18.06 Quiz 2 April 7, 2010 Professor Strang

Your PRINTED name is: ____________________ **1.**

Your recitation number or instructor is ____________________ **2.**

3.

1. (33 points)

(a) Find the matrix P that projects every vector b in R^3 onto the line in the direction of $a = (2, 1, 3)$.

Solution The general formula for the orthogonal projection onto the column space of a matrix **A** is

$$P = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T.$$

Here,

$$\mathbf{A} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad \text{so that} \quad \boxed{\mathbf{P} = \frac{1}{14}\begin{bmatrix} 4 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 9 \end{bmatrix}}$$

Remarks:

- Since we're projecting onto a one-dimensional space, $\mathbf{A}^T\mathbf{A}$ is just a number and we can write things like $P = (\mathbf{A}\mathbf{A}^T)/(\mathbf{A}^T\mathbf{A})$. *This won't work in general.*
- You don't have to know the formula to do this. The i^{th} column of **P** is, pretty much by definition, the projection of e_i ($e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$) onto the line in the direction of a. And this is something you should know how to do without a formula.

RUBRIC: There was some leniency for computational errors, but otherwise there weren't many opportunities for partial credit.

(b) What are the column space and nullspace of P? Describe them geometrically and also *give a basis for each space.*

Solution The column space is the line in R^3 in the direction of $a = (2, 1, 3)$. One basis for it is

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

and there's not really much choice in giving this basis (you can rescale by a non-zero constant).

The nullspace is the plane in R^3 that is perpendicular to $a = (2, 1, 3)$ (i.e., $2x+y+z = 0$.) One basis for it is

$$\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

though there are a lot of different looking choices for it (any two vectors that are perpendicular to a and not in the same line will work).

RUBRIC: 6 points for giving a correct basis, and 4 points for giving the complete geometric description. Note that it is not correct to say e.g., $N(\mathbf{P}) = R^2$. It is correct to say that $N(\mathbf{P})$ is a (2-dimensional) plane in R^3, but this is not a complete geometric description unless you say (geometrically) *which* plane it is: the one perpendicular to a/to the line through a.

(c) What are *all* the eigenvectors of P and their corresponding eigenvalues? (You can use the geometry of projections, not a messy calculation.) The diagonal entries of P add up to ________ .

Solution The diagonal entries of P add up to $\boxed{1 = \text{the sum of the eigenvalues}}$

Since $\mathbf{P}$ is a projection, it's only possible eigenvalues are $\lambda = 0$ (with multiplicity equal to the dimension of the nullspace, here 2) and $\lambda = 1$ (with multiplicity equal to the dimension of the column space, here 1). So, a complete list of eigenvectors and eigenvalues is:

- $\lambda = 0$ with multiplicity 2. The eigenvectors for $\lambda = 0$ are precisely the vectors in the null space. That is, all linear combinations of $\begin{bmatrix} 3 & 0 & -2 \end{bmatrix}^T$ and $\begin{bmatrix} -1 & 2 & 0 \end{bmatrix}^T$.
- $\lambda = 1$ with multiplicity 1. The eigenvectors for $\lambda = 1$ are precisely the vectors in the column space. That is, all multiples of $\begin{bmatrix} 2 & 1 & 3 \end{bmatrix}^T$.

RUBRIC: 2 points for the sum of eigenvalues, 4 points for a full list (with multiplicities) of eigenvalues, and 4 points for a complete description of all eigenvectors. In light of the emphasized "*all*," you'd lose 1 point if you gave two eigenvectors for $\lambda = 0$ and didn't say that all (at least non-zero) linear combinations were also eigenvectors for $\lambda = 0$.

2. (34 points)

(a) $p = A\widehat{x}$ is the vector in $C(A)$ nearest to a given vector b. If A has independent columns, what equation determines $\widehat{x}$? What are all the vectors perpendicular to the error $e = b - A\widehat{x}$? What goes wrong if the columns of A are dependent?

Solution $\widehat{x}$ is determined by the equation $\widehat{x} = (A^TA)^{-1}A^Tb$ (since A has independent columns, A^TA is invertible whether or not A is square). The vectors perpendicular to an arbitrary error vector are the elements of the column space of A. If the columns of A are dependent, A^TA is no longer invertible, and there is no unique nearest vector (i.e. there are multiple solutions).

RUBRIC: 4 points for the determining equation (1 point off for actually inverting A^TA or saying that it was invertible), 3 points for identifying the column space, and three points for identifying the multiple solutions (1 point off if you just say that A^TA is not invertible). Note that you cannot write $A^{-1}B$ as $\frac{B}{A}$: this only works for numbers because multiplication and division are commutative, which is not true for matrices.

(b) Suppose $A = QR$ where Q has orthonormal columns and R is upper triangular invertible. Find $\widehat{x}$ and p in terms of Q and R and b (not A).

Solution Since $Q^TQ = I$ and R is invertible, we obtain

$$\widehat{x} = (A^TA)^{-1}A^Tb = ((QR)^T(QR))^{-1}(QR)^Tb$$

$$= (R^TQ^TQR)^{-1}R^TQ^Tb = R^{-1}(R^T)^{-1}R^TQ^Tb = R^{-1}Q^Tb$$

$$p = (QR)\widehat{x} = QQ^Tb$$

Note that QQ^T is not the identity matrix in general.

RUBIC: 6 points for finding $\widehat{x}$, 4 points for p. One point off from each if the equations are not simplified, more points off for bad form, having variables other than Q, R and b, etc.

(c) If q_1 and q_2 are any orthonormal vectors in R^5, give a formula for the projection p of any vector b onto the plane spanned by q_1 and q_2 *(write p as a combination of q_1 and q_2).*

Solution $p = (q_1^Tb)q_1 + (q_2^Tb)q^2$.

RUBRIC: little partial credit. If you identified the difference between b and p instead, you may have gotten some points.

3. (33 points) This problem is about the n by n matrix A_n that has zeros on its main diagonal and all other entries equal to -1. In MATLAB $A_n = \text{eye}\,(n) - \text{ones}\,(n)$.

(a) Find the determinant of A_n. Here is a suggested approach:

Start by adding all rows (except the last) to the last row, and then factoring out a constant. (You could check $n = 3$ to have a start on part b.)

Solution Following the hint, add all of the rows to the last row (which does not change the determinant). Thus the matrix becomes

$$\begin{bmatrix} 0 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 \\ -1 & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ -(n-1) & -(n-1) & -(n-1) & \cdots & -(n-1) \end{bmatrix}.$$

Next, pull out the factor of $-(n-1)$ from the last row. As the determinant is linear in each row separately, we get

$$\begin{vmatrix} 0 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 \\ -1 & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ -(n-1) & -(n-1) & -(n-1) & \cdots & -(n-1) \end{vmatrix} = (1-n) \begin{vmatrix} 0 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 \\ -1 & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix}.$$

Next, add the last row back to each of the other rows (which again keeps the determinant the same). So now we want to find

$$(1-n) \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix}.$$

This matrix is lower triangular. So its determinant is the product of the entries on its diagonal. Thus the above quantity is $(1-n)$.

Alternately, one can find the determinant of the matrix by finding all its eigenvalues. As $A_n = I - \texttt{ones}(n)$, we know that $N(A_n - I) = N(-\texttt{ones}(n))$. The latter nullspace has dimension $n-1$. Thus 1 is an eigenvalue of multiplicity $n-1$, and the corresponding eigenvectors are all the nonzero vectors whose entries add up to 0.

In addition, all of the rows of A_n add up to $1-n$. So $1-n$ is an eigenvalue with eigenvector $(1, 1, \ldots, 1)$. Thus we have found all of the eigenvectors and eigenvalues. The determinant is the product of the eigenvalues, so it is $1^{n-1} \cdot (1-n)$ or $1-n$.

RUBRIC: 2 points for following the hint, 2 points for pulling out the factor of $(1-n)$ correctly, 2 points for adding the last row to the other rows, 2 points for the correct answer.

(b) For any invertible matrix A, the $(1,1)$ entry of A^{-1} is the ratio of ________________ .

So the $(1,1)$ entry of A_4^{-1} is ________________ .

Solution Cramer's rule gives $A^{-1} = \frac{1}{|A|}C^{\mathrm{T}}$ where C is the cofactor matrix, whose (i,j) entry is $(-1)^{i+j}|M_{ij}|$ where M_{ij} is the submatrix obtained by deleting row i and column j of the (arbitrary) invertible matrix A. Thus the entry with $i = j = 1$ is $|M_{11}|/|A|$.

In the case where $A = A_n$, the submatrix M_{11} is A_{n-1}; so the desired formula is $|A_{n-1}|/|A_n|$. Now, $|A_n| = 1 - n$ by part (a). So $|A_4| = -3$ and $|A_3| = -2$. Thus the $(1,1)$ entry of A_4^{-1} is $2/3$.

RUBRIC: 5 points for the correct ratio, 5 points for the correct application to the current problem. If the wrong ratio was given, then no credit was given for applying it.

(c) Find *two orthogonal eigenvectors* with $A_3 x = x$. (So $\lambda = 1$ is a double eigenvalue.)

Solution In solution 2 of part (a) above, we saw that the eigenvectors are all the nonzero vectors whose entries add up to 0. Two obvious such vectors are $(1, -1, 0)$ and $(0, 1, -1)$, but there are many more linearly independent pairs.

However, $(1, -1, 0)$ and $(0, 1, -1)$ are not orthogonal! So we must find another pair. We can use the Gram–Schmidt process to get orthogonal vectors, or we can just try to guess two orthogonal vectors whose entries add up to 1. For example, $(1, -1, 0)$ and $(1, 1, -2)$ work. (Note that the vectors are not required to have unit length.)

RUBRIC: up to 5 points for a correct method, 2 points for finding linearly independent vectors, 3 points for orthogonality.

(d) What is the third eigenvalue of A_3 and a corresponding eigenvector?

Solution In solution 2 of part (a) above, we saw that the third eigenvalue is -2 and a corresponding eigenvector is $(1, 1, 1)$.

Another way to proceed is to notice that the trace of A_3 is 0. However, the trace is the sum of the eigenvalues, and two of them are 1. So the third must be -2. Alternatively, in part (a), we saw that $|A_3| = -2$. However, the determinant is the product of the eigenvalues, and two of them are 1. So the third must be -2.

A third way to proceed is to find the characteristic polynomial of A_3, which is $\lambda^3 - 3\lambda + 2$. Since 1 is a double root, we can find the third root by dividing twice by $\lambda - 1$.

RUBRIC: 5 points for the eigenvalue, 5 points for a corresponding eigenvector.

18.06 Quiz 2 Professor Strang April 6, 2011

Your PRINTED name is ______________________________ 1.

Your Recitation Instructor (and time) is ______________________ 2.

Instructors: (Pires)(Hezari)(Sheridan)(Yoo) 3.

Please show enough work so we can see your method and give due credit.

1. **(8 pts. each)** Suppose a_1 and a_2 are orthogonal unit vectors in R^5.

   (a) What are the requirements on a matrix P to be a projection matrix? Verify that $P = a_1a_1^T + a_2a_2^T$ satisfies those requirements.

   (b) If a_3 is in R^5, what combination of a_1 and a_2 is closest to a_3?

   (c) Find a combination c of a_1, a_2, a_3 that is perpendicular to a_1 and a_2. *If possible, choose* $c \neq 0$. Describe all cases when $c = 0$ is the only possibility.

   (d) Show that a_1 and a_2 and c are eigenvectors of P (*if* $c \neq 0$) and find their eigenvalues.

1:

a: P is a projection (orthogonal projection!) if

$$\begin{cases} P^2 = P, \\ P^T = P. \end{cases}$$

We know check this for $P = a_1 a_1^T + a_2 a_2^T$.

- $P^2 = (a_1 a_1^T + a_2 a_2^T)(a_1 a_1^T + a_2 a_2^T) = a_1 \underbrace{a_1^T a_1} a_1^T + a_1 \underbrace{a_1^T a_2} a_2^T + a_2 \underbrace{a_2^T a_1} a_1^T + a_2 \underbrace{a_2^T a_2} a_2^T.$

Since $a_1^T a_2 = 0$, $a_2^T a_1 = 0$, and $a_1^T a_1 = a_2^T a_2 = 1$, we get

$P^2 = a_1 a_1^T + a_2 a_2^T = P.$

- $P^T = (a_1 a_1^T + a_2 a_2^T)^T = (a_1^T)^T a_1^T + (a_2^T)^T a_2^T = a_1 a_1^T + a_2 a_2^T = P.$

b: The closest combination is $Pa_3 = (a_1^T a_3) a_1 + (a_2^T a_3) a_2$.

c: c = error term = $a_3 - Pa_3 = a_3 - (a_1^T a_3) a_1 - (a_2^T a_3) a_2$.

$c = 0$ only if c is in the plane generated by a_1 and a_2.

d: Since P is the projection on the column space of

$A = [a_1 | a_2]$, we have:

$Pa_1 = a_1 \Longrightarrow \lambda_1 = 1$

$Pa_2 = a_2 \Longrightarrow \lambda_2 = 1$

$Pc = 0 \Longrightarrow \lambda_3 = 0.$

2. **(7 pts. each)**

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 11 & 12 \end{bmatrix}.$$

(a) Find all nonzero terms in the big formula $\det A = \sum \pm a_{1\alpha}\, a_{2\beta}\, a_{3\gamma}\, a_{4\delta}$ and combine them to compute $\det A$.

(b) Find all the pivots of A.

(c) Find the cofactors C_{11}, C_{12}, C_{13}, C_{14} of row 1 of A.

(d) Find column 1 of A^{-1}.

2:

a: $\det A = 1\,(6\cdot(9\cdot 12 - 10\cdot 11)) - 2\,(5\,(9\cdot 12 - 10\cdot 11)) = 8$

b: By row reduction:

A reduces to $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -16 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 0 & -\frac{12}{9} \end{bmatrix}$.

Hence the pivots are: $1, -4, 9, -\frac{12}{9}$.

c:

$$C_{11} = \det \begin{bmatrix} 6 & 7 & 8 \\ 0 & 9 & 10 \\ 0 & 11 & 12 \end{bmatrix} = -12$$

$$C_{12} = -\det \begin{bmatrix} 5 & 7 & 8 \\ 0 & 9 & 10 \\ 0 & 11 & 12 \end{bmatrix} = 10$$

$$C_{13} = \det \begin{bmatrix} 5 & 6 & 8 \\ 0 & 0 & 10 \\ 0 & 0 & 12 \end{bmatrix} = 0$$

$$C_{14} = -\det \begin{bmatrix} 5 & 6 & 7 \\ 0 & 0 & 9 \\ 0 & 0 & 11 \end{bmatrix} = 0.$$

d: From c we get

$$(A^{-1})_{11} = \frac{-12}{8}$$

$$(A^{-1})_{21} = \frac{10}{8}$$

$$(A^{-1})_{31} = 0$$

$$(A^{-1})_{41} = 0.$$

3. **(8 pts. each)** Suppose A is a 2 by 2 matrix and $Ax = x$ and $Ay = -y$ ($x \neq 0$ and $y \neq 0$).

(a) (Reverse engineering) What is the polynomial $p(\lambda) = \det(A - \lambda I)$?

(b) If you know that the first column of A is $(2, 1)$, find the second column:

$$A = \begin{bmatrix} 2 & ? \\ 1 & ? \end{bmatrix}.$$

(c) For that matrix in part (b), find an invertible S and a diagonal matrix Λ so that $A = S\Lambda S^{-1}$.

(d) Compute A^{101}. (If you don't solve parts (b) -(c), use the description of A at the start. In all questions **show enough work** so we can see your method and give due credit.)

(e) If $Ax = x$ and $Ay = -y$ (with $x \neq 0$ and $y \neq 0$) prove that x and y are *independent.*

Start of a proof: Suppose $z = cx + dy = 0$. Then $Az =$ (follow from here.)

3:

a: $p(\lambda) = (1-\lambda)(-1-\lambda) = \lambda^2 - 1$

b: we know that $\operatorname{Tr} A = 1 + (-1) = 0$.
on the other hand if we put $A = \begin{bmatrix} 2 & a_{12} \\ 1 & a_{22} \end{bmatrix}$
then $\operatorname{Tr} A = 2 + a_{22}$. Hence $a_{22} = -2$.
To find a_{12} we note that on one hand
$\det A = 1 \cdot (-1) = -1$ and on the other hand
$\det A = 2a_{22} - a_{12} = -4 - a_{12}$. Therefore $a_{12} = -3$.
So $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$.

(c): It is easy to see that X an eigenvector of $\lambda_1 = 1$
is $X = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and for Y an eigevector of $\lambda_2 = -1$
we have $Y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So we can choose $S = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$.

d: From c we have $A^{101} = S \Lambda^{101} S^{-1} = S \Lambda S^{-1} = A$.
Note that $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and therefore $\Lambda^{101} = \Lambda$.

e: on one hand since $Z = 0$ we have $AZ = 0$.
on the other hand $AZ = A(cX + dY) = cAX + dAY$
$= cX - dY$.

Therefore

$$\begin{cases} Z = cX + dY = 0 \\ AZ = cX - dY = 0 \end{cases} \overset{\text{since } X \neq 0,\ Y \neq 0}{\Longrightarrow} c = d = 0 \Rightarrow X \text{ and } Y \text{ are linearly independent.}$$

18.06 Professor Edelman Quiz 2 November 7, 2012

Your PRINTED name is: ____________________

Grading

1

2

3

4

____

Please circle your recitation:

1	T 9	2-132	Andrey Grinshpun	2-349	3-7578	`agrinshp`
2	T 10	2-132	Rosalie Belanger-Rioux	2-331	3-5029	`robr`
3	T 10	2-146	Andrey Grinshpun	2-349	3-7578	`agrinshp`
4	T 11	2-132	Rosalie Belanger-Rioux	2-331	3-5029	`robr`
5	T 12	2-132	Geoffroy Horel	2-490	3-4094	`ghorel`
6	T 1	2-132	Tiankai Liu	2-491	3-4091	`tiankai`
7	T 2	2-132	Tiankai Liu	2-491	3-4091	`tiankai`

1 (27 pts.)

P is any $n \times n$ Projection Matrix. Compute the ranks of A,B, and C below. Your method must be visibly correct for every such P, not just one example.

a) (8 pts.) $A = (I - P)P$.

Since P is a projection matrix, $P^2 = P$, so $(I-P)P = P - P^2 = P - P = 0$ and has rank 0.

b) (10 pts.) $B = (I - P) - P$. (Hint: Squaring B might be helpful.)

$B^2 = (I - 2P)^2 = I^2 - 4P + 4P^2 = I$. The rank of I is n. The rank of B^2 is at most the rank of B and the rank of B is at most n, so B must have rank n.

c) (9 pts.) $C = (I - P)^{2012} + P^{2012}$.

Note $I - P$ is a projection matrix, so $(I - P)^{2012} = I - P$ and $P^{2012} = P$, so the above simplifies to I, which has rank n.

2 (22 pts.)

Consider a 4×4 matrix

$$A = \begin{pmatrix} 0 & x & y & z \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{pmatrix}.$$

a) (17 pts.) Compute $|A|$, the determinant of A, in simplest form.

The answer is $\det A = -x^2 - y^2 - z^2$. But before we discuss how to get this answer, I'd like to call your attention to that fact that the expression $-x^2 - y^2 - z^2$ is symmetric in the three variables x, y, z. That is to say, if we swap the roles of any two of these variables, the expression as a whole is unchanged. Why might we have predicted that $\det A$ has this property? Well, if we swap rows 2 and 3 of A, and then swap *columns* 2 and 3 of the result, we end up with

$$A' = \begin{pmatrix} 0 & y & x & z \\ y & 1 & 0 & 0 \\ x & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{pmatrix},$$

which is the same as A, but with the roles of x and y swapped. In performing one row swap and one column swap, we have multiplied the determinant by $(-1)^2 = 1$, so A' has the same determinant as A. From this we conclude that $\det A$, whatever it is, must be an expression that's symmetric in x and y. Similar considerations show that it's symmetric in all three variables x, y, z.

Anyway, let's actually compute $\det A$. Here were some of the most common ways from the students' tests:

- By cofactor expansion (p. 260) in the first row (or the first column), using the big formula (p. 257) or any other method for each 3×3 minor:

$$\det A = 0 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - x \begin{vmatrix} x & 0 & 0 \\ y & 1 & 0 \\ z & 0 & 1 \end{vmatrix} + y \begin{vmatrix} x & 1 & 0 \\ y & 0 & 0 \\ z & 0 & 1 \end{vmatrix} - z \begin{vmatrix} x & 1 & 0 \\ y & 0 & 1 \\ z & 0 & 0 \end{vmatrix}$$
$$= 0 - x(x) + y(-y) - z(z)$$
$$= -x^2 - y^2 - z^2.$$

Note the alternating + and − signs in the cofactors:

$$C_{11} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \; C_{12} = - \begin{vmatrix} x & 0 & 0 \\ y & 1 & 0 \\ z & 0 & 1 \end{vmatrix}, \; C_{13} = \begin{vmatrix} x & 1 & 0 \\ y & 0 & 0 \\ z & 0 & 1 \end{vmatrix}, \; C_{14} = - \begin{vmatrix} x & 1 & 0 \\ y & 0 & 1 \\ z & 0 & 0 \end{vmatrix}.$$

In general, the formula is $C_{ij} = (-1)^{i+j} \det M_{ij}$.

- By cofactor expansion in the *second* row (or the second column), using the big formula or any other method for each 3×3 minor:

$$\det A = -x \begin{vmatrix} x & y & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & y & z \\ y & 1 & 0 \\ z & 0 & 1 \end{vmatrix} - 0 \begin{vmatrix} 0 & x & z \\ y & 0 & 0 \\ z & 0 & 1 \end{vmatrix} + 0 \begin{vmatrix} 0 & x & y \\ y & 0 & 1 \\ z & 0 & 0 \end{vmatrix}$$
$$= -x(1) + 1(-y^2 - z^2)$$
$$= -x^2 - y^2 - z^2.$$

The cofactor C_{22}, for example, can be calculated using the big formula for 3×3 matrices:

$$\begin{vmatrix} 0 & y & z \\ y & 1 & 0 \\ z & 0 & 1 \end{vmatrix} = 0 \cdot 1 \cdot 1 + y \cdot 0 \cdot z + z \cdot y \cdot 0 - 0 \cdot 0 \cdot 0 - y \cdot y \cdot 1 - z \cdot 1 \cdot z = -y^2 - z^2.$$

- By the big formula (pp. 258–259) for 4×4 matrices. The big formula has 24 terms (one for each 4×4 permutation matrix), but only three of them are nonzero:

$$\det A = x^2 \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} + y^2 \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} + z^2 \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}.$$

These three permutation matrices all have determinant -1, because they are one row exchange away from the identity matrix, so

$$\det A = -x^2 - y^2 - z^2.$$

- By performing row operations to reach an upper triangular matrix. First exchange row 1 with another row to put a pivot in the top-left corner; to make the future computations simpler, let's swap row 1 with row 4:

$$|A| = -\begin{vmatrix} z & 0 & 0 & 1 \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ 0 & x & y & z \end{vmatrix};$$

here we have a $-$ sign because row exchanges negate the determinant (rule 2, p. 246). Now subtract x/z times row 1 from row 2, and y/z times row 1 from row 3:

$$|A| = -\begin{vmatrix} z & 0 & 0 & 1 \\ 0 & 1 & 0 & -x/z \\ 0 & 0 & 1 & -y/z \\ 0 & x & y & z \end{vmatrix};$$

remember that such operations do not affect the determinant (rule 5, p. 247). Finally subtract x times row 2 and y times row 3 from row 4:

$$|A| = -\begin{vmatrix} z & 0 & 0 & 1 \\ 0 & 1 & 0 & -x/z \\ 0 & 0 & 1 & -y/z \\ 0 & 0 & 0 & z + x^2/z + y^2/z \end{vmatrix}.$$

Now we can mutiply the diagonal entries (rule 7, p. 247) to find that

$$|A| = -z \cdot 1 \cdot 1 \cdot (z + x^2/z + y^2/z) = -x^2 - y^2 - z^2.$$

- By performing row operations to reach a *lower* triangular matrix. From row 1 of A, we subtract x times row 2, y times row 3, and z times row 4. These operations do not change the determinant, so

$$|A| = \begin{vmatrix} -x^2 - y^2 - z^2 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{vmatrix} = -x^2 - y^2 - z^2.$$

In other words, we may factorize A as

$$A = \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -x^2 - y^2 - z^2 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{pmatrix},$$

so the product rule (rule 9, p. 248) says

$$|A| = \begin{vmatrix} 1 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} -x^2 - y^2 - z^2 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{vmatrix} = -x^2 - y^2 - z^2.$$

b) (5 pts.) For what values of x, y, z is A singular?

A square matrix is singular if and only if its determinant equals zero. So we are asked to find all triples (x, y, z) such that

$$\det A = -x^2 - y^2 - z^2 = 0,$$

or in other words

$$x^2 + y^2 + z^2 = 0.$$

So far, we have been talking about real numbers x, y, z in this course, so the left-hand side is just the square of the distance from (x, y, z) to the origin in $\mathbb{R}^3$. Since only the origin is at a distance 0 from the origin, the matrix A is singular if and only if $x = y = z = 0$.

3 (22 pts.)

The 3×3 matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ has QR decomposition

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = Q \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}.$$

a) (7 pts.) What is r_{11} in terms of the variables $a, b, c, d, e, f, g, h, i$? (but not any of the elements of Q.)

You should probably remember that r_{11} is the norm of the first column of the matrix on the left, which we will call A. But let's rederive it. So, when we do a QR decomposition, we always start with the first column of our matrix, here the vector $(a\ d\ g)^T$, and we normalize it to obtain the first column of Q: $q_1 = (a\ d\ g)^T / \left\|(a\ d\ g)^T\right\|$. Now, if we look at the first column of $A = QR$, we have $(a\ d\ g)^T = r_{11} \cdot q_1 + 0 \cdot q_2 + 0 \cdot q_3 = r_{11} \cdot q_1 = r_{11} \cdot (a\ d\ g)^T / \left\|(a\ d\ g)^T\right\|$, which implies that $r_{11} = \left\|(a\ d\ g)^T\right\| = \sqrt{a^2 + d^2 + g^2}$.

b) (15 pts.) Solve for x in the equation,

$$Q^T x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

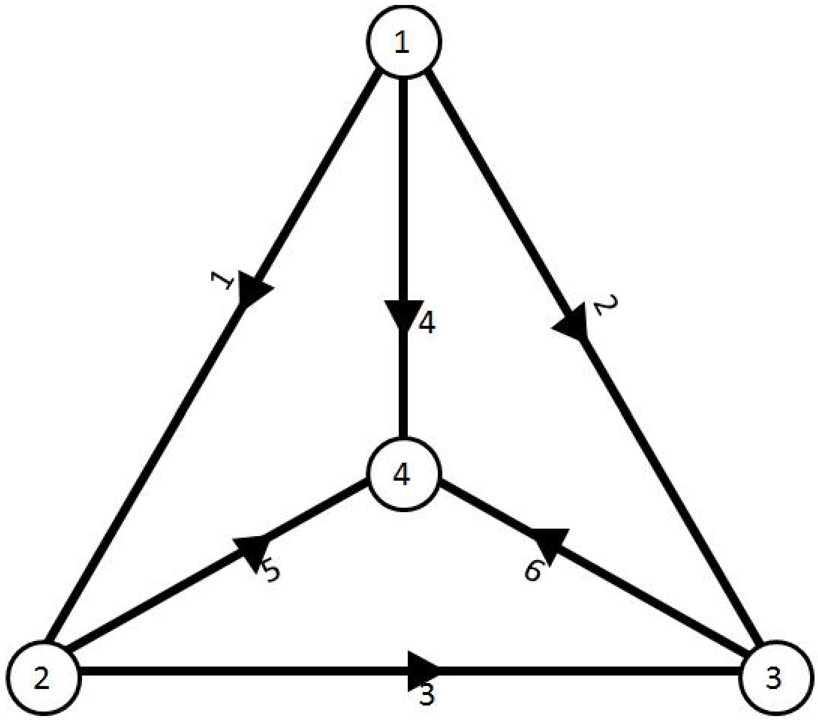
expressing your answer possibly in terms of r_{11}, r_{22}, r_{33} and the variables $a, b, c, d, e, f, g, h, i$, (but not any of the elements of Q.)

Look at the product

$$Q^T x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

row by row: we take the first row of Q^T, that is, the first column of Q, and take its dot product with x to obtain 1. We also take the second and third row of Q^T, that is, the second and third column of Q, and take their dot products with x to obtain 0. This means that x is perpendicular to the last 2 columns of Q. But because Q has orthonormal columns and we are in R^3, this can only mean that x is a multiple of the first column, say $x = zq_1$ for some real number z. But remember we said that $q_1^T x = 1$, which means $q_1^T zq_1 = zq_1^T q_1 = 1$, but we know $q_1^T q_1 = 1$ because the columns of Q have norm 1. So clearly $z = 1$ and x is q_1, which we found in the previous question. So $x = q_1 = (a\ d\ g)^T / \left\|(a\ d\ g)^T\right\| = (a\ d\ g)^T / r_{11}$.

4 (29 pts.)



a) (15 pts.) Use loops or otherwise to find a basis for the left nullspace of the incidence matrix A for the graph above. We will start you off, one basis vector is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

The incidence matrix is 6 by 4. Since the graph is connected, the nullspace has dimension 1, it is the line generated by $(1, 1, 1, 1)^T$, therefore, the matrix has rank 3. It follows that the left nullspace has dimension $6 - 3 = 3$.

We use the result of page 425 of the book :

A basis of the left nullspace of the incidence matrix is given by a set of independant loops. In this case, we need to find 3 independant loops in the graph. It is easy to check that the 3 small loops are independant :

A basis of the left nullspace is :

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

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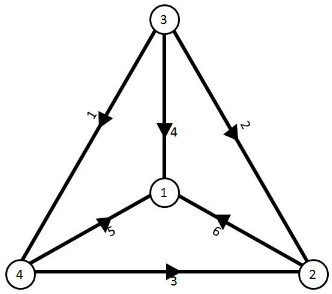

There are 24 ways to relabel the four nodes in the graph in part(a). Edge labels remain unchanged. One of the 24 ways is pictured above. This produces 24 incidence matrices A.

b) (7 pts.) Is the row space of A independent of the labelling? Argue convincingly either way.

Yes it is independent. Indeed, the incidence matrix of a connected graph with 4 nodes has the line generated by $(1, 1, 1, 1)^T$ as its nullspace whatever the graph is. In particular, we see that the nullspace is independant of the labelling. Since the row space is the orthogonal of the nullspace it is also independant of the labelling.

c) (7 pts.) Is the column space of A independent of the labelling? Argue convincingly either way.

Yes, it is independent. Relabelling the nodes has the effect of permuting the columns of the incidence matrix. The columns space is the space of linear combinations of the columns of the matrix, therefore, it is independent of the way the columns are ordered in the matrix.

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18.06 Professor Strang Quiz 2 April 11th, 2012

Grading

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Your PRINTED name is: ____________________

Please circle your recitation:

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r02	T 11	36-153	Rune Haugseng	`haugseng`
r03	T 12	4-159	Jennifer Park	`jmypark`
r04	T 12	36-153	Rune Haugseng	`haugseng`
r05	T 1	4-153	Dimiter Ostrev	`ostrev`
r06	T 1	4-159	Uhi Rinn Suh	`ursuh`
r07	T 1	66-144	Ailsa Keating	`ailsa`
r08	T 2	66-144	Niels Martin Moller	`moller`
r09	T 2	4-153	Dimiter Ostrev	`ostrev`
r10	ESG		Gabrielle Stoy	`gstoy`

1 (40 pts.)

(a) Find the projection p of the vector b onto the plane of a_1 and a_2, when

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad a_1 = \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -1 \\ 7 \\ 1 \\ -7 \end{bmatrix}.$$

Solution. Observe that $a_1^T a_2 = 0$. Thus

$$p = \frac{a_1^T b}{a_1^T a_1} a_1 + \frac{a_2^T b}{a_2^T a_2} a_2 = \frac{8}{100} a_1 - \frac{8}{100} a_2 = \begin{bmatrix} 4/25 \\ 0 \\ 0 \\ 28/25 \end{bmatrix}.$$

□

(b) What projection matrix P will produce the projection $p = Pb$ for every vector b in $\mathbb{R}^4$?

Solution. Let A be the 4×2 matrix with columns a_1, a_2. P is given by $P = A(A^T A)^{-1} A^T$. Notice that

$$A^T A = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}.$$

(a_1 and a_2 are orthogonal and of same length.)

Thus

$$P = \frac{1}{100} A A^T = \frac{1}{100} \begin{bmatrix} 2 & 0 & 0 & 14 \\ 0 & 98 & 14 & 0 \\ 0 & 14 & 2 & 0 \\ 14 & 0 & 0 & 98 \end{bmatrix}.$$

□

(c) What is the determinant of $I - P$? Explain your answer.

Solution. $I - P$ is the matrix of the projection to the orthgonal complement of $C(A)$, i.e. $N(A^T)$. In particular, $I - P$ has rank the dimension of $N(A^T)$, which is 3. Thus $I - P$ is singular, and $\det(I - P) = 0$. □

(d) <u>What are all nonzero eigenvectors of P with eigenvalue</u> $\lambda = 1$?

How is the number of independent eigenvectors with $\lambda = 0$ of a square matrix A connected to the rank of A?

(You could answer (c) and (d) even if you don't answer (b).)

Solution. The non-zero eigenvectors with eigenvalue $\lambda = 1$ are all the non-zero linear combinations of a_1 and a_2, i.e. all the non-zero vectors in the plane spanned by a_1 and a_2.

Suppose A is a $n \times n$ matrix, with rank r.

$$\begin{aligned}\#\text{ independent zero-eigenvectors of } A &= \#\text{ independent vectors in } N(A)\\ &= \text{dimension of } N(A) = n - r\end{aligned}$$

□

2 (30 pts.)

(a) Suppose the matrix A factors into $A = PLU$ with a permutation matrix P, and 1's on the diagonal of L (lower triangular) and pivots $d_1, \ldots, d_n$ on the diagonal of U (upper triangular).

What is the determinant of A? EXPLAIN WHAT RULES YOU ARE USING.

Solution. Use

$$\det(A) = \det(P) \cdot \det(L) \cdot \det(U)$$

where we make two uses of the rule $\det(MN) = \det(M)\det(N)$, for any two $n \times n$ matrices M and N. We will compute each of the determinants on the right-hand side.

The determinant of a triangular matrix is the product of its diagonal entries; this is true whether the matrix is upper or lower triangular. Thus

$$\det(L) = 1 \qquad \text{and} \qquad \det(U) = d_1 \cdot d_2 \cdot \ldots \cdot d_n.$$

The determinant changes sign whenever two rows are swapped. Thus

$$\det(P) = \begin{cases} +1 & \text{if } P \text{ is even (even \# of row exchanges)} \\ -1 & \text{if } P \text{ is odd (odd \# of row exchanges)} \end{cases}$$

and so

$$\det(A) = \pm d_1 \cdot d_2 \cdot \ldots \cdot d_n$$

where the sign depends on the parity of P.

□

(b) Suppose the first row of a new matrix A consists of the numbers $1, 2, 3, 4$. Suppose the cofactors C_{ij} of that first row are the numbers $2, 2, 2, 2$.

(Cofactors already include the $\pm$ signs.)

Which entries of A^{-1} does this tell you and what are those entries?

Solution. Using the cofactor expansion in the first row gives

$$\begin{aligned}\det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14} \\ &= 1 \times 2 + 2 \times 2 + 3 \times 2 + 4 \times 2 \\ &= 20\end{aligned}$$

As $A^{-1} = C^T / \det(A)$, where C is the cofactor matrix, this data gives us the entries of the first column of A^{-1}; they are all $2/20 = 1/10$. □

(c) What is the determinant of the matrix $M(x)$? For which values of x is the determinant equal to zero?

$$M(x) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & x \\ 1 & 1 & 4 & x^2 \\ 1 & -1 & 8 & x^3 \end{bmatrix}$$

Solution. **Solution no. 1.**

From, for instance, the 'Big Formula', we know that $\det(M)$ is a cubic polynomial in x. Say

$$\det(M) = ax^3 + bx^2 + cx + d.$$

We can calculate d by setting $x = 0$. Using the cofactor expansion in the last column, we get that

$$d = -\begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & 4 \\ 1 & -1 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 6 \end{vmatrix} = -12.$$

We will determine the other coefficients of $\det(M)$ by finding three roots for it. x is a root of $\det(M)$ if and only if $M(x)$ is a singular matrix. Now, notice that

$$\begin{aligned} (1,1,1) &= (x, x^2, x^3) \quad \text{for } x = 1 \\ (1,-1,1) &= (x, x^2, x^3) \quad \text{for } x = -1 \\ (2,4,8) &= (x, x^2, x^3) \quad \text{for } x = 2. \end{aligned}$$

Thus $M(x)$ is singular for $x = 1, -1$ and 2; moreover, this implies that

$$\det(M) = a(x-1)(x+1)(x-2).$$

As $d = 2a$, we must have $a = -6$. Thus

$$\det(M) = -6(x-1)(x+1)(x-2) = -6x^3 + 12x^2 + 6x - 12.$$

The values of x for which $M(x)$ is singular are $1, -1$ and 2.

Solution no. 2.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & x \\ 1 & 1 & 4 & x^2 \\ 1 & -1 & 8 & x^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & x-1 \\ 0 & 0 & 3 & x^2-1 \\ 0 & -2 & 7 & x^3-1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & x-1 \\ 0 & 0 & 3 & x^2-1 \\ 0 & 0 & 6 & x^3-x \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 1 & x-1 \\ 0 & 3 & x^2-1 \\ 0 & 6 & x^3-x \end{vmatrix} = -2\begin{vmatrix} 3 & x^2-1 \\ 6 & x^3-x \end{vmatrix} = -6x^3 + 12x^2 + 6x - 12$$

In the first step, subtract the first row from the second, third and fourth rows. In the second step, subtract the second row from the fourth. For the third and fourth steps, use the cofactor expansion in the first column.

We factorize $\det(M)$ by guessing roots, trying small integers; we find that $1, -1$ and 2 are all roots, which gives

$$\det(M) = -6(x-1)(x+1)(x-2).$$

The values of x for which $M(x)$ is singular are $1, -1$ and 2. □

3 (30 pts.)

(a) Starting from independent vectors a_1 and a_2, use Gram-Schmidt to find formulas for two orthonormal vectors q_1 and q_2 (combinations of a_1 and a_2):

Solution.

$$q_1 = \frac{a_1}{||a_1||}$$

$$q_2 = \frac{a_2 - (a_2^T q_1)q_1}{||a_2 - (a_2^T q_1)q_1||} = \left(a_2 - \frac{(a_2^T a_1)}{a_1^T a_1} a_1\right) / \, ||a_2 - \frac{(a_2^T a_1)}{a_1^T a_1} a_1||$$

□

(b) The connection between the matrices $A = [a_1 \; a_2]$ and $Q = [q_1 \; q_2]$ is often written $A = QR$. From your answer to Part (a), what are the entries in this matrix R?

Solution. Re-arranging the expressions above gives

$$a_1 = q_1 ||a_1||$$

$$a_2 = (a_2^T q_1)q_1 + ||a_2 - (a_2^T q_1)q_1|| q_2$$

and thus

$$R = \begin{bmatrix} a_1^T q_1 & a_2^T q_1 \\ a_1^T q_2 & a_2^T q_2 \end{bmatrix} = \begin{bmatrix} ||a_1|| & a_2^T q_1 \\ 0 & ||a_2 - (a_2^T q_1)q_1|| \end{bmatrix}$$

□

(c) The least squares solution $\widehat{x}$ to the equation $Ax = b$ comes from solving what equation? If $A = QR$ as above, show that $R\widehat{x} = Q^T b$.

Solution. $\widehat{x}$ comes from solving $A^T A\widehat{x} = A^T b$.

Suppose we have $A = QR$. Notice that:

- $Q^T Q = I$, so $A^T A = (QR)^T QR = R^T Q^T QR = R^T R$.
- As a_1 and a_2 are independent, R is invertible. Thus R^T is also invertible.

Thus we have

$$\begin{aligned} A^T A\widehat{x} &= A^T b \\ \Leftrightarrow R^T R\widehat{x} &= R^T Q^T b \\ \Leftrightarrow R\widehat{x} &= Q^T b. \end{aligned}$$

□

Grading

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Your PRINTED name is:________________________

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Please circle your recitation:

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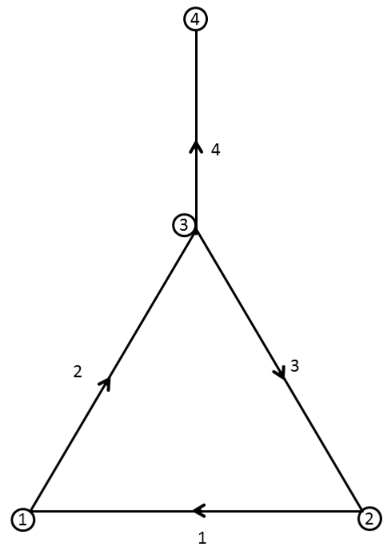
2	T 10	2-132	Niels Moeller	2-588	3-4110	moller
3	T 10	2-146	Kestutis Cesnavicius	2-089	2-1195	kestutis
4	T 11	2-132	Niels Moeller	2-588	3-4110	moller
5	T 12	2-132	Yan Zhang	2-487	3-4083	yanzhang
6	T 1	2-132	Taedong Yun	2-342	3-7578	tedyun

1 (30 pts.)

Consider the directed graph with four vertices and four edges pictured below:



1. (7 pts) The 4×4 incidence matrix (following class conventions) of this directed graph is:

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

2. (7 pts) Find the determinant of the incidence matrix. (The easy way or the hard way)

det(A)=0. This can be done by direct expansion or appeal conceptually to show matrix is not invertible. Can use that the sum of all columns is 0, that we know from book/class that rank is $4 - 1 = 3$, or that the nullspace includes $(1, 1, 1, 1)^T$ (note this is equivalent to the columns summing to 0 condition) and thus is nonempty.

3. (8 pts) Find a basis for the column space of the incidence matrix (Note this can be done with or without the answer in part 1.)

   We need $4-1=3$ basis vectors. Any three columns of A form a basis, as would any three independent vectors whose first three components sum to 0.

4. (8 pts) Consider whether or not it is possible to have an incidence matrix for a graph with n nodes and n edges that is invertible. If it is possible, draw the directed graph, if not possible, argue briefly why not.

   Impossible, as the ones vector is in the nullspace of every incidence matrix for every graph. As in problem 2, can also argue from book/class knowledge, or explicitly show that the column sums are 0. You can also show that the rows corresponding to a loop must sum to 0 and we must have a loop.

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2 (20 pts.)

1. (10 pts) Project the function $\sin(x) + \cos(x)$ defined on the interval $[0, 2\pi]$ onto the three dimensional space of functions spanned by $\cos x$, $\cos 2x$, and $\cos 3x$. Express the (hint: very simple) answer in simplest form. Briefly explain your answer.

   The projection is $\cos x$. We know $\sin x$ is orthogonal to the space and projects to 0, while $\cos x$ is already in the space.

2. (10 pts) Write down all $n \times n$ permutation matrices that are also projection matrices. (Explain briefly.)

   Since $P^2 = P$, multiplying both sides by P^{-1}, we get $P = I$ is the only projection, permutation. Note P^{-1} exists and you need it to exist; it is P^T, or you can note that it has nonzero determinant and is thus invertible.

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3 (15 pts.)

1. (10 pts) What are all possible values for the determinant of a projection matrix? (Please explain briefly.)

   Since $P^2 = P$, $\det(P)^2 = \det(P)$so that only 0 or 1 are possible.

2. (5 pts) What are all possible values for the determinant of a permutation matrix? (Please explain briefly.)

   Starting with I, a permutation matrix is obtained through row exchanges, therefore we can get only ± 1.

4 (35 pts.)

1. (20 pts) The matrix A is 2000×2000 and $A^TA = I$. Let v be the vector $[1, 2, 3, \dots, 2000]^T$. Let v_1 be the projection of v onto the space spanned by the first 1000 columns of A. Let v_2 be the projection of v onto the space spanned by the remaining 1000 columns of A. What is $v_1 + v_2$ in simplest form? Why? Give an example of a 2000×2000 A, where $A^TA \neq I$, and where $v_1 + v_2$ gives a different answer.

   $v_1 + v_2 = v$ since projections onto orthogonal complements add to the identity. Here's something more explicit: let A take block form $\begin{bmatrix} A_1 & A_2 \end{bmatrix}$. Then v_1 and v_2 are projections onto the column spaces of A_1 and A_2 respectively. Note that since $A^TA = I$, we have:

   $$I = AA^T = A_1A_1^T + A_2A_2^T.$$

   Adding the two projections (recall in the orthogonal case this is just $(A_1A_1^Tv + A_2A_2^Tv)$ gives $Iv = v$ by the above equation.
   An easy example where we get a different answer is if A is the zero matrix, where we have $v_1 + v_2 = 0$ always for every v.

2. (15 pts) In a matrix A, (which may not be invertible) the cofactors from the first row are $C_{11}, C_{12}, \dots, C_{1n.}$. Prove that the vector $C = (C_{11}, C_{12}, \dots, C_{1n.})$ is orthogonal to every row of A from row 2 to row n. Hint: the dot product of C with row i $(i = 2, \dots, n)$ is the determinant of what matrix?

   Take the matrix A and replace row 1 with row i. This matrix has two equal rows hence 0 determinant. The determinant expansion by cofactors is the desired dot product.

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18.06 Professor Edelman Quiz 2 November 6, 2013

Grading

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Your PRINTED name is: ______________________

Please circle your recitation:

1	T 9	Dan Harris	E17-401G	3-7775	`dmh`
2	T 10	Dan Harris	E17-401G	3-7775	`dmh`
3	T 10	Tanya Khovanova	E18-420	4-1459	`tanya`
4	T 11	Tanya Khovanova	E18-420	4-1459	`tanya`
5	T 12	Saul Glasman	E18-301H	3-4091	`sglasman`
6	T 1	Alex Dubbs	32-G580	3-6770	`dubbs`
7	T 2	Alex Dubbs	32-G580	3-6770	`dubbs`

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1 (25 pts.)

Compute the determinant of

a) (10 pts.) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1806 & 1806 & 0 \\ 2013 & 2014 & 2015 \end{bmatrix}$

b) (15 pts.)

The $n \times n$ matrix A_n has ones in every element off the diagonal, and also $a_{11} = 1$ as well. The rest of the diagonal elements are 0: $a_{22} = a_{33} = \ldots = a_{nn} = 0$. For example

$$A_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Write the determinant of A_n in terms of n in simplest form. Argue briefly but convincingly your answer is right.

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2 (30 pts.)

Let $Q = [q_1\ q_2\ q_3]$ be an $m \times 3$ real matrix with $m > 3$ and $Q^TQ = I_3$, the 3×3 identity.

Let $P = QQ^T$.

a) (7 pts.) What are all possible values of $\det(P)$?

b) (7 pts.) What are all the eigenvalues of the $m \times m$ matrix P including multiplicities?

c) (8 pts.) Find one eigenvalue, eigenvector pair of the non-symmetric $m \times m$ matrix $q_1 q_2^T$.

d) (8 pts.) What are the four fundamental subspaces of $M = I - P$ in terms of the column space of P?

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3 (20 pts.)

Let A be a 4×4 general matrix and x a scalar variable. Circle your answers and provide a very brief explanation.

a) (5 pts.) What kind of polynomial in x best describes $\det(A - xI)$?

constant   linear   quadratic   cubic (degree 3)   quartic (degree 4)

b) (5 pts.) What kind of polynomial in A_{11} best describes $\det(A - xI)$?

constant   linear   quadratic   cubic (degree 3)   quartic (degree 4)

c) (5 pts.) What kind of polynomial in x best describes $\det(xA)$?

constant   linear   quadratic   cubic (degree 3)   quartic (degree 4)

d) (5 pts.) What kind of polynomial in x best describes $\det(A(x))$, where

$$A(x) = \begin{bmatrix} xA_{11} & xA_{12} & xA_{13} & xA_{14} \\ A_{21}+x & A_{22}+x & A_{23}+x & A_{24}+x \\ A_{31}-x & A_{32}-x & A_{33}-x & A_{34}-x \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}$$

constant   linear   quadratic   cubic (degree 3)   quartic (degree 4)

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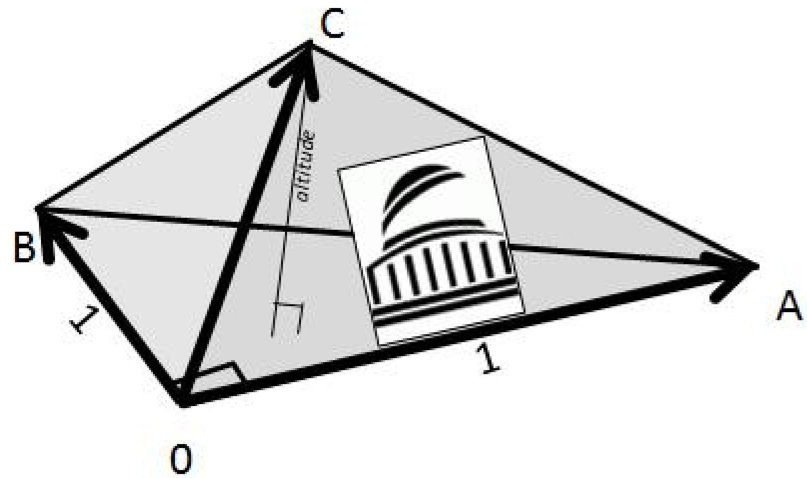

4 (20 pts.)

In R^3 an artist plans an MIT triangular pyramid artwork with one vertex at the origin. The other three vertices are at the tips of vectors A, B and C.

The triangular base of the pyramid $(0, A, B)$ is an isosceles right triangle, The vectors A and B are unit vectors orthogonal to each other.

The other vector C is not in any especially convenient position.

a) (12 pts.) Write an expression for L the length of the altitude of the top of the pyramid to the base in terms of A, B and C.

b) (8 pts.) Write an expression for the volume of the pyramid.

18.06 F13 EXAM 2 SOLUTIONS

1. a) Subtracting 1806 × 1st row from 2nd row and 2013 × 1st row from 3rd row, matrix becomes

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1806 \\ 0 & 1 & 2 \end{pmatrix}$$

Then det = 1806 clearly. (Or any correct method).

b) Subtract 1st row from each other row to get

$$\begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & & -1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -1 \end{pmatrix}$$

So $\det A_n = (-1)^{n-1}$. (Or any correct method).

2. a) P is projection onto a 3 dimensional subspace of an $m > 3$ - dimensional vector space. So $\det(P) = 0$.

b) The eigenvalues are 1 (multiplicity 3)
0 (multiplicity $m-3$).

c) An eigenvector is q_1 with eigenvalue $q_2^T q_1 = q_2 \cdot q_1$.

d) P is symmetric, so M is symmetric.
Left nullspace = right nullspace = column space of P
Column space = row space = orthogonal complement of column space of P

3. a) Quartic, since the diagonal term in the big formula is $(a_{11}-x)(a_{22}-x)(a_{33}-x)(a_{44}-x)$.

b) Linear, since A_{11} appears at most once in any term of the big formula.

c) Quartic, since

$$\det(xA) = x^4 \det(A).$$

d) Quadratic: we can add the second row to the third row to eliminate some xs. Then each term of the big formula is quadratic in x.

4. a) Let $M = (A\ B)$ be the 3×2 matrix with columns A and B. Since A and B are orthonormal, the projection onto the (A, B)-plane is given by

$$MM^T.$$

Thus the length L is the length of $C - MM^T C$

$$L = \|C - MM^T C\|.$$

b) The volume of a pyramid is

$$\frac{1}{3}(\text{base area})(\text{altitude length})$$

The area of the OAB face is $\frac{1}{2}$, so the volume is

$$V = \frac{1}{6}L = \frac{1}{6}\|C - MM^T C\|.$$

18.06 Quiz 2 Professor Strang April 10, 2013

Please PRINT your name ______________________________ **1.**

2.

Please Circle your Recitation: **3.**

r1	T	10	36-156	Russell Hewett	r7	T	1	36-144	Vinoth Nandakumar
r2	T	11	36-153	Russell Hewett	r8	T	1	24-307	Aaron Potechin
r3	T	11	24-407	John Lesieutre	r9	T	2	24-307	Aaron Potechin
r4	T	12	36-153	Stephen Curran	r10	T	2	36-144	Vinoth Nandakumar
r5	T	12	24-407	John Lesieutre	r11	T	3	36-144	Jennifer Park
r6	T	1	36-153	Stephen Curran					

(1) **(40 pts)**

(a) If P projects every vector b in $\mathbb{R}^5$ to the nearest point in the subspace spanned by $a_1 = (1, 0, 1, 0, 4)$ and $a_2 = (2, 0, 0, 0, 4)$, what is the rank of P and **why?**

(b) If these two vectors are the columns of the 5 by 2 matrix A, which of the four fundamental subspaces for A is the nullspace of P?

(c) By Gram-Schmidt find an orthonormal basis for the column space of A (spanned by a_1 and a_2).

(d) If P is any (symmetric) projection matrix, show that $Q = I - 2P$ is an orthogonal matrix.

(2) **(30 pts.)**

(a) Find the determinant of the matrix A

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \end{bmatrix}.$$

(b) The absolute value of $\det A$ tells you the volume of a box in $\mathbb{R}^4$. Describe that box (2 points – describe a different box with the same volume).

(c) Suppose you remove row 3 and column 4 of an invertible 5 by 5 matrix A. If that reduced matrix is not invertible, what fact does that tell you about A^{-1}?

(3) **(30 pts.)** This 4 by 4 Hadmard matrix is an orthogonal matrix. Its columns are orthogonal unit vectors.

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}$$

(a) What projection matrix P_4 (give numbers) will project every b in $\mathbb{R}^4$ onto the line through q_4?

(b) What projection matrix P_{123} will project every b in $\mathbb{R}^4$ onto the subspace spanned by q_1, q_2, and q_3? Remember that those columns are orthogonal.

(c) Suppose A is the 4 by 3 matrix whose columns are q_1, q_2, q_3. Find the least-squares solution $\widehat{x}$ to the four equations

$$Ax = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = b.$$

What is the error vector e?

SOLUTIONS TO EXAM 2

Problem 1 (30 pts)

(a) The rank of P is 2. Any vector perpendicular to the subspace spanned by a_1 and a_2 is in the nullspace of P, and the orthogonal complement of the subspace spanned by a_1 and a_2 is 3-dimensional (that is, there are three independent vectors that project to 0 by P). This is exactly the nullspace of P, and since

$$\text{rank } P = \dim C(P) = 5 - \dim \text{Nullspace } P,$$

the rank of P is $5 - 3 = 2$.

(b) The nullspace of P is the left nullspace of A. Indeed, we have

$$\begin{aligned} Pv = 0 &\Leftrightarrow a_1^T v = 0 \text{ and } a_2^T v = 0 \\ &\Leftrightarrow v^T a_1 = 0 \text{ and } v^T a_2 = 0 \\ &\Leftrightarrow vA = 0. \end{aligned}$$

(c) Gram-Schmidt gives

$$q_1 = \frac{a_1}{||a_1||} = \frac{(1,0,1,0,4)^T}{\sqrt{1^2+0^2+1^2+0^2+4^2}} = \frac{1}{3\sqrt{2}}(1,0,1,0,4)^T$$

and

$$\begin{aligned} q_2 &= \frac{a_2 - \frac{a_2^T q_1}{q_1^T q_1} q_1}{||a_2 - \frac{a_2^T q_1}{q_1^T q_1} q_1||} = \frac{a_2 - a_2^T q_1 q_1}{||a_2 - a_2^T q_1 q_1||} = \frac{(2,0,0,0,4)^T - (1,0,1,0,4)^T}{||(2,0,0,0,4)^T - (1,0,1,0,4)^T||} \\ &= \frac{1}{\sqrt{2}}(1,0,-1,0,0)^T, \end{aligned}$$

and q_1 and q_2 form an orthonormal basis for the column space of A.

(d) Since P is a projection matrix, we have $P = P^T$. To show that Q is an orthogonal matrix, we need to check that $QQ^T = I$. We have

$$\begin{aligned} QQ^T &= (I-2P)(I-2P)^T \\ &= (I-2P)(I^T - 2P^T) \\ &= (I-2P)(I-2P) (\text{ } I \text{ and } P \text{ are symmetric}) \\ &= I - 4P + 4P^2 \end{aligned}$$

Since for a projection matrix we have $P^2 = P$, this product is equal to $QQ^T = I$, as required.

Problem 2 (30 pts)

(a) We will find the determinant by doing row operations:

$$\det \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$
$$= -\det \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$
$$= -\det \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

and the last matrix has determinant $(1)\cdot(2)\cdot(3)\cdot(4) = 24$, so the original matrix has determinant -24.

(b) $\det A$ tells the volume of a box in $\mathbb{R}^4$ whose sides are given by the vectors $(1,1,0,0)^T$, $(2,2,2,0)^T$, $(0,3,3,3)^T$, and $(0,0,4,4)^T$. Another box with the same volume would be a box whose sides are given by the vectors $(1,0,0,0)^T$, $(2,2,0,0)^T$, $(0,3,3,0)^T$, and $(0,4,0,4)^T$. (these are obtained from A via row operations, and so the absolute value of the determinants do not change!)

(c) The formula for A^{-1} says that (see page 270 of the textbook!)

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det A}$$

where C_{ji} is the cofactor given by removing row j and column i. From the problem, this matrix is not invertible, so its determinant is 0, meaning that $C_{ij} = 0$. This means that the $(4,3)$-entry of A^{-1} is also 0.

Problem 3 (30 pts)

(a) Letting

$$A = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix},$$

the projection matrix that projects every $b \in R^4$ onto the column space of A (which is the line through q_4) is given by the formula

$$A(A^TA)^{-1}A^T = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

(b) Letting

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

the projection matrix that projects every $b \in R^4$ onto the column space of A (which is the subspace spanned by q_1, q_2 and q_3) is given by the formula

$$A(A^TA)^{-1}A^T = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 3 & 1 & 1 & -1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ -1 & 1 & 1 & 3 \end{bmatrix}$$

(c) We must solve the new system

$$A^TA\hat{x} = A^Tb.$$

Since $A^TA = I$, we have

$$\hat{x} = A^Tb = \begin{bmatrix} 5 \\ -1 \\ -2 \end{bmatrix}.$$

Then $A\hat{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, and $e = b - A\hat{x} = 0$.

18.06 **EXAM 2** November 5, 2014

Your PRINTED name is: ____________________

Please circle your recitation:

R01	T 9	E17-136	Darij Grinberg
R02	T 10	E17-136	Darij Grinberg
R03	T 10	24-307	Carlos Sauer
R04	T 11	24-307	Carlos Sauer
R05	T 12	E17-136	Tanya Khovanova
R06	T 1	E17-139	Michael Andrews
R07	T 2	E17-139	Tanya Khovanova

Grading

1	
2	
3	
4	
Total:	

Each problem is 25 points, and each of its five parts (a)–(e) is 5 points.

In all problems, write all details of your solutions. Just giving an answer is not enough to get a full credit. Explain how you obtained the answer.

Problem 1. (a) Do Gram-Schmidt orthogonalization for the vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.
(Find an orthogonal basis. Normalization is not required.)

(b) Find the $A = QR$ decomposition for the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$.

(c) Find the projection of the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ onto the line spanned by the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(d) Find the projection of the vector $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ onto the plane $x + y + z = 0$ in $\mathbb{R}^3$.

(e) Find the least squares solution $\widehat{\mathbf{x}}$ for the system $\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 10 \\ 0 \end{pmatrix}$.

Problem 2. Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$. (a) Calculate the determinant $\det(A)$.

(b) Explain why A is an invertible matrix. Find the entry $(2,3)$ of the inverse matrix A^{-1}.

(c) Notice that all sums of entries in rows of A are the same. Explain why this implies that $(1,1,1)^T$ is an eigenvector of A. What is the corresponding eigenvalue λ_1?

(d) Find two other eigenvalues λ_2 and λ_3 of A.

(e) Find the projection matrix P for the projection onto the column space of A.

Problem 3.

(a) Calculate the area of the triangle on the plane $\mathbb{R}^2$ with the vertices $(1,0)$, $(0,1)$, $(3,3)$ using determinants.

(b) Find all values of x for which the matrix $A = \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix}$ has an eigenvalue equal to 2.

(c) Diagonalize the matrix $B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$.

(d) Calculate the power B^{2014} of the matrix $B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$.

(e) Let Q be any matrix which is symmetric and orthogonal. Find Q^{2014}. Explain your answer.

Problem 4. Consider the Markov matrix $A = \begin{pmatrix} 0 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 1/3 & 1/2 \\ 1/2 & 1/3 & 0 & 1/2 \\ 0 & 1/3 & 1/3 & 0 \end{pmatrix}$.

(a) Three of the eigenvalues of A are $1, 0, -1/3$. Find the fourth eigenvalue of A.

(b) Find the determinant $\det(A)$.

(c) Find the eigenvector of the transposed matrix A^T with the eigenvalue $\lambda_1 = 1$.

(d) Find the eigenvector of the matrix A with the eigenvalue $\lambda_1 = 1$. (Hint: Notice that nonzero entries in each column of A are the same.)

(e) Find the limit of $A^k\,(1\ 0\ 0\ 0)^T$ as $k \to +\infty$.

If needed, you can use this extra sheet for your calculations.

If needed, you can use this extra sheet for your calculations.

Exam Solutions

Problem 1

(a) Do Gram-Schmidt orthogonalization for the vectors $a_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $a_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $a_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

(b) Find the $A = QR$ decomposition for the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$.

(c) Find the projection of the vector $(1, 0, 0)^T$ onto the line spanned by the vector $(1, 1, 1)^T$.

(d) Find the projection of the vector $(1, -1, 0)^T$ onto the plane $x + y + z = 0$ in $\mathbb{R}^3$.

(e) Find the least squares solution $\hat{x}$ for the system $\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 10 \\ 0 \end{pmatrix}$.

Solutions:

(a) a_1 and a_2 are already orthogonal so $b_1 = a_1$ and $b_2 = a_2$.

$$b_3 = a_3 - \frac{a_3 \cdot b_1}{b_1 \cdot b_1} b_1 - \frac{a_3 \cdot b_2}{b_2 \cdot b_2} b_2 = a_3 - 2a_1 - 2a_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

(b) Gram-Schmidt orthogonalization on $a_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $a_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ gives $b_1 = a_1$ and $b_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ so $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Inspection gives $R = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

(c) $$\frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}}{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} /3.$$

(d) The vector already lies in the plane so projection does nothing: $(1, -1, 0)^T$.

(e) We must solve $\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \hat{x} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 10 \\ 0 \end{pmatrix}$, i.e. $\begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix} \hat{x} = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$.

So $\hat{x} = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 10 \\ 10 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Problem 2

Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.

(a) Calculate $\det(A)$.

(b) Explain why A is an invertible matrix. Find the $(2,3)$ entry of the inverse matrix A^{-1}.

(c) Notice that all sums of entries in rows of A are the same. Explain why this implies that $(1,1,1)^T$ is an eigenvector of A. What is the corresponding eigenvalue λ_1.

(d) Find two other eigenvalues λ_2 and λ_3 of A.

(e) Find the projection matrix P for the projection onto the column space of A.

Solutions:

(a) Using row operations we see that $\det(A) = \det \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & -3 \end{pmatrix}$. Moreover, using the cofactor formula, $\det \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & -3 \end{pmatrix} = \det \begin{pmatrix} 1 & -1 \\ -1 & -3 \end{pmatrix} = -3 - 1 = -4$.

(b) $\det(A) = -4 \neq 0$. Matrices with non-zero determinants are invertible. The $(2,3)$ entry of A^{-1} is given by

$$\frac{C_{3,2}}{\det A} = \frac{1}{4} \det \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = -1/4.$$

(c) $A(1,1,1)^T = 4(1,1,1)^T$ shows directly that $(1,1,1)^T$ is an eigenvector for A with eigenvalue $\lambda_1 = 4$.

(d) We have $\lambda_1 + \lambda_2 + \lambda_3 = \mathrm{tr}(A) = 4$ and $\lambda_1\lambda_2\lambda_3 = \det(A) = -4$. Remembering that $\lambda_1 = 4$ this gives $\lambda_2 + \lambda_3 = 0$ and $\lambda_2\lambda_3 = -1$. Up to reordering, this system of equations has a unique solution, $\lambda_2 = 1$, $\lambda_3 = -1$.

(e) Since $\det(A) \neq 0$, A is invertible and so the column space of A is all of $\mathbb{R}^3$. The projection matrix onto $\mathbb{R}^3$ is the identity I.

Problem 3

(a) Calculate the area of a triangle on the plane $\mathbb{R}^2$ with the vertices $(1, 0)$, $(0, 1)$, $(3, 3)$ using the determinant.

(b) Find all values of x for which the matrix $A = \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix}$ has an eigenvlue equal to 2.

(c) Diagonalize the matrix $B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$.

(d) Calculate the power B^{2014} of the matrix $B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$.

(e) Let Q be any matrix which is symmetric and orthogonal. Find Q^{2014}. Explain your answer.

Solutions:

(a) Translation by $(-1, 0)$ is an isometry and so it is equivalent to find the area of a triangle with the vertices $(0, 0)$, $(-1, 1)$, $(2, 3)$. This is given by

$$\frac{1}{2}\left| \det \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix} \right| = \frac{5}{2}.$$

(b) A has an eigenvalue equal to 2 if and only if the matrix $A - 2I$ is singular. Thus, A has an eigenvalue equal to 2 if and only if $\det(A - 2I) = 0$. But

$$\det(A - 2I) = \det \begin{pmatrix} -1 & x \\ 1 & -1 \end{pmatrix} = 1 - x.$$

So $\det(A - 2I) = 0$ if and only if $1 - x = 0$, i.e. $x = 1$.

(c) Since B is diagonal its eigenvalues can be read off from the diagonal $\lambda_1 = 1$ and $\lambda_2 = -1$. We find corresponding eigenvectors $(1, 0)^T$ and $(1, -1)^T$. So $B = S\Lambda S^{-1}$, where

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

By chance we have $S = S^{-1}$.

(d) $B^2 = S\Lambda^2 S^{-1} = SIS^{-1} = I$, so $B^{2014} = (B^2)^{1007} = I$.

(e) Since Q is orthogonal we have $Q^T Q = I$. Since Q is symmetric we have $Q^T = Q$. Thus

$$Q^2 = QQ = Q^T Q = I \text{ and } Q^{2014} = (Q^2)^{1007} = I.$$

Problem 4

Consider the Markov matrix $A = \begin{pmatrix} 0 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 1/3 & 1/2 \\ 1/2 & 1/3 & 0 & 1/2 \\ 0 & 1/3 & 1/3 & 0 \end{pmatrix}$.

(a) Three of the eigenvalues are $1, 0, -1/3$. Find the fourth eigenvalue of A.

(b) Find the determinant $\det(A)$.

(c) Find the eigenvector of the transposed matrix A^T with eigenvalue $\lambda_1 = 1$.

(d) Find the eigenvector of the matrix A with the eigenvalue $\lambda_1 = 1$. (Hint: notice that the nonzero entries in each column of A are the same.)

(e) Find the limit of $A^k(1,0,0,0)^T$ as $k \longrightarrow +\infty$.

Solutions:

(a) Since $\mathrm{tr}(A) = 0$ the sum of the eigenvalues are 0. Thus, the fourth eigenvalue must be $-2/3$.

(b) The determinant is the product of the eigenvalues, which is 0.

(c) $(1,1,1,1)A = (1,1,1,1)$ and so the eigenvector of A^T with eigenvalue $\lambda_1 = 1$ is $(1,1,1,1)^T$.

(d) The Markov matrix A corresponds to a random walk on the graph with four nodes 1, 2, 3, 4 connected by the edges $(1,2)$, $(1,3)$, $(2,3)$, $(2,4)$, $(3,4)$. The degrees of the nodes are 2, 3, 3, 2. Thus the vector $(2,3,3,2)^T$ is an eigenvector with eigenvalue $\lambda_1 = 1$.

(e) Let $v_1 = (2,3,3,2)^T$ and let v_2, v_3 and v_4 be eigenvectors for $0, -1/3, -2/3$, respectively. Then there exist $c_1, \dots, c_4 \in \mathbb{R}$ with

$$(1,0,0,0)^T = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4.$$

Thus

$$A^k(1,0,0,0)^T = c_1 v_1 + \frac{(-1)^k c_3}{3^k} v_3 + \frac{(-2)^k c_4}{3^k} v_4 \longrightarrow c_1 v_1, \text{ as } k \longrightarrow +\infty$$

To find c_1 we recall that $(1,1,1,1)A = (1,1,1,1)$. By induction we obtain

$$(1,1,1,1)A^k = (1,1,1,1)$$

and so $(1,1,1,1)A^k(1,0,0,0)^T = (1,1,1,1)(1,0,0,0)^T = 1$. Letting $k \longrightarrow +\infty$ we obtain

$$(1,1,1,1)c_1 v_1 = 1$$

so that $c_1 = 1/((1,1,1,1)v_1) = 1/10$. The answer to the question is $(2,3,3,2)^T/10$.

18.06 Exam II   Professor Strang   April 7, 2014

Your PRINTED Name is: ______________________________

Please circle your section:

R01	T	10	36-144	Qiang Guang
R02	T	10	35-310	Adrian Vladu
R03	T	11	36-144	Qiang Guang
R04	T	11	4-149	Goncalo Tabuada
R05	T	11	E17-136	Oren Mangoubi
R06	T	12	36-144	Benjamin Iriarte Giraldo
R07	T	12	4-149	Goncalo Tabuada
R08	T	12	36-112	Adrian Vladu
R09	T	1	36-144	Jui-En (Ryan) Chang
R10	T	1	36-153	Benjamin Iriarte Giraldo
R11	T	1	36-155	Tanya Khovanova
R12	T	2	36-144	Jui-En (Ryan) Chang
R13	T	2	36-155	Tanya Khovanova
R14	T	3	36-144	Xuwen Zhu
ESG	T	3		Gabrielle Stoy

Grading 1:

2:

3:

4:

____

1. (24 points total)

(a) (6 points) What matrix P projects every vector in $\mathbf{R}^3$ onto the line that passes through origin and $a = (3, 4, 5)$?

(b) (6 points) What is the nullspace of that matrix P?

(c) (6 points) What is the row space of P^2?

(d) (6 points) What is the determinant of P?

2. (25 points total)

(a) (11 points) Suppose $\widehat{x}$ is the best least squares solution to $Ax = b$ and $\widehat{y}$ is the best least squares solution to $Ay = c$.

Does this tell you the best least squares solution $\widehat{z}$ to $Az = b + c$? If so, what is the best $\widehat{z}$ and *why*?

(b) (7 points) If Q is an m by n matrix with orthonormal columns, find the best least squares solution $\widehat{x}$ to $Qx = b$.

(c) (7 points) If $A = QR$, where R is square invertible and Q is the same as in (b), find the least squares solution to $Ax = b$.

3. (25 points total)

(a) (17 points) Find the determinant of this matrix A (with an unknown x in 4 entries).

$$A = \begin{bmatrix} x & 1 & 0 & 0 \\ 2 & x & 2 & 0 \\ 0 & 3 & x & 3 \\ 0 & 0 & 4 & x \end{bmatrix} \qquad B = \begin{bmatrix} x & 1 & 0 & 1 \\ 2 & x & 2 & 0 \\ 0 & 3 & x & 3 \\ 0 & 0 & 4 & x \end{bmatrix}$$

You could use the big formula or the cofactor formula or possibly the pivot formula.

(b) (5 points) Find the determinant for matrix B which has an additional 1 in the corner. What new contribution to the determinant does this 1 make?

(c) (3 points) If M is any 3 by 3 matrix, let $f(x) = \det(xM)$. Find the derivative of f at $x = 1$.

4. (26 points total)

(a) (6 points) Find the projection p of the vector b onto the column space of A.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 2 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

(b) (7 points) Use Gram-Schmidt to find an orthogonal basis q_1, q_2 for the column space of A.

(c) (6 points) Find the projection p of the same vector b onto the column space of the new matrix Q with columns q_1 and q_2.

(d) (7 points) True or False: The best least squares solution $\widehat{x}$ to $Ax = b$ is the same as the best least squares solution $\widehat{y}$ to $Qy = b$. Explain why.

Scrap Paper

18.06 Exam II   Professor Strang   April 7, 2014

Solutions

1. (24 points total)

(a) (6 points) What matrix P projects every vector in $\mathbf{R}^3$ onto the line that passes through origin and $a = (3, 4, 5)$?

(b) (6 points) What is the nullspace of that matrix P?

(c) (6 points) What is the row space of P^2?

(d) (6 points) What is the determinant of P?

Solution.

(a) The projection of the vector $(1, 0, 0)$ onto the line $a = (3, 4, 5)$ is $(9/50, 12/50, 15/50)$. Similarly, the projections of vectors $(0, 1, 0)$ and $(0, 0, 1)$ are $(12/50, 16/50, 20/50)$ and $(15/50, 20/50, 25/50)$ correspondingly. These are the columns of the projection matrix:

$$P = \begin{bmatrix} 9/50 & 12/50 & 15/50 \\ 12/50 & 16/50 & 20/50 \\ 15/50 & 20/50 & 25/50 \end{bmatrix} = \begin{bmatrix} 9/50 & 6/25 & 3/10 \\ 6/25 & 8/25 & 2/5 \\ 3/10 & 2/5 & 1/2 \end{bmatrix}.$$

(b) The nullspace of P is 2-dimensional. It can be generated by the following two vectors orthogonal to $a = (3, 4, 5)$: $(-5/3, 0, 1)$ and $(-4/3, 1, 0)$.

(c) Row space of P^2 is the same as row space of P, since $P^2 = P$. Row space of P is generated by $a = (3, 4, 5)$.

(d) The projection is onto 1-dimensional space, therefore, the rank of matrix P must equal to 1. Therefore, the determinant of P is 0.

2. (25 points total)

(a) (11 points) Suppose $\widehat{x}$ is the best least squares solution to $Ax = b$ and $\widehat{y}$ is the best least squares solution to $Ay = c$.

Does this tell you the best least squares solution $\widehat{z}$ to $Az = b + c$? If so, what is the best $\widehat{z}$ and *why*?

(b) (7 points) If Q is an m by n matrix with orthonormal columns, find the best least squares solution $\widehat{x}$ to $Qx = b$.

(c) (7 points) If $A = QR$, where R is square invertible and Q is the same as in (b), find the least squares solution to $Ax = b$.

Solution.

(a) Denote by P the projection onto the column space of A. We have $A\widehat{x} = Pb$ and $A\widehat{y} = Pc$. That means $A\widehat{x} + A\widehat{y} = Pb + Pc = P(b + c)$. It follows that $\widehat{x} + \widehat{y}$ is the least squares solution for $A\widehat{z} = b + c$.

(b) The least squares solution can be written as $\widehat{x} = (Q^TQ)^{-1}Q^Tb$. As Q is orthonormal, $Q^TQ = I$. Therefore, $\widehat{x} = Q^Tb$. Alternatively, solving least squares means finding a solution to $Q^TQ\widehat{x} = Q^Tb$. As $Q^TQ = I$, we see that $\widehat{x} = Q^Tb$.

(c) The least squares solution can be written as $\widehat{x} = (A^TA)^{-1}A^Tb = (R^TQ^TQR)^{-1}R^TQ^Tb$. As Q is orthonormal, $Q^TQ = I$. Therefore, $\widehat{x} = (R^TR)^{-1}R^TQ^Tb$. As R is invertible, we get $\widehat{x} = (R^TR)^{-1}R^TQ^Tb = R^{-1}(R^T)^{-1}R^TQ^Tb = R^{-1}Q^Tb$.

3. (25 points total)

(a) (17 points) Find the determinant of this matrix A (with an unknown x in 4 entries).

$$A = \begin{bmatrix} x & 1 & 0 & 0 \\ 2 & x & 2 & 0 \\ 0 & 3 & x & 3 \\ 0 & 0 & 4 & x \end{bmatrix} \qquad B = \begin{bmatrix} x & 1 & 0 & 1 \\ 2 & x & 2 & 0 \\ 0 & 3 & x & 3 \\ 0 & 0 & 4 & x \end{bmatrix}$$

You could use the big formula or the cofactor formula or possibly the pivot formula.

(b) (5 points) Find the determinant for matrix B which has an additional 1 in the corner. What new contribution to the determinant does this 1 make?

(c) (3 points) If M is any 3 by 3 matrix, let $f(x) = \det(xM)$. Find the derivative of f at $x = 1$.

Solution.

(a) Using the cofactor method we can expand the determinant of A as:

$$= x \det\left(\begin{bmatrix} x & 2 & 0 \\ 3 & x & 3 \\ 0 & 4 & x \end{bmatrix}\right) - 1 \det\left(\begin{bmatrix} 2 & 2 & 0 \\ 0 & x & 3 \\ 0 & 4 & x \end{bmatrix}\right).$$

We can calculate the 3 by 3 determinants by using any formula. The first one has determinant $x^3 - 18x$, and the second one $2x^2 - 24$. The determinant of A is $x^4 - 20x^2 + 24$.

(b)

By the cofactor formula one more term is added, which is equal

$$-1 \det\left(\begin{bmatrix} 2 & x & 2 \\ 0 & 3 & x \\ 0 & 0 & 4 \end{bmatrix}\right).$$

The 3 by 3 matrix is triangular, so its determinant is the product of the diagonal elements and is equal to 24. So $\det(B) = \det(A) - 24 = x^4 - 20x^2$.

(c)

For a 3 by 3 matrix $f(x) = \det(xM) = x^3 \det(M)$. The derivative $f'(x) = 3x^2 \det(M)$.

4. (26 points total)

(a) (6 points) Find the projection p of the vector b onto the column space of A.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 2 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

(b) (7 points) Use Gram-Schmidt to find an orthogonal basis q_1, q_2 for the column space of A.

(c) (6 points) Find the projection p of the same vector b onto the column space of the new matrix Q with columns q_1 and q_2.

(d) (7 points) True or False: The best least squares solution $\widehat{x}$ to $Ax = b$ is the same as the best least squares solution $\widehat{y}$ to $Qy = b$. Explain why.

Solution.

(a) By the formula, the projection is $A(A^TA)^{-1}A^Tb$:

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 2 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 2 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 9 & 9 \\ 9 & 14 \end{bmatrix}^{-1} \begin{bmatrix} 11 \\ 12 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 14/45 & -0.2 \\ -0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -13/45 & 0.4 \\ 2/9 & 0 \\ 19/45 & -0.2 \end{bmatrix} \begin{bmatrix} 11 \\ 12 \end{bmatrix} = \begin{bmatrix} 73/45 \\ 22/9 \\ 101/45 \end{bmatrix}.$$

(b) $q_1 = (1, 2, 2)$—the first column of A. The projection of $(3, 2, 1)$ onto $(1, 2, 2)$ is $(1, 2, 2)$, with an error vector $e = (2, 0, -1)$. Thus $q_2 = (2, 0, -1)$.

(c) Columns q_1 and q_2 span the same space as columns of A. Thus the projection must be the same as before.

(d) Matrices A and Q span the same column space. Dentoe the projection of b onto that space as p. The solution $\widehat{x}$ satisfies the equation: $A\widehat{x} = p$, the solution $\widehat{y}$ satisfies the equation $Q\widehat{y} = p$. Now $A = QR$, which means $\widehat{y} = R\widehat{x}$.

18.06 Exam II   Professor Strang   April 10, 2015

Your PRINTED Name is: ______________________

Please CIRCLE your section:

R01	T10	26-302	Dmitry Vaintrob
R02	T10	26-322	Francesco Lin
R03	T11	26-302	Dmitry Vaintrob
R04	T11	26-322	Francesco Lin
R05	T11	26-328	Laszlo Lovasz
R06	T12	36-144	Michael Andrews
R07	T12	26-302	Netanel Blaier
R08	T12	26-328	Laszlo Lovasz
R09	T1pm	26-302	Sungyoon Kim
R10	T1pm	36-144	Tanya Khovanova
R11	T1pm	26-322	Jay Shah
R12	T2pm	36-144	Tanya Khovanova
R13	T2pm	26-322	Jay Shah
R14	T3pm	26-322	Carlos Sauer
ESG			Gabrielle Stoy

Grading 1:

2:

3:

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1. (33 points) Suppose we measure $b = 0, 0, 0, 1, 0, 0, 0$ at times $t = -3, -2, -1, 0, 1, 2, 3$.

(a) To fit these 7 measurements by a straight line $C + Dt$, what 7 equations $Ax = b$ would we want to solve?

(b) Find the least squares solution $\widehat{x} = (\widehat{C}, \widehat{D})$.

(c) The projection of that vector b in $\mathbf{R}^7$ onto the column space of A is what vector p?

2. (34 points) Suppose $q_1 = (c, d, e)$ and $q_2 = (f, g, h)$ are **orthonormal column vectors in $\mathbf{R}^3$**. They span a subspace S.

(a) Find the $(1, 1)$ entry in the projection matrix P that projects each vector in $\mathbf{R}^3$ onto that subspace S.

(b) For this projection matrix P, describe 3 independent eigenvectors (vectors for which Px is a number λ times x). What are the 3 eigenvalues of P? What is its determinant?

(c) For some vectors v and w in $\mathbf{R}^3$ the Gram-Schmidt orthonormalization process (applied to v and w) will produce those particular vectors q_1 and q_2. **Describe** the vectors v and w that lead to this q_1 and q_2.

3. (34 points)

(a) If q_1, q_2, q_3 are orthonormal vectors in $\mathbf{R}^3$, what are the possible determinants of this matrix A with columns $2q_1$ and $3q_2$ and $5q_3$? **Why?**

$$A = \begin{bmatrix} 2q_1 & 3q_2 & 5q_3 \end{bmatrix}$$

(b) For a matrix A, suppose the cofactor C_{11} of the first entry a_{11} is **zero**. What information does that give about A^{-1}? Can this inverse exist?

(c) Find the 3 eigenvalues of this matrix A and find all of its eigenvectors. Why is the diagonalization $S^{-1}AS = \Lambda$ not possible?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Scrap Paper

18.06 Exam II Solutions Professor Strang April 10, 2015

1. (33 points) Suppose we measure $b = 0, 0, 0, 1, 0, 0, 0$ at times $t = -3, -2, -1, 0, 1, 2, 3$.

(a) To fit these 7 measurements by a straight line $C + Dt$, what 7 equations $Ax = b$ would we want to solve?

Solution. We want to solve the following 7 equations: $C - 3D = 0$, $C - 2D = 0$, $C - D = 0$, $C = 1$, $C + D = 0$, $C + 2D = 0$, and $C + 3D = 0$.

(b) Find the least squares solution $\widehat{x} = (\widehat{C}, \widehat{D})$.

Solution. First we need to find the projection of b onto the plane generated by two vectors: $(1, 1, 1, 1, 1, 1, 1)$ and $(-3, -2, -1, 0, 1, 2, 3)$. As b is perpendicular to the second vector, we only need to find the projection of b on the line generated by the first vector, which is $(1/7, 1/7, 1/7, 1/7, 1/7, 1/7, 1/7)$. Now we need to solve the seven equations: $C - 3D = 1/7$, $C - 2D = 1/7$, $C - D = 1/7$, $C = 1/7$, $C + D = 1/7$, $C + 2D = 1/7$, and $C + 3D = 1/7$, and $C = 1/7$ and $D = 0$.

Alternatively, we can denote by A the matrix that has these two vectors as its two columns, then $A^T A = \left[\begin{smallmatrix} 7 & 0 \\ 0 & 28 \end{smallmatrix}\right]$ and $A^T b = (1, 0)$. The two equations corresponding to $A^T A\widehat{x} = A^T b$ are $7C = 1$ and $28D = 0$, resulting in the same solution $C = 1/7$ and $D = 0$.

(c) The projection of that vector b in $\mathbf{R}^7$ onto the column space of A is what vector p?

Solution. If we used the first method above, we already calculated the projection as $(1/7, 1/7, 1/7, 1/7, 1/7, 1/7, 1/7)$. If we used the second method, the projection is $A\widehat{x} = (1/7, 1/7, 1/7, 1/7, 1/7, 1/7, 1/7)^T$.

2. (34 points) Suppose $q_1 = (c, d, e)$ and $q_2 = (f, g, h)$ are **orthonormal column vectors in $\mathbf{R}^3$**. They span a subspace S.

(a) Find the $(1, 1)$ entry in the projection matrix P that projects each vector in $\mathbf{R}^3$ onto that subspace S.

Solution. Denote by Q the matrix with columns q_1 and q_2: $Q = \left[\begin{smallmatrix} c & f \\ d & g \\ e & h \end{smallmatrix}\right]$. The projection matrix $P = Q(Q^T Q)^{-1}Q^T$. As the column vectors are orthonormal, we know that $Q^T Q$ is the 2-by-2 identity matrix. Thus, $P = QQ^T$, and the first entry is $c^2 + f^2$.

(b) For this projection matrix P, describe 3 independent eigenvectors (vectors for which Px is a number λ times x). What are the 3 eigenvalues of P? What is its determinant?

Solution. The projection matrix P projects onto a 2d plane. That means its eigenvalues are $(1, 1, 0)$ and the determinant is 0. The eigenvector corresponding to the eigenvalue 0 is orthogonal to the projection plane, that is orthogonal to both vectors q_1 and q_2. The independent vectors corresponding to value 1 are any two independent vectors in the projection plane. We can choose q_1 and q_2 as such vectors.

(c) For some vectors v and w in $\mathbf{R}^3$ the Gram-Schmidt orthonormalization process (applied to v and w) will produce those particular vectors q_1 and q_2. **Describe** the vectors v and w that lead to this q_1 and q_2.

Solution. Vector v is on the same line as q_1 and in the same direction. Therefore, $v = aq_1$, where a is a positive number. The second vector w has to be in the same plane as q_1 and q_2, on the same side of the line drawn through q_1 as q_2 and has to be independent of v.

3. (34 points)

(a) If q_1, q_2, q_3 are orthonormal vectors in $\mathbf{R}^3$, what are the possible determinants of this matrix A with columns $2q_1$ and $3q_2$ and $5q_3$? **Why?**

$$A = \begin{bmatrix} 2q_1 & 3q_2 & 5q_3 \end{bmatrix}$$

Solution. The determinant of the matrix $Q = [q_1\ q_2\ q_3]$ has to be 1 or -1. This is because $Q^TQ = I$, which means that $\det Q^T \cdot \det Q = 1$, that is, $\det Q^2 = 1$. When we multiply a column by a number, the determinant is multiplied by the same number. Thus, the determinant of A is either 30 or -30.

(b) For a matrix A, suppose the cofactor C_{11} of the first entry a_{11} is **zero**. What information does that give about A^{-1}? Can this inverse exist?

Solution. The cofactor C_{11} being zero does not give us enough information to decide whether the inverse exists or not. For example, in the matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ this cofactor is zero and the inverse does not exist, and in the matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ this cofactor is zero and the inverse exists. If this inverse exists, then we know that the entry $(1, 1)$ in this inverse is zero.

(c) Find the 3 eigenvalues of this matrix A and find all of its eigenvectors. Why is the diagonalization $S^{-1}AS = \Lambda$ not possible?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution. The eigenvalues of this matrix are $(2, 2, 2)$. But the rank of $A - 2I$ is 1. That means, you can only find two independent eigenvectors. When the number of independent eigenvectors is smaller than the size of the matrix, then the diagonalization is not possible because you cannot build the square matrix of eigenvectors S.