the trace is $\sum_i \lambda_i = 0$, and the determinant is $\prod_i \lambda_i = 4$.

2 (a) B is similar to A when $B = M^{-1}AM$, with M invertible. The exponential of A is

$$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots.$$

Every power B^k of B is similar to the same power A^k of A:

$$B^k = M^{-1}AMM^{-1}AM \cdots M^{-1}AM = M^{-1}A^kM.$$

Then

$$e^B = I + B + \frac{1}{2}B^2 + \cdots = M^{-1}\left(I + A + \frac{1}{2}A^2 + \cdots\right)M = M^{-1}e^AM.$$

It is also OK to show this using $e^A = Se^{\Lambda}S^{-1}$, although that assumes that the matrices are diagonalizable.

(b) The exponential of A is

$$e^A = Se^{\Lambda}S^{-1} = S\begin{bmatrix} e^0 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^4 \end{bmatrix}S^{-1}.$$

But this is an eigenvalue decomposition of e^A, so the eigenvalues are $1, e^2, e^4$.
More generally, the eigenvalues of e^A are the exponentials of the eigenvalues of A, and

$$\det(e^A) = e^{\lambda_1}e^{\lambda_2}\cdots e^{\lambda_n} = e^{\lambda_1+\lambda_2+\cdots+\lambda_n} = e^{\operatorname{tr}(A)}.$$

3 (a) For A to be symmetric, U has to be equal to V (notice V^T in the matrices):

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}.$$

Together with the restrictions on θ, α this requires that $\theta = \alpha$. A is then a positive definite symmetric matrix, since it is similar to $\begin{bmatrix}9 & 0\\ 0 & 4\end{bmatrix}$.

(b) The eigenvalues of A^TA are the square of the singular values, that is, 81 and 16. The eigenvectors of A^TA are the columns of V, that is, $(\cos\alpha, \sin\alpha)$ and $(-\sin\alpha, \cos\alpha)$. This can also be shown by multiplying $A^TA = V\Sigma^2V^T$ and identifying this as the eigenvalue decomposition of A^TA.

4 (a) A is singular, so one eigenvalue is 0. It is also a Markov matrix, so another eigenvalue is 1 (Motivation: Each column of A sums to 1, so each column of $A - I$ sums to 0. $A - I$ then has an eigenvalue 0, and A has an eigenvalue 1). The last eigenvalue is 0.5 since $\text{trace}(A) = \sum_i \lambda_i = 1.5$.

The eigenvectors are found by solving the following systems:

$$\lambda_1 = 1: \qquad (A - \lambda_1 I)\boldsymbol{x}_1 = \begin{bmatrix}-.5 & .5 & .5\\ .25 & -.5 & 0\\ .25 & 0 & -.5\end{bmatrix}\boldsymbol{x}_1 = 0 \Longrightarrow \boldsymbol{x}_1 = (2, 1, 1),$$

$$\lambda_2 = 0.5: \qquad (A - \lambda_2 I)\boldsymbol{x}_2 = \begin{bmatrix}0 & .5 & .5\\ .25 & 0 & 0\\ .25 & 0 & 0\end{bmatrix}\boldsymbol{x}_2 = 0 \Longrightarrow \boldsymbol{x}_2 = (0, 1, -1),$$

$$\lambda_3 = 0: \qquad (A - \lambda_3 I)\boldsymbol{x}_3 = \begin{bmatrix}.5 & .5 & .5\\ .25 & .5 & 0\\ .25 & 0 & .5\end{bmatrix}\boldsymbol{x}_3 = 0 \Longrightarrow \boldsymbol{x}_3 = (2, -1, -1).$$

(b) Write the initial value as a linear combination of the eigenvectors:

$$\boldsymbol{u}_0 = (6, 0, 6) = 3\boldsymbol{x}_1 - 3\boldsymbol{x}_2.$$

The distribution after k steps is then

$$\boldsymbol{u}_k = A^k\boldsymbol{u}_0 = 3\lambda_1^k\boldsymbol{x}_1 - 3\lambda_2^k\boldsymbol{x}_2 = 3\boldsymbol{x}_1 - 3\cdot 0.5^k\boldsymbol{x}_2 \to 3\boldsymbol{x}_1 = (6, 3, 3) \text{ as } k \to \infty.$$

18.06 Exam 3 Solutions

1. a) $M = \begin{bmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{bmatrix}$.

   b) It is the eigenvector for M corresponding to eigenvalue 1, $(0.5, 0.3)$.

   c) After many years, the percentage of people drinking two kinds of coffee will converge to one that is proportional to the steady state vector. So people drinking regular coffee will be about $5/8 = 62.5\%$.

2. a) Column vectors of Q are normalized eigenvectors of A, denote column vectors of Q by v_1, v_2, v_3. Then

$$\begin{aligned} (A-(-2)I)v_1 &= 0 \Rightarrow v_1 = (0,1,0) \\ (A-4I)v_2 &= 0 \Rightarrow v_2 = (-1,0,2)/\sqrt{5} \\ (A-(-1)I)v_3 &= 0 \Rightarrow v_3 = (2,0,1)/\sqrt{5} \end{aligned}$$

$$Q = \begin{bmatrix} 0 & -1/\sqrt{5} & 2/\sqrt{5} \\ 1 & 0 & 0 \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

   b) Λ is the matrix for L under the basis $\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$

3. First we find out the eigenvalues of A^TA corresponding to w_i respectively.

$$\begin{aligned} A^TAw_1 &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = 3w_1 \\ A^TAw_2 &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = w_2 \\ A^TAw_3 &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \end{aligned}$$

So $\lambda_1 = 3, \sigma_1 = \sqrt{3}, \lambda_2 = 1, \sigma_2 = 1$, and $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Column vectors of V are normalized eigenvectors, $v_1 = w_1/|w_1|, v_2 = w_2/|w_2|, v_3 = w_3/|w_3|$,
$V = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$

Column vectors of U satisfies, $u_1 = Av_1/\sigma_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $u_2 = Av_2/\sigma_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

Finally the answer is

$$SVD(A) = U\Sigma V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

4. a)

$$L(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$L(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = L(\frac{1}{2}\begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \frac{1}{2}\begin{bmatrix} 8 \\ -6 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \end{bmatrix} = -8\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So the matrix for L with respect to the standard basis for $\mathbf{R}^2$ is $A = \begin{bmatrix} 4 & 0 \\ 5 & -8 \end{bmatrix}$

b) The change of basis is equal to the inverse of basis matrix, i.e. $P = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1/2 \end{bmatrix}$.

c)

$$B = P^{-1}AP = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 5 & -8 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} -1 & 14 \\ \frac{5}{2} & -3 \end{bmatrix}$$

18.06 Professor Strang Quiz 3 December 5, 2005

Your PRINTED name is: ____________________

Grading

1

2

3

Please circle your recitation:

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4)	T 10	2-132	P-O. Persson	2-363A	3-4989	`persson`
5)	T 11	2-131	P-O. Persson	2-363A	3-4989	`persson`
6)	T 11	2-132	P. Pylyavskyy	2-333	3-7826	`pasha`
7)	T 12	2-132	T. Lawson	4-182	8-6895	`tlawson`
8)	T 12	2-131	P. Pylyavskyy	2-333	3-7826	`pasha`
9)	T 1	2-132	A. Chan	2-588	3-4110	`alicec`
10)	T 1	2-131	D. Chebikin	2-333	3-7826	`chebikin`
11)	T 2	2-132	A. Chan	2-588	3-4110	`alicec`
12)	T 3	2-132	T. Lawson	4-182	8-6895	`tlawson`

1 (30 pts.) a) Find the eigenvalues and eigenvectors of the Markov matrix

$$A = \begin{bmatrix} .9 & .4 \\ .1 & .6 \end{bmatrix}$$

b) What is the limiting value of $A^k \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as the power k goes to infinity ?

c) What does it mean to say that "A *is similar to* B" ?

Is that 2 by 2 matrix A similar (yes or no) to its transpose B ?

$$B = \begin{bmatrix} .9 & .1 \\ .4 & .6 \end{bmatrix}$$

Give a reason for your answer.

$\mathcal{X}$

2 (40 pts.) This 4 by 4 matrix H is a Hadamard matrix:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Key Properties

$H^{\mathrm{T}} = H$ and $H^2 = 4I$

a) Figure out the eigenvalues of H. Explain your reasoning.

b) Figure out H^{-1} and the determinant of H. Explain your reasoning.

c) This matrix S contains three eigenvectors of H. Find a 4th eigenvector x_4 and explain your reasoning:

$$S = \begin{bmatrix} 1 & 1 & 0 & \\ 1 & 0 & -1 & \\ 1 & 0 & 1 & \\ -1 & 1 & 0 & \end{bmatrix}$$

d) Find the solution to $du/dt = Hu$ given that $u(0) =$ third column of S.

$\mathcal{XX}$

3 (30 pts.) Suppose A is a 3 by 3 symmetric matrix with eigenvalues $2, 5, 7$ and corresponding eigenvectors x_1, x_2, x_3.

a) Suppose x is a combination $x = c_1 x_1 + c_2 x_2 + c_3 x_3$. *Find* Ax. Now find $x^{\mathrm{T}} A x$ using the symmetry of A. Prove that $x^{\mathrm{T}} A x > 0$ (unless $x = 0$).

b) Suppose those eigenvectors have length 1 (unit vectors). Show that $B = 2x_1 x_1^{\mathrm{T}} + 5x_2 x_2^{\mathrm{T}} + 7x_3 x_3^{\mathrm{T}}$ has the same eigenvectors and eigenvalues as A. Is B necessarily the same matrix as A (yes or no)?

c) For which numbers b does this matrix have 3 positive eigenvalues?

$$A = \begin{bmatrix} 2 & b & 3 \\ b & 2 & b \\ 3 & b & 4 \end{bmatrix}$$

Note: The SVD will be on the final when you have more time to digest it.

xxx

18.06 Professor Strang Quiz 3 December 5, 2005

Grading

1

2

3

Your PRINTED name is: ____________________

Please circle your recitation:

1)	M 2	2-131	P. Lee	2-087	2-1193	`lee`
2)	M 2	2-132	T. Lawson	4-182	8-6895	`tlawson`
4)	T 10	2-132	P-O. Persson	2-363A	3-4989	`persson`
5)	T 11	2-131	P-O. Persson	2-363A	3-4989	`persson`
6)	T 11	2-132	P. Pylyavskyy	2-333	3-7826	`pasha`
7)	T 12	2-132	T. Lawson	4-182	8-6895	`tlawson`
8)	T 12	2-131	P. Pylyavskyy	2-333	3-7826	`pasha`
9)	T 1	2-132	A. Chan	2-588	3-4110	`alicec`
10)	T 1	2-131	D. Chebikin	2-333	3-7826	`chebikin`
11)	T 2	2-132	A. Chan	2-588	3-4110	`alicec`
12)	T 3	2-132	T. Lawson	4-182	8-6895	`tlawson`

1 (30 pts.) a) Find the eigenvalues and eigenvectors of the Markov matrix

$$A = \begin{bmatrix} .9 & .4 \\ .1 & .6 \end{bmatrix}$$

Solution: Any Markov matrix has eigenvalue $\lambda_1 = 1$; since the trace of A is 1.5, and the eigenvalues of a matrix add up to its trace, the second eigenvalue is $\lambda_2 = .5$. To find the corresponding eigenvectors v_1 and v_2, we look at $A - \lambda_1 I$ and $A - \lambda_2 I$:

$$(A - \lambda_1 I)v_1 = (A - I)v_1 = \begin{bmatrix} -.1 & .4 \\ .1 & -.4 \end{bmatrix} v_1 = 0 \quad \Rightarrow \quad v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix};$$

$$(A - \lambda_2 I)v_2 = (A - .5I)v_2 = \begin{bmatrix} .4 & .4 \\ .1 & .1 \end{bmatrix} v_2 = 0 \quad \Rightarrow \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix};$$

b) What is the limiting value of $A^k \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as the power k goes to infinity?

Solution: We have

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = v_1 + v_2,$$

so

$$A^k \begin{bmatrix} 3 \\ 2 \end{bmatrix} = A^k v_1 + A^k v_2 = v_1 + (.5)^k v_2.$$

Since $(.5)^k$ goes to 0 as k goes to infinity, the limiting value of $A^k \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is $v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Another argument: the steady state eigenvector of A is $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, so the limit of A^k as k goes to infinity is the Markov matrix whose both columns are multiples of $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, i.e.

$$A^\infty = \begin{bmatrix} .8 & .2 \\ .8 & .2 \end{bmatrix},$$

and the limiting value of $A^k \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is

$$A^\infty \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

c) What does it mean to say that "*A is similar to B*"?

Is that 2 by 2 matrix A similar (yes or no) to its transpose B?

$$B = \begin{bmatrix} .9 & .1 \\ .4 & .6 \end{bmatrix}$$

Give a reason for your answer.

Solution: Matrices A and B are similar if there exists an invertible matrix M such that $A = M^{-1}BM$. Equivalently, A and B are similar if their Jordan form is the same.

The matrix A^T has the same eigenvalues $\lambda_1 = 1$ and $\lambda_2 = .5$ as A, so both are similar to the same Jordan matrix

$$J = \begin{bmatrix} 1 & \\ & .5 \end{bmatrix}.$$

Thus A is similar to A^T.

2 (40 pts.) This 4 by 4 matrix H is a Hadamard matrix:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Key Properties

$H^{\mathrm{T}} = H$ and $H^2 = 4I$

a) Figure out the eigenvalues of H. Explain your reasoning.

Solution: Suppose $Hv = \lambda v$ for some non-zero vector v. Then $H^2 v = \lambda^2 v = (4I)v = 4v$, so $\lambda^2 = 4$, and thus every eigenvalue of H is equal to either 2 or -2. The trace of H is 0, hence the sum of the eigenvalues of H is 0. We conclude that H has eigenvalues $\lambda = 2, 2, -2, -2$.

b) Figure out H^{-1} and the determinant of H. Explain your reasoning.

Solution: From $H^2 = 4I$ we obtain

$$H^{-1} = \frac{1}{4} H.$$

The determinant of a matrix is the product of its eigenvalues:

$$\det H = 2 \cdot 2 \cdot (-2) \cdot (-2) = 16.$$

c) This matrix S contains three eigenvectors of H. Find a 4th eigenvector x_4 and explain your reasoning:

$$S = \begin{bmatrix} 1 & 1 & 0 & \\ 1 & 0 & -1 & \\ 1 & 0 & 1 & \\ -1 & 1 & 0 & \end{bmatrix}$$

Solution: The first two eigenvectors correspond to $\lambda = 2$, so the missing eigenvector corresponds to $\lambda = -2$. Denote the unknown eigenvector

v_4 by $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. Then

$$(H+2I)v_4 = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3a+b+c+d \\ a+b+c-d \\ a+b+c-d \\ a-b-c+3d \end{bmatrix} = 0$$

The third component of $(H+2I)v_4$ is equal to the second, and the fourth is the sum of the first two, hence we can choose v_4 to be any vector satisfying $3a+b+c+d=0$ and $a+b+c-d=0$ which is not a multiple of the third eigenvector $(0,-1,1,0)$. For example, we can choose

$$v_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}.$$

Note that since H is symmetric and the three given eigenvectors are pairwise orthogonal, any non-zero vector perpendicular to them is automatically a fourth eigenvector (and v_4 above is in fact such a vector). On the other hand, v_4 doesn't have to be orthogonal to the three given eigenvectors: we could have chosen any vector $c_1(0,-1,1,0) + c_2(1,-1,-1,-1)$ with $c_2 \neq 0$.

d) Find the solution to $du/dt = Hu$ given that $u(0) =$ third column of S.

Solution: Let v_3 be the third column of S. It is an eigenvector corresponding to $\lambda_3 = -2$, so $u = e^{-2t}v_3$ is a solution to $du/dt = Hu$, and in fact it gives $u(0) = v_3$, so it is the desired solution.

3 (30 pts.) Suppose A is a 3 by 3 symmetric matrix with eigenvalues $2, 5, 7$ and corresponding eigenvectors x_1, x_2, x_3.

a) Suppose x is a combination $x = c_1x_1 + c_2x_2 + c_3x_3$. *Find* Ax. Now find $x^{\mathrm{T}}Ax$ using the symmetry of A. Prove that $x^{\mathrm{T}}Ax > 0$ (unless $x = 0$).

Solution: We write

$$Ax = c_1Ax_1 + c_2Ax_2 + c_3Ax_3 = 2c_1x_1 + 5c_2x_2 + 7c_3x_3$$

and

$$x^TAx = (c_1x_1^T + c_2x_2^T + c_3x_3^T)(2c_1x_1 + 5c_2x_2 + 7c_3x_3) =$$

$$= 2c_1^2x_1^Tx_1 + 5c_2^2x_2^Tx_2 + 7c_3^2x_3^Tx_3$$

(opening the parentheses, we use the fact that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal, and hence $x_i^Tx_j = 0$ for $i \neq j$). Since $x_i^Tx_i = ||x_i||^2 > 0$ and $c_i^2 > 0$ unless $c_i = 0$, we conclude that $x^TAx > 0$ unless $c_1 = c_2 = c_3 = 0$, i.e. $x = 0$.

b) Suppose those eigenvectors have length 1 (unit vectors). Show that $B = 2x_1x_1^{\mathrm{T}} + 5x_2x_2^{\mathrm{T}} + 7x_3x_3^{\mathrm{T}}$ has the same eigenvectors and eigenvalues as A. Is B necessarily the same matrix as A (yes or no)?

Solution: We have

$$Bx_1 = 2x_1x_1^Tx_1 + 5x_2x_2^Tx_1 + 7x_3x_3^Tx_1 = 2x_1$$

because $x_1^Tx_1 = ||x_1||^2 = 1$ and $x_i^Tx_j = 0$ for $i \neq j$. Thus x_1 is an eigenvector of B with eigenvalue $\lambda_1 = 2$. Similarly, we can show that $Bx_2 = 5x_2$ and $Bx_3 = 7x_3$. Since both A and B have diagonalization

$$\begin{bmatrix} & & \\ x_1 & x_2 & x_3 \\ & & \end{bmatrix} \begin{bmatrix} 2 & & \\ & 5 & \\ & & 7 \end{bmatrix} \begin{bmatrix} & & \\ x_1 & x_2 & x_3 \\ & & \end{bmatrix}^{-1},$$

they are the same matrix.

c) For which numbers b does this matrix have 3 positive eigenvalues?

$$A = \begin{bmatrix} 2 & b & 3 \\ b & 2 & b \\ 3 & b & 4 \end{bmatrix}$$

Solution: A has 3 positive eigenvalues if and only if it is positive-definite. To test for positive-definiteness, we check the three upper-left determinants to see when they are positive. The 1 by 1 upper-left determinant is 2, which is positive. The 2 by 2 upper-left determinant is $4 - b^2$, which is positive whenever $-2 < b < 2$. Finally, we compute the 3 by 3 upper-left determinant, or $\det A$:

$$\det A = 2\det \begin{bmatrix} 2 & b \\ b & 4 \end{bmatrix} - b\det \begin{bmatrix} b & b \\ 3 & 4 \end{bmatrix} + 3\det \begin{bmatrix} b & 2 \\ 3 & b \end{bmatrix} =$$

$$= 2(8 - b^2) - b(4b - 3b) + 3(b^2 - 6) = -2,$$

which is always negative. Since $\det A < 0$ regardless of the value of b, we conclude that A cannot have 3 positive eigenvalues.

Note: The SVD will be on the final when you have more time to digest it.

xxx

18.06 Professor Strang Quiz 3 May 4, 2005

Your PRINTED name is: ____________	**Grading**
	1
	2
	3

Please circle your recitation:

1)	M 2	2-131	A. Chan	2-588	3-4110	`alicec`
2)	M 3	2-131	A. Chan	2-588	3-4110	`alicec`
3)	M 3	2-132	D. Testa	2-586	3-4102	`damiano`
4)	T 10	2-132	C.I. Kim	2-273	3-4380	`ikim`
5)	T 11	2-132	C.I. Kim	2-273	3-4380	`ikim`
6)	T 12	2-132	W.L. Gan	2-101	3-3299	`wlgan`
7)	T 1	2-131	C.I. Kim	2-273	3-4380	`ikim`
8)	T 1	2-132	W.L. Gan	2-101	3-3299	`wlgan`
9)	T 2	2-132	W.L. Gan	2-101	3-3299	`wlgan`

1 (37 pts.) (a) (16 points) Find the three eigenvalues and *all* the real eigenvectors of A. *It is a symmetric Markov matrix with a repeated eigenvalue.*

$$A = \begin{bmatrix} \frac{2}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{2}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{2}{4} \end{bmatrix}.$$

(b) (9 points) Find the limit of A^k as $k \to \infty$. (You may work with $A = S\Lambda S^{-1}$ without computing every entry.)

(c) (6 points) Choose any positive numbers r, s, t so that

$$\begin{array}{lll} A - rI & \text{is} & \text{positive definite} \\ A - sI & \text{is} & \text{indefinite} \\ A - tI & \text{is} & \text{negative definite} \end{array}$$

(d) (6 points) Suppose this A equals $B^{\mathrm{T}}B$. What are the singular values of B?

2 (41 pts.) (a) (14 points) Complete this 2 by 2 matrix A (depending on a) so that its eigenvalues are $\lambda = 1$ and $\lambda = -1$:

$$A = \begin{bmatrix} a & 1 \\ & \end{bmatrix}$$

(b) (9 points) How do you know that A has two independent eigenvectors?

(c) (9 points) Which choices of a give orthogonal eigenvectors and which don't?

(d) (9 points) Explain why any two choices of a lead to matrices A that are *similar* (with the same Jordan form).

3 (22 pts.) Suppose the 3 by 3 matrix A has independent eigenvectors in $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2$, $Ax_3 = \lambda_3 x_3$. (Those λ's might not be different.)

(a) (11 points) Describe the general form of every solution $u(t)$ to the differential equation $\frac{du}{dt} = Au$. (The answer $e^{At}u(0)$ does not use the λ's and x's.)

(b) (11 points) Starting from any vector u_0 in $\mathbf{R}^3$, suppose $u_{k+1} = Au_k$. What are the conditions on the x's and λ's to guarantee that $u_k \to 0$ (as $k \to \infty$)? Why?

Exam 3, Friday May 4th, 2005

Solutions

Question 1. (a) The characteristic polynomial of the matrix A is

$$-\lambda^3 + \frac{3}{2}\lambda^2 - \frac{9}{16}\lambda + \frac{1}{16} = -(\lambda - 1)\left(\lambda - \frac{1}{4}\right)^2$$

and thus the eigenvalues of A are 1 with multiplicity one and $\frac{1}{4}$ with multiplicity two. Clearly the eigenvectors with eigenvalue 1 are the non-zero multiples of the vector $(1, 1, 1)^T$. The remaining eigenvectors are all non-zero vector of the orthogonal complement of $(1, 1, 1)^T$: they are the vectors

$$\begin{pmatrix} a \\ b \\ -a-b \end{pmatrix}, \quad \text{with } (a, b) \neq (0, 0)$$

and an orthogonal basis for this vector space is

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

(b) For the matrix S we may choose the orthogonal matrix

$$S = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{pmatrix}$$

and the matrix Λ is then the diagonal matrix with entries 1, $\frac{1}{4}$, and $\frac{1}{4}$ along the diagonal.

We have

$$\lim_{k\to\infty} A^k = S\left(\lim_{k\to\infty} \Lambda^k\right) S^{-1} = S \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} S^T$$

and therefore

$$\lim_{k\to\infty} A^k = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 \end{pmatrix} S^T = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(c) Any $r < \frac{1}{4}$ is such that $A - rI$ is positive definite. Since we want r to be positive, we may choose $r = \frac{1}{8}$.
Any $\frac{1}{4} < s < 1$ is such that $A - sI$ is indefinite. We may choose $s = \frac{1}{2}$.
Any $1 < t$ is such that $A - tI$ is negative definite. We may choose $t = 2$.
(d) The singular values of B are 1, $\frac{1}{2}$ and $\frac{1}{2}$.

Question 2. The trace of A equals the sum of the eigenvalues, which is zero. We deduce that the entry in the second row and second column is $-a$. Similarly, the determinant of A equals the product of the eigenvalues, which is -1. We deduce that the entry in the second row and first column is $1-a^2$. Thus we have

$$A = \begin{pmatrix} a & 1 \\ 1-a^2 & -a \end{pmatrix}$$

(b) The matrix A has two independent eigenvectors since it has two distinct eigenvalues.
(c) The only choices of a giving orthogonal eigenvectors are the ones for which A is symmetric. This implies $a = 0$. If $a \neq 0$, then A does not have orthogonal eigenvectors.
(d) For any choice of a the matrix A has exactly one eigenvalue 1 and exactly one eigenvalue -1. Thus the Jordan canonical form of A is always

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

independently of what a is.

Question 3. (a) The general solution to the differential equation $\frac{du}{dt} = Au$ is

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + c_3 e^{\lambda_3 t} x_3$$

where c_1, c_2, c_3 are arbitrary constants.
(b) Since the vectors x_1, x_2, x_3 are independent, they form a basis for $\mathbb{R}^3$. It follows that we may write any vector $u_0 \in \mathbb{R}^3$ as a linear combination of these vectors: $u_0 = a_1x_1 + a_2x_2 + a_3x_3$. Repeatedly applying the matrix A we obtain

$$u_k = A^k u_0 = \lambda_1^k a_1 x_1 + \lambda_2^k a_2 x_2 + \lambda_3^k a_3 x_3$$

If we want the limit as k goes to infinity of the vectors u_k to be zero, then all the limits $\lim \lambda_i^k$ must be zero. It follows that we necessarily have $-1 < \lambda_i < 1$, for all i's.

18.06 Professor Strang Quiz 3 December 4, 2006

	Grading
Your PRINTED name is: ______________________	**1**
	2
	3

Please circle your recitation:

1)	T 10	2-131	K. Meszaros	2-333	3-7826	`karola`
2)	T 10	2-132	A. Barakat	2-172	3-4470	`barakat`
3)	T 11	2-132	A. Barakat	2-172	3-4470	`barakat`
4)	T 11	2-131	A. Osorno	2-229	3-1589	`aosorno`
5)	T 12	2-132	A. Edelman	2-343	3-7770	`edelman`
6)	T 12	2-131	K. Meszaros	2-333	3-7826	`karola`
7)	T 1	2-132	A. Edelman	2-343	3-7770	`edelman`
8)	T 2	2-132	J. Burns	2-333	3-7826	`burns`
9)	T 3	2-132	A. Osorno	2-229	3-1589	`aosorno`

1 (34 pts.) (a) If a square matrix A has all n of its *singular values* equal to 1 in the SVD, what basic classes of matrices does A belong to? (Singular, symmetric, orthogonal, positive definite or semidefinite, diagonal)

(b) Suppose the (orthonormal) columns of H are eigenvectors of B:

$$H = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \qquad H^{-1} = H^{\mathrm{T}}$$

The eigenvalues of B are $\lambda = 0, 1, 2, 3$. Write B as the product of 3 specific matrices. Write $C = (B + I)^{-1}$ as the product of 3 matrices.

(c) Using the list in question (a), which basic classes of matrices do B and C belong to? (Separate question for B and C)

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2 (33 pts.) (a) Find three eigenvalues of A, and an eigenvector matrix S:

$$A = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Explain why $A^{1001} = A$. Is $A^{1000} = I$? Find the three diagonal entries of e^{At}.

(c) The matrix $A^{\mathrm{T}}A$ (for the same A) is

$$A^{\mathrm{T}}A = \begin{bmatrix} 1 & -2 & -4 \\ -2 & 4 & 8 \\ -4 & 8 & 42 \end{bmatrix}.$$

How many eigenvalues of $A^{\mathrm{T}}A$ are positive? zero? negative? (Don't compute them but explain your answer.) Does $A^{\mathrm{T}}A$ have the same eigenvectors as A?

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3 (33 pts.) Suppose the n by n matrix A has n orthonormal eigenvectors $q_1, \ldots, q_n$ and n positive eigenvalues $\lambda_1, \ldots, \lambda_n$. Thus $Aq_j = \lambda_j q_j$.

(a) What are the eigenvalues and eigenvectors of A^{-1}? *Prove that your answer is correct.*

(b) Any vector b is a combination of the eigenvectors:

$$b = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n \, .$$

What is a quick formula for c_1 using orthogonality of the q's?

(c) The solution to $Ax = b$ is also a combination of the eigenvectors:

$$A^{-1} b = d_1 q_1 + d_2 q_2 + \cdots + d_n q_n \, .$$

What is a quick formula for d_1? You can use the c's even if you didn't answer part (b).

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18.06 Professor Strang Quiz 3 December 4, 2006

Grading

1

2

3

Your PRINTED name is: SOLUTIONS

Please circle your recitation:

1)	T 10	2-131	K. Meszaros	2-333	3-7826	`karola`
2)	T 10	2-132	A. Barakat	2-172	3-4470	`barakat`
3)	T 11	2-132	A. Barakat	2-172	3-4470	`barakat`
4)	T 11	2-131	A. Osorno	2-229	3-1589	`aosorno`
5)	T 12	2-132	A. Edelman	2-343	3-7770	`edelman`
6)	T 12	2-131	K. Meszaros	2-333	3-7826	`karola`
7)	T 1	2-132	A. Edelman	2-343	3-7770	`edelman`
8)	T 2	2-132	J. Burns	2-333	3-7826	`burns`
9)	T 3	2-132	A. Osorno	2-229	3-1589	`aosorno`

1 (34 pts.) (a) If a square matrix A has all n of its *singular values* equal to 1 in the SVD, what basic classes of matrices does A belong to? (Singular, symmetric, orthogonal, positive definite or semidefinite, diagonal)

(b) Suppose the (orthonormal) columns of H are eigenvectors of B:

$$H = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \qquad H^{-1} = H^{\mathrm{T}}$$

The eigenvalues of B are $\lambda = 0, 1, 2, 3$. Write B as the product of 3 specific matrices. Write $C = (B + I)^{-1}$ as the product of 3 matrices.

(c) Using the list in question (a), which basic classes of matrices do B and C belong to? (Separate question for B and C)

Solution.

(a) If $\sigma = I$ then $A = UV^{\mathrm{T}}$ = product of orthogonal matrices = orthogonal matrix.

2nd proof: All $\sigma_i = 1$ implies $A^{\mathrm{T}}A = I$. So A is orthogonal.

(A is *never* singular, and it won't always be symmetric — take $U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $V = I$, for example. This also shows it can't be diagonal, or positive definite or semidefinite.)

(b) $B = H\Lambda H^{-1}$ with $\Lambda = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 2 & \\ & & & 3 \end{bmatrix}$

$(B+I)^{-1} = H(\Lambda+I)^{-1}H^{-1}$ with (same eigenvectors) $(\Lambda+I)^{-1} = \begin{bmatrix} 1 & & & \\ & 1/2 & & \\ & & 1/3 & \\ & & & 1/4 \end{bmatrix}$

(c) B is singular, symmetric, positive semidefinite.

C is symmetric positive definite.

2 (33 pts.) (a) Find three eigenvalues of A, and an eigenvector matrix S:

$$A = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Explain why $A^{1001} = A$. Is $A^{1000} = I$? Find the three diagonal entries of e^{At}.

(c) The matrix $A^{\mathrm{T}}A$ (for the same A) is

$$A^{\mathrm{T}}A = \begin{bmatrix} 1 & -2 & -4 \\ -2 & 4 & 8 \\ -4 & 8 & 42 \end{bmatrix}.$$

How many eigenvalues of $A^{\mathrm{T}}A$ are positive? zero? negative? (Don't compute them but explain your answer.) Does $A^{\mathrm{T}}A$ have the same eigenvectors as A?

Solution.

(a) The eigenvalues are $-1, 0, 1$ since A is triangular.

$$\lambda = -1 \text{ has } x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \lambda = 0 \text{ has } x = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \lambda = 1 \text{ has } x = \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix}.$$

Those vectors x are the columns of S (upper triangular!).

(b) $A = A\Lambda S^{-1}$ and $A^{1001} = S\Lambda^{1001}S^{-1}$. Notice $\Lambda^{1001} = \Lambda$, $A^{1000} \neq I$ (A is singular) ($0^{1000} = 0 \neq 1$).

e^{At} has e^{-1t}, $e^{0t} = 1$, e^t on its diagonal. *Proof using series:*

$\sum_0^\infty (At)^n/n!$ has triangular matrices so the diagonal has $\sum(-t)^n/n! = e^{-t}$, $\sum 0^n/n! = 1$, $\sum t^n/n! = e^t$.

Proof using $S\Lambda S^{-1}$:

$$e^{At} = Se^{\Lambda t}S^{-1} = \begin{bmatrix} 1 & \times & \times \\ 0 & 1 & \times \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & & \\ & 1 & \\ & & e^t \end{bmatrix} \begin{bmatrix} 1 & \times & \times \\ 0 & 1 & \times \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) $A^{\mathrm{T}}A$ has 2 positive eigenvalues (it has rank 2, its eigenvalues can never be negative).

One eigenvalue is zero because $A^{\mathrm{T}}A$ is singular. And $3 - 2 = 1$.

(Or: $A^{\mathrm{T}}A$ is symmetric, so the eigenvalues have the same signs as the pivots.

Do elimination: the pivots are 1, 0, and $42 - 16 = 26$.)

3 (33 pts.) Suppose the n by n matrix A has n orthonormal eigenvectors $q_1, \ldots, q_n$ and n positive eigenvalues $\lambda_1, \ldots, \lambda_n$. Thus $Aq_j = \lambda_j q_j$.

(a) What are the eigenvalues and eigenvectors of A^{-1}? *Prove that your answer is correct.*

(b) Any vector b is a combination of the eigenvectors:

$$b = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n \, .$$

What is a quick formula for c_1 using orthogonality of the q's?

(c) The solution to $Ax = b$ is also a combination of the eigenvectors:

$$A^{-1} b = d_1 q_1 + d_2 q_2 + \cdots + d_n q_n \, .$$

What is a quick formula for d_1? You can use the c's even if you didn't answer part (b).

Solution.

(a) A^{-1} has eigenvalues $\dfrac{1}{\lambda_j}$ with the same eigenvectors

$$Aq_j = \lambda_j q_j \longrightarrow q_j = \lambda_j A^{-1} q_j \longrightarrow A^{-1} q_j = \frac{1}{\lambda_j}\, q_j\,.$$

(b) Multiply $b = c_1 q_1 + \cdots + c_n q_n$ by q_1^{T}.

Orthogonality gives $q_1^{\mathrm{T}} b = c_1 q_1^{\mathrm{T}} q_1$ so $c_1 = \dfrac{q_1^{\mathrm{T}} b}{q_1^{\mathrm{T}} q_1} = q_1^{\mathrm{T}} b$.

(c) Multiplying b by A^{-1} will multiply each q_i by $\dfrac{1}{\lambda_i}$ (part (a)). So c_i becomes

$$d_1 = \frac{c_1}{\lambda_1} \quad \left(= \frac{q_1^{\mathrm{T}} b}{\lambda_1 q_1^{\mathrm{T}} q_1} \text{ or } \frac{q_1^{\mathrm{T}} b}{\lambda_1} \right).$$

18.06 **QUIZ 3** May 07, 2007

Your PRINTED name is: SOLUTIONS

Please circle your recitation:

(1)	M 2	2-131	A. Osorno
(2)	M 3	2-131	A. Osorno
(3)	M 3	2-132	A. Pissarra Pires
(4)	T 11	2-132	K. Meszaros
(5)	T 12	2-132	K. Meszaros
(6)	T 1	2-132	Jerin Gu
(7)	T 2	2-132	Jerin Gu

Grading

1	
2	
3	
4	
Total:	

Problem 1 *(25 points)*

(a) Compute the singular value decomposition $A = U\Sigma V^T$ for $A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \\ 1 & 0 \end{pmatrix}$.

(b) Find orthonormal bases for all four fundamental subspaces of A.

Solution 1

(a) $A^TA = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, with eigenvalues 3 and 1. Thus $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$.

v_1 is a normal eigenvector corresponding to 3, so $v_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$.

v_2 is a normal eigenvector corresponding to 1, so $v_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.

$u_1 = \frac{1}{\sqrt{3}}Av_1 = \begin{pmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$; $u_2 = Av_2 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$.

For u_3 we find a basis for the nullspace of A^T: $u_3 = \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$.

Thus $A = \underbrace{\begin{pmatrix} -1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{V^T}$.

(b) Rowspace: v_1 and v_2, i.e. $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$. Nullspace: 0.

Columnspace: u_1 and u_2, i.e. $\begin{pmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$. Left nullspace: u_3, i.e. $\begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$.

Problem 2 *(25 points)*

(a) Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$.

(b) Find 3 linearly independent eigenvectors of A.

(c) Write down a diagonal matrix that is similar to A.

(d) Diagonalize the matrix A as $A = Q\Lambda Q^T$ with orthogonal matrix Q.

Solution 2

(a) Notice that $A^T = A$ and $A^2 = A^T A = 9I$. So if λ is an eigenvalue of A, $\lambda^2 = 9$. Thus $\lambda = \pm 3$.

The trace of A is 3, so $\lambda_1 + \lambda_2 + \lambda_3 = 3$. We get then that $\lambda_1 = 3$, $\lambda_2 = 3$ and $\lambda_3 = -3$.

(b) For $\lambda = 3$, we need to find vectors in the nullspace of $\begin{pmatrix} -2 & 2 & 2 \\ 2 & -2 & -2 \\ 2 & -2 & -2 \end{pmatrix}$: $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

For $\lambda = -3$, we find the nullspace of $\begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & -2 \\ 2 & -2 & 4 \end{pmatrix}$: $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

(c) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ since A is diagonalizable.

(d) We need to find orthonormal eigenvectors. We can do this by Gram-Schmidt or by inspection.

$$A = \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}}_{Q^T}.$$

Problem 3 *(25 points)*

Consider the matrix $A = \begin{pmatrix} a & 2 & 1 \\ 2 & a & 1 \\ 1 & 1 & 2 \end{pmatrix}$, where a is a real number.

(a) For which values of the parameter a is the matrix A positive definite?

(b) For which values of the parameter a is the matrix $-A$ positive definite?

(c) For which values of the parameter a is the matrix A singular?

Solution 3

(a) We will use the upper left determinants:

$a > 0$

$a^2 - 4 > 0 \Rightarrow a > 2$

$2(a+1)(a-2) > 0$ which is always true if $a > 2$.

So the only condition we have is $a > 2$.

(b) Again using upper left determinants:

$-a > 0 \Rightarrow a < 0$

$a^2 - 4 > 0 \Rightarrow a < -2$

$-2(a+1)(a-2) > 0$ which is never true if $a < -2$.

So $-A$ is never positive definite.

(c) $det(A) = 2(a+1)(a-2)$, so A is singular if $a = -1$ or $a = 2$.

Problem 4 *(25 points)*

(a) Find the steady state for the Markov matrix $A = \begin{pmatrix} 0.2 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.3 \\ 0.4 & 0.4 & 0.4 \end{pmatrix}$.

(b) Calculate the limit of $A^n \begin{pmatrix} 0 \\ 20 \\ 0 \end{pmatrix}$ as $n \to \infty$.

Solution 4

(a) The steady state is the eigenvector corresponding to the eigenvalue 1. As a convention, we take it so that the sum of the components is 1.

To find it, we need to look at the nullspace of the matrix $A - I = \begin{pmatrix} -0.8 & 0.4 & 0.3 \\ 0.4 & -0.8 & 0.3 \\ 0.4 & 0.4 & -0.6 \end{pmatrix}$.

The steady state is $\begin{pmatrix} 0.3 \\ 0.3 \\ 0.4 \end{pmatrix}$.

(b) The limit of $A^n \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ as $n \to \infty$ is the steady state, so the limit of $A^n \begin{pmatrix} 0 \\ 20 \\ 0 \end{pmatrix}$ as $n \to \infty$ is $20 \cdot \begin{pmatrix} 0.3 \\ 0.3 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 8 \end{pmatrix}$.

18.06 Professor Johnson Quiz 3 December 3, 2007

SOLUTIONS

1 (20 pts.) True or false. Explain why if *false*, or give an example if *true*.

(a) There exist matrices $A \neq 0$ that are simultaneously Hermitian ($A = A^H$) and unitary ($A^H = A^{-1}$).

(b) There exist matrices $A \neq 0$ that are simultaneously anti-Hermitian ($A = -A^H$) and unitary ($A^H = A^{-1}$).

(c) There exist matrices $A \neq 0$ that are simultaneously Hermitian ($A = A^H$) and anti-Hermitian ($A = -A^H$).

(d) There exist matrices A that are simultaneously Hermitian and Markov.

Solution:

(a) True. For example, $A = I, -I$, or more generally, $A = S\Lambda S^H$, where S is any unitary matrix, and Λ is a diagonal matrix whose diagonal entries are ± 1.

(b) True. For example, the 1×1 matrix $A = i, -i$, or more generally, $A = S\Lambda S^H$, where S is any unitary matrix, and Λ is a diagonal matrix whose diagonal entries are $\pm i$.

(c) False. If A is Hermitian then all the eigenvalues are real, and if it is anti-Hermitian then the eigenvalues are imaginary, and the eigenvalues cannot be at the same time real and imaginary unless they are zero. The only Hermitian matrix whose eigenvalues are all 0 is the zero matrix, but $A \neq 0$.

(d) True, e.g. $A = I$. All 2×2 examples are of the form $\begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}$ with $0 \leq a \leq 1$.

2 (30 pts.) Suppose we form a sequence of real numbers f_k defined by the recurrence $f_{k+1} = f_k - f_{k-1} + f_{k-2}$, starting with the initial conditions $f_0 = 2$, $f_1 = 1$ and $f_2 = 0$.

(a) Define a 3-component vector $\vec{g}_k = (f_k, f_{k-1}, f_{k-2})^T$ and a 3×3 matrix A so that the recurrence is $\vec{g}_{k+1} = A\vec{g}_k$.

(b) If you constructed A correctly, the three eigenvalues should be 1 and $\pm i$ [I'm giving you these so you *don't* have to solve a cubic equation], and the latter two eigenvectors should be $(-1, \pm i, 1)^T$. Check that you have these $\pm i$ eigenvalues and eigenvectors, and find the $\lambda = 1$ eigenvector.

(c) Give an explicit formula for f_k for any k. (By "explicit," I mean involving elementary arithmetic and powers of complex numbers only. Formulas involving A^k are not acceptable.)

(d) Is there any choice of initial conditions that will make $|f_k|$ diverge as $k \to \infty$? Explain.

$\boxed{\text{Solution}}$

(a) This recurrence gives $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. That is, the first row of A gives $f_{k+1} = f_k - f_{k-1} + f_{k-2}$, while the second and third rows of A just give $f_k = f_k$ and $f_{k-1} = f_{k-1}$ (copying the first and second rows of $\vec{g}_k$ to the second and third rows of $\vec{g}_{k+1}$.

(b) We need to find the nullspace of $A - \lambda I$, via elimination to obtain row-reduced echelon form. In each case, it will be convenient to swap the first two rows, which will make the first pivot 1 and will not change the nullspace. For $\lambda_1 = 1$:

$$A - I = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & -1 & 0 \\ 0 & \boxed{-1} & 1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & -1 & 0 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

for which the nullspace vector is $\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

To check the provided $\pm i$ eigenvectors, we just multiply them by A:

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ \pm i \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \mp i + 1 \\ -1 \\ \pm i \end{pmatrix} = \begin{pmatrix} \mp i \\ -1 \\ \pm i \end{pmatrix} = \pm i \begin{pmatrix} -1 \\ \pm i \\ 1 \end{pmatrix}.$$

For your edification, if we had to solve for the $\pm i$ eigenvectors we would do it by elimination too, of course. For $\lambda_2 = i$: $A - iI = \begin{pmatrix} 1-i & -1 & 1 \\ 1 & -i & 0 \\ 0 & 1 & -i \end{pmatrix} \to \begin{pmatrix} \boxed{1} & -i & 0 \\ 1-i & -1 & 1 \\ 0 & 1 & -i \end{pmatrix} \to$ $\begin{pmatrix} \boxed{1} & -i & 0 \\ 0 & \boxed{i} & 1 \\ 0 & 1 & -i \end{pmatrix} \to \begin{pmatrix} \boxed{1} & -i & 0 \\ 0 & \boxed{i} & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -i \\ 0 & 0 & 0 \end{pmatrix}$, for which the nullspace vector is $\vec{x}_2 = \begin{pmatrix} -1 \\ i \\ 1 \end{pmatrix}$. For $\lambda_3 = -i$, the eigenvector is just the complex conjugate $\vec{x}_3 = \begin{pmatrix} -1 \\ -i \\ 1 \end{pmatrix}$.

(c) We have to expand the initial vector in the eigenvectors (note that the initial vector is $\vec{g}_2$, not $\vec{g}_0$, here). There are several ways to do this. First, we can do this by inspection: you might guess that you have to add $\vec{x}_2$ and $\vec{x}_3$ to cancel the i factors, and once you guess this the other coefficients are easy:

$$\vec{g}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2} \left[\begin{pmatrix} -1 \\ i \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -i \\ 1 \end{pmatrix} \right].$$

More explicitly, we can solve the linear system $S\vec{c} = \vec{g}_2$ for the coefficients $\vec{c}$, where S is the matrix of eigenvectors. Via elimination on the augmented matrix, we obtain $\begin{pmatrix} \boxed{1} & -1 & -1 & 0 \\ 1 & i & -i & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \to$

$$\begin{pmatrix} \boxed{1} & -1 & -1 & 0 \\ 0 & \boxed{1+i} & 1-i & 1 \\ 0 & 2 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & -1 & -1 & 0 \\ 0 & \boxed{2} & 2 & 2 \\ 0 & 1+i & 1-i & 1 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & -1 & -1 & 0 \\ 0 & \boxed{2} & 2 & 2 \\ 0 & 0 & \boxed{-2i} & -i \end{pmatrix},$$ where we have swapped rows to keep the pivots real (which simplifies the algebra somewhat). The resulting trangular system is easily solved for $\vec{c} = (1, 1/2, 1/2)^T$.

Common mis-step: Many students correctly wrote out the solution as $A^k\vec{g}_2 = S\Lambda^k S^{-1}\vec{g}_2$, but then got stuck because they tried to directly compute S^{-1}, which is painful. In linear algebra, explicitly inverting a matrix is usually a mistake, if what we want at the end is a vector! We have emphasized that you instead should solve the linear system (i.e. expand the initial vector in the eigenvectors). (On the other hand, if you just stopped at $S\Lambda^k S^{-1}$, you only lost a few points.)

Anothe common mistake: Many studends wrote $A^k = S\Lambda^k S^{-1}$, but then wrote $S^{-1} = S^H$. This is not true unless S is unitary, i.e. it has orthonormal rows. This is not true here, and there is no reason for it to be true since A is not Hermitian or unitary, etc.

To get $\vec{g}_{k+2} = A^k\vec{g}_2$, we just multiply each eigenvector by λ^k, and take the third row to get f_k:

$$f_k = 1 + \frac{1}{2}\left[i^k + (-i)^k\right] = 1 + \cos(k\pi/2).$$

(This is just the sequence $2, 1, 0, 1, 2, 1, 0, 1, 2, 1, \ldots$ repeated over and over.)

(d) No, because all of the eigenvalues have $|\lambda| = 1$, hence their powers don't blow up. (However, as one may check, the matrix is not unitary.)

3 (30 pts.) (a) Suppose $A = e^{iB}$ where B is Hermitian; what is $A^H A$? Hence A is a _____________ matrix.

(b) For the recurrence relation $\vec{f}_{k+1} = e^{iB}\vec{f}_k$, what is $\|\vec{f}_k\|^2/\|\vec{f}_0\|^2$? [Hint: part (a) is useful.]

(c) Compute $\vec{f}_k$ explicitly [i.e. no matrix exponentials or powers of matrices] for $B = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$ and $\vec{f}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The eigenvectors of this B are $\vec{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{x}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ with eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -5$, respectively.

(d) Check that your answer from (b) is true for your answer from (c).

$\boxed{\text{Solution:}}$

(a) $A^H = e^{(iB)^H} = e^{-iB^H} = e^{-iB}$. Hence $A^H A = e^{-iB}e^{iB} = e^{-iB+iB} = e^0 = I$. (Note that iB and $-iB$ obviously commute, which is why we can combine the exponentials like this.) Hence A is unitary.

Common mistake: many students forgot to take the complex conjugate, i.e. forgetting to replace i with $-i$.

(b) As in class, $\vec{f}_k = A^k \vec{f}_0$. Hence

$$\|\vec{f}_k\|^2 = \vec{f}_k^H \vec{f}_k = \vec{f}_0^H (A^k)^H A^k \vec{f}_0 = \vec{f}_0^H A^H A^H \cdots A^H A \cdots A A \vec{f}_0 = \vec{f}_0^H \vec{f}_0 = \|\vec{f}_0\|^2$$

[using the result from part (a) to cancel the $A^H A$ factors in the middle], and hence $\|\vec{f}_k\|^2/\|\vec{f}_0\|^2 = 1$. Equivalently, the product of unitary matrices is unitary, so A^k is unitary, so it preserves lengths.

(c) We first have to expand the initial condition in terms of the eigenvectors. This is easy enough to do by inspection here:

$$\vec{f}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 2 \\ -1 \end{pmatrix}}{5}.$$

Alternatively, we could solve the 2×2 system $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for the coefficients $c_1 = 1/5$ and $c_2 = 2/5$. Or, we could use the orthogonality to get $c_j = \vec{f}_0 \cdot \vec{x}_j / \|\vec{x}_j\|^2$. Once this is done, we use the fact that $\vec{f}_k = A^k \vec{f}_0 = e^{iBk}\vec{f}_0$ to multiply each eigenvector by $e^{i\lambda k}$:

$$\vec{f}_k = \frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{i5k} + 2\begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-i5k}}{5} = \frac{\begin{pmatrix} e^{i5k} + 4e^{-i5k} \\ 2e^{i5k} - 2e^{-i5k} \end{pmatrix}}{5}.$$

(d) This is simplest if we don't combine the terms above and instead use the orthogonality to eliminate the $\vec{x}_1 \cdot \vec{x}_2$ and $\vec{x}_2 \cdot \vec{x}_1$ cross terms:

$$\|\vec{f}_k\|^2 = \vec{f}_k^H \vec{f}_k = \frac{\left\|\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\|^2 \left|e^{i5k}\right|^2 + 2^2 \left\|\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right\|^2 \left|e^{-i5k}\right|^2}{5^2} = \frac{5 + 4 \cdot 5}{25} = 1 = \|\vec{f}_0\|^2.$$

Alternatively, we can explicitly write out

$$\begin{aligned} |e^{i5k} + 4e^{-i5k}|^2 + |2e^{i5k} - 2e^{-i5k}|^2 &= (e^{i5k} + 4e^{-i5k})(e^{-i5k} + 4e^{i5k}) + (2e^{i5k} - 2e^{-i5k})(2e^{-i5k} - 2e^{i5k}) \\ &= (1 + 4e^{-i10k} + 4e^{i10k} + 16) + (4 - 4e^{-i10k} - 4e^{i10k} + 4) \\ &= 25, \end{aligned}$$

so again $\|\vec{f}_k\|^2 = 25/25 = 1 = \|\vec{f}_0\|^2$.

4 (20 pts.) Some 3×3 real matrix A has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 2$, with the corresponding eigenvectors $\vec{x}_1 = (1,0,0)^T$, $\vec{x}_2 = (0,1,2)^T$, and $\vec{x}_3 = (0,1,1)^T$.

(a) Give a basis for: (i) the nullspace $N(A)$, (ii) the column space $C(A)$, and (iii) the row space $C(A^H)$.

(b) Find all solutions $\vec{x}$ to $A\vec{x} = \vec{x}_2 - 3\vec{x}_3$.

(c) Is A (i) real-symmetric, (ii) orthogonal, (iii) Markov, or (iv) none of the above?

Solution:

(a) The nullspace is just the span of the $\lambda = 0$ eigenvector $\vec{x}_1$. If we act A on any vector, we only get multiples of the $\lambda \neq 0$ eigenvectors, so $C(A)$ is the span of $\vec{x}_2$ and $\vec{x}_3$. The row space is the orthogonal complement of the nullspace, and here this is spanned by (e.g.) the vectors $(0,1,0)^T$ and $(0,0,1)^T$.

(b) The right hand side is clearly in the column space. Since we have expanded the right hand side in the $\lambda \neq 0$ eigenvectors, we can get a particular solution just by dividing them by the corresponding eigenvalues: remember, A acts just like a number on these vectors. Hence a particular solution is $\vec{x}_p = \vec{x}_2/1 - 3\vec{x}_3/2 = (0, -1/2, 1/2)^T$. To get all the solutions we must add the nullspace, obtaining $\vec{x} = (a, -1/2, 1/2)^T$ for any constant a.

Equivalently, expand $\vec{x}$ in the eigenvectors, $\vec{x} = a\vec{x}_1 + b\vec{x}_2 + c\vec{x}_3$, and plug in to $A\vec{x} = b\vec{x}_2 + 2c\vec{x}_3 = \vec{x}_2 - 3\vec{x}_3$ to find $a =$ arbitrary, $b = 1$, and $c = -3/2$.

(c) (iv) None of the above. It's clearly not Markov or orthogonal since there is a $\lambda = 2$ eigenvalue. Although the eigenvalues are real, it's not real-symmetric since the eigenvectors are not orthogonal.

Practice 18.06 Exam 3 questions

List of potential topics:

Material from exams 1 and 2. Eigenvalues and eigenvectors, characteristic polynomials and nullspaces of $A - \lambda I$. Similar matrices and diagonalization. Complex vs. real linear algebra, adjoints vs. transposes. Hermitian ($A^H = A$), anti-Hermitian ($A^H = -A$), and unitary matrices ($A^H = A^{-1}$), and their eigenvalues/eigenvectors. Markov matrices. Linear recurrences $\mathbf{x}_{n+1} = A\mathbf{x}_n$ and powers of matrices. Differential equations $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ and matrix exponentials. Hermitian operators on functions, eigenfunctions, Fourier series, and equations written in terms of these (exponentials, inverses, etc. of operators) [see online handouts]. Positive-definite and positive-semidefinite matrices. The singular value decomposition (SVD) $A = U\Sigma V^H$ and the pseudoinverse $A^+ = V\Sigma^+ U^H$.

Definitely not on exam 3: finite-difference approximations, sparse matrices and iterative methods, non-diagonalizable matrices, generalized eigenvectors, or Jordan forms. Also *not* fast Fourier transforms or the discrete Fourier transform (which were on the original syllabus but were skipped).

Key ideas: for an eigenvector, any complicated matrix or operator acts just like a number λ, and we can do anything we want (inversion, powers, exponentials...) using that number. To act on an arbitrary vector, we expand that vector in the eigenvectors (in the usual case where the eigenvectors form a basis), and then treat each eigenvector individually. Finding eigenvectors and eigenvalues is complicated, though, so we try to infer as much as we can about their properties from the structure of the matrix/operator (Hermitian, Markov, etcetera).

The actual exam will be four or five questions, so this is about two to three (hard) exams worth of potential questions. (Some of these questions were rejected because they were a bit too hard/long for an exam.)

Problem 1

Suppose A is a square matrix with $A^H = u^2 A$, where u is some complex number with $|u| = 1$.

(a) Show that $B = zA$ is Hermitian for some complex number z. What is z?

(b) What can you conclude about the eigenvalues and eigenvectors of A?

Solution:

(a) If $B = zA$, then $B^H = \bar{z}A^H = \bar{z}u^2 A = (\bar{z}u^2/z)B$. Then, to make $B^H = B$, we must have $z/\bar{z} = u^2$, which is solvable since $|z/\bar{z}| = 1 = |u^2|$. The magnitude of z is arbitrary, so let us choose $|z| = 1$ in which case $\bar{z} = 1/z$ and thus we have $z^2 = u^2$ and hence $z = u$. That is, $B = uA$ works.

(b) The eigenvectors with distinct eigenvalues are orthogonal, and the eigenvectors form a basis (A is diagonalizable since B is). The eigenvalues of B are real because it is Hermitian, so the eigenvalues of $A = B/u = \bar{u}B$ are real numbers multiplied by $\bar{u}$.

Problem 2

In an ordinary eigenproblem we solve $A\mathbf{x} = \lambda\mathbf{x}$ to find the eigenvectors $\mathbf{x}$ and eigenvalues λ, and if $A = A^H$ (Hermitian) we find that λ is real and eigenvectors with different eigenvalues are orthogonal.

Now, suppose that instead we are looking for solutions of $A\mathbf{x} = \lambda B\mathbf{x}$ where we have matrices A and B on both sides of the equation. Suppose that both A and B are Hermitian, and B is positive-definite.

(a) Show that the "eigenvalues" λ in $A\mathbf{x} = \lambda B\mathbf{x}$ are real. (Hint: take the dot product of both sides with $\mathbf{x}$.)

(b) Show that two solutions $A\mathbf{x}_1 = \lambda_1 B\mathbf{x}_1$ and $A\mathbf{x}_2 = \lambda_2 B\mathbf{x}_2$ with $\lambda_1 \neq \lambda_2$ satisfy $\mathbf{x}_1 \cdot (B\mathbf{x}_2) = 0$. (Hint: take the dot product of both sides of one equation with $\mathbf{x}_1$.)

Solution

(a) $\mathbf{x} \cdot (A\mathbf{x}) = \lambda\mathbf{x} \cdot (B\mathbf{x})$ but it also $= (A\mathbf{x}) \cdot \mathbf{x} = \bar{\lambda}(B\mathbf{x}) \cdot \mathbf{x} = \bar{\lambda}\mathbf{x} \cdot (B\mathbf{x})$ since A and B are Hermitian (we can move them from one side to the other of the dot product). Therefore, $\lambda\mathbf{x} \cdot (B\mathbf{x}) = \bar{\lambda}\mathbf{x} \cdot (B\mathbf{x})$ and hence $\lambda = \bar{\lambda}$ is real [$\mathbf{x} \cdot (B\mathbf{x}) > 0$ since B is positive-definite].

(b) $\mathbf{x}_1 \cdot (A\mathbf{x}_2) = \lambda_2\mathbf{x}_1 \cdot (B\mathbf{x}_2) = (A\mathbf{x}_1) \cdot \mathbf{x}_2 = \lambda_2(B\mathbf{x}_1) \cdot \mathbf{x}_2 = \lambda_2\mathbf{x}_1 \cdot (B\mathbf{x}_2)$, hence $(\lambda_2 - \lambda_1)\mathbf{x}_1 \cdot (B\mathbf{x}_2) = 0$, hence $\mathbf{x}_1 \cdot (B\mathbf{x}_2) = 0$ since $\lambda_1 \neq \lambda_2$.

Problem 3

True or false: any Markov matrix A is also positive-semidefinite. Explain why if true, or give a counter-example if false.

Solution*:* False. All the eigenvalues of a Markov matrix have $|\lambda| \leq 1$ but complex λ and $\lambda < 0$ are also possible. For example, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a real-symmetric Markov matrix with eigenvalues $\lambda = \pm 1$. The entries of a Markov matrix are all non-negative, but that should not be confused with positive-semidefinite.

Problem 4

True or false: Explain why if true, or give a counter-example if false.

(a) Any diagonalizable matrix with real eigenvalues is Hermitian.

(b) The product of two Hermitian matrices is Hermitian.

(c) The product of two unitary matrices is unitary.

(d) The sum of two Hermitian matrices is Hermitian.

(e) The sum of two unitary matrices is unitary.

Solutions:

(a) False. Say $A = S\Lambda S^{-1}$ with Λ real and distinct $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$; A is not Hermitian if S is any invertible matrix with non-orthogonal columns (non-orthogonal eigenvectors).

(b) False, since $(AB)^H = B^H A^H = BA \neq AB$ unless A and B commute. For example, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ give $AB = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, which is not Hermitian.

(c) True: $(AB)^H = B^H A^H = B^{-1}A^{-1} = (AB)^{-1}$ if A and B are unitary.

(d) True: $(A + B)^H = A^H + B^H = A + B$ if A and B are Hermitian.

(e) False, since $(A + B)^{-1} \neq A^{-1} + B^{-1}$ in general. For example, I is unitary but $I + I = 2I$ is not.

Problem 5

Cal Q. Luss, a Harvard student, doesn't like the definition of Markov matrices. He suggests instead that we use "Markoffish" matrices: real matrices A whose columns sum to 1 like for Markov matrices, but negative entries are allowed.

(a) Show that Markoffish matrices still have a $\lambda = 1$ eigenvalue (hint: consider the eigenvalues of A^T).

(b) Show that the product of two Markoffish matrices is a Markoffish matrix.

(c) For Markov matrices, from (b) we concluded that $|\lambda| > 1$ eigenvalues were not allowed. Is that still true for Markoffish matrices? Explain why if true, or give a counter-example if false.

Solution

(a) As in class, the fact that the sum of each column is one is equivalent to the statement that $A^T\mathbf{x} = \mathbf{x}$ for $\mathbf{x} = (1, 1, 1, \ldots)^T$. Hence, $\lambda = 1$ is still an eigenvalue, since A and A^T have the same eigenvalues.

(b) We can show this by the same explicit summation argument as in class. Or we can use the fact that A being Markoffish is equivalent to $\mathbf{x}^T A = \mathbf{x}^T$ for the $\mathbf{x}$ from (a). Thus, if we have two Markoffish matrices A and B then $\mathbf{x}^T(AB) = (\mathbf{x}^T A)B = \mathbf{x}^T B = \mathbf{x}^T$ and hence AB is Markoffish.

(c) No, this is no longer true; before, the fact that A^n was Markov meant that it could not blow up and hence $|\lambda| \leq 1$, but now A^n is Markoffish and can blow up: its entries can be arbitrarily large and negative. For example, take the Markoffish matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, whose eigenvalues satisfy $\lambda^2 - 4\lambda + 3 = 0$ and hence $\lambda = 3$ and $\lambda = 1$, one of which is > 1.

Problem 6

Consider the vector space of real functions $f(x)$ on $x \in [0, 1]$ with $f(0) = f(1) = 1$, and define the dot product $f \cdot g = \int_0^1 x\, f(x)\, g(x)\, dx$.

(a) Is the second-derivative operator d^2/dx^2 still Hermitian under this inner product? Why or why not?

(b) Show that the operator A defined by $Af = \frac{1}{\sqrt{x}} \frac{d^2}{dx^2}[\sqrt{x} f(x)]$ is Hermitian under this inner product.

(c) What can you conclude about the eigenfunctions and eigenvalues of A?

(d) Show that $-A$ is positive definite ($f \cdot (-Af) > 0$ for all $f \neq 0$). What does this tell you about the eigenvalues of A?

(e) What does your answer to (d) tell you about the solution to $\frac{\partial f}{\partial t} = \frac{1}{\sqrt{x}} \frac{\partial^2}{\partial x^2}[\sqrt{x} f(x, t)] = Af$ with some initial condition $f(x, 0) = g(x)$, as $t \to \infty$? (Hint: expand $g(x)$ in the eigenfunctions, as in class, and write a series solution for $f(x, t)$. You can assume that the eigenfunctions form a basis for the space.)

Solution

(a) No. If we integrate by parts twice in $\int xfg''$, as in classwe get $\int (xf)''g \neq \int xf''g$.

(b) Plugging in, we find $f \cdot Ag = \int \sqrt{x} f (\sqrt{x} g)''$. Integrating by parts twice as above, we get $\int (\sqrt{x} f)'' \sqrt{x} g = (Af) \cdot g$ and hence A is Hermitian.

(c) The eigenvalues must be real as usual, and eigenfunctions for distinct eigenvalues must be orthogonal. [For this operator, you can actual solve analytically for the eigenfunctions $\sin(n\pi x)/\sqrt{x}$ and the eigenvalues $-(n\pi)^2$. However, this is *not* necessary to solve the problem.]

(d) Integrating by parts once: $f \cdot (-Af) = \int \sqrt{x} f (\sqrt{x} f)'' = \int |(\sqrt{x} f)'|^2 \geq 0$. It only $= 0$ if $\sqrt{x} f(x)$ is a constant, but to satisfy the boundary conditions $f(0) = f(1) = 0$ we must therefore have $f(x) = 0$. Hence,

$f \cdot (-Af) > 0$ for all $f \neq 0$, and $-A$ is positive-definite. Hence the eigenvalues of $-A$ are positive, and hence the eigenvalues of A are negative.

(e) Call the eigenfunctions $f_n(x)$ and the corresponding eigenvalues $\lambda_n < 0$. Then, we write $g(x) = \sum_n c_n f_n(x)$ for some coefficients $c_n = f_n \cdot g / \|f_n\|$. We have the equation $\frac{\partial f}{\partial t} = Af$, so formally the solution is $f(x,t) = e^{At} f(x,0) = e^{At} g(x)$ as for the diffusion equation in class. The exponential of A is a bit weird, but as usual we know that it is just a number when it acts on an eigenfunction:

$$f(x,t) = e^{At} g(x) = e^{At} \sum_n c_n f_n(x) = \sum_n c_n e^{At} f_n(x) = \sum_n c_n e^{\lambda_n} f_n(x).$$

But since all the λ_n eigenvalues are negative, every term goes exponentially to zero and hence $f(x,t) \to 0$ as $t \to \infty$.

Problem 7

Suppose that you have a system of differential equations $B^{-1} \frac{d\mathbf{x}}{dt} = -B^H \mathbf{x}$ with some initial condition $\mathbf{x}(0)$, for some invertible matrix B.

(a) Find $\mathbf{x}(t)$ for $B = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{x}(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$.

(b) What happens to your $\mathbf{x}(t)$ from (a) as $t \to \infty$?

(c) Argue that your answer from (b) is true in general, for *any* invertible $n \times n$ matrix B and any initial condition $\mathbf{x}(0)$.

Solutions:

(a) We have $\frac{d\mathbf{x}}{dt} = -BB^H \mathbf{x}$, and hence $\mathbf{x}(t) = e^{-BB^H t} \mathbf{x}(0)$. To solve this we must find the eigenvectors of BB^H and expand $\mathbf{x}(0)$ in terms of them. $BB^H = \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}$, whose eigenvalues solve $\lambda^2 - 6\lambda + 4 = 0$, and hence $\lambda_\pm = 3 \pm \sqrt{5}$ by the quadratic equation. If we look for eigenvectors $\mathbf{v}$ of the form $\mathbf{v} = \begin{pmatrix} 1 \\ u \end{pmatrix}$, we find $5 + u = \lambda_\pm$ and hence $u = -2 \pm \sqrt{5}$. By inspection, we can then write $\mathbf{x}(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2+\sqrt{5} \end{pmatrix} + \begin{pmatrix} 1 \\ -2-\sqrt{5} \end{pmatrix}$. Hence,

$$\mathbf{x}(t) = e^{-BB^H t} \mathbf{x}(0) = \begin{pmatrix} 1 \\ -2+\sqrt{5} \end{pmatrix} e^{-(3+\sqrt{5})t} + \begin{pmatrix} 1 \\ -2-\sqrt{5} \end{pmatrix} e^{-(3\pm\sqrt{5})t}$$

(b) $3 \pm \sqrt{5} > 0$, so the $\mathbf{x}(t)$ function decays exponentially towards 0.

(c) This is true in general since $BB^H = (B^H)^H (B^H)$ is always positive-definite as shown in class; B has full column rank (it is invertible) so it is not merely semidefinite. This means that the eigenvalues of BB^H are all real and positive, and hence the eigenvalues of $-BB^H$ are negative, and hence $e^{-BB^H t}$ is exponentially decaying for all the eigenvectors.

Problem 8

Suppose A and B are Hermitian $n \times n$ matrices. What can you say about the eigenvalues and eigenvectors of $C = i(AB - BA)$? (Hint: what is C^H?)

Solution: $C^H = -i(B^H A^H - A^H B^H) = -i(BA - AB) = i(AB - BA) = C$, so C is Hermitian and has real eigenvalues, orthogonal eigenvectors for distinct eigenvalues, and is diagonalizable.

Problem 9

Suppose that A is an 6×4 matrix with full column rank (rank $= 4$), and has the SVD

$$A = U\Sigma V^H = U \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \sigma_3 & \\ & & & \sigma_4 \\ & & & 0 \\ & & & 0 \end{pmatrix} V^H.$$

Recall the definition of the pseudo-inverse: $A^+ = V\Sigma^+ U^H$, where Σ^+ is the transpose of Σ with the non-zero entries (σ) inverted ($1/\sigma$). Show (by explicit multiplication etc.) that the pseudoinverse A^+ equals $(A^H A)^{-1} A^H$. [That is, the $A^+\mathbf{b}$ is equivalent to the least-squares solution to $A\mathbf{x} = \mathbf{b}$, as we discussed in class.]

Solution:

Given this A, we find

$$A^H A = V\Sigma^T \Sigma V^H = V \begin{pmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \sigma_3^2 & \\ & & & \sigma_4^2 \end{pmatrix} V^H,$$

and hence the inverse is given by inverting the eigenvalues σ^2:

$$(A^H A)^{-1} = V \begin{pmatrix} \sigma_1^{-2} & & & \\ & \sigma_2^{-2} & & \\ & & \sigma_3^{-2} & \\ & & & \sigma_4^{-2} \end{pmatrix} V^H V,$$

and hence

$$(A^HA)^{-1}A^H = V\begin{pmatrix} \sigma_1^{-2} & & & \\ & \sigma_2^{-2} & & \\ & & \sigma_3^{-2} & \\ & & & \sigma_4^{-2} \end{pmatrix} V^H V \begin{pmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \sigma_3 & & & \\ & & & \sigma_4 & 0 & 0 \end{pmatrix} U^H$$

$$= V\begin{pmatrix} \sigma_1^{-2} & & & \\ & \sigma_2^{-2} & & \\ & & \sigma_3^{-2} & \\ & & & \sigma_4^{-2} \end{pmatrix} \begin{pmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \sigma_3 & & & \\ & & & \sigma_4 & 0 & 0 \end{pmatrix} U^H$$

$$= V\begin{pmatrix} \sigma_1^{-1} & & & & & \\ & \sigma_2^{-1} & & & & \\ & & \sigma_3^{-1} & & & \\ & & & \sigma_4^{-1} & 0 & 0 \end{pmatrix} U^H = V\Sigma^+U^H$$

$$= A^+.$$

Problem 10

Suppose that an $m \times n$ matrix has the SVD $A = U\Sigma V^H$. Recall the definition of the pseudo-inverse: $A^+ = V\Sigma^+U^H$, where Σ^+ is the transpose of Σ with the non-zero entries (σ) inverted ($1/\sigma$). Show that $(A^H)^+ = (A^+)^H$.

Solution:

The adjoint is $A^H = V\Sigma^T U^H$, which immediately tells us the SVD of A^H. Hence $(A^H)^+ = U(\Sigma^T)^+V^H$. In comparison, $(A^+)^H = U(\Sigma^+)^H V^H$. Clearly, the two are equal if $(\Sigma^T)^+ = (\Sigma^+)^T$, which is obviously true since inverting the singular values ($\Sigma \to \Sigma^+$ means $\sigma \to 1/\sigma$) can clearly be interchanged with transposition (swapping rows and columns) without changing the result.

Problem 11

Consider the vector space of twice differentiable real functions $f(x)$ on $x \in [0,1]$ with $f(0) = f(1) = 1$, and define the dot product $f \cdot g = \int_0^1 f(x)\,g(x)\,dx$ as in class. Now, define a sequence of functions $f_k(x)$ $[k = 0, 1, 2, \ldots]$ by the recurrence relation $Af_{k+1}(x) = f_k(x)$ with $A = \frac{d^2}{dx^2}$.

(a) Suppose that the initial function $f_0(x)$ in the recurrence has the Fourier sine series:

$$f_0(x) = \frac{4}{\pi^2}\sin(\pi x) - \frac{4}{(3\pi)^2}\sin(3\pi x) + \frac{4}{(5\pi)^2}\sin(5\pi x) - \cdots = \frac{4}{\pi^2}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{(2\ell+1)^2}\sin[(2\ell+1)\pi x].$$

Give an explicit Fourier sine series for $f_k(x)$. [Recall from class: for this vector space and dot product, $\sin(n\pi x)$ is an eigenfunction of the Hermitian operator $\frac{d^2}{dx^2}$, with eigenvalue $-(n\pi)^2$.]

(b) Suppose we replace $\frac{d^2}{dx^2}$ with $c^2\frac{d^2}{dx^2}$ for some real number c. For what values of c (if any) does $\|f_k(x)\|^2$ diverge as $k \to \infty$? How does your answer depend on the initial function $f_0(x)$, if at all?

Solutions:

(a) The solution to this recurrence, like for matrix recurrences, is $f_k(x) = A^{-k} f_0(x)$. To compute A^{-k}, we just multiply the eigenfunctions by λ^{-k} as usual. Hence

$$f_k(x) = (-1)^k \left[\frac{4}{\pi^{2+2k}} \sin(\pi x) - \frac{4}{(3\pi)^{2+2k}} \sin(3\pi x) + \frac{4}{(5\pi)^{2+2k}} \sin(5\pi x) - \cdots \right] = \frac{4(-1)^k}{\pi^{2+2k}} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+1)^{2+2k}} \sin[(2\ell+1)\pi x],$$

using the fact that the eigenvalues are $-(n\pi)^2$.

(b) For f_k to diverge, A^{-1} should have an eigenvalue $|\lambda| > 1$, and hence A should have eigenvalues $|\lambda| < 1$. If $A = c^2 \frac{d^2}{dx^2}$, the eigenvalues of A are $-(cn\pi)^2$. This has magnitude less than 1 for $n = 1$ when $|c| < 1/\pi$, for $n = 2$ when $|c| < 1/2\pi$, and so forth. So, f_k diverges when $|c| < 1/\pi$. However, this does depend on our initial function f_0: if the $n = 1$ term is not present, then $|c|$ must be smaller; in general, if the first non-zero eigenfunction in the sine series for f_0 is $n = m$, then we must have $|c| < 1/m\pi$ to make f_k diverge.

18.06 Professor Edelman Quiz 3 November 21, 2008

Grading

1

Your PRINTED name is: ____________________ **2**

3

4

Please circle your recitation: ____________

1)	T 10	2-131	J.Yu	2-348	4-2597	`jyu`
2)	T 10	2-132	J. Aristoff	2-492	3-4093	`jeffa`
3)	T 10	2-255	Su Ho Oh	2-333	3-7826	`suho`
4)	T 11	2-131	J. Yu	2-348	4-2597	`jyu`
5)	T 11	2-132	J. Pascaleff	2-492	3-4093	`jpascale`
6)	T 12	2-132	J. Pascaleff	2-492	3-4093	`jpascale`
7)	T 12	2-131	K. Jung	2-331	3-5029	`kmjung`
8)	T 1	2-131	K. Jung	2-331	3-5029	`kmjung`
9)	T 1	2-136	V. Sohinger	2-310	4-1231	`vedran`
10)	T 1	2-147	M Frankland	2-090	3-6293	`franklan`
11)	T 2	2-131	J. French	2-489	3-4086	`jfrench`
12)	T 2	2-147	M. Frankland	2-090	3-6293	`franklan`
13)	T 2	4-159	C. Dodd	2-492	3-4093	`cdodd`
14)	T 3	2-131	J. French	2-489	3-4086	`jfrench`
15)	T 3	4-159	C. Dodd	2-492	3-4093	`cdodd`

1 (30 pts.) The complex matrix

$$A = \begin{bmatrix} a & c+di \\ c-di & b \end{bmatrix},$$

where a, b, c, and $d \neq 0$ are real numbers.

In (a) and (b) below circle the one **best** answer to the questions:

(a) This matrix is necessarily: symmetric? Hermitian? unitary? Markov?

(b) The two eigenvalues are necessarily: real? positive? zero? complex conjugates?

(c) The sum of the two eigenvalues is ________________.

(d) The product of the two eigenvalues in terms of a, b, c, and d but not i is ____________________.

(e) In terms of an eigenvalue λ (whose value you need not derive), write down an eigenvector of A.

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2 (32 pts.) The real matrix

$$A = \begin{bmatrix} x & 3/5 \\ y & z \end{bmatrix}.$$

The answers to the questions below involve alternative equations or inequalities involving x,y, and z that characterize all matrices of a certain type. Write down the relations. For (a) through (c), credit is only given for the complete description in reasonably clear and simple form.

(a) When is A positive definite? (Write two inequalities.)

(b) When is A Markov? (Perhaps write two or more inequalities, and two equalities.)

(c) When is A singular? (Write one equality)

(d) Write down one such A that is orthogonal. (There are four possible A and you are asked to write down one.)

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3 (13 pts.) The 4x4 Fourier matrix F has eigenvalues $-2, 2, 2i, -2i$. Preferably without any explicit computation (or even knowledge of the matrix itself) what is the matrix F^4? How do you know it has that particular Jordan form?

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4 (25 pts.) In terms of x ($0 < x < 1$) complete

$$A = \begin{bmatrix} x & \\ & \end{bmatrix},$$

so that A is a 2x2 matrix that is both Markov and singular.

What is A^{2008}?

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SOLUTIONS TO QUIZ 3

Problem 1. (6 points each)

$A = \begin{pmatrix} a & c+di \\ c-di & b \end{pmatrix}$

a) This matrix is clearly hermetian.

b) Thus, the two eigenvalues are real.

c) The sum of the eigenvalues is $tr(A) = a+b$.

d) The product of the eigenvalues is $det(A) = ab-(c+di)(c-di) = ab-(c^2+d^2)$.

e) We need to solve $\begin{pmatrix} a-\lambda & c+di \\ c-di & b-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$. We see that $\begin{pmatrix} -(c+di) \\ a-\lambda \end{pmatrix}$ is one such (note that this solves the top row equation, and the other by singularity of the matrix).

Problem 2. (8 points each)

$A = \begin{pmatrix} x & 3/5 \\ y & z \end{pmatrix}$.

a) A is positive definite if $x > 0$ and $det(A) = xz - 3y/5 > 0$.

b) A is Markov if $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x+y = 1$ and $z = 2/5$.

c) A is singular if $0 = det(A) = xz - 3y/5$.

d) Well, we need $(3/5)^2 + z^2 = 1$, so $z^2 = 16/25$, so set $z = 4/5$. So set $x = -4/5$ and $y = 3/5$. Flip the signs around to get the other possibilities.

Problem 3. (13 points)

As F has four distinct eigenvlaues, it is diagonalizable, i.e., $F = S\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2i & 0 \\ 0 & 0 & 0 & -2i \end{pmatrix}S^{-1}$

. Thus $F^4 = S\begin{pmatrix} (-2)^4 & 0 & 0 & 0 \\ 0 & 2^4 & 0 & 0 \\ 0 & 0 & (2i)^4 & 0 \\ 0 & 0 & 0 & (-2i)^4 \end{pmatrix}S^{-1} = S\begin{pmatrix} 16 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 16 \end{pmatrix}S^{-1} = \begin{pmatrix} 16 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 16 \end{pmatrix}$.

As this is already a Jordan matrix, this is the Jordan form of F^4. The underlying reason is that F is diagonalizable, hence $\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2i & 0 \\ 0 & 0 & 0 & -2i \end{pmatrix}$ is its Jordan form.

Problem 4. (25 points)

$A = \begin{pmatrix} x & ? \\ ? & ? \end{pmatrix}$. As A is supposed to be Markov, we must have $A = \begin{pmatrix} x & y \\ 1-x & 1-y \end{pmatrix}$, and A is singular implies $x(1-y) - y(1-x) = 0$, therefore $0 = x - xy - y + xy = x - y$, so $y = x$. Thus $A = \begin{pmatrix} x & x \\ 1-x & 1-x \end{pmatrix}$. As A is Markov, we know that $\lambda_1 = 1$ is an eigenvalue. As A is singular, we know that the product of the eigenvalues is 0. Therefore, $\lambda_2 = 0$ is another eigenvalue, and so A is diagonalizable; $A = S\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}S^{-1}$. Thus $A^{2008} = S\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{2008}S^{-1} = S\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}S^{-1} = A$.

18.06 Professor Strang Quiz 3 May 2, 2008

Grading

1

2

3

Your PRINTED name is: ____________________

Please circle your recitation:

1)	M 2	2-131	A. Ritter	2-085	2-1192	afr
2)	M 2	4-149	A. Tievsky	2-492	3-4093	tievsky
3)	M 3	2-131	A. Ritter	2-085	2-1192	afr
4)	M 3	2-132	A. Tievsky	2-492	3-4093	tievsky
5)	T 11	2-132	J. Yin	2-333	3-7826	jbyin
6)	T 11	8-205	A. Pires	2-251	3-7566	arita
7)	T 12	2-132	J. Yin	2-333	3-7826	jbyin
8)	T 12	8-205	A. Pires	2-251	3-7566	arita
9)	T 12	26-142	P. Buchak	2-093	3-1198	pmb
10)	T 1	2-132	B. Lehmann	2-089	3-1195	lehmann
11)	T 1	26-142	P. Buchak	2-093	3-1198	pmb
12)	T 1	26-168	P. McNamara	2-314	4-1459	petermc
13)	T 2	2-132	B. Lehmann	2-089	2-1195	lehmann
14)	T 2	26-168	P. McNamara	2-314	4-1459	petermc

1 (40 pts.) The (real) matrix A is

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & x & 3 \\ 2 & 3 & 6 \end{bmatrix}.$$

(a) What can you tell me about the eigenvectors of A?
What is the sum of its eigenvalues?

(b) For which values of x is this matrix A positive definite?

(c) For which values of x is A^2 positive definite? **Why**?

(d) If R is any **rectangular** matrix, *prove* from $x^{\mathrm{T}}(R^{\mathrm{T}}R)x$ that $R^{\mathrm{T}}R$ is positive semidefinite (or definite). What condition on R is the test for $R^{\mathrm{T}}R$ to be positive definite?

$\boxed{\text{Solution}}$ (10+10+10+10 points)

a) Since A is a symmetric matrix (no matter what x is), its eigenvectors may be chosen orthonormal (5 points). The sum of the eigenvalues is the same as the trace of A, that is, the sum of the diagonal entries: $tr(A) = 7 + x$.

b) In this case, the easiest tests for positive definiteness are the pivot test and the determinant test. I'll use the determinant test.
A matrix A is positive definite when every one of the top-left determinants is positive (3 points for correct defn.). In this case, the three determinants are 1, $x - 1$, and

$$\det(A) = 1(6x - 9) - (6 - 6) + 2(3 - 2x) = 2x - 3. \tag{1}$$

(6 points). All of these are positive precisely when $x > 3/2$ (1 point).

c) Perhaps the clearest way to think about this is by using the eigenvalues. Suppose A has eigenvalues $\lambda_1, \lambda_2, \lambda_3$. (They are all real because A is symmetric.) Then the eigenvalues of A^2 are $\lambda_1^2, \lambda_2^2, \lambda_3^2$ (5 points). These are all positive so long as the eigenvalues are non-zero. So, A^2 is positive definite except when A has an eigenvalue of 0, or equivalently, except when A is not invertible (3 points). We found in part that $\det(A) = 0$ only when $x = 3/2$. Thus, the final answer is that A^2 is positive definite except when $x = 3/2$ (2 points).

One could also find A^2 explicitly and use the determinant or pivot test. In practice this turned out to lead to a lot of mistakes. However, you could notice that the top left entry of A^2 is 6, the 2 by 2 determinant is $6(10 + x^2) - (7 + x)^2 = 5x^2 - 14x + 11 > 0$, and the 3 by 3 determinant is $\det(A^2) = \det(A)^2$. The only way that any of these could be non-positive is if $\det(A) = 0$.

A final approach is to follow the steps for part d) below.

d) We use the $x^T Ax$ test for positive (semi)definiteness. We have

$$x^T R^T Rx = (Rx)^T Rx = Rx \cdot Rx \tag{2}$$

This is just the length of the vector Rx. This length is positive when Rx is not the zero vector and is 0 when Rx is the 0 vector. In particular, since this number is always at least 0, $R^T R$ is definitely positive semidefinite (6 points). It is positive definite when this number is positive for any nonzero x. That is, we need for Rx to only be the 0 vector when x is the 0 vector. This is equivalent to saying that R has trivial nullspace, or R has full column rank (4 points).

2 (30 pts.) The **cosine** of a matrix is defined by copying the series for $\cos x$ (which always converges):

$$\cos A = I - \frac{1}{2!}A^2 + \frac{1}{4!}A^4 - \cdots$$

(a) Suppose $Ax = \lambda x$. Show that x is an eigenvector of $\cos A$. Find the eigenvalue.

(b) Find the eigenvalues of $A = \frac{\pi}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The eigenvectors are $(1,1)$ and $(1,-1)$. From the eigenvalues and eigenvectors of $\cos A$, find that matrix $\cos A$.

(c) The second derivative of the series for $\cos(At)$ is $-A^2\cos(At)$. So $u(t) = \boldsymbol{\cos(At)u(0)}$ is a short formula for the solution of

$$\frac{d^2u}{dt^2} = -A^2 u \quad \text{starting from} \quad u(0) \quad \text{with} \quad u'(0) = 0\,.$$

Now construct that $u(t) = \cos(At)u(0)$ by the usual three steps when A is diagonalizable: $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2$, $Ax_3 = \lambda_3 x_3$.

1. Expand $u(0) = c_1x_1 + c_2x_2 + c_3x_3$ in the eigenvectors.
2. Multiply those eigenvectors by ____, ____, ____.
3. Add up the solution $u(t) = c_1$ ____ $x_1 + c_2$ ____ $x_2 + c_3$ ____ x_3.

$\boxed{\text{Solution}}$ (10+10+10 points)

a) Suppose that $Ax = \lambda x$. Then

$$\begin{aligned} \cos(A)x &= Ix - \frac{1}{2!}A^2x + \frac{1}{4!}A^4x - \ldots && (3)\\ &= x - \frac{1}{2!}\lambda^2 x + \frac{1}{4!}\lambda^4 x - \ldots && (4)\\ &= \left(1 - \frac{1}{2!}\lambda^2 + \frac{1}{4!}\lambda^4 - \ldots\right)x && (5)\\ &= \cos(\lambda)x && (6) \end{aligned}$$

So x is an eigenvector of $\cos(A)$ with eigenvalue $\cos(\lambda)$.

b) We define

$$A = \frac{\pi}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{7}$$

We know that $(1,1)$ and $(1,-1)$ are eigenvectors of A. We can find the eigenvalues simply by acting by A:

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{\pi}{2} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \pi \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{8}$$

So A has eigenvalue $\lambda_1 = \pi$. Similarly,

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{9}$$

So A has eigenvalue $\lambda_2 = 0$ (4 points). Just as for any other function $(A^2, e^A, A^{-1}, \ldots)$, this means that $\cos(A)$ has eigenvectors $(1,1)$ with eigenvalue $\cos(\pi) = -1$ and $(1,-1)$ with eigenvalue $\cos(0) = 1$ (3 points). We can put these into the diagonalization formula to find $\cos(A)$:

$$\cos(A) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \tag{10}$$

(3 points)

c) This problem is modeled after what happens for e^{At}. After expanding $u(0)$, step 2 involves multiplying the eigenvectors by $\cos(\lambda_1 t)$, $\cos(\lambda_2 t)$, and $\cos(\lambda_3 t)$. So the final answer is

$$u(t) = c_1 \cos(\lambda_1 t) x_1 + c_2 \cos(\lambda_2 t) x_2 + c_3 \cos(\lambda_3 t) x_3 \tag{11}$$

(10 points) Some common mistakes were forgetting to include the t, using the function e instead of cos, or putting in something entirely different for the coefficients.

3 (30 pts.) Suppose the vectors x, y give an orthonormal basis for $\mathbf{R}^2$ and $A = xy^{\mathrm{T}}$.

(a) Compute the rank of A and the rank of $A^2 = (xy^{\mathrm{T}})(xy^{\mathrm{T}})$. Use this information to find the eigenvalues of A.

(b) Explain why this matrix B is **similar** to A (*and write down what similar means*):

$$B = \begin{bmatrix} & x^{\mathrm{T}} & \\ & y^{\mathrm{T}} & \end{bmatrix} A \begin{bmatrix} x & & y \end{bmatrix}$$

(c) The eigenvalues of Q are $\lambda_1 = e^{i\theta} = \cos\theta + i\sin\theta$ and $\lambda_2 = e^{-i\theta} = \cos\theta - i\sin\theta$:

$$\text{Rotation matrix} \quad Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Find the eigenvectors of Q. Are they perpendicular?

a) Any matrix given by $A = xy^T$ for two non-zero vectors x,y will have rank 1. Every row will be a multiple of y^T, and every column will be a multiple of x, meaning that it must have rank 1. Alternatively, we note that x is in the nullspace of A, and that y is not, so that A must have rank exactly 1. (3 points)

Note that $A^2 = (xy^T)(xy^T) = x(y^Tx)y^T$ is the zero matrix, since $y^Tx = 0$ (they are perpendicular vectors). So A^2 has rank 0. (3 points)

If the eigenvalues of A are λ_1 and λ_2, then the eigenvalues of A^2 are λ_1^2 and λ_2^2. Since A^2 only has the eigenvalue 0, both λ_1 and λ_2 must be 0. (4 points)

b) Two square matrices A and B are similar if there is some invertible matrix M such that $B = MAM^{-1}$ (5 points). Similarity is *not* the same thing as having equal eigenvalues; this only works if both A and B are diagonalizable matrices, and in fact our A is not diagonalizable. To be more precise, similarity implies that A and B have equal eigenvalues, but the converse is not true.

In this case we check that A and B are similar by showing that the other factors are inverses.

$$\begin{bmatrix} x^{\mathrm{T}} \\ y^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} x^T x & x^T y \\ y^T x & y^T y \end{bmatrix} \tag{12}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{13}$$

The last step is true because x and y are perpendicular, and both of unit length (5 points).

c) Given the eigenvalues of Q, we find the eigenvectors using $N(Q - \lambda I)$. We start with $\lambda_1 = \cos\theta + i\sin\theta$:

$$Q - (\cos\theta + i\sin\theta)I = \begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix} \tag{14}$$

and this matrix has nullspace generated by $(1, -i)$ or equivalently $(i, 1)$. Similarly, for $\lambda_2 = \cos\theta - i\sin\theta$ we find

$$Q - (\cos\theta - i\sin\theta)I = \begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix} \tag{15}$$

which has nullspace generated by $(1, i)$. (8 points)

Every orthogonal matrix has perpendicular eigenvectors. We check in this specific case:

$$\begin{bmatrix} i \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}^H \begin{bmatrix} 1 \\ i \end{bmatrix} \tag{16}$$

$$= \begin{bmatrix} -i & 1 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} \tag{17}$$

$$= 0 \tag{18}$$

(2 points) Make sure to take $x_1^H x_2$ and not $x_1^T x_2$.

18.06 Professor Postnikov Quiz 3 November 23, 2009

Your PRINTED name is: SOLUTIONS

Please circle your recitation:

(R01)	T10	2-132	HwanChul Yoo
(R02)	T11	2-132	HwanChul Yoo
(R03)	T12	2-132	David Shirokoff
(R04)	T1	2-131	Fucheng Tan
(R05)	T1	2-132	David Shirokoff
(R06)	T2	2-131	Fucheng Tan
(R07)	T2	2-146	Leonid Chindelevitch
(R08)	T3	2-146	Steven Sivek

Grading	
1	
2	
3	
Total:	

Problem 1. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$.

(A) Find the eigenvalues and the eigenvectors of A.

(B) Solve the differential equation $\dfrac{d\mathbf{u}(t)}{dt} = A\,\mathbf{u}(t)$ with the initial condition $\mathbf{u}(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

(C) Find a symmetric matrix B which is similar to A.

(D) Find the singular values σ_1 and σ_2 of A.

(A): $\det(A - \lambda I) = (1-\lambda)(-1-\lambda)$. Eigenvalues are $1, -1$

$\begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \rightsquigarrow$ eigenvector for 1 is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow$ eigenvector for -1 is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(B) $\dfrac{du(t)}{dt} = Au(t) \implies u(t) = Ce^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + De^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$u(0) = C\begin{pmatrix} 1 \\ 0 \end{pmatrix} + D\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \implies \begin{matrix} C = 1 \\ D = -1 \end{matrix}$

$$\boxed{\therefore\ u(t) = e^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

(C) A is diagonalizable (2 distinct eigenvals / vectors)

So A is similar to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(D) $A^TA = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $\det(A^TA - \lambda I) = (1-\lambda)(2-\lambda) - 1 = 0$

$= \lambda^2 - 3\lambda + 1$

$\lambda = \dfrac{3 \pm \sqrt{9-4}}{2}$

$\sigma_1, \sigma_2 = \sqrt{\dfrac{3 \pm \sqrt{5}}{2}}$

Problem 2. Consider the matrix

$$A = \begin{pmatrix} 1 & t & 0 \\ t & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix},$$

which depends on a parameter t.

(A) Find all values of the parameter t when the matrix A is positive definite.

(B) Suppose that $t = 0$. Find a 3×3 matrix R such that $A = R^T R$.

(C) Suppose that $t = 0$. Verify directly that A satisfies the energy-based definition of a positive definite matrix, as follows. For a vector $\mathbf{x} = (x, y, z)^T$, write out $\mathbf{x}^T A \mathbf{x}$; show that this can be written as a sum of squares; and deduce that $\mathbf{x}^T A \mathbf{x} > 0$ for any non-zero $\mathbf{x}$.

(A) $1 > 0$, $1 - t^2 > 0$, $1 \cdot (2-1) - t \cdot (2t) = 1 - 2t^2 > 0$

$\Rightarrow t^2 < \frac{1}{2}$ or $-\sqrt{\frac{1}{2}} < t < \sqrt{\frac{1}{2}}$.

(B) From problem 1, we saw $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

So take

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

(C) $x^T A x = x^T R^T R x = (Rx)^T Rx = \|Rx\| > 0 \quad \forall x$

because R is nonsingular (has 3 pivots)

Problem 3. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.

(A) Indicate which of the following statements are true and which are false:

(1) A is symmetric; (2) A is orthogonal;

(3) A is invertible; (4) $\frac{1}{3}A$ is a Markov matrix

(B) Find the eigenvalues and the eigenvectors of A. (Hint: Part (A) might help you.)

(C) Find an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^T$.

(D) Calculate the limit $\mathbf{u}_\infty$ of $\mathbf{u}_k = (\frac{1}{3}A)^k \mathbf{u}_0$ as $k \to \infty$, for $\mathbf{u}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

(A) (1) Yes!

(2) No: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \neq 0$, and vectors aren't unit length.

(3) No: $2 \cdot \text{row } 1 = \text{row } 2 + \text{row } 3$

(4) Yes! Colums all add to 3.

(B) A is singular, so 0 is an eigenvalue.

$\frac{1}{3}A$ is Markov $\Rightarrow$ 1 is eigenval of $\frac{1}{3}A \Rightarrow 3$ is eigenval of A.

$tr(A) = 1+2+2 = 3+0+\lambda \Rightarrow \lambda = 2$ is other eigenvalue.

$0 \rightsquigarrow \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ $\quad 2: \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ $\quad 3: \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (row sums are $= 3$)

(C) Since A is symmetric, eigenvects are orthogonal: must normalize!

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(c) continued:

$$Q = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 0 & & \\ & 2 & \\ & & 3 \end{pmatrix}$$

D. Calculate lim u_∞ : $u_k = \left(\frac{1}{3}A\right)^k u_0$ $\quad u_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$$\tfrac{1}{3}A = Q\,\tfrac{1}{3}\Lambda Q^T$$

$$\left(\tfrac{1}{3}A\right)^k = Q\begin{pmatrix} 0 & & \\ & \frac{2}{3} & \\ & & 1 \end{pmatrix}^k Q^T \xrightarrow[k\to\infty]{} Q\begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}Q^T$$

$$\therefore\ u_\infty = Q\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}Q^T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = Q\begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= Q\begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}.$$

18.06 Spring 2009 Exam 3 Practice

General comments

Exam 2 covers the first 31 lectures of 18.06, mainly focusing on lectures 19–31 (eigenproblems). The topics covered are (very briefly summarized):

1. All of the topics from exams 1 and 2, although of course these are not the focus of the exam.

2. Determinants: their properties, how to compute them (simple formulas for 2×2 and 3×3, usually by elimination for matrices $> 3\times 3$), their relationship to linear equations (zero determinant = singular), their use for eigenvalue problems.

3. Eigenvalues and eigenvectors: their definition $\boxed{A\vec{x} = \lambda\vec{x}}$, their properties, the fact that for an eigenvector the matrix (or *any function of the matrix*) acts just like a number. Computing λ from the characteristic polynomial $\det(A-\lambda I)$ and $\vec{x}$ from $N(A-\lambda I)$; zero eigenvalues $\lambda = 0$ just correspond to $N(A)$. Understand (from the definition) why, if A has an eigenvalue λ, then A^k has an eigenvalue λ^k, αA has an eigenvalue $\alpha\lambda$, and $A+\beta I$ has an eigenvalue $\lambda+\beta$, all with the *same* eigenvector.

4. Diagonalization $A = S\Lambda S^{-1}$: where it comes from, its use in understanding properties of matrices and eigenvalues. **The basic idea that, to solve a problem involving A, you first expand your vector in the basis of the eigenvectors (S), then for each eigenvector you treat A as just a number λ, then at the end you add up the solutions.**

5. Similar matrices: A and $B = MAM^{-1}$ have the same eigenvalues for any invertible matrix M, and if $A\vec{x} = \lambda\vec{x}$ then $B\vec{y} = \lambda\vec{y}$ for $\vec{y} = M\vec{x}$. Similar matrices have the same trace (sum of the eigenvalues) and determinant (product of the eigenvalues).

6. Using eigenvalues/eigenvectors to solve problems involving matrix powers, such as linear recurrences (e.g. Fibonacci). Multiplying by A many times tends towards the eigenvector for the largest $|\lambda|$. Markov matrices: what the defining properties are, and the consequences (a steady state with $\lambda = 1$, all other solutions decay away, the sum of the components of the vector is conserved, a unique steady state if all entries of the matrix are > 0). $A^n = S\Lambda^n S^{-1}$.

7. Using eigenvalues/eigenvectors to solve linear systems of differential equations $\frac{d\vec{u}}{dt} = A\vec{u}$ with initial conditions $\vec{u}(0)$. Practical scheme: expand $\vec{u}(0)$ in eigenvector basis and multiply each term by $e^{\lambda t}$. Formal solution: $e^{At}\vec{u}(0)$, and the meaning of the matrix exponential $e^A = Se^\Lambda S^{-1}$ and how to compute it and manipulate it.

8. If $A = A^T$ (real-symmetric), then the eigenvalues are real and the eigenvectors are orthogonal (or can be chosen orthogonal), and A is diagonalizable as $A = Q\Lambda Q^T$ for an orthogonal Q. If $A = B^TB$ where B has full column rank, then A is positive definite: all $\lambda > 0$ and all pivots > 0 and $\vec{y}^TA\vec{y} > 0$ for any $\vec{y} \neq 0$; connection to minimization problems (like least-squares).

9. Complex matrices, for which we replace $\vec{x}^T$ and A^T by and $\vec{x}^H = \overline{\vec{x}^T}$ $A^H = \overline{A^T}$ (and why). What to do if you get a complex λ: consequences for matrix powers (recurrence relations) and differential equations are oscillating solutions, using $e^{i\theta} = \cos\theta + i\sin\theta$.

10. Defective matrices and generalized eigenvectors: what to do if A is not diagonalizable, especially for a practical problem like $A^k\vec{u}$ or $e^{At}\vec{u}$. (Note that real-symmetric, real-orthogonal, Hermitian, and unitary matrices are never defective, nor are $n\times n$ matrices with n distinct eigenvalues.)

11. Singular value decomposition $A = U\Sigma V^T$ and their relationship to eigenvectors/eigenvalues of A^TA and AA^T.

The central concept from this part of the course is highlighted in boldface above. Once you have an eigenvector, *any* operation involving the matrix just becomes that operation with the single number λ. And single numbers are easy to handle. So, we try to find the eigenvectors and then express every vector in that basis (aside from rare defective cases), at which point problems become easy (ideally)! Also, you should be able to recognize and reason about how and why special forms of the matrix A (symmetric, Markov, singular, etcetera) give you additional information about the eigenvectors and eigenvalues.

Defective matrices and SVDs will see at most limited coverage on the exam, perhaps one part of a problem each, at most.

Some practice problems

The 18.06 web site has exams from previous terms that you can download, with solutions. I've listed a few practice exam problems that I like below, but there are plenty more to choose from. The exam will consist of 3 or 4 questions (perhaps with several parts each), and you will have one hour. You can find the solutions to these problems on the 18.06 web site (in the section for old exams/psets).

1. (Fall 2002 exam 3.) **(a)** What are the eigenvalues of the 5×5 matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$? Please look at A, not at $\det(A-\lambda I)$. **(b)** Solve $\frac{d\vec{u}}{dt} = A\vec{u}$ starting from $\vec{u}(0) = (0,1,1,1,2)^T$. (First split $\vec{u}(0)$ as the sum of two eigenvectors of A.) **(c)** Using part (a), what are the *eigenvalues* and *trace* and *determinant* of the matrix B which is the same as A except that it has zeros on its diagonal?

2. (Fall 2002 exam 3.) **(a)** if A is similar to B show that e^A is similar to e^B. (Hint: first write down the definitions of "similar" and e^A.) **(b)** If A has 3 eigenvalues $\lambda = 0,2,4$, find the eigenvalues of e^A. **(c)** Explain this connection with determinants: $\det(e^A) = e^{\text{trace of } A}$.

3. (Fall 2002 exam 3.) Companies in the US, Asia, and Europe have assets of \$12 trillion. At the start, \$6 trillion are in the US and \$6 trillion are in Europe. Each year, half the US money stays home, $1/4$ each goes to Asia and Europe. For Asia and Europe, half stays home and half is sent to the US, hence

$$\begin{pmatrix} \text{US} \\ \text{Asia} \\ \text{Europe} \end{pmatrix}_{\text{year } k+1} = \begin{pmatrix} .5 & .5 & .5 \\ .25 & .5 & 0 \\ .25 & 0 & .5 \end{pmatrix} \begin{pmatrix} \text{US} \\ \text{Asia} \\ \text{Europe} \end{pmatrix}_{\text{year } k+1}.$$

(a) The eigenvalues and eigenvectors of this *singular* matrix A are what? **(b)** The limiting distribution of the \$12 trillion after many many years is US=?, Asia=?, Europe=?

4. (Fall 2002 exam 2.) If you know that $\det A = 6$, what is $\det B$ for B given by:

$$A = \begin{pmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{pmatrix} \qquad B = \begin{pmatrix} \text{row 3 + row 2 + row 1} \\ \text{row 2 + row 1} \\ \text{row 1} \end{pmatrix}$$

5. (Spring 2004 exam 3.) For the symmetric matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$, you are given that one of the eigenvalues is $\lambda = 1$ with a line of eigenvectors $\vec{x} = (c,c,0)$. **(a)** That line is the nullspace of what matrix constructed from A? **(b)** Find (in any way) the other two eigenvalues of A and two corresponding eigenvectors. **(c)** The diagonalization $A = S\Lambda S^{-1}$ has an especially nice form because $A = A^T$. Write all entries in the nice symmetric diagonalization of A. **(d)** Give a reason why e^A *is* or *is not* a symmetric positive-definite matrix.

6. (Spring 2004 exam 3.) **(a)** Find the eigenvalues and eigenvectors (depending on c) of $A = \begin{pmatrix} 0.3 & c \\ 0.7 & 1-c \end{pmatrix}$. For which c is the matrix A *not diagonalizable*?[1] **(b)** What is the largest range of (real) values of c so that A^n approaches a limiting matrix A^∞ as $n \to \infty$? **(c)** What is that limit of A^n (still depending on c)?

7. (Spring 2005 exam 3.) **(a)** Find all the eigenvalues and all the eigenvectors of the following A. *It is a symmetric Markov matrix with a repeated eigenvalue.*

$$A = \begin{pmatrix} 2/4 & 1/4 & 1/4 \\ 1/4 & 2/4 & 1/4 \\ 1/4 & 1/4 & 2/4 \end{pmatrix}.$$

(b) Find the limit of A^k as $k \to \infty$. **(c)** Choose any positive numbers r, s, and t so that $A - rI$ is positive-definite, $A - sI$ is indefinite, and $A - tI$ is negative definite. **(d)** Suppose that this $A = B^T B$. What are the singular values σ_i in the SVD of B?

8. (Spring 2005 exam 3.) **(a)** Complete this 2×2 matrix A, depending on the real number a, so that its eigenvalues are $\lambda = 1$ and $\lambda = -1$. $A = \begin{pmatrix} a & 1 \\ ? & ? \end{pmatrix}$. **(b)** How do you know that A has two independent eigenvectors? **(c)** Which choices of a give orthogonal eigenvectors and which don't?

9. (Spring 2005 exam 3.) Suppose that the 3×3 matrix A has 3 independent eigenvectors $\vec{x}_{1,2,3}$ and corresponding eigenvalues $\lambda_{1,2,3}$. (The λ's might not be different.) **(a)** Describe the general form of every solution $\vec{u}(t)$to the differential equation $\frac{d\vec{u}}{dt} = A\vec{u}$ in terms of the λ's and $\vec{x}$'s. (The answer $e^{At}\vec{u}(0)$ is not sufficient.) **(b)** Starting from any vector $\vec{u}_0$, suppose $\vec{u}_{k+1} = A\vec{u}_k$. What are the conditions on the $\vec{x}$'s and λ's to guarantee that $\vec{u}_k \to 0$ as $k \to \infty$? Why?

10. (Fall 2005 exam 3.) This 4×4 matrix H is a special matrix called a "Hadamard matrix:"

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

It has two key properties: $H^T = H$, and $H^2 = 4I$. **(a)** Figure out the eigenvalues of H and explain your reasoning. **(b)** Figure out H^{-1} and $\det H$. Explain your reasoning. **(c)** This matrix S contains three eigenvectors of H. Find a 4-th eigenvector $\vec{x}_4$ and explain your reasoning.

$$S = \begin{pmatrix} 1 & 1 & 0 & ? \\ 1 & 0 & -1 & ? \\ 1 & 0 & 1 & ? \\ -1 & 1 & 0 & ? \end{pmatrix}.$$

e**(d)** Find the solution to $d\vec{u}/dt = H\vec{u}$ given that $\vec{u}(0)$ is the 3rd column of S.

11. (Fall 2005 exam 3.) Suppose A is a 3×3 symmetric matrix with eigenvalues $2, 5, 7$ and corresponding eigenvectors $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$. **(a)** Suppose $\vec{x}$ is a linear combination $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$. Find $A\vec{x}$. Now find $\vec{x}^T A\vec{x}$ using the symmetry of A. Explain why $\vec{x}^T A\vec{x} > 0$ unless $\vec{x} = 0$.

12. (Fall 2006 exam 3.) **(a)** Find all three eigenvalues of A, and an eigenvector matrix S. $A = \begin{pmatrix} -1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{pmatrix}$. **(b)** Explain why $A^{1001} = A$. Is $A^{1000} = I$? **(c)** The matrix $A^T A$ for this A is $A^T A = \begin{pmatrix} 1 & -2 & -4 \\ -2 & 4 & 8 \\ -4 & 8 & 42 \end{pmatrix}$.How many

[1] The solutions are a little tricky. A is *not* a Markov matrix because c may be < 0. However, its columns sum to 1, and that was enough to give us an eigenvalue $\lambda = 1$ in our analysis of Markov matrices. In the non-diagonalizable case, the solution's formula for part **(b)** is incorrect. As we know from lecture 30, for a repeated eigenvalue $\lambda = 1$ that is defective, there is a term in A^n that goes as 1^n and another term that goes as $n\,1^{n-1}$. Since the latter blows up, the defective case does not have a finite A^∞ limit.

eigenvalues of $A^T A$ are positive? zero? negative? Does $A^T A$ have the same eigenvectors as A? (Don't compute anything, but explain your answers.)

13. (Fall 2006 exam 3.) Suppose the $n \times n$ matrix A has n orthonormal eigenvectors $\vec{q}_1, \ldots, \vec{q}_n$ and n positive eigenvalues $\lambda_1, \ldots, \lambda_n$. That is, $A\vec{q}_j = \lambda_j \vec{q}_j$. **(a)** What are the eigenvalues and eigenvectors of A^{-1}? **(b)** Any vector $\vec{b}$ can be written as a combination of the eigenvectors $\vec{b} = c_1\vec{q}_1 + c_2\vec{q}_2 + \cdots + c_n\vec{q}_n$ for some coefficients c_j. What is a quick formula for c_1 using the orthogonality of the $\vec{q}$'s? **(c)** The solution to $A\vec{x} = \vec{b}$ is also a combination of the eigenvectors $A^{-1}\vec{b} = d_1\vec{q}_1 + d_2\vec{q}_2 + \cdots + d_n\vec{q}_n$. What is a quick formula for d_1. (You can write it in terms of the c's even if you didn't answer part b.)

14. (Fall 2007 exam 3.) Suppose that we form a sequence of real numbers f_k defined by the recurrence relation $f_{k+1} = f_k - f_{k-1} + f_{k-2}$, starting with the initial numbers $f_0 = 2$, $f_1 = 1$, and $f_2 = 2$. **(a)** Define a 3-component vector $\vec{g}_k = (f_k, f_{k-1}, f_{k-2})$ and a 3×3 matrix A so that $\vec{g}_{k+1} = A\vec{g}_k$. **(b)** If you constructed A correctly, the three eigenvalues should be 1 and $\pm i$, and the latter two eigenvectors should be $(-1, \pm i, 1)$. Check that you have these $\pm i$ eigenvalues and eigenvectors, and find the $\lambda = 1$ eigenvector. **(c)** Give an explicit formula for f_k for any k (formulas involving A^k are not acceptable; elementary arithmetic and powers of complex numbers only). **(d)** Is there any choice of initial conditions (f_0, f_1, and f_2) that will make $|f_k|$ diverge as $k \to \infty$? Explain.

15. (Fall 2007 exam 3.) Some 3×3 real matrix A has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 2$, with corresponding eigenvectors $\vec{x}_1 = (1,0,0)$, $\vec{x}_2 = (0,1,2)$, and $\vec{x}_3 = (0,1,1)$. **(a)** Give a basis for the nullspace $N(A)$, the column space $C(A)$, and the row space $C(A^T)$. **(b)** Find *all* solutions (the complete solution) $\vec{x}$ to $A\vec{x} = \vec{x}_2 - 3\vec{x}_3$. **(c)** Is A real-symmetic, orthogonal, Markov, or none of the above?

18.06 Quiz 3 Solution

Hold on Friday, 1 May 2009 at 11am in Walker Gym.
Total: 65 points.

Problem 1:

For each part, give **as much information as possible** about the **eigenvalues** of the matrix A described in that part. (Each part describes a *different* matrix A. A may be complex.)

(a) The recurrence $\mathbf{u}_{k+1} = A\mathbf{u}_k$ has a solution where $\|\mathbf{u}_k\| \to 0$ as $k \to \infty$ for one initial vector $\mathbf{u}_0$, but also has a solution with $\|\mathbf{u}_k\| \to \infty$ as $k \to \infty$ for a *different* choice of the initial vector $\mathbf{u}_0$.

(b) The equation $(A^2 - 4I)\mathbf{x} = \mathbf{b}$ has no solution for some right-hand side $\mathbf{b}$.

(c) $A = e^{B^{\mathrm{T}}B}$ for some real matrix B with full column rank.

(d) $A = B^{\mathrm{T}}B$ for a 4×3 real matrix B, and the matrix BB^{T} has eigenvalues $\lambda = 3, 2, 1, 0$. (Hint: think about the SVD of B.)

$\boxed{\text{Solution}}$ (20 points = 5+5+5+5)

(a) (There was a bug in this problem: in the first condition, we should have required the initial vector $\mathbf{u}_0$ to be nonzero.) The first condition implies that A has an eigenvalue with absolute value $|\lambda| < 1$. The second condition implies that either A has an eigenvalue with absolute value $|\lambda| > 1$, or A is defective for 2 eigenvalues λ with $|\lambda| = 1$.

(b) The condition says that $A^2 - 4I$ is singular. But we know that, if $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A, then the eigenvalues of $A^2 - 4I$ are $\lambda_1^2 - 4, \ldots, \lambda_n^2 - 4$. The condition $A^2 - 4I$ being singular says that one of $\lambda_i^2 - 4$ is zero, and hence $\lambda_i = 2$ or -2. That is to say A has an eigenvalue 2 or -2.

(c) Since B has full column rank, the eigenvalues of $B^{\mathrm{T}}B$ are postive real numbers λ_i. Hence, we know $A = e^{B^{\mathrm{T}}B}$ has eigenvalues e^{λ_i}; they are real numbers bigger than 1.

(d) Since BB^{T} and $B^{\mathrm{T}}B$ have the same set of *nonzero* eigenvalues. So $B^{\mathrm{T}}B$ must have eigenvalues $3, 2, 1$. Moreover, since B is a 4×3 matrix, $B^{\mathrm{T}}B$ is a 3×3 matrix. Hence, $3, 2, 1$ are exactly all the eigenvalues.

Problem 2: You are given the matrix

$$A = \begin{pmatrix} 0.5 & 0.2 & 0.2 \\ 0.1 & 0.5 & 0.5 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}.$$

(i) What are the eigenvalues of A? [*Hint:* Very little calculation required! You should be able to see two eigenvalues by inspection of the form of A, and the third by an easy calculation. You *shouldn't* need to compute $\det(A - \lambda I)$ unless you really want to do it the hard way.]

(ii) The vector $\mathbf{u}(t)$ solves the system

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}$$

for some initial condition $\mathbf{u}(0)$. If you are told that $\mathbf{u}(t)$ approaches some constant vector as $t \to \infty$, give as much true information as possible regarding the initial condition $\mathbf{u}(0)$.

[*Note:* be sure you understand that this is *not the same thing* as solving the recurrence $\mathbf{u}_{k+1} = A\mathbf{u}_k$! Imagine how you would find $\mathbf{u}(t)$ if you knew what $\mathbf{u}(0)$ was.]

$\boxed{\text{Solution}}$ (20 points = 10+10)

(i) First, the last two columns of A are the same. Hence A is singular and it must have an eigenvalue $\lambda_1 = 0$. Also, we observe that A is a Markov matrix. This means that $\lambda_2 = 1$ is an eigenvalue of A. Finally, we know the trace of A is the sum of its three eigenvalues. So, $\mathrm{Tr}(A) = 0.5 + 0.5 + 0.3 = 1.3$ and the last eigenvalue is $\lambda_3 = 1.3 - 1 - 0 = 0.3$.

(ii) We can write $\mathbf{u}(0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ using three eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, which correspond to $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 0.3$, respectively. We know that this system has solution $\mathbf{u}(t) = c_1\mathbf{v}_1 + c_2e^t\mathbf{v}_2 + c_3e^{0.3t}\mathbf{v}_3$. So, if either one of c_2 and c_3 is nonzero, the system would blow up as $t \to \infty$. Therefore, the only possibility for $\mathbf{u}(t)$ to approaches some constant is to have $c_2 = c_3 = 0$, that is to say that $\mathbf{u}(0)$ is a multiple of the eigenvector $\mathbf{v}_1 = (0, -1, 1)^{\mathrm{T}}$. In this case, $\mathbf{u}(t) = \mathbf{u}(0) = c_1\mathbf{v}_1$ is a constant.

Problem 3:

The 3×3 matrix A has three independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ with corresponding eigenvalues λ_1, λ_2, and λ_3 (that is, $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for $i = 1, 2, 3$).

If

$$\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

for some coefficients c_1, c_2, and c_3, then write (in terms of λ_i, c_i, and $\mathbf{v}_i$) a formula for the solution $\mathbf{x}$ of

$$A^2\mathbf{x} + 2A\mathbf{x} - 3I\mathbf{x} = \mathbf{b}$$

(you can assume that a solution exists for any $\mathbf{b}$).

$\boxed{\text{Solution}}$ (10 points)

Using the eigenvalues $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, we have

$$\begin{aligned}
\mathbf{x} &= (A^2 + 2A - 3I)^{-1}\mathbf{b} \\
&= (A^2 + 2A - 3I)^{-1}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) \\
&= \frac{c_1}{\lambda_1^2 + 2\lambda_1 - 3}\mathbf{v}_1 + \frac{c_2}{\lambda_2^2 + 2\lambda_2 - 3}\mathbf{v}_2 + \frac{c_3}{\lambda_3^2 + 2\lambda_3 - 3}\mathbf{v}_3.
\end{aligned}$$

Problem 4: A is a 3×3 real-symmetric matrix. Two of its eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$ with eigenvectors $\mathbf{v}_1 = (1,1,1)^{\mathrm{T}}$ and $\mathbf{v}_2 = (1,-1,0)^{\mathrm{T}}$, respectively. The third eigenvalue is $\lambda_3 = 0$.

(I) Give an eigenvector $\mathbf{v}_3$ for the eigenvalue λ_3. (*Hint:* what must be true of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$?)

(II) Using your result from (I), write the matrix e^A as the product of three matrices, and explicitly give the three matrices. (You need not work out the arithmetic, but your answer should contain no matrix inverses or matrix exponentials. *If you find yourself doing a lot of arithmetic, you are forgetting a useful property of this matrix!*)

$\boxed{\text{Solution}}$ (15 points = 7+8)

(I) For a real-symmetric matrix, its eigenvectors are orthogonal to each other. So, by inspection, in order for $\mathbf{v}_3$ to be perpendicular to $\mathbf{v}_2$, we need its first two components same. Hence, we should take $\mathbf{v}_3$ to be $(1,1,-2)^{\mathrm{T}}$. To easy the second part, we can normalize the eigenvectors

$$\mathbf{q}_1 = \mathbf{v}_1/\|\mathbf{v}_1\| = (1,1,1)^{\mathrm{T}}/\sqrt{3},$$
$$\mathbf{q}_2 = \mathbf{v}_2/\|\mathbf{v}_2\| = (1,-1,0)^{\mathrm{T}}/\sqrt{2},$$
$$\mathbf{q}_3 = \mathbf{v}_3/\|\mathbf{v}_3\| = (1,1,-2)^{\mathrm{T}}/\sqrt{6}.$$

Alternatively, we can use Gram-Schmidt to find (a multiple of) $\mathbf{v}_3$ as follows. We start with $\mathbf{v} = (1,0,0)$,

$$\mathbf{v}_3 = \mathbf{v} - (\mathbf{v}\cdot\mathbf{q}_1)\mathbf{q}_1 - (\mathbf{v}\cdot\mathbf{q}_2)\mathbf{q}_2 = (1,0,0)^{\mathrm{T}} - \frac{1}{3}(1,1,1)^{\mathrm{T}} - \frac{1}{2}(1,-1,0)^{\mathrm{T}} = (\frac{1}{6}, \frac{1}{6}, -\frac{1}{3})^{\mathrm{T}}.$$

(II) We can write

$$A = S\Lambda S^{-1} = Q\Lambda Q^{\mathrm{T}}$$
$$= \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix}.$$

Hence,

$$e^A = Qe^{\Lambda}Q^{\mathrm{T}}$$
$$= \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} e & 0 & 0 \\ 0 & 1/e & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix}.$$