5 November 1997 **Profs. S. Lee and A. Kirillov**

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Hour Exam II for Course 18.06: Linear Algebra

Recitation Instructor:

Your Name: **SOLUTIONS**

Recitation Time:

Lecturer:

Grading

1. **32**
2. **16**
3. **28**
4. **24**

TOTAL: 100

Do all your work on these pages.
No calculators or notes.
Please work carefully, and check your intermediate results whenever possible.
Point values (total of 100) are marked on the left margin.

[10] **1a.** Give a vector $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ that makes $\underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{s}, \underbrace{\begin{bmatrix} 5 \\ 11 \\ -8 \end{bmatrix}}_{t}, \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{v}$ an orthogonal basis for the vector space $\mathbf{R}^3$.

$v = \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}$, or any nonzero multiple of it.

Are the vectors s and t orthogonal to each other? YES.

Next, we need to find a nonzero vector orthogonal to s and t.

For example, the special solution to $\begin{bmatrix} 1 & 1 & 2 \\ 5 & 11 & -8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

is orthogonal to s and t.

The 3 linearly independent (orthogonal!) vectors s, t, v span a 3-dimensional subspace of the vector space $\mathbf{R}^3$ (all of $\mathbf{R}^3$). The vectors are a basis for $\mathbf{R}^3$.

[10] **1b.** Given that $\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}}_{B} A = \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}}_{BA}$, find $\det(A)$.

$\det(A) = 27$.

$$\underbrace{\det(B)}_{-1} \ \det(A) = \underbrace{\det(BA)}_{-27}.$$

[12] **1c.** Can you find a matrix A such that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis for the <u>left</u> nullspace of A

and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis for the nullspace of A?

If 'yes', give a matrix A.
If 'no', briefly explain why the matrix A cannot exist.

`YES. For example:` $A = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$

`The dimension and number of components for each basis indicates that` A `has` $m = 3$ `rows,` $n = 3$ `columns, and rank` $r = 2$.

`The left nullspace indicates that the sum of the rows of` A `is` $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$.

`The nullspace indicates that the sum of the columns of` A `is` $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

`The example chose the first 2 columns of` A `to be linearly independent vectors that are orthogonal to the left nullspace of` A.

`Column 3 is chosen so that` $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ `is in the nullspace of` A`; the rank remains` 2.

[16] **2.** Find an orthogonal basis for the column space of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 4 & 6 \\ 1 & 4 & 6 \end{bmatrix}$.

> `Using the Gram-Schmidt process, we obtain:` $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 2 \\ 2 \end{bmatrix}$.

`The orthogonal vectors are denoted as` A, B, C:

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$B = \begin{bmatrix} 0 \\ 4 \\ 4 \\ 4 \end{bmatrix} - \left(\tfrac{12}{4}\right) \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{A} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$C = \begin{bmatrix} 0 \\ 0 \\ 6 \\ 6 \end{bmatrix} - \left(\tfrac{12}{4}\right) \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{A} - \left(\tfrac{12}{12}\right) \underbrace{\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{B} = \begin{bmatrix} 0 \\ -4 \\ 2 \\ 2 \end{bmatrix}.$$

3. Let $A = \begin{bmatrix} 1 & -2 & -5 & 1 \\ 2 & -4 & -10 & 3 \end{bmatrix}$.

[16] **3a.** Find the solution to $Ax = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ that is closest to $\begin{bmatrix} 5 \\ 5 \\ 11 \\ 11 \end{bmatrix}$.

$$x = \begin{bmatrix} 7 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Gaussian elimination reveals that the nullspace of A spans a 2-dimensional subspace of $\mathbf{R}^4$.

In particular, the special solutions $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ give a basis for this subspace.

The vector $\begin{bmatrix} 5 \\ 5 \\ 11 \\ 11 \end{bmatrix}$ is closest to $c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, where $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ satisfies

$$\underbrace{\begin{bmatrix} 2 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\begin{bmatrix} 5 & 10 \\ 10 & 26 \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 11 \\ 11 \end{bmatrix}}_{\begin{bmatrix} 15 \\ 36 \end{bmatrix}}.$$

The coefficients are $c_1 = 1$ and $c_2 = 1$.

[12] **3b.** Give an ortho<u>normal</u> basis for the nullspace of $A = \begin{bmatrix} 1 & -2 & -5 & 1 \\ 2 & -4 & -10 & 3 \end{bmatrix}$.

$$q_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad q_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

The special solutions to $A = \begin{bmatrix} 1 & -2 & -5 & 1 \\ 2 & -4 & -10 & 3 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ give a <u>linearly independent</u> basis for the nullspace of A (see Problem 3a.).

Next, we make an orthogonal basis for the nullspace.

The orthogonal vectors are denoted as A, B:

$$A = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$B = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{10}{5}\right) \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{A} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

We make an ortho<u>normal</u> basis by scaling these vectors to have unit length.

$\|A\| = \sqrt{5}$ and $\|B\| = \sqrt{6}$.

So, $q_1 = A/\|A\|$ and $q_2 = B/\|B\|$.

4. Let $A = \begin{bmatrix} 5 & -12 \\ 2 & -5 \end{bmatrix}$.

[8] **4a.** Find the eigenvalues of A.

$$\boxed{\lambda_1 = 1\,,\ \ \lambda_2 = -1\,.}$$

$$\det(A - \lambda I) = \underbrace{(5-\lambda)(-5-\lambda) - (-24)}_{-25+\lambda^2+24} = \lambda^2 - 1 = 0\,.$$

[8] **4b.** Find an eigenvector for each eigenvalue of A.

$$\boxed{x_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix},\ \ x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.}$$

$$\underbrace{\begin{bmatrix} 4 & -12 \\ 2 & -6 \end{bmatrix}}_{(A-\lambda_1 I)} \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{x_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad \underbrace{\begin{bmatrix} 6 & -12 \\ 2 & -4 \end{bmatrix}}_{(A-\lambda_2 I)} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{x_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

[8] **4c.** Find A^{99}. Recall that $A = \begin{bmatrix} 5 & -12 \\ 2 & -5 \end{bmatrix}$.

$$\boxed{A^{99} = \begin{bmatrix} 5 & -12 \\ 2 & -5 \end{bmatrix} = A\,.}$$

`In terms of eigenvalues and eigenvectors,`

$$A = \underbrace{\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}}_{S^{-1}} \text{ and } A^{99} = \underbrace{\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 1^{99} & 0 \\ 0 & (-1)^{99} \end{bmatrix}}_{\Lambda^{99}} \underbrace{\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}}_{S^{-1}}.$$

A^{99} `is the same as` A `since` $\underbrace{\begin{bmatrix} 1^{99} & 0 \\ 0 & (-1)^{99} \end{bmatrix}}_{\Lambda^{99}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\Lambda}$.

18.06 Professor Strang Quiz 2 April 4, 1997

1. **(a)** The system $Ax = b$ has a solution if and only if b is orthogonal to what subspace?

(b) Find the determinant of the 4×4 matrix A whose entries are $a_{ij} =$ smaller of i^2 and j^2.

(c) What is the relation between the determinant of A and the pivots? *Why is this true*?

2. At $t = 1, 2, 3$ we are given values b_1, b_2, b_3. The idea is to fit the best straight line $b = C + Dt$ to those three points.

(a) What three equations in two unknowns will have a solution if the three points lie exactly on a line?

(b) Under what condition $m\,b_1 + n\,b_2 + p\,b_3 = 0$ (**find** m, n, p) will the three points lie on a line? (You could use elimination or your answer to Question 1(a).)

(c) Find the best line $\bar{C} + \bar{D}t$ if the values are $(b_1, b_2, b_3) = (0, 0, 1)$.

(d) What 3×3 matrix P projects every vector onto the plane containing the column vectors $(1, 1, 1)$ and $(1, 2, 3)$?

3. **(a)** Suppose q_1, q_2, q_3 are orthonormal vectors in $\mathbb{R}^6$. Under *what condition on the vector* v will there be a fourth orthonormal vector q_4 that is a combination of v, q_1, q_2, q_3?

(b) Give a formula for that fourth orthonormal vector q_4.

(c) Suppose $q_1, \ldots, q_n$ is an orthonormal basis for $\mathbf{R}^n$. Define the $n \times n$ matrix $A = q_1 q_1^T + \cdots + q_n q_n^T$. What does Aq_1 equal? What does Aq_i equal? What is A?

18.06 Solutions Quiz 2 April 4, 1997

1. (a) The left nullspace (The nullspace of A^T)

(b)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 4 & 4 \\ 1 & 4 & 9 & 9 \\ 1 & 4 & 9 & 16 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & 3 & 8 & 8 \\ 0 & 3 & 8 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

$$\text{determinant} = 1 \cdot 3 \cdot 5 \cdot 7 = \boxed{105}$$

(c) The determinant is $\pm$ product of the pivots .

The sign is $(-1)^{\text{number of row exchanges}}$

Reason: Row exchanges reverse sign
Subtracting multiples of row i from j does not change determinant

Det of triangular matrix U = product of pivots on diagonal.

2. (a)

$$\begin{aligned} C + D &= b_1 \\ C + 2D &= b_2 \\ C + 3D &= b_3 \end{aligned} \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

(b) Elimination gives (by subtracting equation 1):

$$\begin{aligned} D &= b_2 - b_1 \\ 2D &= b_3 - b_1 \end{aligned} \quad \text{then} \quad \begin{gathered} 0 = (b_3 - b_1) - 2(b_2 - b_1) \\ \boxed{0 = b_3 - 2b_2 + b_1} \end{gathered}$$

Other method:

A basis for the left nullspace of A is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ since $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Then by Question 1(a), b should be orthogonal to this vector which means $b_1 - 2b_2 + b_3 = 0$.

(c)

$$A^T A = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \qquad A^T b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\text{Solve} \quad \begin{aligned} 3\bar{C} + 6\bar{D} &= 1 \\ 6\bar{C} + 14\bar{D} &= 3 \end{aligned} \quad \rightarrow \quad \begin{aligned} &2\bar{D} = 1 \\ &\boxed{\bar{D} = \tfrac{1}{2} \quad \bar{C} = -\tfrac{2}{3}} \end{aligned}$$

(d)

$$\begin{aligned} P &= A(A^T A)^{-1} A^T \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \frac{\begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}}{6} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 8 & 2 & -4 \\ -3 & 0 & 3 \end{bmatrix} \\ &\boxed{= \tfrac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}} \qquad \text{check} \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \text{is still in the nullspace} \end{aligned}$$

3. (a) The vectors v, q_1, q_2, q_3 must be linearly independent

(b)

$$q_4 = \frac{v - (v^T q_1) q_1 - (v^T q_2) q_2 - (v^T q_3) q_3}{\| v - (v^T q) q_1 - (v^T q_2) q_2 - (v^T q_3) q_3 \|}$$

Always OK to write $q^T v$ instead of $v^T q$ (for real vectors)

(c)

$$\begin{aligned} Aq_1 &= q_1 \underset{\overset{\downarrow}{1}}{q_1^T q_1} + q_2 \underset{\overset{\downarrow}{0}}{q_2^T q_1} + \cdots + q_n \underset{\overset{\downarrow}{0}}{q_n^T q_1} \\ &= q_1 \end{aligned}$$

Similarly $Aq_i = q_i$. Then $\boxed{A = I}$ (since q's are a basis for $\mathbf{R}^n$).

Other method: (columns × rows)

$$A = [q_1 \ \ldots \ q_n] \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} = QQ^T = QQ^{-1} = I$$

18.06 Professor Edelman Quiz 2 October 23, 1998

Your name is ____.

Please circle your recitation:

1)	M2	2-131	Darren Crowdy	crowdy@math	2-335	3-7905
2)	M2	2-132	Yue Lei	yuelei@math	2-586	3-4102
3)	M3	2-131	Darren Crowdy	crowdy@math	2-335	3-7905
4)	T10	2-131	Sergiu Moroianu	bebe@math	2-491	3-4091
5)	T10	2-132	Gabrielle Stoy	stoy@math	2-235	3-4984
6)	T11	2-131	Sergiu Moroianu	bebe@math	2-491	3-4091
7)	T11	2-132	Gabrielle Stoy	stoy@math	2-235	3-4984
8)	T12	2-132	Anda Degeratu	anda@math	2-229	3-1589
9)	T12	2-131	Edward Goldstein	egold@math	2-092	3-6228
10)	T1	2-131	Anda Degeratu	anda@math	2-229	3-1589
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This exam is harder than quiz 1, but in many cases there is an easy solution to the problem that does not require much work.

1. (a.) (20 pts) Find the dimensions and bases for the four fundamental subspaces of

$$M = \begin{bmatrix} 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

1. (b.) (10 pts) Find orthonormal bases for the same four subspaces.

2. (a.) (10 pts) Find the QR factorization of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

(b.) (10 pts) What is A^{-1}?

3. (20 pts) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $P = A(A^TA)^{-1}A^T = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$.

What is P^3? (Hint: This problem requires no arithmetic.)

4. (30 pts) Let A be an $n \times n$ matrix

(a.) If the row space of A is R^n then the column space of A is ____________________ ?

(b.) If the nullspace of A is R^n then the column space of A is ____________________ ?

(c.) If the left nullspace of A is R^n then the column space of A is ____________________ ?

The next three questions consider a square matrix A whose column space is orthogonal to the row space.

(d.) Give an example of a square matrix A such that the column space is orthogonal to the row space.

(e.) If the column space of an $n \times n$ matrix A is orthogonal to the row space there is an inequality relating the rank r to n. What is the strongest possible inequality? (Hint: $r \leq n$ is a true inequality, but is not the strongest and hence will be considered an incorrect answer. Only the right answer will be given credit.)

(f.) If the column space is orthogonal to the row space, then $\det(A) =$ ____________________ ?

18.06 Prof. Edelman

Solutions to Selected Problems from Quiz #2 November 9, 1998

(2)

(a) The columns of A are orthogonal and of norm 2. Hence

$$A = Q \cdot (2I), \text{ where } Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{2}A.$$

This means $R = 2I$.

(b) $A^{-1} = (Q(2I))^{-1} = (2I)^{-1}Q^{-1} = \frac{1}{2}I \cdot Q^T = \frac{1}{4}A^T = \frac{1}{4}A$ since $A^T = A$.

(4) A is an $n \times n$ matrix.

(a) $R(A) = \mathbb{R}^n \Rightarrow \text{rank } (A) = n \Rightarrow \dim C(A) = n \Rightarrow C(A) = \mathbb{R}^n$.

(b) $N(A) = \mathbb{R}^n \Rightarrow \text{rank } (A) = 0 \Rightarrow C(A) = Z = \text{zero vector space.}$

(c) $N(A^T) = \mathbb{R}^n \Rightarrow \text{rank } (A) = 0 \Rightarrow C(A) = Z$

(d) Example 1 : $A = [0]$.

Example 2:

$$A = \left[\begin{array}{c|c} 0 & 0_m \\ \hline I_k & 0 \end{array}\right]$$

I_k is the $k \times k$ identity matrix.
0_m is the $m \times m$ zero matrix.

(e)

$$\left.\begin{array}{l} C(A) \perp R(A) \\ N(A) = (R(A))^\perp \end{array}\right\} \Rightarrow C(A) \subseteq N(A)$$

Hence $r + r = \dim C(A) + \dim R(A) \le \dim N(A) + \dim R(A) = n$,

i.e. $\boxed{2r \le m}$.

Example 2 above with $k = \left[\frac{n}{2}\right]$, $m = n - k$ shows that $2r \le n$ is the strongest inequality.

(f) From (e), $r \le \frac{n}{2} < n \Rightarrow A$ is singular $\Rightarrow \det(A) = 0$.

18.06 Professor Strang Exam 2 April 8, 1998

Your name is: ______________________________

Grading 1
2
3
4

Please circle your recitation:

1)	M2	2-132	M. Nevins	2-588	3-4110	monica@math
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4)	T12	2-132	A. Edelman	2-380	3-7770	edelman@math
5)	T12	2-131	Z. Spasojevic	2-101	3-4470	zoran@math
6)	T1	2-131	Z. Spasojevic	2-101	3-4770	zoran@math
7)	T2	2-132	Y. Ma	2-333	3-7826	yanyuan@math

1 (a) **(15)** Find an orthonormal basis for the subspace S of $\mathbf{R}^4$ spanned by these three vectors:

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \qquad a_3 = a_1 + a_2$$

(b) **(15)** Find the closest vector p in that subspace S to the vector

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

2 (a) **(15)** Start with the same subspace S. Find a basis (not necessarily orthonormal) for its orthogonal complement $S^{\perp}$ (the space of all vectors perpendicular to S).

(b) **(10)** Find the closest vector q in $S^{\perp}$ to the same vector b.

3 (a) **(10)** Find the determinant of this matrix A_4:

$$A_4 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

(b) **(10)** How many terms are *nonzero* out of the 24 terms in the big formula

$$\det A = \Sigma(\pm) a_{1\alpha} a_{2\beta} a_{3\gamma} a_{4\omega}$$

and what are those nonzero terms?

Suppose the matrices A_n all follow the same pattern as A_4, with 2's on the main diagonal and 1's on the *second* diagonals above and below. Thus

$$A_1 = \begin{bmatrix} 2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

(c) **(10)** Use cofactors along row 1 of A_n to *find the relation between* $\det A_n$, $\det A_{n-1}$ and $\det M_{n-2}$.

That mysterious matrix M_{n-2} is not the same as A_{n-2}. Start with $n = 4$, and use cofactor to find M_{n-2} when this submatrix is 2 by 2. Decribe M_{n-2} for larger n.

4 (a) **(10)** Give the formula for the projection matrix P onto the column space of a matrix A. Where does the formula assume that A has independent columns?

(b) **(5)** The two properties of all these projection matrices are $P^2 = P$ and $P^T = P$. Suppose v^T is the first row of P and v_1 is the first entry in that row. Prove that $v^T v = v_1$.

18.06 Professor Strang Exam 2 Solutions April 8, 1998

1. (a) $\|a_1\| = 2$ so $q_1 = a_1/2$. Then subtract from a_2 its projection onto a_1:

$$B = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \frac{8}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

This also has length $\|B\| = 2$ so $q_2 = B/2$. The vector $a_3 = a_1 + a_2$ does not affect the dimension of S or its basis.

(b) $$p = QQ^Tb = \tfrac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}.$$

2. (a) The orthogonal complement of S is the nullspace of A:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \end{bmatrix} \rightarrow U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \end{bmatrix}.$$

The special solutions give a basis for $S^\perp$ (you may find another basis!):

$$\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) Since b is split into perpendicular pieces $p + q$, we know immediately that

$$q = b - p = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix}.$$

3. (a)

$$A_4 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \rightarrow U = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{2} \end{bmatrix}$$

The product of the pivots is 9.

(b) There are four nonzero terms: 16, -4, -4 and 1.

(c) $\det A_n = 2\det A_{n-1} - \det M_{n-2}$. The matrix M_{n-2} starts with $\begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}$ in its upper left corner and after that it continues like A_{n-2}. With $n = 4$ we only see that 2 by 2 corner from the cofactor rule used twice (which removes rows 1, 3 and columns 1, 3).

$$\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

Note for the future: by continuing on M_{n-2} I finally arrived at

$$\det A_n = 2\det A_{n-1} - 2\det A_{n-3} + \det A_{n-4}\,.$$

4. (a) $P = A(A^TA)^{-1}A^T$: the matrix A^TA is invertible if and only if A has independent columns.

(b) The properties give $PP^T = P$. Compare the $(1,1)$ entry on both sides of this equation to find $v^Tv = v_1$.

18.06 Professor Strang Quiz 2 April 12, 1999

Your name is: ______________________________ **Grading** 1 2 3 4

Please circle your recitation:

1)	Mon	2–3	2-131	S. Kleiman
2)	Mon	3–4	2-131	S. Hollander
3)	Tues	11–12	2-132	S. Howson
4)	Tues	12–1	2-132	S. Howson
5)	Tues	12–1	2-131	S. Kleiman
6)	Tues	1–2	2-131	S. Kleiman
7)	Tues	2–3	2-132	S. Howson

1 (30 pts.) (a) Compute the determinant of

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 5 & 0 \\ 1 & 3 & 9 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

(b) Find an orthogonal basis (orthonormal is even better) for the column space of A. Start from a basis and use Gram-Schmidt (and common sense).

(c) If you change the 1 in the upper left corner of A to 2, what is the change in the determinant (I would use cofactors).

2 (24 pts.) An experiment at the nine times $t = -4, -3, -2, -1, 0, 1, 2, 3, 4$ yields the consistent result $b = 0$ except at the last time ($t = 4$) we get $b = 10$. We want the best straight line $b = C + Dt$ to fit these nine data points by least squares.

(a) Write down the equations $Ax = b$ with unknowns C and D that would be solved if a straight line exactly fit the data (it doesn't).

(b) Find the best least squares value of C and D.

(c) This problem is really projecting the vector $b = (0, 0, 0, 0, 0, 0, 0, 0, 10)$ onto a certain subspace. Give a basis for that subspace and give the projection p of b onto the subspace.

3 (22 pts.) Suppose an m by n matrix Q has orthonormal columns.

(a) What is the rank of Q?

(b) Give an expression with no inverses for the projection matrix P onto the column space of Q.

(c) Check that your formula for P satisfies the two requirements for a projection matrix.

4 (24 pts.) (a) Suppose Q is an orthogonal matrix and $Qx = \lambda x$. Compare the lengths of λx and Qx (using $(Qx)^T(Qx)$) to reach a conclusion about λ.

(b) The Hadamard matrix H has orthogonal columns:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Project the vector $b = (1, 2, 3, 4)$ onto the line spanned by the *last* column. Then project b onto the subspace spanned by all four columns.

(c) Find the eigenvalues of $H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

18.06 Professor Strang Quiz 2 Solutions April 12, 1999

1 (a) $|A| = 2$ times the 3 by 3 determinant $= 2$ times $0 = 0$.

(b) A has rank 3 so we want three orthogonal basis vectors A, B, C:

$$\begin{aligned} A &= \text{first column } (1,1,1,0) \\ B &= \text{(second column)} - \text{(projection onto first column)} \\ &= (-1,1,3,0) - (1,1,1,0)\cdot\frac{3}{3} \\ &= (-2,0,2,0) \quad \text{check: orthogonal to first column} \\ C &= \text{last column } (0,0,0,2) \end{aligned}$$

To orthogonalize divide by lengths:

$$\begin{aligned} q_1 &= \frac{A}{\|A\}} = \frac{A}{\sqrt{3}} \\ &= (1,1,1,0)/\sqrt{3} \end{aligned}$$

$$\begin{aligned} q_2 &= \frac{B}{2\sqrt{2}} \\ &= (-1,0,1,0)/\sqrt{2} \end{aligned}$$

$$\begin{aligned} q_3 &= \frac{C}{2} \\ &= (0,0,0,1) \end{aligned}$$

(c) Adding 1 to the a_{11} entry will add its cofactor to the determinant:

$$\text{Cofactor } C_{11} = \begin{vmatrix} 1 & 5 & 0 \\ 3 & 9 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -12\,.$$

2 (a)

$$\begin{bmatrix} 1 & -4 \\ 1 & -3 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}$$

(b)

$$A^T A = \begin{bmatrix} 9 & 0 \\ 0 & 60 \end{bmatrix} \qquad A^T b = \begin{bmatrix} 10 \\ 40 \end{bmatrix}$$

Solve $A^T A \begin{bmatrix} C \\ D \end{bmatrix} = A^b$ to find $C = \frac{10}{9}$, $D = \frac{40}{60}$.

(c) The columns of A are a basis for the subspace. The projection is

$$p = C \text{ (column 1)} + D \text{ (column 2)}.$$

3 (a) Q has rank n (the n orthonormal) columns are independent).

(b) $P = Q(Q^T Q)^{-1} Q^T = QQ^T$.

(c) Check $P^T = P$: $(QQ^T)^T = QQ^T$.
Check $P^2 = P$: $(Q^T Q)Q^T = QQ^T$.

4 (a) The length of λ_x is $|\lambda| \|x\|$.
The length squared of Q_x is $(Qx)^T(Qx) = x^T Q^T Q x = x^T x = x^T x$.
Thus $|\lambda| \|x\| = \|x\|$ and $|\lambda| = 1$.
Note: We did not use the correct notation when λ and x are complex. The reasoning stays the same.

(b) Projection onto the last column:

$$p = a \frac{a^T b}{a^T a} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} 0 = \text{zero vector.}$$

Projection onto column space (which is all of R^4) is b itself.

(c) $|H_2 - \lambda I| = \begin{vmatrix} 1 - \lambda & 1 \\ 1 & -1 - \lambda \end{vmatrix} = \lambda^2 - 2 = 0.$
The eigenvalues are $\sqrt{2}$ and $-\sqrt{2}$. Check trace $= 0$ and determinant $= -2$.

18.06 Quiz 2 November 10, 1999 Closed Book

Your name is: ______________________________ **Grading** 1 2 3 4

Please circle your recitation:

1)	M 2	2-131	W. Fong	2)	M 2	2-132	L. Nave
3)	M 3	2-131	W. Fong	4)	T 10	2-131	H. Matzinger
5)	T 10	2-132	P. Clifford	6)	T 11	2-131	H. Matzinger
7)	T 11	2-132	P. Clifford	8)	T 12	2-132	M. Skandera
9)	T 12	2-131	V. Kac	10)	T 1	2-131	H. Matzinger
11)	T 2	2-132	M. Skandera				

1 (25 pts.) (a) Find equations (**do not solve**) for the coefficients C, D, E in $b = C + Dt + Et^2$, the parabola which best fits the four points $(t, b) = (0, 0), (1, 1), (1, 3)$ and $(2, 2)$.

(b) In solving this problem you are projecting the vector $b =$ ____________ onto the subspace spanned by ____________. The projection in terms of C, D, E is $p =$ ____________.

2 (28 pts.) Let

$$A = \begin{bmatrix} 3 & 4 & 6 \\ 0 & 1 & 0 \\ -1 & -2 & -2 \end{bmatrix}.$$

(a) Find the eigenvalues of the singular matrix A.

(b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

(c) By expressing $(1, 1, 1)$ as a combination of eigenvectors or by diagonalizing $A = S\Lambda S^{-1}$, compute

$$A^{99} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

3 (25 pts.) Start with two vectors (the columns of A):

$$a_1 = \begin{bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{bmatrix} \quad \text{and} \quad a_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(a) With $q_1 = a_1$ find an orthonormal basis q_1, q_2 for the space spanned by a_1 and a_2 (column space of A).

(b) What shape is the matrix R in $A = QR$ and why is $R = Q^T A$? Here Q has columns q_1 and q_2. Compute the matrix R.

(c) Find the projection matrices P_A and P_Q onto the column spaces of A and Q.

4 (22 pts.) (a) If Q is an orthogonal matrix (square with orthonormal columns), show that $\det Q = 1$ or -1.

(b) How many of the 24 terms in $\det A$ are nonzero, and what is $\det A$?

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

18.06 Quiz 2 Solution November 10, 1999 Closed Book

1 (25 pts.) (a) Find equations (**do not solve**) for the coefficients C, D, E in $b = C + Dt + Et^2$, the parabola which best fits the four points $(t, b) = (0,0), (1,1), (1,3)$ and $(2,2)$.

Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b}$ has no solution. We need to look for its least square solution and solve the system

$$A^T A \begin{bmatrix} C \\ D \\ E \end{bmatrix} = A^T \mathbf{b}, \qquad \text{i.e.,} \qquad \begin{bmatrix} 4 & 4 & 6 \\ 4 & 6 & 10 \\ 6 & 10 & 18 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 12 \end{bmatrix}$$

(b) In solving this problem you are projecting the vector $\underline{\mathbf{b} = (0,1,3,2)}$ onto the subspace spanned by <u>the column vectors of A</u>. The projection in terms of C, D, E is

$$P = A\hat{x} = A \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} C \\ C + D + E \\ C + D + E \\ C + 2D + 4E \end{bmatrix}$$

2 (28 pts.) Let

$$A = \begin{bmatrix} 3 & 4 & 6 \\ 0 & 1 & 0 \\ -1 & -2 & -2 \end{bmatrix}.$$

(a) Find the eigenvalues of the singular matrix A.

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 4 & 6 \\ 0 & 1-\lambda & 0 \\ -1 & -2 & -2-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)\lambda,$$

so the eigenvalues of A are $0, 1, 1$.

(b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

$$A\mathbf{x} = \begin{bmatrix} 3 & 4 & 6 \\ 0 & 1 & 0 \\ -1 & -2 & -2 \end{bmatrix} \mathbf{x} = \mathbf{0} \quad \text{has special solution} \quad \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$(A - I)\mathbf{x} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \\ -1 & -2 & -3 \end{bmatrix} \mathbf{x} = \mathbf{0} \quad \text{has special solutions} \quad \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

So one such basis is

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

(c) By expressing $(1, 1, 1)$ as a combination of eigenvectors or by diagonalizing $A = S\Lambda S^{-1}$, compute

$$A^{99} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

First method:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 6 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} - 5 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

So

$$A^{99} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = A^{99}(6v_1) + A^{99}(v_2) + A^{99}(-5v_3) = 0 + v_2 - 5v_3 = \begin{bmatrix} 13 \\ 1 \\ -5 \end{bmatrix}$$

Second method:

$$A = S \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} S^{-1},$$

$$A^{99} = S \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{99} S^{-1} = S \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} S^{-1} = A.$$

So

$$A^{99} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 1 \\ -5 \end{bmatrix}$$

3 (25 pts.) Start with two vectors (the columns of A):

$$a_1 = \begin{bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{bmatrix} \quad \text{and} \quad a_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(a) With $q_1 = a_1$ find an orthonormal basis q_1, q_2 for the space spanned by a_1 and a_2 (column space of A).

$$b_2 = a_2 - a_2 \cdot q_1\, q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \cos\theta \begin{bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{bmatrix} = \begin{bmatrix} 1 - \cos^2\theta \\ 0 \\ -\cos\theta\sin\theta \end{bmatrix},$$

$$q_2 = \frac{b_2}{|b_2|} = \begin{bmatrix} \sin\theta \\ 0 \\ -\cos\theta \end{bmatrix}.$$

(b) What shape is the matrix R in $A = QR$ and why is $R = Q^T A$? Here Q has columns q_1 and q_2. Compute the matrix R.

R is a 2×2 upper triangular matrix.

$$A = QR \quad \Rightarrow \quad Q^T A = Q^T QR \quad \Rightarrow \quad Q^T A = IR = R$$

$$R = \begin{bmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{bmatrix}$$

(c) Find the projection matrices P_A and P_Q onto the column spaces of A and Q.

$$\text{Since } C(A) = C(Q), \quad P_A = P_Q = QQ^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If you notice that the second entry of both a_1 and a_2 are zero, then you know you are looking for the projection matrix onto the xz-plane. You can obtain the answer without doing any matrix multiplication.

4 (22 pts.) (a) If Q is an orthogonal matrix (square with orthonormal columns), show that $\det Q = 1$ or -1.

$$Q^TQ = I$$
$$\Rightarrow \quad |Q^TQ| = |I|$$
$$\Rightarrow \quad |Q^T||Q| = 1$$
$$\Rightarrow \quad |Q||Q| = 1 \quad \text{because } |A^T| = |A|$$
$$\Rightarrow \quad |Q| = \pm 1.$$

(b) How many of the 24 terms in $\det A$ are nonzero, and what is $\det A$?

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

There are four nonzero terms in $\det A$:

$$\begin{bmatrix} \mathbf{1} & 0 & 1 & 0 \\ 0 & \mathbf{1} & 0 & 1 \\ 1 & 0 & \mathbf{-1} & 0 \\ 0 & -1 & 0 & \mathbf{1} \end{bmatrix}, \begin{bmatrix} \mathbf{1} & 0 & 1 & 0 \\ 0 & 1 & 0 & \mathbf{1} \\ 1 & 0 & \mathbf{-1} & 0 \\ 0 & \mathbf{-1} & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 1 \\ \mathbf{1} & 0 & -1 & 0 \\ 0 & -1 & 0 & \mathbf{1} \end{bmatrix}, \begin{bmatrix} 1 & 0 & \mathbf{1} & 0 \\ 0 & 1 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & -1 & 0 \\ 0 & \mathbf{-1} & 0 & 1 \end{bmatrix}$$

Each of the four terms is equal to -1, so det A = -4.

18.06 Professor Strang Quiz 2 October 30, 2000

Your name is: ______________________________

Please circle your recitation:

1)	M2	2-131	Holm	2-181	3-3665	tsh@math
2)	M2	2-132	Dumitriu	2-333	3-7826	dumitriu@math
3)	M3	2-131	Holm	2-181	3-3665	tsh@math
4)	T10	2-132	Ardila	2-333	3-7826	fardila@math
5)	T10	2-131	Czyz	2-342	3-7578	czyz@math
6)	T11	2-131	Bauer	2-229	3-1589	bauer@math
7)	T11	2-132	Ardila	2-333	3-7826	fardila@math
8)	T12	2-132	Czyz	2-342	3-7578	czyz@math
9)	T12	2-131	Bauer	2-229	3-1589	bauer@math
10)	T1	2-132	Ingerman	2-372	3-4344	ingerman@math
11)	T1	2-131	Nave	2-251	3-4097	nave@math
12)	T2	2-132	Ingerman	2-372	3-4344	ingerman@math
13)	T2	1-150	Nave	2-251	3-4097	nave@math

1 (36 pts.) Suppose Q is a 4 by 3 matrix with orthonormal columns q_1, q_2, q_3.

(a) Starting from a vector v (not in the column space of Q), give a formula for the fourth orthonormal vector q_4 that is produced by Gram-Schmidt from q_1, q_2, q_3, v.

(b) Describe the nullspace of Q (the same 4 by 3 matrix) and the nullspace of Q^{T}. (You can answer even if you didn't find the particular formula for q_4 in part a.) Describe also the nullspaces of $Q^{\mathrm{T}}Q$ and QQ^{T}

(c) Suppose $b = q_1 + 2q_2 + 3q_3 + 4q_4$. Find the least squares solution $\bar{x}$ to $Qx = b$. What is the projection p of this b onto the column space of Q?

2 (24 pts.) (a) Fitting the best (least squares) straight line through the points $(t, b) = (2, 3), (3, 5)$, and $(4, K)$ is the same as solving what system of equations $Ax = b$ by least squares? Is there any value of K for which this system $Ax = b$ has an exact solution?

(b) For general A and B, under what condition does the equation $Ax = b$ have $\bar{x} = 0$ as its least squares solution? In the example of part (a), prove that there is or there isn't a value of K so that $\bar{x} = 0$ is the least squares solution.

3 (40 pts.)

(a) Suppose A is a 4 by 4 matrix. If you add 1 to the entry a_{14} in the northeast corner, how much will the determinant change?

(b) Explain why the determinant of every projection matrix is either 0 or 1.

(c) Find the determinant of the "circulant matrix"

$$A = \begin{bmatrix} 0 & b & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & b \\ b & 0 & a & 0 \end{bmatrix}.$$

Math 18.06 Exam 2 Solutions

1 (36 pts.) (a) $q_4^* = v - (q_1^T v)q_1 - (q_2^T v)q_2 - (q_3^T v)q_3$

$q_4 = \frac{q_4^*}{\|q_4^*\|}$

(b) The nullspace of Q is just the zero vector (Q has a pivot in every column). The nullspace of Q^T has dimension one and consists of all scalar multiples of q_4 (because we know q_4 is orthogonal to q_1, q_2 and q_3).

The nullspace of $Q^T Q = I$ is just the zero vector. The nullspace of QQ^T again has dimension one and is all scalar multiples of q_4.

(c) $Q^T Q\bar{x} = Q^T b$ is the same as $\bar{x} = Q^T b$, so

$$\bar{x} = \begin{bmatrix} q_1^T(q_1 + 2q_2 + 3q_3 + 4q_4) \\ q_2^T(q_1 + 2q_2 + 3q_3 + 4q_4) \\ q_3^T(q_1 + 2q_2 + 3q_3 + 4q_4) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

The projection $p = Q\bar{x} = q_1 + 2q_2 + 3q_3$

2 (24 pts.) (a) $\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} x = \begin{bmatrix} 3 \\ 5 \\ K \end{bmatrix}$

$Ax = b$ has an exact solution when b is in the column space. This happens when $K = 7$.

(b) $\bar{x} = 0$ is the least squares solution when b is in the nullspace of A^T

For $\begin{bmatrix} 3 \\ 5 \\ K \end{bmatrix}$ to be in the nullspace of A^T, K would have to be -8 and $\frac{-21}{4}$, which is impossible.

3 (40 pts.)

(a) The determinant will have the cofactor of a_{14} added to it. In the second part of the question, the determinant will double.

(b) We know $P^2 = P$, so $(det(P))^2 = det(P)$, so $det(P) = 0$ or 1.

(c) Using cofactors by the first row, $det(C) = (-b)(-b)(a^2-b^2)+(-a)(a)(a^2-b^2) = -(a^2-b^2)^2$

(d) 24 terms using a_{11} + 24 terms using a_{22} − 6 terms using both = 42 total

18.06 Exam 2 April 12, 2000 Closed Book

Your name is: ____________________________

Please circle your recitation:

1)	M 2	2–131	P. Clifford	2)	M 3	2–131	P. Clifford
3)	T 11	2–132	T. de Piro	4)	T 12	2–132	T. de Piro
5)	T 1	2–131	T. Bohman	6)	T 1	2–132	T. Pietraho
7)	T 2	2–132	T. Pietraho	8)	T 2	2–131	T. Bohman

Note: Make sure your exam has 4 problems.

Problem		Points possible
1	______	30
2	______	16
3	______	30
4	______	24
Total	______	100

Note: Some problems are worth more than others.

1 (30 pts) Let

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}.$$

(a) Find orthonormal vectors q_1, q_2, and q_3 so that q_1 and q_2 form a basis for the column space of A.

(b) Which of the four fundamental subspaces contains q_3?

(c) Find the projection matrix P projecting onto the left nullspace (not the column space!) of A.

(d) Find the least squares solution to $Ax = (1, 2, 7)$.

Note: You must show your work to receive credit for this problem.

2 (16 pts) Compute the determinant of

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

Note: You must show your work to receive credit for this problem.

3 (30 pts) Consider this sequence: $G_0 = 0$, $G_1 = 1$ and $G_{k+2} = (G_k + G_{k+1})/2$. (So G_{k+2} is the average of the previous two numbers G_k and G_{k+1}.) This problem will find the limit of G_k as $k \to \infty$.

(a) Find a matrix A which satisfies

$$\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = A \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}.$$

(b) Find the eigenvalues and eigenvectors of A.

(c) Write $A^k = S\Lambda^k S^{-1}$, where Λ is a diagonal matrix. You do **not** need to multiply this out to get a single matrix.

(d) Find the limit as $k \to \infty$ of the numbers G_k.

Note: You must show your work to receive credit for this problem.

4 (24 pts) Suppose A is a 3×3 matrix with eigenvalues 0, 1, and 2. Find the following:

(a) the rank of A.

(b) the determinant of $A^{\mathrm{T}}A$.

(c) the determinant of $A + I$.

(d) the eigenvalues of $(A + I)^{-1}$.

Note: You must show your work to receive credit for this problem.

18.06 Exam 2 Solutions April 25, 2000

1 (30 pts.)

(a) $q_1 = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, $q_2 = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $q_3 = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$

Find q_3 by finding the left nullspace of A then normalising, or by taking $q_1 \times q_2$, or by guessing a vector independent of q_1 and q_2 and using Gram Schmidt.

(b) q_3 is in the left nullspace of A, since it is orthogonal to both columns of A

(c) $P = q_3(q_3^T q_3)^{-1} q_3^T = \frac{1}{9}\begin{bmatrix} 4 & -4 & -2 \\ -4 & 4 & 2 \\ -2 & 2 & 1 \end{bmatrix}$

(d) $\hat{x} = (A^T A)^{-1} A^T \vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

2 (16 pts.) Big formula: det of $A = 16 - 4 - 4 - 4 + 1 = 5$

$$\text{Row reduce: det of } A = \text{det of} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix} = 5$$

3 (30 pts.) (a) $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$

(b) $\lambda_1 = 1$ with e-vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -\frac{1}{2}$ with e-vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(c) $A^k = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1^k & 0 \\ 0 & (-\frac{1}{2})^k \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$

(d) $A^\infty = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$

$$\begin{pmatrix} G_\infty \\ G_\infty \end{pmatrix} = A^\infty \begin{pmatrix} G_1 \\ G_0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

So $G_k \to \frac{2}{3}$

4 (24 pts.)

(a) $n - r =$ dimension of the nullspace of $A =$ the number of e-vals of A which are 0. So $r = 2$

(b) $det(A^T A) = det(A^T)det(A) = det(A)det(A)$, and $det(A) = 0 * 1 * 2 = 0$, so $det(A^T A) = 0$

(c) When we add I to a matrix, it increases the e-vals by 1. So the e-vals of $A + I$ are $1, 2, 3$, and $det(A + I) = 1 * 2 * 3 = 6$

(d) If A has e-val λ, then A^{-1} has e-val $\frac{1}{\lambda}$. So the e-vals of $(A + I)^{-1}$ are $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}$

18.06 Strang, Edelman, Huhtanen Quiz 2 November 5, 2001

Your name is: ______________________________

Please circle your recitation:

Recitations

#	Time	Room	Instructor	Office	Phone	Email @math
Lect. 1	MWF 12	4-270	M Huhtanen	2-335	3-7905	huhtanen
Lect. 2	MWF 1	4-370	A Edelman	2-380	3-7770	edelman
Rec. 1	M 2	2-131	D. Sheppard	2-342	3-7578	sheppard
2	M 2	2-132	M. Huhtanen	2-335	3-7905	huhtanen
3	M 3	2-131	D. Sheppard	2-342	3-7578	sheppard
4	T 10	2-132	A. Lachowska	2-180	3-4350	anechka
5	T 10	2-131	S. Kleiman	2-278	3-4996	kleiman
6	T 11	2-131	M. Honsen	2-490	3-4094	honsen
7	T 11	2-132	A. Lachowska	2-180	3-4350	anechka
8	T 12	2-131	M. Honsen	2-490	3-4094	honsen
9	T 1	2-132	A. Lachowska	2-180	3-4350	anechka
10	T 1	2-131	S. Kleiman	2-278	3-4996	kleiman
11	T 2	2-132	F. Latour	2-090	3-6293	flatour

1 (36 pts.) Let A be the square matrix

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} q_1^T + \begin{bmatrix} -1 \\ a \\ -1 \end{bmatrix} q_2^T,$$

where q_1 and q_2 are orthonormal vectors in $\mathbf{R}^3$. (12p)

(a) Find x such that

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(b) Choose a such that the column space of A has dimension 1. (8p)

(c) If

$$q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } q_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and $a = 0$, solve

$$Ay = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

in the least squares sense. (16p)

2 (28 pts.) Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

(a) By Gram-Schmidt, factor A into QR where Q is orthogonal and R is upper triangular. (16p such that 10p from Q and 6p from R)

(b) Find the inverse of R and then give the inverse of A by using $A = QR$. (12p such that 4p from R^{-1} and 4p from Q^{-1} and 4p from A^{-1})

3 (36 pts.) (a) Let u, v and w be linearly independent. How is the matrix A with columns u, v, w related to the matrix B with columns $u+v$, $u-v$, $u-2v+w$? Show that those three columns are linearly independent. (12p)

(b) Using Cramer's rule, find b_3 such that $x_3 = 0$ for the solution of

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ b_3 \end{bmatrix}.$$

(12p)

(c) Using rules for the determinant (so do not compute it with any of the 3 formulas), show the steps and rules that lead to

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$$

(12p)

18.06 Midterm Exam 2, Spring, 2001

Name ______________________ Optional Code ______________

Recitation Instructor ______________ Email Address ______________

Recitation Time ________________

This midterm is closed book and closed notes. No calculators, laptops, cell phones or other electronic devices may be used during the exam.
There are 3 problems. Good luck.

1. (40pts.) Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & -1 \\ 9 & 5 & 1 \\ 9 & 8 & 7 \end{pmatrix}$$

   (a) Find the rank of A.

   (b) Find a basis for the row space of A, and find a basis for the nullspace of A. What is the dimension of the nullspace of A?

   (c) What can you say about the relation between the rank and the dimension of the nullspace of A?

   (d) Verify that all vectors in your basis of the nullspace are orthogonal to all vectors in your basis of the row space.

2. (30pts.) Let $a, b \in \mathbb{R}$, and let

$$A = \begin{pmatrix} 1 & 2 & 3 & a \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & b \end{pmatrix}.$$

   (a) What are the dimensions of the four subspaces associated with the matrix A? This will of course depend on the values of a and b, and you should distinguish all different cases.

   (b) For $a = b = 1$, give a basis for the column space of A. Is this also a basis for $\mathbb{R}^3$? Justify your answer.

3. (30pts.) An experiment at the seven times $t = -3, -2, -1, 0, 1, 2, 3$ yields the consistent result $b = 0$, except at the last time ($t = 3$), when we get $b = 28$. We want the best straight line $b = C + Dt$ to fit these seven data points by least squares.

   (a) Write down the equation $A\mathbf{x} = \mathbf{b}$ with unknowns C and D that would be solved if a straight line exactly fit the data.

(b) Use the method of least squares to find the best fit values for C and D.

(c) This problem is really that of projecting the vector $\mathbf{b} = (0,0,0,0,0,0,28)^T$ onto a certain subspace. Give a basis for that subspace, and give the projection $\mathbf{p}$ of $\mathbf{b}$ onto that subspace.

18.06 Solutions to Midterm Exam 2, Spring, 2001

1. (40pts.) Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & -1 \\ 9 & 5 & 1 \\ 9 & 8 & 7 \end{pmatrix}$$

(a) Find the rank of A.

- After doing row operations on the matrix A, we obtain

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since there are two non-zero pivots, A has rank 2.

(b) Find a basis for the row space of A, and find a basis for the nullspace of A. What is the dimension of the nullspace of A?

- A basis for the row space of A is given by the two pivot rows: the first two rows, since we did not have to exchange rows during row operations; $\{(1,0,-1),(3,1,-1)\}$.

The solutions of $A\mathbf{x} = \mathbf{0}$ are given by $\mathbf{x} = \alpha(1,-2,1)^T$ where $\alpha \in \mathbb{R}$. Hence, a basis for the nullspace is $(1,-2,1)^T$. Since the basis of the nullspace contains one vector, dim $N(A) = 1$.

(c) What can you say about the relation between the rank and the dimension of the nullspace of A?

- dim $N(A)$+rk(A) = number of columns of A. Here, 1+2=3.

(d) Verify that all vectors in your basis of the nullspace are orthogonal to all vectors in your basis of the row space.

- $$(1,0,-1).(1,-2,1) = 0, \quad \text{and} \quad (3,1,-1).(1,-2,1) = 0.$$

2. (30pts.) Let $a, b \in \mathbb{R}$, and let

$$A = \begin{pmatrix} 1 & 2 & 3 & a \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & b \end{pmatrix}.$$

(a) What are the dimensions of the four subspaces associated with the matrix A? This will of course depend on the values of a and b, and you should distinguish all different cases.

- After doing row operations on the matrix A, we find

$$A = \begin{pmatrix} 1 & 2 & 3 & a \\ 0 & 2 & 4 & a \\ 0 & 0 & 0 & a - 2b \end{pmatrix}.$$

If $a = 2b$, then there are two non-zero pivots, and so $\text{rk}(A) = \dim \text{col}(A) = \dim \text{row}(A) = 2$. Also, $\dim \text{null}(A) = 4 - 2 = 2$ and $\dim \text{left-null}(A) = 3 - 2 = 1$.

If $a \neq 2b$, then there are three non-zero pivots, and so $\text{rk}(A) = \dim \text{col}(A) = \dim \text{row}(A) = 3$. Also, $\dim \text{null}(A) = 4 - 3 = 1$ and $\dim \text{left-null}(A) = 3 - 3 = 0$.

(b) For $a = b = 1$, give a basis for the column space of A. Is this also a basis for $\mathbb{R}^3$? Justify your answer.

- A basis is given by the columns of A which lead to non-zero pivots,

$$\text{basis} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \right\}$$

Since $\mathbb{R}^3$ has dimension 3, and these are 3 linearly independent vectors in $\mathbb{R}^3$, they are indeed are basis for $\mathbb{R}^3$.

3. (30pts.) An experiment at the seven times $t = -3, -2, -1, 0, 1, 2, 3$ yields the consistent result $b = 0$, except at the last time ($t = 3$), when we get $b = 28$. We want the best straight line $b = C + Dt$ to fit these seven data points by least squares.

(a) Write down the equation $A\mathbf{x} = \mathbf{b}$ with unknowns C and D that would be solved if a straight line exactly fit the data.

•

$$\begin{pmatrix} 1 & -3 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 28 \end{pmatrix}$$

(b) Use the method of least squares to find the best fit values for C and D.

• $A^TA = \begin{pmatrix} 7 & 0 \\ 0 & 28 \end{pmatrix}$, and $A^T\mathbf{b} = \begin{pmatrix} 28 \\ 84 \end{pmatrix}$. Solving $(A^TA)\mathbf{x} = (A^T\mathbf{b})$, we find that $\mathbf{x}_{\text{sol}} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

(c) This problem is really that of projecting the vector $\mathbf{b} = (0, 0, 0, 0, 0, 0, 28)^T$ onto a certain subspace. Give a basis for that subspace, and give the projection $\mathbf{p}$ of $\mathbf{b}$ onto that subspace.

• A basis is given by the columns of A, $\{(1, 1, 1, 1, 1, 1, 1)^T, (-3, -2, -1, 0, 1, 2, 3)^T\}$. The projection is given by

$$\mathbf{p} = A\mathbf{x}_{\text{sol}} = \begin{pmatrix} -5 \\ -2 \\ 1 \\ 3 \\ 7 \\ 10 \\ 13 \end{pmatrix}.$$

18.06 Midterm Exam 2 (Make-up), Spring, 2001

Name ______________________ Optional Code ______________

Recitation Instructor ______________ Email Address ______________

Recitation Time ________________

This midterm is closed book and closed notes. No calculators, laptops, cell phones or other electronic devices may be used during the exam.

There are 3 problems. Good luck.

1. (30pts.)

   (a) Can $\mathbf{v} = (1, 2, 3)$ be in the nullspace and also in the column space of A? Give an example to prove yes, or a reason to prove no.

   (b) In $\mathbb{R}^4$, find the projections of $\mathbf{b} = (1, 2, 2, 7)$ onto the line through $\mathbf{a} = (1, 1, 1, 1)$ and also onto the plane $x_1 + x_2 + x_3 + x_4 = 0$.

   (c) If you can solve $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b}$ must be perpendicular to ______________________.

2. (40pts.) Consider the matrix

$$A = \begin{pmatrix} 1 & -2 & 1 & -1 \\ 3 & -6 & 2 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

(a) Find the rank of A.

(b) Find a basis for the row space of A, and find a basis for the nullspace of A. What is the dimension of the nullspace of A?

(c) What can you say about the relation between the rank and the dimension of the nullspace of A?

(d) Verify that all vectors in your basis of the nullspace are orthogonal to all vectors in your basis of the row space.

3. (30pts.) We look for the line $y = C + Dt$ closest to the 3 points, $(t, y) = (0, -1)$ and $(1, 2)$ and $(2, -1)$.

(a) If the line went through all those points (it doesn't), what three equations would need to be solved?

(b) Find the best C and D by the least squares method.

(c) Explain the result you get for C and D: How is the vector $\mathbf{b} = (-1, 2, -1)$ related to the plane you are projecting onto?

(d) What is the length of the error vector $\mathbf{e}$ ($=$ distance to plane $= \|\mathbf{b} - A\mathbf{x}\|$)?

18.06 Professor Strang/Ingerman Quiz 2 October 25, 2002

Your name is: ____________________

Please circle your recitation:

1)	M2	2-131	P.-O. Persson	2-088	2-1194	persson
2)	M2	2-132	I. Pavlovsky	2-487	3-4083	igorvp
3)	M3	2-131	I. Pavlovsky	2-487	3-4083	igorvp
4)	T10	2-132	W. Luo	2-492	3-4093	luowei
5)	T10	2-131	C. Boulet	2-333	3-7826	cilanne
6)	T11	2-131	C. Boulet	2-333	3-7826	cilanne
7)	T11	2-132	X. Wang	2-244	8-8164	xwang
8)	T12	2-132	P. Clifford	2-489	3-4086	peter
9)	T1	2-132	X. Wang	2-244	8-8164	xwang
10)	T1	2-131	P. Clifford	2-489	3-4086	peter
11)	T2	2-132	X. Wang	2-244	8-8164	xwang

1 (40 pts.) (a) Find the projection matrix P_C onto the column space of A (after looking closely at the matrix!)

$$A = \begin{bmatrix} 3 & 3 & 6 \\ 1 & 1 & 2 \end{bmatrix}$$

(b) Find the 3 by 3 projection matrix P_R onto the row space of A. What is the closest vector in the row space to the vector $\boldsymbol{b} = (1, 0, 0)$?

(c) Multiply $P_C A$ and then $P_C A P_R$. Your answers should be a little surprising—can you explain?

(d) Find a basis for the subspace of all vectors orthogonal to the row space of A.

2 (30 pts.) (a) Choose c and the last column of Q so that you have an orthogonal matrix:

$$Q = c \begin{bmatrix} 1 & -1 & -1 & \mathrm{x} \\ -1 & 1 & -1 & \mathrm{x} \\ -1 & -1 & -1 & \mathrm{x} \\ -1 & -1 & 1 & \mathrm{x} \end{bmatrix}$$

(b) Project $\boldsymbol{b} = (1, 1, 1, 1)$ onto the first column of Q. Then project $\boldsymbol{b}$ onto the plane spanned by the first two columns.

(c) Suppose the last column of the 4 by 4 matrix (where the x's are) was changed to $(1, 1, 1, 1)$. Call this new matrix A. If Gram-Schmidt is applied to the 4 columns of A, what would be the 4 outputs $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4$? (Don't do a lot of calculations... please.)

3 (30 pts.) (a) If you multiply all $n!$ permutations together into a single P, is the product odd or even? (Answer might depend on n.)

(b) If you know that $\det A = 6$, what is the determinant of B?

$$\det A = \begin{vmatrix} \text{row } 1 \\ \text{row } 2 \\ \text{row } 3 \end{vmatrix} = 6 \qquad \det B = \begin{vmatrix} \text{row } 3 + \text{row } 2 + \text{row } 1 \\ \text{row } 2 + \text{row } 1 \\ \text{row } 1 \end{vmatrix} = ?$$

(c) Prove $\det A = 0$ for the 5 by 5 *all-ones matrix* (all $a_{ij} = 1$) in ***two ways***:

(1) Using Properties 1–10 of determinants

(2) Using the "big formula" = sum of 120 terms.

Course 18.06, Fall 2002: Quiz 2, Solutions

1 (a) The columns of A are linearly dependent, but the column space is spanned by $A = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Use this matrix in the formula for the projection matrix:

$$P_C = A(A^TA)^{-1}A^T = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \frac{1}{10} \begin{bmatrix} 3 & 1 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

(b) As before but with $A = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$:

$$P_R = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

The vector in the row space of A closest to $\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ is

$$P_R\boldsymbol{b} = \frac{1}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

(c)

$$P_CA = \frac{1}{10} \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 6 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 6 \\ 1 & 1 & 2 \end{bmatrix} = A$$

$$P_CAP_R = \begin{bmatrix} 3 & 3 & 6 \\ 1 & 1 & 2 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 6 \\ 1 & 1 & 2 \end{bmatrix} = A$$

The two multiplications project the columns/rows of A onto the column/row space of A. This does not change the matrix.

(d) All vectors orthogonal to the row space of A are in the null space of A. A basis for the nullspace of A is (note dimension 2):

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

2 (a) One choice for the last column is

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

and the normalization constant is

$$c = \frac{1}{2}$$

(b) The projection of $\boldsymbol{b} = (1,1,1,1)$ onto $\boldsymbol{q}_1$ is

$$\boldsymbol{q}_1\boldsymbol{q}_1^T\boldsymbol{b} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The projection of $\boldsymbol{b} = (1,1,1,1)$ onto the plane spanned by $\boldsymbol{q}_1, \boldsymbol{q}_2$ is

$$\boldsymbol{q}_1\boldsymbol{q}_1^T\boldsymbol{b} + \boldsymbol{q}_2\boldsymbol{q}_2^T\boldsymbol{b} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

(c) The first three outputs will be the first three columns of A, since they are already orthogonal and normalized. The last column becomes

$$\boldsymbol{v} = \boldsymbol{b} - \boldsymbol{q}_1^T\boldsymbol{b}\,\boldsymbol{q}_1 - \boldsymbol{q}_2^T\boldsymbol{b}\,\boldsymbol{q}_2 - \boldsymbol{q}_3^T\boldsymbol{b}\,\boldsymbol{q}_3 = \boldsymbol{b} + \boldsymbol{q}_1 + \boldsymbol{q}_2 + \boldsymbol{q}_3 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{q}_4 = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

3 (a) Half of the $n!$ permutations are even and half of them are odd. Multiplying even permutations gives an even permutation. Multiplying odd permutations an odd number of times gives an odd permutation, and an even number of times gives an even permutation. So the problem is equivalent to asking if

$$\frac{n!}{2} = \frac{n \cdot n-1 \cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2}$$

is an even or an odd number. The answer is that it is even when $n \geq 4$, it is odd when $n = 2$ or $n = 3$, and in the special case $n = 1$ there is only one even permutation, so that product is even.

(b) You can get the matrix B from A by:

* Adding row 1 and row 2 to row 3.
* Adding row 1 to row 2.
* Exchanging row 1 and row 3.

The first two operations do not change the determinant, and the third changes the sign. Therefore, $\det B = -6$.

(c) (1) For example Property 4: ***If two rows of A are equal, then det A=0***.

(2) Half of the 120 terms are $+1$, and half of them are -1. The sum is zero.

18.06 Exam 2 #1 Solutions

1. The row echelon form of A is $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So we find that a basis for $R(A^T)$ is $\{(1,2,-1,4),(0,1,-2,3)\}$, and a basis for $N(A)$ is $\{(-3,2,1,0),(2,-3,0,1)\}$. Similarly, row echelon form of A^T is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So a basis for $C(A)$ is $\{(1,0,-1),(0,1,2)\}$ and a basis for $N(A^T)$ is $\{(1,-2,1)\}$.

2. a) Using the row operation $R4 - R1$ gives

$$\begin{vmatrix} -1 & 2 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & 1 & 2 & 0 \\ -1 & -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & 1 & 2 & 0 \\ -0 & -3 & 0 & 0 \end{vmatrix}.$$

Using a cofactor expansion about the fourth column gives

$$\begin{aligned} \begin{vmatrix} -1 & 2 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & 1 & 2 & 0 \\ -0 & -3 & 0 & 0 \end{vmatrix} &= (-1)\begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 2 \\ 0 & 3 & 0 \end{vmatrix} \\ &= (-1)(3)\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} = -12. \end{aligned}$$

(here, we computed the 3×3 determinant by expanding about the third row.

b) Using Cramer's rule,

$$A^{-1}(1,4) = \frac{C_{4,1}}{\det(A)} = \frac{(-1)^{4+1}\begin{vmatrix} 2 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix}}{\det(A)} = \frac{-3}{-12} = \frac{1}{4}$$

c)

$$\begin{aligned} \det(2A^2A^T(A^{-1})^3) &= \det(2I)\det(A)^2\det(A^T)\det(A^{-1})^3 \\ &= 2^4 \cdot \det(A)^2\det(A)\det(A)^{-3} = 16 \end{aligned}$$

3. A basis for the space in question is $\{(1,1,0,0),(2,0,1,0),(-1,0,0,1)\}$. To get a orthogonal basis, we need to do the Gram-Schmidt algorithm. Start with $v_1 = (1,1,0,0), v_2 = (2,0,1,0), v_3 = (-1,0,0,1)$,

$$\begin{aligned} \tilde{v_1} &= v_1 = (1,1,0,0) \\ \tilde{v_2} &= v_2 - \frac{(\tilde{v_1}, v_2)}{|\tilde{v_1}|^2}\tilde{v_1} = (1,-1,1,0) \\ \tilde{v_3} &= v_3 - \frac{(\tilde{v_1}, v_3)}{|\tilde{v_1}|^2}\tilde{v_1} - \frac{(\tilde{v_2}, v_3)}{|\tilde{v_2}|^2}\tilde{v_2} = (-\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, 1) \end{aligned}$$

Then $\{\tilde{v_1}, \tilde{v_2}, \tilde{v_3}\}$ is a set of orthogonal basis, to make them orthonormal, just multiply each $\tilde{v_i}$ by the reciprocal of its norm. So an orthonormal basis for the subspace in the problem is $\{\frac{\tilde{v_1}}{|\tilde{v_1}|}, \frac{\tilde{v_2}}{|\tilde{v_2}|}, \frac{\tilde{v_3}}{|\tilde{v_3}|}\} = \{\frac{1}{\sqrt{2}}(1,1,0,0), \frac{1}{\sqrt{3}}(1,-1,1,0), \frac{1}{\sqrt{42}}(-1,1,2,6)\}$.

4. a) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 4 \end{bmatrix}$, $\hat{x} = \begin{bmatrix} C \\ D \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, $\hat{x}$ is the solution to the linear equation $A^T A\hat{x} = A^T b$, and the least squares line is $y = C + Dx$.

   b) $P = B(B^T B)^{-1} B^T$.

18.06 Professor A.J. de Jong Exam 2 April 9, 2003

Your name is: ______________________________

Please circle your recitation:

Important: Briefly explain all of your answers.

1 (29 pts.)

(a) Compute the determinant of the following matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 & 2 \end{pmatrix}$$

Mention the method used for each step in the calculation.

(b) Give a basis for each of the four fundamental subspaces associated to the following matrix

$$\begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

2 (29 pts.)

(a) Apply the Gram-schmidt algorithm to the columns of the matrix A below. (Use the order in which they occur in the matrix!) Use this to write $A = QR$, where Q is a matrix with orthonormal columns, and R is upper triangular.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & -1 \end{pmatrix}.$$

(b) Compute the matrix of the projection onto the column space of A. What is the distance of the point $(1, 1, 1, 0)$ to this column space?

3 (14 pts.) Show that the following determinant is zero for any values of a, b and c:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}$$

4 (28 pts.) Let A be the matrix

$$\begin{pmatrix} 7 & 5 \\ 3 & -7 \end{pmatrix}.$$

(a) Find matrices S and Λ such that A has a factorization of the form

$$A = S\Lambda S^{-1},$$

where S is invertible and Λ is diagonal: $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$.

(b) Find a matrix B such that $B^3 = A$. (Hint: First find such a matrix for Λ. Then use the formula above.)

18.06 Professor A.J. de Jong Exam 2 April 9, 2003

Your name is: ______________________________

Please circle your recitation:

Important: Briefly explain all of your answers.

1 (29 pts.)

(a) Compute the determinant of the following matrix

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = -1$$

We expanded the determinant along row 1 then subtracted row 1 from rows 3 and 4 and then expanded the determinant along the 1st column. The last 3x3 determinant was computed directly.

(b) Give a basis for each of the four fundamental subspaces associated to the following matrix

$$\begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We switched rows 1 and 2 then subtracted row 1 from row 3 and then subtracted row 2 from row 3.

First two rows (0, 1, -1 , 0) and (1, 0, -1, 0) is a basis for the row space.

First two columns (0, 1, 1) and (1, 0, -1) is a basis for the column space.

Solving $Av = 0$ we get $x_1 = x_2 = x_3$. Thus,(1, 1, 1, 0) and (0, 0, 0, 1) is a basis for the $Null(A)$ space.

Solving $A^t v = 0$ we get $x_1 = -x_2 = x_3$. Thus, (1, -1, 1) is a basis for the $Null(A^t)$ space.

2 (29 pts.)

(a) Apply the Gram-schmidt algorithm to the columns of the matrix A below. (Use the order in which they occur in the matrix!) Use this to write $A = QR$, where Q is a matrix with orthonormal columns, and R is upper triangular.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & -1 \end{pmatrix}.$$

$q_1 = a_1 = (1, 0, 0, -1)$.
$q_2 = a_2 - \frac{(a_2 \cdot q_1)}{(q_1 \cdot q_1)} q_1 = (0, 1, 0, -1) - (1/2, 0, 0, -1/2) = (-1/2, 1, 0, -1/2)$.
Normalize $q_1 = (1/\sqrt{2}, 0, 0, -1/\sqrt{2})$, $q_2 = (-1/\sqrt{6}, 2/\sqrt{6}, 0, -1/\sqrt{6})$. $Q = [q_1, q_2]$, $R = Q^t A$.

$$A = QR = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 2/\sqrt{6} \\ 0 & 0 \\ -1/\sqrt{2} & -1/\sqrt{6} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} \end{pmatrix}.$$

(b) Compute the matrix of the projection onto the column space of A. What is the distance of the point $(1, 1, 1, 0)$ to this column space?

$$P = QQ^t = \begin{pmatrix} 2/3 & -1/3 & 0 & -1/3 \\ -1/3 & 2/3 & 0 & -1/3 \\ 0 & 0 & 0 & 0 \\ -1/3 & -1/3 & 0 & 2/3 \end{pmatrix}.$$

If $b = (1, 1, 1, 0)$ then its projection is $p = Pb = (1/3, 1/3, 0, -2/3)$. The distance $d = \|b - p\| = \|(2/3, 2/3, 1, 2/3)\| = \sqrt{21}/3$.

3 (14 pts.) Show that the following determinant is zero for any values of a, b and c:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a+b+c & b+c+a & c+a+b \end{vmatrix} = 0$$

We added row 2 to row 3. The determinant is 0 since rows 1 and 3 are multiples of each other.

4 (28 pts.) Let A be the matrix

$$\begin{pmatrix} 7 & 5 \\ 3 & -7 \end{pmatrix}$$

(a) Find matrices S and Λ such that A has a factorization of the form

$$A = S\Lambda S^{-1},$$

where S is invertible and Λ is diagonal: $\Lambda = \text{diag}(\lambda_1, \lambda_2)$.

$det(A - \lambda I) = 0$. The eigenvalues: $\lambda^2 - 64 = 0$, $\lambda_1 = 8$, $\lambda_2 = -8$. Eigenvectors: $(A - \lambda_1 I)v_1 = \begin{pmatrix} -1 & 5 \\ 3 & -15 \end{pmatrix} v_1 = 0$ then $v_1 = (5, 1)$,

$(A - \lambda_2 I)v_2 = \begin{pmatrix} 15 & 5 \\ 3 & 1 \end{pmatrix} v_2 = 0$ then $v_2 = (1, -3)$.

$S = \begin{pmatrix} 5 & 1 \\ 1 & -3 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 8 & 0 \\ 0 & -8 \end{pmatrix}$, $S^{-1} = (-1/16) \begin{pmatrix} -3 & -1 \\ -1 & 5 \end{pmatrix}$.

(b) Find a matrix B such that $B^3 = A$. (Hint: First find such a matrix for Λ. Then use the formula above.)

$B = S\Lambda^{1/3}S^{-1} = \begin{pmatrix} 5 & 1 \\ 1 & -3 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \cdot (-1/16) \begin{pmatrix} -3 & -1 \\ -1 & 5 \end{pmatrix} = 1/4 \begin{pmatrix} 7 & 5 \\ 3 & -7 \end{pmatrix}$.

18.06 Fall 2004 Quiz 2 November 15, 2004

Your name is:

Please circle your recitation:

1. M2 A. Brooke-Taylor
2. M2 F. Liu
3. M3 A. Brooke-Taylor
4. T10 K. Cheung
5. T10 Y. Rubinstein
6. T11 K. Cheung
7. T11 V. Angeltveit
8. T12 V. Angeltveit
9. T12 F. Rochon
10. T1 L. Williams
11. T1 K. Cheung
12. T2 T. Gerhardt

Grading:

Question	Points	Maximum
Name + rec		5
1		22
2		20
3		25
4		28
Total:		100

Remarks:
Do all your work on these pages.
No calculators or notes.
Putting your name and recitation section correctly is worth 5 points.
The exam is worth a total of 100 points.

1. (22 pts.)

(a) For the following 3×3 matrix A, compute its determinant by using the cofactor formula and expanding along the third column. Show the values of the 3 cofactors you compute.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -2 \\ 1 & -4 & 1 \end{bmatrix}.$$

Solution:

$$\begin{aligned} \det(A) &= 3(\det\left(\begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}\right)) - 2(-\det\left(\begin{bmatrix} 1 & 2 \\ 1 & -4 \end{bmatrix}\right)) + 1(\det\left(\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}\right)) \\ &= 3(4-2) - 2(-(-4-2)) + 1(2-(-2)) \\ &= 3(2) - 2(6) + 1(4) \\ &= -2. \end{aligned}$$

The cofactors are of course the terms in the brackets in the penultimate line.

(b) Consider the matrix B obtained from A by adding rows 1 and 3 to row 2. Should $\det(B)$ equal $\det(A)$? Why?

Solution: Yes. Adding a multiple of one row to another does not change the determinant.

Compute $\det(B)$ directly from B.
Solution:

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 1 & -4 & 1 \end{bmatrix}$$

We'll expand along the second row because it looks easiest.

$$\begin{aligned} \det(B) &= 1(-(2+12)) + 0(\text{whatever}) + 2(-(-4-2)) \\ &= -14 + 12 \\ &= -2 \end{aligned}$$

Sure enough, this is the same as the answer for part (a).

2. (20 pts.) Give all values of x for which A has an eigenvalue equal to 2.

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & x & 2 \\ x & -2 & 3 \end{bmatrix}.$$

Solution: The matrix A will have 2 as an eigenvalue if and only $\det(A - 2I) = 0$. Therefore, to achieve this, it is necessary and sufficient that x satisfy

$$\begin{aligned} 0 &= \det(A - 2I) \\ &= \det\left(\begin{bmatrix} 1 & 2 & -1 \\ 2 & x-2 & 2 \\ x & -2 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 1 & 2 & -1 \\ 2 & x-2 & 2 \\ x+1 & 0 & 0 \end{bmatrix}\right) \quad \text{(after adding row 1 to row 3)} \\ &= (x+1)(4 + x - 2) \\ &= (x+1)(x+2). \end{aligned}$$

Hence, A has 2 as an eigenvalue if and only if x equals -1 or -2.

3. (25 pts.)

(a) Use the Gram-Schmidt procedure to find an **orthonormal** basis of $C(A)$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Solution:

$$u_1 = a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$u_2 = a_2 - \frac{a_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\begin{aligned} u_3 &= a_3 - \frac{a_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{a_3 \cdot u_2}{u_2 \cdot u_2} u_2 \\ &= \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} - \frac{-2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{1} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

Now

$$q_1 = \frac{1}{\|u_1\|} u_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \qquad q_2 = \frac{1}{\|u_2\|} u_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \qquad q_3 = \frac{1}{\|u_3\|} u_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}.$$

So an orthonormal basis for $C(A)$ is

$$\left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} \right\}.$$

(b) Find the projection matrix P for projecting onto $C(A)$.

Solution: Recall that with the factorization $A = QR$, the projection matrix $P = A(A^TA)^{-1}A^T$ becomes QQ^T. Hence, we have

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(c) Check your answer for P by computing Pa_1 where a_1 is the first column of A.

Solution: We have

$$Pa_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = a_1$$

as expected.

4. (28 pts.) For each of the following statements, determine if it is *always* true. If so, answer yes. Otherwise, answer no. Just circle yes or no. You do not need to justify your answer. However, we will be giving 4 points for a correct answer, 0 points for no answer at all, and **-2 points for an incorrect answer.**

(a) **Yes — No.** Let F be the vector space of all 3×3 matrices. Let T be the transformation that maps $A \in F$ to $\det(A) + \text{trace}(A)$. Then T is a linear transformation.

Solution: No (Reason: $T(2I) \neq 2T(I)$, for example)

(b) **Yes — No.** Let A be the $m \times n$ (edge-node) incidence matrix of a connected graph with n vertices and m edges. Then the left nullspace of A has dimension $m - n + 1$.

Solution: Yes (Reason: the rank of A is $n - 1$.)

(c) **Yes — No.** An orthogonal matrix can have an eigenvalue equal to 0.

Solution: No (Reason: an orthogonal matrix is non-singular as the columns are perpendicular and so certainly independent.)

(d) **Yes — No** Let $\hat{x}$ be the least-squares solution to $Ax = b$. Then $b - A\hat{x}$ is orthogonal to the column space of A.

Solution: Yes (By definition!)

(e) **Yes — No.** Let P be the projection matrix for projecting onto a subspace F. Then $I - P$ is the projection matrix for projecting onto $F^{\perp}$.

Solution: Yes (Reason: the sum of the projections must be the identity)

(f) **Yes — No.** There are values for a, b, c, d, e, f such that the following matrix A satisfies $A^2 = 2A$:

$$A = \begin{bmatrix} 1 & a & b \\ c & 1 & d \\ e & f & 1 \end{bmatrix}.$$

Solution: No (Reason: multiplying by any eigenvector, and then considering a non-zero entry of it and dividing out, we get that *any* eigenvalue must satisfy $\lambda^2 = 2\lambda$, ie, $\lambda = 0$ or 2. But the sum of the eigenvalues of A is $\text{trace}(A) = 3$ which cannot be obtained by adding 0's and 2's.)

(g) **Yes — No.** Let 0 denote the 5×5 matrix whose entries are all 0. If $A^{10} = 0$ then $A = 0$.

Solution: No (Counterexample: $A = 0$ except for $A_{1,10} = 1$.)

18.06 Professor Strang Quiz 2 November 2, 2005

Grading

1

2

3

Your PRINTED name is: ____________________

Please circle your recitation:

1)	M 2	2-131	P. Lee	2-087	2-1193	`lee`
2)	M 2	2-132	T. Lawson	4-182	8-6895	`tlawson`
4)	T 10	2-132	P-O. Persson	2-363A	3-4989	`persson`
5)	T 11	2-131	P-O. Persson	2-363A	3-4989	`persson`
6)	T 11	2-132	P. Pylyavskyy	2-333	3-7826	`pasha`
7)	T 12	2-132	T. Lawson	4-182	8-6895	`tlawson`
8)	T 12	2-131	P. Pylyavskyy	2-333	3-7826	`pasha`
9)	T 1	2-132	A. Chan	2-588	3-4110	`alicec`
10)	T 1	2-131	D. Chebikin	2-333	3-7826	`chebikin`
11)	T 2	2-132	A. Chan	2-588	3-4110	`alicec`
12)	T 3	2-132	T. Lawson	4-182	8-6895	`tlawson`

1 (30 pts.) The matrix A has a varying $1 - x$ in the $(1,2)$ position:

$$A = \begin{bmatrix} 2 & \mathbf{1 - x} & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{bmatrix}$$

(a) When $x = 1$ compute $\det A$. What is the $(1,1)$ entry in the inverse when $x = 1$?

(b) When $x = 0$ compute $\det A$.

(c) How do the properties of the determinant say that $\det A$ is a linear function of x? For any x compute $\det A$. For which x's is the matrix singular?

χ

2 (30 pts.) This matrix Q has orthonormal columns q_1, q_2, q_3:

$$Q = \begin{bmatrix} .1 & .5 & a \\ .7 & .5 & b \\ .1 & -.5 & c \\ .7 & -.5 & d \end{bmatrix}.$$

(a) What equations must be satisfied by the numbers a, b, c, d? Is there a unique choice for those numbers, apart from multiplying them all by -1?

(b) Why is $P = QQ^{\mathrm{T}}$ a projection matrix? (Check the two properties of projections.) Why is QQ^{T} a singular matrix? Find the determinants of $Q^{\mathrm{T}}Q$ and QQ^{T}.

(c) Suppose Gram-Schmidt starts with those same first two columns and with the third column $a_3 = (1, 1, 1, 1)$. What third column would it choose for q_3? You may leave a square root not completed (if you want to).

$\chi\chi$

3 (40 pts.) Our measurements at times $t = 1, 2, 3$ are $b = 1, 4$, and b_3. We want to fit those points by the nearest line $C + Dt$, using least squares.

(a) Which value for b_3 will put the three measurements on a straight line? *Which line is it*? Will least squares choose that line if the third measurement is $b_3 = 9$? (Yes or no).

(b) What is the linear system $Ax = b$ that would be solved exactly for $x = (C, D)$ if the three points do lie on a line? Compute the projection matrix P onto the column space of A. Remember the inverse

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

(c) What is the rank of that projection matrix P? How is the column space of P related to the column space of A? (You can answer with or without the entries of P computed in (b).)

(d) Suppose $b_3 = 1$. Write down the equation for the best least squares solution $\widehat{x}$, and show that the best straight line is horizontal.

$\chi\chi\chi$

18.06 Professor Strang Quiz 2 November 2, 2005

Grading

Your PRINTED name is: ____________________

1

2

3

Please circle your recitation:

1)	M 2	2-131	P. Lee	2-087	2-1193	lee
2)	M 2	2-132	T. Lawson	4-182	8-6895	tlawson
4)	T 10	2-132	P-O. Persson	2-363A	3-4989	persson
5)	T 11	2-131	P-O. Persson	2-363A	3-4989	persson
6)	T 11	2-132	P. Pylyavskyy	2-333	3-7826	pasha
7)	T 12	2-132	T. Lawson	4-182	8-6895	tlawson
8)	T 12	2-131	P. Pylyavskyy	2-333	3-7826	pasha
9)	T 1	2-132	A. Chan	2-588	3-4110	alicec
10)	T 1	2-131	D. Chebikin	2-333	3-7826	chebikin
11)	T 2	2-132	A. Chan	2-588	3-4110	alicec
12)	T 3	2-132	T. Lawson	4-182	8-6895	tlawson

1 (30 pts.) The matrix A has a varying $1-x$ in the $(1,2)$ position:

$$A = \begin{bmatrix} 2 & \boldsymbol{1-x} & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{bmatrix}$$

(a) When $x = 1$ compute $\det A$. What is the $(1,1)$ entry in the inverse when $x = 1$?

Solution: We expand along the first row, then use elimination, and then expand along the first column:

$$\det \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{bmatrix} = 2 \cdot \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = 2 \cdot \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{bmatrix} =$$

$$= 2 \cdot \det \begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix} = 2 \cdot 2 = 4.$$

We use the cofactor formula to find $A^{-1}(1,1)$:

$$A^{-1}(1,1) = \frac{(-1)^{1+1}}{\det A} \cdot \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = \frac{2}{4} = \frac{1}{2}.$$

(b) When $x = 0$ compute $\det A$.

Solution: Subtract the second column from the first, and then expand along the first column:

$$\det \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 3 & 9 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = 2.$$

(c) How do the properties of the determinant say that $\det A$ is a linear function of x? For any x compute $\det A$. For which x's is the matrix singular?

Solution: To find $\det A(x)$, we perform the same steps as in the previous part:

$$\det \begin{bmatrix} 2 & 1-x & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{bmatrix} = \det \begin{bmatrix} 1+x & 1-x & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 3 & 9 \end{bmatrix} =$$

$$= (1+x) \cdot \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = 2 + 2x.$$

The matrix $A(x)$ is singular when $\det A(x) = 0$, i.e. when $x = -1$.

2 (30 pts.) This matrix Q has orthonormal columns q_1, q_2, q_3:

$$Q = \begin{bmatrix} .1 & .5 & a \\ .7 & .5 & b \\ .1 & -.5 & c \\ .7 & -.5 & d \end{bmatrix}.$$

(a) What equations must be satisfied by the numbers a, b, c, d? Is there a unique choice for those numbers, apart from multiplying them all by -1?

Solution: We need the vector (a, b, c, d) to be a unit vector orthogonal to the other two columns of Q. Hence the numbers a, b, c, and d must satisfy the following equations:

$$a^2 + b^2 + c^2 + d^2 = 1;$$

$$.1a + .7b + .1c + .7d = 0;$$

$$.5a + .5b - .5c - .5d = 0.$$

The last two equations define a two-dimensional subspace of $\mathbb{R}^4$. Taking two independent vectors in this subspace and normalizing them gives two good choices of a, b, c, and d that are not merely negatives of each other. Hence the choice of the last column of Q is not unique.

(b) Why is $P = QQ^{\mathrm{T}}$ a projection matrix? (Check the two properties of projections.) Why is QQ^{T} a singular matrix? Find the determinants of $Q^{\mathrm{T}}Q$ and QQ^{T}.

Solution: P is a projection matrix if it satisfies two conditions: P is symmetric, and $P^2 = P$. And indeed, we have

$$P^{\mathrm{T}} = (QQ^{\mathrm{T}})^{\mathrm{T}} = (Q^{\mathrm{T}})^{\mathrm{T}}Q^{\mathrm{T}} = QQ^{\mathrm{T}} = P;$$

$$P^2 = Q(Q^{\mathrm{T}}Q)Q^{\mathrm{T}} = QQ^{\mathrm{T}} = P$$

since $Q^{\mathrm{T}}Q = I$ by orthonormality of Q. (Another valid argument is that P is the matrix for projection onto the column space of Q, as can be easily checked using the projection matrix formula.)

The matrix $P = QQ^{\mathrm{T}}$ is singular because both Q and Q^{T} have rank 3, and hence their product P has rank at most 3 (in fact, exactly 3). Since P is a 4 by 4 matrix, it is not full rank and hence singular.

Since QQ^{T} is singular, we have $\det QQ^{\mathrm{T}} = 0$. Also, $\det Q^{\mathrm{T}}Q = \det I = 1$.

(c) Suppose Gram-Schmidt starts with those same first two columns and with the third column $a_3 = (1, 1, 1, 1)$. What third column would it choose for q_3? You may leave a square root not completed (if you want to).

Solution: The Gram-Schmidt process first produces a vector Q_3 orthogonal to both q_1 and q_2:

$$Q_3 = a_3 - (q_1 \cdot a_3)q_1 - (q_2 \cdot a_3)q_2$$

(remember that q_1 and q_2 are unit vectors). We have $q_2 \cdot a_3 = 0$ and $q_1 \cdot a_3 = 1.6$, so

$$Q_3 = (1, 1, 1, 1) - 1.6 \cdot (.1, .7, .1, .7) = (.84, -.12, .84, -.12).$$

Normalizing, we get

$$q_3 = \frac{Q_3}{||Q_3||} = (.7, -.1, .7, -.1).$$

3 (40 pts.) Our measurements at times $t = 1, 2, 3$ are $b = 1, 4$, and b_3. We want to fit those points by the nearest line $C + Dt$, using least squares.

(a) Which value for b_3 will put the three measurements on a straight line? *Which line is it*? Will least squares choose that line if the third measurement is $b_3 = 9$? (Yes or no).

Solution: The three data points lie on the same line when $b_3 = 7$. This line is $-2 + 3t$. If $b_3 = 9$, the least squares method will NOT choose this line. (A quick way to see this is from the fact that the line chosen by least squares will give the average of the given b's at the time equal to the average of the given t's; in this case, the best fit line would take the value $(1 + 3 + 9)/3 = 13/3$ at $t = (1 + 2 + 3)/3 = 2$, whereas our line gives 4 at $t = 2$.)

(b) What is the linear system $Ax = b$ that would be solved exactly for $x = (C, D)$ if the three points do lie on a line? Compute the projection matrix P onto the column space of A. Remember the inverse

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Solution: The linear system for $x = (C, D)$ would be the following:

$$Ax = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} x = \begin{bmatrix} 1 \\ 4 \\ b_3 \end{bmatrix}.$$

We compute the projection matrix P onto the column space of A using the projection matrix formula:

$$P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = \frac{1}{6} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} =$$

$$\frac{1}{6}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 8 & 2 & -4 \\ -3 & 0 & 3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}.$$

(c) What is the rank of that projection matrix P? How is the column space of P related to the column space of A? (You can answer with or without the entries of P computed in (b).)

Solution: The column space of P is the space consisting of all the vectors Pb, i.e. all the projections of vectors in $\mathbb{R}^3$ onto the column space of A, which is precisely the column space of A. Thus the rank of P is equal to the rank of A, which is 2.

(d) Suppose $b_3 = 1$. Write down the equation for the best least squares solution $\widehat{x}$, and show that the best straight line is horizontal.

Solution: The equation for the best least squares solution $\widehat{x}$ is $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$, where $b = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$. Writing out this system, we get

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}\widehat{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}.$$

The solution to this system is $\widehat{x} = \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, so the best fit line is the horizontal line $b = 2$.

18.06 Professor Strang Quiz 2 April 1, 2005

Grading

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Your PRINTED name is: ______________________

Please circle your recitation:

1)	M 2	2-131	A. Chan	2-588	3-4110	alicec
2)	M 3	2-131	A. Chan	2-588	3-4110	alicec
3)	M 3	2-132	D. Testa	2-586	3-4102	damiano
4)	T 10	2-132	C.I. Kim	2-273	3-4380	ikim
5)	T 11	2-132	C.I. Kim	2-273	3-4380	ikim
6)	T 12	2-132	W.L. Gan	2-101	3-3299	wlgan
7)	T 1	2-131	C.I. Kim	2-273	3-4380	ikim
8)	T 1	2-132	W.L. Gan	2-101	3-3299	wlgan
9)	T 2	2-132	W.L. Gan	2-101	3-3299	wlgan

1 (17 pts.) If the output vectors from Gram-Schmidt are

$$q_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad \text{and} \quad q_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

describe all possible input vectors a_1 and a_2.

2 (15 pts.) If a and b are nonzero vectors in $\mathbf{R}^n$, what number x minimizes the squared length $\|b - xa\|^2$?

3 (17 pts.) Find the projection p of the vector $b = (1, 2, 6)$ onto the plane $x + y + z = 0$ in $\mathbf{R}^3$. (You may want to find a basis for this 2-dimensional subspace, even an orthogonal basis.)

4 (17 pts.) Find the determinants of A and A^{-1} and the $(1,2)$ entry of A^{-1} if

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 3 \\ 1 & 3 & 1 & 7 \end{bmatrix}.$$

5 (17 pts.) By recursion or cofactors or otherwise(!) compute the determinant of this 5 by 5 circulant matrix C:

$$C = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

6 (17 pts.) Suppose P_1 is the projection matrix onto the 1-dimensional subspace spanned by the first column of A. Suppose P_2 is the projection matrix onto the 2-dimensional column space of A. After thinking a little, compute the product $P_2 P_1$.

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Exam 2, Friday April 1st, 2005

Solutions

Question 1. The vector a_1 can be any non-zero positive multiple of q_1. The vector a_2 can be any multiple of q_1 plus any non-zero positive multiple of q_2:

$$\begin{array}{rcl} a_1 & = & cq_1 \\ a_2 & = & c_1 q_1 + c_2 q_2 \end{array} \quad , \text{ with } c, c_1 > 0.$$

Question 2. We want to find the least squares solution to the equation

$$ax = b$$

and we know that it is enough to multiply both sides by a^T and solve the resulting system:

$$a^T a x = a^T b \qquad \Longrightarrow \qquad x = \frac{a \cdot b}{a \cdot a}$$

Question 3. The vectors $(-1,1,0)^T$ and $(-1,0,1)^T$ form a basis for the subspace $x + y + z = 0$. Let A be the matrix whose columns are the two vectors found above. Thus the projection matrix P onto the subspace $x + y + z = 0$ is

$$\begin{aligned} P &= A\left(A^T A\right)^{-1} A^T = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \\ &= \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \\ &= \frac{1}{3} \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = \\ &= \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \end{aligned}$$

The projection of $(1,2,6)^T$ onto the plane $x + y + z = 0$ is thus simply

$$p = P \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$$

Question 4. Looking at the first row of A we deduce that

$$\det A = \det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 1 & 3 & 9 \end{pmatrix} = 9 - 6 = 3$$

Of course, $\det(A^{-1}) = \frac{1}{3}$. Finally

$$\left(A^{-1}\right)_{12} = \frac{-C_{21}}{\det A} = \frac{-1}{3}\det\begin{pmatrix}0 & 1 & 0\\ 2 & 1 & 3\\ 3 & 1 & 9\end{pmatrix} = \frac{-(-9)}{3} = 3$$

Question 5. (a) The column space of QQ^T is at most two dimensional, since the matrix QQ^T is 4×4, it cannot have rank four. Thus $\det QQ^T = 0$.

Similarly, the matrix $[\ Q\ Q\ ]$ has dependent columns, and therefore $\det[\ Q\ Q\ ] = 0$.

(b) Using the projection formula,

$$p = Q(Q^TQ)^{-1}Q^Tb = QIQ^Tb = QQ^Tb$$

(c) The error vector $e = b - p$ is contained in the left null-space of Q, the nullspace of Q^T. To check this, we compute

$$Q^Te = Q^T\left(b - QQ^Tb\right) = Q^Tb - Q^TQQ^Tb = Q^Tb - IQ^Tb = 0$$

Question 6. The product P_2P_1 is projection onto the column space of P_1, followed by the projection onto the column space of P_2. Since the column space of P_2 contains the column space of P_1, the second projection does not change the vectors anymore. Thus

$$P_2P_1 = P_1 = \begin{pmatrix}1\\2\\0\\1\end{pmatrix}\left(\begin{pmatrix}1 & 2 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\2\\0\\1\end{pmatrix}\right)^{-1}\begin{pmatrix}1 & 2 & 0 & 1\end{pmatrix} = \frac{1}{6}\begin{pmatrix}1 & 2 & 0 & 1\\ 2 & 4 & 0 & 2\\ 0 & 0 & 0 & 0\\ 1 & 2 & 0 & 1\end{pmatrix}$$

18.06 Professor Strang Quiz 2 November 1, 2006

Grading

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Your PRINTED name is: ____________________

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Please circle your recitation:

1)	T 10	2-131	K. Meszaros	2-333	3-7826	karola
2)	T 10	2-132	A. Barakat	2-172	3-4470	barakat
3)	T 11	2-132	A. Barakat	2-172	3-4470	barakat
4)	T 11	2-131	A. Osorno	2-229	3-1589	aosorno
5)	T 12	2-132	A. Edelman	2-343	3-7770	edelman
6)	T 12	2-131	K. Meszaros	2-333	3-7826	karola
7)	T 1	2-132	A. Edelman	2-343	3-7770	edelman
8)	T 2	2-132	J. Burns	2-333	3-7826	burns
9)	T 3	2-132	A. Osorno	2-229	3-1589	aosorno

1 (24 pts.) Suppose q_1, q_2, q_3 are orthonormal vectors in $\mathbb{R}^3$. Find **all possible values** for these 3 by 3 determinants and explain your thinking in 1 sentence each.

(a) $\det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} =$

(b) $\det \begin{bmatrix} q_1 + q_2 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} =$

(c) $\det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ times $\det \begin{bmatrix} q_2 & q_3 & q_1 \end{bmatrix} =$

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2 (24 pts.) Suppose we take measurements at the 21 equally spaced times $t = -10, -9, \ldots, 9, 10$. All measurements are $b_i = 0$ except that $b_{11} = 1$ at the middle time $t = 0$.

(a) Using least squares, what are the best $\widehat{C}$ and $\widehat{D}$ to fit those 21 points by a straight line $C + Dt$?

(b) You are projecting the vector b onto what subspace? (*Give a basis.*) Find a nonzero vector perpendicular to that subspace.

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3 (9 + 12 + 9 pts.) The Gram-Schmidt method produces orthonormal vectors q_1, q_2, q_3 from independent vectors a_1, a_2, a_3 in $\mathbb{R}^5$. Put those vectors into the columns of 5 by 3 matrices Q and A.

(a) Give formulas using Q and A for the projection matrices P_Q and P_A onto the column spaces of Q and A.

(b) ***Is $\boldsymbol{P_Q = P_A}$ and why ? What is $\boldsymbol{P_Q}$ times $\boldsymbol{Q}$? What is* det $\boldsymbol{P_Q}$ *?***

(c) Suppose a_4 is a new vector and a_1, a_2, a_3, a_4 are independent. Which of these (if any) is the new Gram-Schmidt vector q_4? (P_A and P_Q from above)

1. $\dfrac{P_Q a_4}{\|P_Q a_4\|}$ **2.** $\dfrac{a_4 - \frac{a_4^{\mathrm{T}} a_1}{a_1^{\mathrm{T}} a_1} a_1 - \frac{a_4^{\mathrm{T}} a_2}{a_2^{\mathrm{T}} a_2} a_2 - \frac{a_4^{\mathrm{T}} a_3}{a_3^{\mathrm{T}} a_3} a_3}{\|\text{ norm of that vector }\|}$ **3.** $\dfrac{a_4 - P_A a_4}{\|a_4 - P_A a_4\|}$

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4 (22 pts.) Suppose a 4 by 4 matrix has the same entry $\times$ throughout its first row and column. The other 9 numbers could be anything like $1, 5, 7, 2, 3, 99, \pi, e, 4$.

$$A = \begin{bmatrix} \times & \times & \times & \times \\ \times & & \text{any numbers} & \\ \times & & \text{any numbers} & \\ \times & & \text{any numbers} & \end{bmatrix}$$

(a) The determinant of A is a polynomial in $\times$. What is the largest possible degree of that polynomial? **Explain your answer**.

(b) If those 9 numbers give the identity matrix I, what is $\det A$? Which values of $\times$ give $\det A = 0$?

$$A = \begin{bmatrix} \times & \times & \times & \times \\ \times & 1 & 0 & 0 \\ \times & 0 & 1 & 0 \\ \times & 0 & 0 & 1 \end{bmatrix}$$

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18.06 Professor Strang Quiz 2 November 1, 2006

Grading

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Your PRINTED name is: SOLUTIONS

Please circle your recitation:

1)	T 10	2-131	K. Meszaros	2-333	3-7826	karola
2)	T 10	2-132	A. Barakat	2-172	3-4470	barakat
3)	T 11	2-132	A. Barakat	2-172	3-4470	barakat
4)	T 11	2-131	A. Osorno	2-229	3-1589	aosorno
5)	T 12	2-132	A. Edelman	2-343	3-7770	edelman
6)	T 12	2-131	K. Meszaros	2-333	3-7826	karola
7)	T 1	2-132	A. Edelman	2-343	3-7770	edelman
8)	T 2	2-132	J. Burns	2-333	3-7826	burns
9)	T 3	2-132	A. Osorno	2-229	3-1589	aosorno

1 (24 pts.) Suppose q_1, q_2, q_3 are orthonormal vectors in $\mathbb{R}^3$. Find **all possible values** for these 3 by 3 determinants and explain your thinking in 1 sentence each.

(a) $\det\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} =$

(b) $\det\begin{bmatrix} q_1 + q_2 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} =$

(c) $\det\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ times $\det\begin{bmatrix} q_2 & q_3 & q_1 \end{bmatrix} =$

Solution.

(a) The determinant of any square matrix with orthonormal columns ("orthogonal matrix") is ± 1.

(b) Here are two ways you could do this:

(1) The determinant is *linear in each column*:

$$\begin{aligned}
\det\begin{bmatrix} q_1 + q_2 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} &= \det\begin{bmatrix} q_1 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} + \det\begin{bmatrix} q_2 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} \\
&= \det\begin{bmatrix} q_1 & q_2 + q_3 & q_3 \end{bmatrix} + \det\begin{bmatrix} q_2 & q_3 & q_3 + q_1 \end{bmatrix} \\
&= \det\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} + \det\begin{bmatrix} q_2 & q_3 & q_1 \end{bmatrix}
\end{aligned}$$

Both of these determinants are equal (see (c)), so the total determinant is ± 2.

(2) You could also *use row reduction*. Here's what happens:

$$\begin{aligned}
\det\begin{bmatrix} q_1+q_2 & q_2+q_3 & q_3+q_1 \end{bmatrix} &= \det\begin{bmatrix} q_1+q_2 & -q_1+q_3 & q_3+q_1 \end{bmatrix} \\
&= \det\begin{bmatrix} q_1+q_2 & -q_1+q_3 & 2q_3 \end{bmatrix} \\
&= 2\det\begin{bmatrix} q_1+q_2 & -q_1+q_3 & q_3 \end{bmatrix} \\
&= 2\det\begin{bmatrix} q_1+q_2 & -q_1 & q_3 \end{bmatrix} \\
&= 2\det\begin{bmatrix} q_2 & -q_1 & q_3 \end{bmatrix} \\
&= 2\det\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}
\end{aligned}$$

Again, whatever $\det\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ is, this determinant will be twice that, or ± 2.

(c) The second matrix is an *even* permutation of the columns of the first matrix (swap q_1/q_2 then swap q_2/q_3), so it has the *same* determinant as the first matrix. Whether the first matrix has determinant $+1$ or -1, the product will be $+1$.

2 (24 pts.) Suppose we take measurements at the 21 equally spaced times $t = -10, -9, \ldots, 9, 10$. All measurements are $b_i = 0$ except that $b_{11} = 1$ at the middle time $t = 0$.

(a) Using least squares, what are the best $\widehat{C}$ and $\widehat{D}$ to fit those 21 points by a straight line $C + Dt$?

(b) You are projecting the vector b onto what subspace? (*Give a basis.*) Find a nonzero vector perpendicular to that subspace.

Solution.

(a) If the line went exactly through the 21 points, then the 21 equations

$$\begin{bmatrix} 1 & -10 \\ 1 & -9 \\ \vdots & \vdots \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 10 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

would be exactly solvable. Since we can't solve this equation $Ax = b$ exactly, we look for a least-squares solution $A^{\mathrm{T}} A\hat{x} = A^{\mathrm{T}} b$.

$$\begin{bmatrix} 21 & 0 \\ 0 & 770 \end{bmatrix} \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So the line of best fit is the horizontal line $\widehat{C} = \frac{1}{21}$, $\widehat{D} = 0$.

(b) We are projecting b onto the column space of A above (basis: $\begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} -10 & \ldots & 10 \end{bmatrix}^{\mathrm{T}}$). There are lots of vectors perpendicular to this subspace; one is the error vector $e = b - P_A b = \frac{1}{21} \begin{bmatrix} (\text{ten } -1\text{'s}) & 20 & (\text{ten } -1\text{'s}) \end{bmatrix}^{\mathrm{T}}$.

3 (9 + 12 + 9 pts.) The Gram-Schmidt method produces orthonormal vectors q_1, q_2, q_3 from independent vectors a_1, a_2, a_3 in $\mathbb{R}^5$. Put those vectors into the columns of 5 by 3 matrices Q and A.

(a) Give formulas using Q and A for the projection matrices P_Q and P_A onto the column spaces of Q and A.

(b) ***Is $\boldsymbol{P_Q = P_A}$ and why ? What is $\boldsymbol{P_Q}$ times $\boldsymbol{Q}$? What is* det $\boldsymbol{P_Q}$ *?***

(c) Suppose a_4 is a new vector and a_1, a_2, a_3, a_4 are independent. Which of these (if any) is the new Gram-Schmidt vector q_4? (P_A and P_Q from above)

$$\mathbf{1.}\ \frac{P_Q a_4}{\|P_Q a_4\|} \qquad \mathbf{2.}\ \frac{a_4 - \frac{a_4^{\mathrm{T}} a_1}{a_1^{\mathrm{T}} a_1} a_1 - \frac{a_4^{\mathrm{T}} a_2}{a_2^{\mathrm{T}} a_2} a_2 - \frac{a_4^{\mathrm{T}} a_3}{a_3^{\mathrm{T}} a_3} a_3}{\|\text{ norm of that vector }\|} \qquad \mathbf{3.}\ \frac{a_4 - P_A a_4}{\|a_4 - P_A a_4\|}$$

Solution.

(a) $P_A = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ and $P_Q = Q(Q^{\mathrm{T}}Q)^{-1}Q^{\mathrm{T}} = QQ^{\mathrm{T}}$.

(b) $P_A = P_Q$ because both projections project onto the same subspace. *(Some people did this the hard way, by substituting $A = QR$ into the projection formula and simplifying. That also works.)* The determinant is zero, because P_Q is singular (like all non-identity projections): all vectors orthogonal to the column space of Q are projected to 0.

(c) Answer: choice 3. (Choice 2 is tempting, and would be correct if the a_i were replaced by the q_i. But the a_i are not orthogonal!)

4 (22 pts.) Suppose a 4 by 4 matrix has the same entry $\times$ throughout its first row and column. The other 9 numbers could be anything like $1, 5, 7, 2, 3, 99, \pi, e, 4$.

$$A = \begin{bmatrix} \times & \times \quad \times \quad \times \\ \times & \text{any numbers} \\ \times & \text{any numbers} \\ \times & \text{any numbers} \end{bmatrix}$$

(a) The determinant of A is a polynomial in $\times$. What is the largest possible degree of that polynomial? **Explain your answer**.

(b) If those 9 numbers give the identity matrix I, what is $\det A$? Which values of $\times$ give $\det A = 0$?

$$A = \begin{bmatrix} \times & \times & \times & \times \\ \times & 1 & 0 & 0 \\ \times & 0 & 1 & 0 \\ \times & 0 & 0 & 1 \end{bmatrix}$$

Solution.

(a) Every term in the big formula for $\det(A)$ takes one entry from each row and column, so we can choose at most two $\times$'s and the determinant has degree 2.

(b) You can find this by cofactor expansion; here's another way:

$$\det(A) = \times \det \begin{bmatrix} 1 & \times & \times & \times \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \times \det \begin{bmatrix} 1-3\times & \times & \times & \times \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \times(1-3\times) \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \times(1 - 3\times).$$

This is zero when $\times = 0$ or $\times = \frac{1}{3}$.

18.06 **QUIZ 2** April 06, 2007

Your PRINTED name is: SOLUTIONS

Please circle your recitation:

(1)	M 2	2-131	A. Osorno
(2)	M 3	2-131	A. Osorno
(3)	M 3	2-132	A. Pissarra Pires
(4)	T 11	2-132	K. Meszaros
(5)	T 12	2-132	K. Meszaros
(6)	T 1	2-132	Jerin Gu
(7)	T 2	2-132	Jerin Gu

Grading

1
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Total:

Problem 1 *(25 points)*

(a) Compute the determinant of the matrix $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 4 & 1 & 1 \end{pmatrix}$

(b) Compute the determinant of the matrix $B = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 4 & 1 & 1 \\ 5 & 1 & 1 & 1 \end{pmatrix}$

(c) Show that the matrix B from (b) is invertible and calculate the entry $(1, 4)$ of the inverse matrix B^{-1}.

Solution 1

(a) Using the 3×3 "big formula": 3+4+2-24-1-1=-17.

(b) $det \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 4 & 1 & 1 \\ 5 & 1 & 1 & 1 \end{pmatrix} = det \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & -4 & -4 & -9 \end{pmatrix} = 74.$

(c) Since $detB = 74 \neq 0$, B is invertible.

By cofactors, the (1,4) entry of B^{-1} is $(-1)^5 C_{4,1} = \frac{-(-17)}{74} = \frac{17}{74}$.

Problem 2 *(25 points)*

(a) Compute the projection of the vector $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ onto the column space of $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 0 & -2 \end{pmatrix}$.

(Hint: first check whether A has linearly independent columns.)

(b) Find the least-square solution $\hat{x}$ for the system

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(c) Find the projection of the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ onto the column space of $\begin{pmatrix} 10000 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 0 & -2 \end{pmatrix}$.

(Hint: No computations!)

Solution 2

(a) The third column is in the span of the first two columns. So in order to calculate the projection we need to use the matrix $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The projection is $p = B(B^T B)^{-1} B^T b = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$.

(b) $\hat{x} = (B^T B)^{-1} B^T b = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$.

(c) The columns are linearly independent so the projection is $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$.

Problem 3 *(25 points)*

Consider the basis $a_1 = (1,0,1,0)^T$, $a_2 = (1,1,1,1)^T$, $a_3 = (0,0,2,0)^T$, $a_4 = (0,0,0,2)^T$ of $\mathbb{R}^4$. Transform this basis into an orthogonal basis using the Gram-Schmidt process.

In other words, find the orthogonal basis b_1, b_2, b_3, b_4 of $\mathbb{R}^4$ such that

$$\begin{aligned} b_1 &= a_1, \\ b_2 &= a_2 - (\text{some coefficient})\, b_1, \\ b_3 &= a_3 - (\text{some linear combination of } b_1, b_2), \\ b_4 &= a_4 - (\text{some linear combination of } b_1, b_2, b_3). \end{aligned}$$

Solution 3

$$b_1 = a_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$b_2 = a_2 - \frac{b_1^T a_2}{b_1^T b_1} b_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$b_3 = a_3 - \frac{b_1^T a_3}{b_1^T b_1} b_1 - \frac{b_2^T a_3}{b_2^T b_2} b_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$b_4 = a_4 - \frac{b_1^T a_4}{b_1^T b_1} b_1 - \frac{b_2^T a_4}{b_2^T b_2} b_2 - \frac{b_3^T a_4}{b_3^T b_3} b_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

It was important to use b_2 and not a_2 to compute b_3, and similarly, use b_2 and b_3 and not a_2 and a_3 to compute b_4.

Problem 4 *(25 points)*

Consider the matrix $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$.

(a) Show that the columns of A are orthogonal to each other.

(b) Calculate the determinant of A.

(c) Calculate the inverse matrix A^{-1}.

Solution 4

(a) Check the 6 dot products between the columns are all zero.

(b) $det \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = det \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 4.$

(c) The easiest way of computing this inverse is to use part (a):

$$A^T A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 2I$$

Thus $A^{-1} = \frac{1}{2}A^T$.

18.06 Professor Johnson Quiz 1 October 3, 2007

SOLUTIONS

1 (30 pts.) A given circuit network (directed graph) which has an $m \times n$ incidence matrix A (rows = edges, columns = nodes) and a conductance matrix C [diagonal = inverse of the (positive) resistance of each edge] given by:

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1/R_1 & 0 & 0 & 0 \\ 0 & 1/R_2 & 0 & 0 \\ 0 & 0 & 1/R_3 & 0 \\ 0 & 0 & 0 & 1/R_4 \end{pmatrix}.$$

Suppose the unknowns are the vector $\mathbf{v}$ of voltages at each node, and you are given a vector $\mathbf{d}$ of applied voltage drops across each edge (e.g. if you connect a battery to each edge). In this case, Kirchhoff's laws plus Ohm's law gives the equation:

$$A^T CA\mathbf{v} = A^T C\mathbf{d}.$$

(a) Sketch the network, labelling each edge from 1 to 4 corresponding to each row of A, and each node from ① to ③ corresponding to each column of A, and put an arrow to show the direction of each edge.

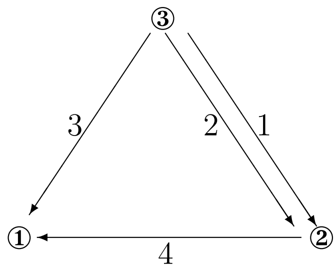
(b) Is $A^T CA\mathbf{v} = A^T C\mathbf{d}$ always solvable for all $\mathbf{d}$? Why or why not? [You can use the fact, from class, that rank$(A^T CA)$ = rank$(A) = n - 1$. Hint: think about the subspaces; little or no calculation is necessary. This is *not* the same as whether $A^T CA\mathbf{v} = \mathbf{s}$ is solvable for all $\mathbf{s}$.]

(c) Solve for $\mathbf{v}$ when $C = I$ (all resistances = 1) and $\mathbf{d} = \begin{pmatrix} 5 & 0 & 0 & 0 \end{pmatrix}^T$. To get a unique solution, set the voltage on node 1 to $v_1 = 0$ ("ground")—this simplifies life to a 3×2 matrix problem, since you then only have 2 unknowns v_2 and v_3. [Recall that the null space of A is the span of $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$, so we can add any constant to the solutions.]

(d) For the same $\mathbf{d}$ as in (c), what is the minimum value (minimum over all $\mathbf{v}$) of $\|A\mathbf{v} - \mathbf{d}\|^2$?

Solution:

(a)



(b) Yes.

Since $\text{rank}(A^TCA) = \text{rank}(A) = n - 1$, and $\text{rank}(A^TCA) \leq \text{rank}(A^TC) \leq \text{rank}(A^T) = \text{rank}(A)$, we see that $\text{rank}(A^TCA) = \text{rank}(A^TC)$. Since the column space of A^TC contains the column space of A^TCA, and both have the same dimension, we see that the two column space are the same. Thus for any $\mathbf{d}$, $A^TC\mathbf{d}$ lies in the column space of A^TCA. In other words, the equation is always solvable.

Alternatively, it is also sufficient to say that the column space $C(A^TC)$ is clearly at least *contained* in $C(A^T) = C(A^TCA)$, with the latter equality because $C(A^T)$ and $C(A^TCA)$ have the same dimension $[\text{rank}(A^TCA) = \text{rank}(A) = \text{rank}(A^T)]$ and $C(A^TCA) \subseteq C(A^T)$. (Recall that $C(AB) \subseteq C(A)$ for any A and B, since $AB\mathbf{x}$ is made of the columns of A.)

Common errors: Many students wrote that, since A^TCA is singular, there isn't always a solution—this is incorrect because the right-hand side is not an arbitrary vector, it is only vectors $A^TC\mathbf{d}$ in $C(A^TC)$. Several students wrote that we must have $A\mathbf{v} = \mathbf{d}$, which is not true since A^TC is not invertible (or even square). Many students wrote that, if you ignore the C, this is just like a least-squares problem and least-squares problems are always solvable—this is on the right track (the A^T on the right-hand side is truly the key here), but the least-squares problems we've studied were only when A has full column rank, which isn't true here.

(c) The equation becomes $A^T A\mathbf{v} = A^T\mathbf{d}$. We calculate:

$$A^T A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 3 & -2 \\ -1 & -2 & 3 \end{pmatrix}, \qquad A^T\mathbf{d} = \begin{pmatrix} 0 \\ 5 \\ -5 \end{pmatrix},$$

so we can solve by elimination:

$$\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -2 & 5 \\ -1 & -2 & 3 & -5 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & -1 & 0 \\ 0 & 5/2 & -5/2 & 5 \\ 0 & -5/2 & 5/2 & -5 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & -1 & 0 \\ 0 & 5/2 & -5/2 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & -1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so the general solution is $\mathbf{v} = \begin{pmatrix} a+1 & a+2 & a \end{pmatrix}^T$, where a is an arbitrary number (the multiple of the nullspace vector). To get $v_1 = 0$, let $a = -1$, and the unique solution is:

$$\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

A faster way: We can set $v_1 = 0$ immediately after constructing $A^T A$, equivalent to deleting the first column (which is multiplied by zero), leaving us with the 3×2 problem:

$$\begin{pmatrix} -1 & -1 & 0 \\ 3 & -2 & 5 \\ -2 & 3 & -5 \end{pmatrix} \to \begin{pmatrix} -1 & -1 & 0 \\ 0 & -5 & 5 \\ 0 & 5 & -5 \end{pmatrix} \to \begin{pmatrix} -1 & -1 & 0 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

which has the solution $v_3 = -1$, $v_2 = 1$ as above.

Another fast way: if we set $v_1 = 0$ at the very beginning, i.e. delete the first column of A to obtain a 4×2 matrix, then $A^T A = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$ and $A^T \mathbf{d} = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$, and the solution is the same as above.

(d) In order to minimize $\|A\mathbf{v} - \mathbf{d}\|^2$, we would solve $A^T A\mathbf{v} = A^T\mathbf{d}$, but this is precisely what we already did in part (c)! So, the minimum value is at $\mathbf{v} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}^T$ from above, and is given by:

$$\left\| A \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \mathbf{d} \right\|^2 = \left\| \begin{pmatrix} 2 \\ 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\|^2 = (-3)^2 + 2^2 + 1^2 + (-1)^2 = 15.$$

2 (30 pts.) Fill in the blanks below: (You don't need to justify your answer.)

(a) The nullspace of AB contains the nullspace of $\underline{\boxed{\text{B}}}$.

If $B\mathbf{x} = \mathbf{0}$ *then* $AB\mathbf{x} = \mathbf{0}$.

(b) Let P be the projection matrix to the row space of a matrix A, then $I - P$ is the projection matrix to $\underline{\boxed{\text{N(A)}}}$.

Reason: $I - P$ *is the projection onto the orthogonal complement, and the orthogonal complement of the row space is the nullspace.*

(c) Suppose A is an $m \times n$ matrix, and the row space of A is n dimensional, then its nullspace is $\underline{\boxed{0}}$ dimensional.

The rank r *of* A *is the dimension of the row space, so* $r = n$, *and the nullspace has dimension* $n - r = n - n = 0$.

(d) Let $\hat{\mathbf{x}}$ be the least-squares solution to $A\mathbf{x} = \mathbf{b}$. Then $\mathbf{b} - A\hat{\mathbf{x}}$ is orthogonal to the $\underline{\boxed{\text{column space}}}$ of A.

The least-squares solution solves $A\hat{\mathbf{x}} = P\mathbf{b}$, *where* P *is the projection onto* $C(A)$, *so* $\mathbf{b} - A\hat{\mathbf{x}} = \mathbf{b} - P\mathbf{b} = (I - P)\mathbf{b}$, *and* $I - P$ *projects onto the complement of* $C(A)$. *Equivalently, the least-squares solution finds the* closest *point* $A\hat{\mathbf{x}}$ *to* $\mathbf{b}$ *in* $C(A)$, *so the difference must be perpendicular to* $C(A)$.

(e) If $A^T = -A$ (A is antisymmetric), and A is $n \times n$ where n is odd, then $\det A = \underline{\boxed{0}}$. .

Reason: $\det A = \det A^T = \det(-A) = (-1)^n \det A = -\det A$ *since* n *is odd, and the only way to have* $\det A = -\det A$ *is if* $\det A = 0$.

(f) If A is symmetric and P is the projection matrix onto the nullspace $N(A)$, then $PA = \underline{\boxed{0}}$.

Since A *is symmetric,* $N(A) = N(A^T) = C(A)^\perp$, *so* P *projects onto the orthogonal complement of* $C(A)$. *Thus,* $PA = 0$ *since* P *projects every column of* A *to zero.*

3 (12 pts.) Construct an example of a least-square curve-fitting problem where the solution (the least-square fit parameters) is *not unique*. (You need not solve it, just write down the $A\mathbf{x} = \mathbf{b}$ equations that you *would* solve by least-squares to minimize $\|A\mathbf{x} - \mathbf{b}\|^2$.)

Solution: The solution is not unique if the matrix A is not of full column rank. (Problem 1(c) is an example of this type.) *There are many possible examples.*

For example, you could have more unknowns than data points. e.g., you could be fitting to a line $C + Dt$, but only have a single point (t_1, b_1)—obviously, a single point is not enough to determine a line uniquely. In terms of matrices, $A = \begin{pmatrix} 1 & t_1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} C & D \end{pmatrix}^T$, and $\mathbf{b} = (b_1)$, which obviously does not have full column rank: there are more columns than rows in A!

Alternatively, you could be fitting a line $C + Dt$ to multiple points (t_1, b_1), (t_1, b_2), (t_1, b_3) etcetera with the same t coordinate—this is enough information to determine C or D but not both. In this case you get a matrix equation of the form:

$$\begin{pmatrix} 1 & t_1 \\ 1 & t_1 \\ \vdots & \vdots \\ 1 & t_1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

which obviously does not have full column rank: (column 2) $= t_1$(column 1) in A.

You could also construct an example where your fit parameters are not really independent, regardless of the data. For example, if you are fitting to $C + Dt + E(3t - 1)$, in which case E does not add any information because $3t - 1$ is a linear combination of C and Dt. Correspondingly, the third column of A will be three times the second column minus the first column.

Perhaps the most trivial example of all is where you have no data points whatsoever, in which case there is no information to constrain the fit. In terms of matrices, though, this is a bit too weird because it would correspond to a matrix A with zero rows, and we usually consider only matrices with positive sizes.

4 (28 pts.) Let $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{pmatrix}$. Ordinary Gram-Schmidt would take the columns of A and produce an orthonormal basis Q with $C(A) = C(Q)$. In this problem, we will modify that process and see what happens.

In particular, suppose that we proceed as in Gram-Schmidt, but we omit the normalization steps—we construct a basis of *orthogonal* vectors spanning $C(A)$ but with lengths $\neq 1$, by subtracting the projections as in Gram-Schmidt but skipping the division by the lengths. Let's call this "unnormalized Gram-Schmidt."

(a) Do "unnormalized Gram-Schmidt" on A to get an orthogonal but not orthonormal basis B for $C(A)$.

(b) Compute two the (4-dimensional) volumes of the two parallelepipeds with edges given by the columns of A and the columns of B.

Recall how ordinary Gram-Schmidt corresponded to multiplying A by a sequence of matrices, leading to the QR decomposition. Now, we want to look at *unnormalized* Gram-Schmidt in the same way, in order to see what it does to the volume (determinant). The next two parts refer to an *arbitrary* $n \times n$ matrix A with independent columns, *not* the 4×4 matrix from parts (a) and (b).

(c) For an arbitrary $n \times n$ matrix A with (independent) columns $\mathbf{a}_1$, $\mathbf{a}_2$, etcetera, write down the matrix M_2 that you would multiply by A in the *first step* of unnormalized Gram-Schmidt to make the *second* column orthogonal to the first. What is $\det M_2$?

(d) Argue that the matrices M_3, M_4, etcetera that you would multiply by in subsequent steps of unnormalized Gram-Schmidt all have the same determinant as M_2. Therefore, the determinant of the final matrix B after unnormalized Gram-Schmidt is $\underline{\boxed{\det(A)}}$?

Solution:

(a) Unnormalized Gram-Schmidt:

$$\begin{aligned}
\mathbf{b}_1 &= \mathbf{a}_1 = \begin{pmatrix} 0 & 0 & 0 & 4 \end{pmatrix}^T, \\
\mathbf{b}_2 &= \mathbf{a}_2 - \frac{\mathbf{b}_1^T \mathbf{a}_2}{\mathbf{b}_1^T \mathbf{b}_1} \mathbf{b}_1 = \begin{pmatrix} 0 & 0 & 3 & 0 \end{pmatrix}^T, \\
\mathbf{b}_3 &= \mathbf{a}_3 - \frac{\mathbf{b}_1^T \mathbf{a}_3}{\mathbf{b}_1^T \mathbf{b}_1} \mathbf{b}_1 - \frac{\mathbf{b}_2^T \mathbf{a}_3}{\mathbf{b}_2^T \mathbf{b}_2} \mathbf{b}_2 = \begin{pmatrix} 0 & 2 & 0 & 0 \end{pmatrix}^T, \\
\mathbf{b}_4 &= \mathbf{a}_4 - \frac{\mathbf{b}_1^T \mathbf{a}_4}{\mathbf{b}_1^T \mathbf{b}_1} \mathbf{b}_1 - \frac{\mathbf{b}_2^T \mathbf{a}_4}{\mathbf{b}_2^T \mathbf{b}_2} \mathbf{b}_2 - \frac{\mathbf{b}_3^T \mathbf{a}_4}{\mathbf{b}_3^T \mathbf{b}_3} \mathbf{b}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T.
\end{aligned}$$

Note that we must subtract off the projections onto the $\mathbf{b}$ vectors, not the $\mathbf{a}$ vectors—the $\mathbf{b}$ vectors span the same space, but are much simpler to project onto because they are orthogonal.

(Why does this work? When creating $\mathbf{b}_3$, for example, we should subtract off the projection of $\mathbf{a}_3$ onto the span of $\mathbf{a}_1$ and $\mathbf{a}_2$, which is some projection matrix. For ordinary Gram-Schmidt, where we have orthonormal vectors $\mathbf{q}$, the projection matrix simplifies to QQ^T and we can just subtract of projections onto each $\mathbf{q}$ individually. Here, the projection onto the span of the previous $\mathbf{b}$ vectors simplifies similarly. One way to think of this is just to realize that $\mathbf{q}_k = \mathbf{b}_k / \|\mathbf{b}_k\|$, and therefore we can apply the ordinary Gram-Schmidt step with this substitution, skipping the normalization. A more complicated way is to realize that $B^T B$, while not the identity as for Q, is a diagonal matrix, and this leads to the same simplified result.)

Thus, an orthogonal but not orthonormal basis B for $C(A)$ is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}.$$

(b) The volume of the parallelepiped with edges given by the columns of A is just the determinant (or rather, its absolute value, but here the determinant is positive anyway):

$$\det(A) = \det \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{pmatrix} = (-1)^2 \det \begin{pmatrix} 4 & 4 & 4 & 4 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 4 \cdot 3 \cdot 2 \cdot 1 = 24,$$

where we have rearranged A into an upper-triangular matrix via two row swaps, and the determinant is then the product of the diagonals. The volume of the parallelepiped with edges given by the columns of B is similarly:

$$\det(B) = \det \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix} = (-1)^2 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 24 = \det(A).$$

(c) We should have

$$\begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{a}_3 & \cdots & \mathbf{a}_n \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \cdots & \mathbf{a}_n \end{pmatrix} M_2.$$

Notice that we must multiply M_2 on the *right* since we are manipulating *columns* of A. Since $\mathbf{b}_1 = \mathbf{a}_1, \mathbf{b}_2 = \mathbf{a}_2 - \frac{\mathbf{b}_1^T \mathbf{a}_2}{\mathbf{b}_1^T \mathbf{b}_1} \mathbf{b}_1$, we must have

$$M_2 = \begin{pmatrix} 1 & -\frac{\mathbf{b}_1^T \mathbf{a}_2}{\mathbf{b}_1^T \mathbf{b}_1} & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Since this is an upper-triangular matrix with 1's on the diagonal, we have $\det M_2 = 1$.

(d) In step k, we only change the vector $\mathbf{a}_k$ to $\mathbf{b}_k$, which is

$$\mathbf{b}_k = \mathbf{a}_k - \text{linear combination of } \mathbf{b}_1, \cdots, \mathbf{b}_{k-1},$$

thus the matrix M_k is an upper triangular matrix with diagonal entries 1 (and only nonzero off-diagonal entries are in the k^{th} column). Therefore, $\det(M_k) = 1$ for all k. This implies, finally, that $\det(B) = \det(AM_2M_3 \cdots M_n) = \det(A)\det(M_1) \cdots \det(M_n) = \det(A)$.

QUIZ 2 SOLUTIONS

1. (10 points). $det(-A^t) = (-1)^{1000} det(A^t) = det(A)$.

2. a) (10 points). The projection matrix of a matrix A is $P = A(A^tA)^{-1}A^t$. So the projection matrix of QA is $(QA)(A^tQ^tQA)^{-1}A^tQ^t = QA(A^tA)^{-1}A^tQ^t = QPQ^t$ where we have used that $Q^tQ = I$.

b) (10 points). By defintion $c - Pc$ is orthogonal to the space $span\{a,b\} = span\{q_1, q_2\}$. So we can choose $q_3 = (c - Pc)/||c - Pc||$.

c) (10 points). Let s_1, s_2, s_3 denote the rows of Q, and r_1, r_2, r_3 denote the columns of R. Then $c = \begin{pmatrix} s_1 \cdot r_3 \\ s_2 \cdot r_3 \\ s_3 \cdot r_3 \end{pmatrix} = Qr_3$. Since orthogonal matrices preserve the lengths of vectors, this implies $||c|| = ||r_3||$.

3. (15 points). Well, the matrix $uu^t = \left(u_iu_j\right)_{1 \le i,j \le n}$; so $I + tuu^t = (\delta_{ij} + tu_iu_j) = Q$. In particular, this matrix is symmetric, so the orthogonality condition reduces to $Q^2 = I$. Writing this condition out gives $I = (I + tuu^t)(I + tuu^t) = I + 2tuu^t + t^2(uu^t)^2$, or equivalently $t(2uu^t + t(uu^t)^2) = 0$. But now we have that $(uu^t)^2 = u(u^tu)u^t = uu^t$ because $u^tu = 1$ (u has length 1). So our equation becomes $t(2+t)(uu^t) = 0$. Clearly $t = 0$ and $t = -2$ are the solutions.

4. (15 points). We have that $C + Dt + (1 - E)t = (C + E) + (D - E)t$. Thus we see that E is a free variable: it is not uniquely determined, and in fact can take any value. Given this, just write down the usual least squares equations but treat $C + E$ and $D - E$ as your variables: the matrix A has two columns: the first consists of n 1's, the second is the vector (t_i). Then solve $A^tA \begin{pmatrix} C+E \\ D-E \end{pmatrix} = A^tb$.

5. a) (15 points). Yes. As A is invertible, it's column space is the full space $\mathbb{R}^n$. The same is true of A^{-1}.

b) (15 points). No. consider $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It has nonzero column space, but its square is 0.

18.06 Professor Strang Quiz 2 April 2, 2008

Your PRINTED name is: SOLUTIONS

Grading

1

2

3

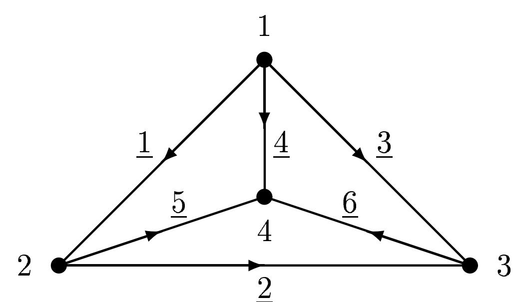
Please circle your recitation:

1)	M 2	2-131	A. Ritter	2-085	2-1192	afr
2)	M 2	4-149	A. Tievsky	2-492	3-4093	tievsky
3)	M 3	2-131	A. Ritter	2-085	2-1192	afr
4)	M 3	2-132	A. Tievsky	2-492	3-4093	tievsky
5)	T 11	2-132	J. Yin	2-333	3-7826	jbyin
6)	T 11	8-205	A. Pires	2-251	3-7566	arita
7)	T 12	2-132	J. Yin	2-333	3-7826	jbyin
8)	T 12	8-205	A. Pires	2-251	3-7566	arita
9)	T 12	26-142	P. Buchak	2-093	3-1198	pmb
10)	T 1	2-132	B. Lehmann	2-089	3-1195	lehmann
11)	T 1	26-142	P. Buchak	2-093	3-1198	pmb
12)	T 1	26-168	P. McNamara	2-314	4-1459	petermc
13)	T 2	2-132	B. Lehmann	2-089	2-1195	lehmann
14)	T 2	26-168	P. McNamara	2-314	4-1459	petermc

1 (33 pts.) (a) If $Ax = b$ and $A^{\mathrm{T}}y = 0$ then b is perpendicular to y. (The column space of A is perpendicular to the nullspace of A^{T}.) **Prove this by computing** $(Ax)^{\mathrm{T}}y$**.**

(b) Write down the 6 by 4 incidence matrix A of this graph (1 and -1 in each row of A). What is the dimension of the column space $C(A)$? Describe the nullpace $N(A)$.

(c) Find one nonzero vector $y = (y_1, y_2, \ldots, y_6)$ that is in the nullspace of A^{T}. (Think loops.) If voltages x_1, x_2, x_3, x_4 are assigned to the nodes (keep the x's as variables not numbers), multiply by A to find Ax. **Check that this Ax is perpendicular to your vector y.** (That's Kirchhoff's Voltage Law.)



Solution (6+13+14 points)

a) We have

$$(Ax)^T y = x^T A^T y = x^T (0) = 0 \tag{1}$$

b) To form the incidence matrix A, for each edge we put a -1 for the node where the edge starts, and a 1 for the node where the edge ends. (5 points)

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{2}$$

Any incidence matrix has a one-dimensional nullspace spanned by the vector consisting of all 1. Thus, $C(A)$ has dimension $r = n - \dim(N(A)) = 4 - 1 = 3$, and the nullspace has the vector $(1, 1, 1, 1)$ as a basis. (8 points) Of course this can also be calculated by hand.

c) To find the left nullspace of an incidence matrix, we traverse around a closed loop, and keep track of the edges with signs. So for example if we go from node 1 to node 2 to node 4 and back to node 1, we find the vector $(1, 0, 0, -1, 1, 0)$ in the left nullspace of A. (7 points) Some other examples are: $(1, 1, -1, 0, 0, 0)$, $(0, 0, 1, -1, 0, 1)$, and $(0, 1, 0, 0, -1, 1)$.
The matrix Ax is (3 points)

$$\begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_2 \\ x_4 - x_3 \end{bmatrix} \tag{3}$$

Taking the dot product of Ax with the first choice of y above yields (4 points)

$$(Ax)^T y = (x_2 - x_1) - (x_4 - x_1) + (x_4 - x_2) = 0 \tag{4}$$

2 (33 pts.) (a) Suppose you want to fit the best straight line $C + Dt$ to the values $b = 1, 1, 1, 2$ at the times $t = 0, 1, 3, 4$. What is the matrix A in the unsolvable system $A \begin{bmatrix} C \\ D \end{bmatrix} = b$? Find the best $\widehat{C}, \widehat{D}$ and the heights p_1, p_2, p_3, p_4 of that line $\widehat{C} + \widehat{D}t$ at the times $t = 0, 1, 3, 4$.

(b) Think of the same problem as a projection onto the column space of A in $\mathbf{R}^4$. What is the error vector $e = b - p$? Show with numbers that e is perpendicular to (what space?).

(c) Use Gram-Schmidt to get orthonormal columns q_1, q_2 from the columns a_1, a_2 of your matrix A.

Solution (11+11+11 points)

a) The matrix A is (4 points)

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \tag{5}$$

To find the best $\widehat{C}, \widehat{D}$, we need to solve the system (2 points)

$$A^T A \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = A^T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \tag{6}$$

or

$$\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 26 & -8 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 12 \end{bmatrix} \tag{8}$$

We obtain the solution $(\widehat{C}, \widehat{D}) = (17/20, 1/5)$. (2 points)

To find the height vector p, we take (3 points)

$$p = A \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \frac{1}{20}[17, 21, 29, 33]^T \tag{9}$$

b) The error vector $e = [1, 1, 1, 2]^T - p = \frac{1}{20}[3, -1, -9, 7]^T$. (3 points) The error vector is perpendicular to the column space of A. (4 points) We check using numbers (4 points):

$$[1, 1, 1, 1]e = \frac{1}{20}(3 - 1 - 9 + 7) = 0 \tag{10}$$

$$[0, 1, 3, 4]e = \frac{1}{20}(0 - 1 - 27 + 28) = 0 \tag{11}$$

c) We use Gram-Schmidt on the columns of A. We set $w_1 = a_1$ (3 points), and then (5 points)

$$w_2 = a_2 - \frac{w_1 \cdot a_2}{w_1 \cdot w_1} w_1 = [-2, -1, 1, 2]^T \tag{12}$$

Finally, we must normalize to obtain $q_1 = \frac{1}{2}[1, 1, 1, 1]^T$ and $q_2 = \frac{1}{\sqrt{10}}[-2, -1, 1, 2]^T$. (3 points)

3 (34 pts.) This question is about the matrix

$$A = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

(a) Compute A^2 and use that to show that the determinant of A is either 1 or -1.

(b) Determine whether $\det A = 1$ or -1.

(c) Find the cofactor C_{11} corresponding to the entry $a_{11} = -\frac{1}{2}$.

(d) Out of the $4! = 24$ terms in the "big formula" for $\det A$, show **four terms** that are $+\frac{1}{16}$. (For each term give the column numbers like $4, 3, 2, 1$ or $2, 1, 4, 3$ as you go down the matrix.)

Solution (9+9+8+8 points)

a) We have (4 points)

$$A^2 = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = I \tag{13}$$

Thus, $\det(A)^2 = \det(I) = 1$, meaning that $\det(A) = \pm 1$. (5 points) In fact A is a symmetric orthogonal matrix, which means that we know many of its properties.

b) Now we actually compute the determinant of A. One could do this using cofactors, row reduction, etc. It turns out to be -1. (9 points for computation) For example, after row-reducing the first column of A we obtain

$$B = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{bmatrix} \tag{14}$$

Then

$$\begin{aligned} \det(A) &= \det(B) && (15) \\ &= \frac{1}{16}\det\begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{bmatrix} && (16) \\ &= \frac{1}{16}(-1)\det\begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} && (17) \\ &= -\frac{1}{2}(1+1) = -1 && (18) \end{aligned}$$

c) Let D be the matrix we get when we cross out row 1 and column 1 of A. Then C_{11} is the determinant of D (there is no negative sign since $(1,1)$ is a "positive" position: $(-1)^{1+1} = 1$). (4 points) The calculation yields $\frac{1}{2}$. (4 points)

A quicker way to do it is to use the inverse formula. We have found that $A = A^{-1}$. We also know that the quantity $C_{11}/\det(A)$ is equal to the top left entry of A^{-1}. Thus, we must have $C_{11} = a_{11}\det(A) = \frac{1}{2}$.

d) Each term of the big formula is a product, where we take one entry from each row and each column. To find the terms that are $+\frac{1}{16}$, we need to have the signs cancel out correctly. Using the notation of the problem, the choices are $(1,2,3,4)$, $(2,1,4,3)$, $(3,4,1,2)$, and $(4,3,2,1)$. (2 points each)

18.06 Quiz 2 Solution

Hold on Wednesday, 1 April 2009 at 11am in Walker Gym.
Total: 70 points.

Problem 1:

(a) If P is the projection matrix onto the *null* space of A, then $P\mathbf{y} - \mathbf{y}$, for any $\mathbf{y}$, is in the ____________________ space of A.

(b) If $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$, then the closest vector to $\mathbf{b}$ in $N(A^{\mathrm{T}})$ is ____________________ (best answer).

(c) If the *rows* of A (an $m \times n$ matrix) are independent, then the dimension of $N(A^{\mathrm{T}}A)$ is __________.

(d) If a matrix U has orthonormal *rows*, then $I =$ __________ and the projection matrix onto the *row* space of U is __________. (Your answers should be the simplest expressions involving U and U^{T} only.)

Solution (20 points = 5+5+5+5)

Answers: (a) row; (b) 0; (c) $n - m$; (d) UU^{T}, $U^{\mathrm{T}}U$.

(a) Since $P\mathbf{y}$ is the projection to the nullspace of A, $P\mathbf{y} - \mathbf{y}$ is orthogonal to the null space; it then must lie in the row space of A.

(b) Since $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$, $\mathbf{b}$ is in the column space $C(A)$ of A. We know that the left nullspace $N(A^{\mathrm{T}})$ is orthogonal to the column space. So the closest vector to $\mathbf{b}$ is 0.

(c) We derived in Problem 7 of Pset 4 that the nullspace of $A^{\mathrm{T}}A$ is the same as the nullspace of A; the latter has dimension $n - m$ because the matrix A is of full row rank m.

Alternatively, we also derived the following in lecture, and it is in the text, and on the practice-exam handout: the ranks of A and $A^{\mathrm{T}}A$ are the same, so both equal to m. Since A has full row rank and $A^{\mathrm{T}}A$ has n columns, $N(A^{\mathrm{T}}A)$ has dimension $n - m$.

(d) Note that $U^{\mathrm{T}} = Q$, a matrix with orthonormal columns. We saw in class that $I = Q^{\mathrm{T}}Q = UU^{\mathrm{T}}$, and the projection matrix onto $C(Q) = C(U^{\mathrm{T}})$ is $QQ^{\mathrm{T}} = U^{\mathrm{T}}U$.

Problem 2: The matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & -7 \\ 2 & 4 & 1 & -5 \\ 1 & 2 & 2 & -16 \end{pmatrix}$$

is converted to row-reduced echelon form by the usual row-elimination steps, resulting in the matrix:

$$R = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(♣) The *minimum* number of columns of A that form a *dependent* set of vectors is ____________. The *maximum* number of columns of A that forms an *independent* set of vectors is ____________.

(♢) Give an *orthonormal* basis for the *row* space of A. (Careful: be sure you start with a basis for the row space, not containing any dependent vectors.) Your answer may contain square roots left as $\sqrt{\textit{some number}}$.

(♠) Given the vector $\mathbf{b} = \begin{pmatrix} 2 & 5 & -9 & 3 \end{pmatrix}^T$, compute the *closest* vector $\mathbf{p}$ to $\mathbf{b}$ in the *row space* $C(A^T)$? (Hint: less calculation is needed if you use your answer from ♢.)

(♡) In terms of your answer $\mathbf{p}$ to ♠ above, what is the closest vector to $\mathbf{b}$ in the *nullspace* $N(A)$? (No calculation required, and you need not have solved ♠: you can leave your answer in terms of $\mathbf{p}$ and $\mathbf{b}$.)

Solution (30 points = 6+10+10+4)

Answers: (♣) 2, 2; (♢) see below; (♠) $\mathbf{p} = (2 \;\; 4 \;\; 0 \; 4)^T$; (♡) $\mathbf{b} - \mathbf{p}$.

(♣) The key point of the problem is that the dependency of columns in R and A is the same. By inspection of R (or A), the first two columns are dependent, so that is the smallest dependent set. R has two pivots, so A is rank 2 and the column space is 2-dimensional, so 2 is the maximum number of independent columns. Equivalently, the maximum number of independent columns is the number of columns in any basis for $C(A)$, such as the 2 pivot columns.

(♢) Note that the (row-reduced) echelon form R has the same row space as A. We may therefore start Gram-Schmidt on the pivot rows of R, which form a basis for the row space of R and A.

$$\mathbf{q}_1 = \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} = (\frac{1}{3}\ \ \frac{2}{3}\ \ 0\ \ \frac{2}{3})^{\mathrm{T}}$$

$$\mathbf{q}_2 = \frac{\mathbf{a}_2 - \mathbf{q}_1^{\mathrm{T}}\mathbf{a}_2\mathbf{q}_1}{\|\mathbf{a}_2 - \mathbf{q}_1^{\mathrm{T}}\mathbf{a}_2\mathbf{q}_1\|} = \frac{(0\ \ 0\ \ 1\ \ -9)^{\mathrm{T}} + 6\cdot(\frac{1}{3}\ \ \frac{2}{3}\ \ 0\ \ \frac{2}{3})^{\mathrm{T}}}{\|(0\ \ 0\ \ 1\ \ -9)^{\mathrm{T}} + 6\cdot(\frac{1}{3}\ \ \frac{2}{3}\ \ 0\ \ \frac{2}{3})^{\mathrm{T}}\|}$$

$$= \frac{(2\ \ 4\ \ 1\ \ -5)^{\mathrm{T}}}{\|(2\ \ 4\ \ 1\ \ -5)^{\mathrm{T}}\|} = (2\ \ 4\ \ 1\ \ -5)^{\mathrm{T}}\Big/\sqrt{46}.$$

REMARK: One can also obtain an orthonormal basis by starting with 2 rows of A since in this case any 2 rows are independent and form a basis. But the pivot rows of R are a nicer basis (more zeros), and the calculations are therefore much simpler.

(♠) The closest vector $\mathbf{p}$ should be given by the projection to the row space. That is

$$\mathbf{p} = \mathbf{p}^{\mathrm{T}}\mathbf{q}_1\mathbf{q}_1 + \mathbf{p}^{\mathrm{T}}\mathbf{q}_2\mathbf{q}_2 = 6\cdot(\frac{1}{3}\ \ \frac{2}{3}\ \ 0\ \ \frac{2}{3})^{\mathrm{T}} + 0 = (2\ \ 4\ \ 0\ 4)^{\mathrm{T}}.$$

(♡) The closest vector $\mathbf{p}$ of $\mathbf{b}$ in the row space is exactly the projection in the row space. But the row space and the nullspace are orthogonal to each other. Then, $\mathbf{b} - \mathbf{p}$ is exactly the orthogonal projection in the nullspace $N(A)$; it is the closest vector to $\mathbf{b}$ in the nullspace.

Problem 3: You are told that the least-square linear fit to three points $(0, b_1)$, $(1, b_2)$, and $(2, b_3)$ is $C + Dt$ for $C = 1$ and $D = -2$. That is, the fit is $1 - 2t$.

In this question, you will work backwards from this fit to reason about the unknown values $\mathbf{b} = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}^T$ at the coordinates $t = 0, 1, 2$.

(i) Write down the explicit equations that $\mathbf{b}$ must satisfy for $1 - 2t$ to be the least-square linear fit. (The points do *not* have to fall exactly on the line.)

(ii) If all the points fall *exactly* on the line $1 - 2t$, then $\mathbf{b} =$ __________. Check that this satisfies your equations in (i).

(iii) More generally, if all the points fall exactly on *any* straight line, then $\mathbf{b}$ is in the ________________ space of what matrix? (Write down the matrix.)

Solution (20 points = 10+5+5)

Answers: (i) see below; (ii) $\mathbf{b} = (1 \ \ -1 \ \ -3)$; (iii) see below.

(i) The system that we would solve if the line passed exactly through all of the points is

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

However, since the line may not pass through all the points this system may have no solution, and instead we find the least-square solution by solving the normal equations:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

That is

$$\begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} b_1 + b_2 + b_3 \\ b_2 + 2b_3 \end{pmatrix}$$

Since the least-square fit is $1 - 2t$, the above linear system has solution $(1 \ \ -2)^{\mathrm{T}}$. Hence, b_1, b_2, b_3 should satisfy

$$\begin{aligned} b_1 + b_2 + b_3 &= -3 \\ b_2 + 2b_3 &= -7 \end{aligned}$$

(ii) If all the points fall exactly on the line $1 - 2t$,

$$b_1 = 1 - 2 \cdot 0 = 1, \quad b_2 = 1 - 2 \cdot 1 = -1, \quad b_3 = 1 - 2 \cdot 2 = -3.$$

We plug the solution back in the relations above and check.

$$1 - 1 - 3 = -3, \quad -1 + 2 \times (-3) = -7.$$

(iii) If all points fall exactly on a straight line, the following system would have a solution.

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

In other words, $\mathbf{b}$ lies in the column space of the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

18.06 Professor Edelman Quiz 2 October 22, 2008

Grading

1

2

3

4

5

Your PRINTED name is: ____________________

Please circle your recitation:

1)	T 10	2-131	J.Yu	2-348	4-2597	jyu
2)	T 10	2-132	J. Aristoff	2-492	3-4093	jeffa
3)	T 10	2-255	Su Ho Oh	2-333	3-7826	suho
4)	T 11	2-131	J. Yu	2-348	4-2597	jyu
5)	T 11	2-132	J. Pascaleff	2-492	3-4093	jpascale
6)	T 12	2-132	J. Pascaleff	2-492	3-4093	jpascale
7)	T 12	2-131	K. Jung	2-331	3-5029	kmjung
8)	T 1	2-131	K. Jung	2-331	3-5029	kmjung
9)	T 1	2-136	V. Sohinger	2-310	4-1231	vedran
10)	T 1	2-147	M Frankland	2-090	3-6293	franklan
11)	T 2	2-131	J. French	2-489	3-4086	jfrench
12)	T 2	2-147	M. Frankland	2-090	3-6293	franklan
13)	T 2	4-159	C. Dodd	2-492	3-4093	cdodd
14)	T 3	2-131	J. French	2-489	3-4086	jfrench
15)	T 3	4-159	C. Dodd	2-492	3-4093	cdodd

1 (10 pts.) The determinant of the 1000 by 1000 matrix A is 12. What is the determinant of $(-A)^T$? (Careful: No credit for the wrong sign.)

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2 (30 pts.)

(a) P is the projection matrix onto the column space of A which has independent columns. Q is a square orthogonal matrix with the same number of rows as A. In simplest form, in terms of P and Q, what is the projection matrix onto the column space of QA?

(b) The vectors a, b, and c are independent. The matrix P is the projection matrix onto the span of a and b. Suppose we apply Gram-Schmidt onto the vectors a,b, and c producing orthonormal vectors q_1, q_2, and q_3. Write the unit vector q_3 in simplest form in terms of P and c only.

(c) The vector a, b and c are independent. The matrix $A = [a\ b\ c]$ has these three vectors as its columns. The QR decomposition writes $A = QR$ where Q is orthogonal and R is 3×3 upper triangular. Write $||c||$ in terms of only the elements of R in simplest form.

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3 (15 pts.) The vector u is a "unit vector" meaning $||u|| = 1$. What are all the possible values of t which guarantee that the matrix $A = I + tuu^T$ is orthogonal?

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4 (15 pts.) Suppose we have obtained from measurements n data points (t_i, b_i), and you are asked to find a best least squares fit function of the form $y = C + Dt + E(1-t)$. Are C,D, and E uniquely determined? Write down a solvable system of equations that gives a solution to the least squares problem.

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5 (30 pts.) (a) If A is invertible, must the column space of A^{-1} be the same as the column space of A?

(b) If A is square, must the column space of A^2 be the same as the column space of A?

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18.06 Professor Postnikov Quiz 2 October 26, 2009

SOLUTIONS ∇ 0

Your PRINTED name is: ____________________

Please circle your recitation:

(R01)	T10	2-132	HwanChul Yoo
(R02)	T11	2-132	HwanChul Yoo
(R03)	T12	2-132	David Shirokoff
(R04)	T1	2-131	Fucheng Tan
(R05)	T1	2-132	David Shirokoff
(R06)	T2	2-131	Fucheng Tan
(R07)	T2	2-146	Leonid Chindelevitch
(R08)	T3	2-146	Steven Sivek

Grading

1	
2	
3	
Total:	

Problem 1. Consider the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{pmatrix}$.

(a) Find an orthogonal basis of the column space of the matrix A.

(b) Find a non-zero vector $\mathbf{v}$ which is orthogonal to the column space of A.

(c) Does this vector $\mathbf{v}$ belong to one of the four fundamental subspaces of A? Which subspace? Explain why.

(d) Find a 3 by 2 matrix Q with $Q^T Q = I$ such that Q has the same column space as the matrix A.

(a) Use Gram-Schmidt on the colums of $A = \begin{pmatrix} | & | \\ a_1 & a_2 \\ | & | \end{pmatrix}$.

$q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$

$q_2 = \frac{a_2 - (a_2^T \cdot q_1) q_1}{\| a_2 - (a_2^T q_1) q_1 \|}$, $a_2^T \cdot q_1 = \frac{9}{\sqrt{3}}$ so $q_2 = \frac{\begin{pmatrix}1\\3\\5\end{pmatrix} - \overbrace{\frac{9}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}}^{3}\begin{pmatrix}1\\1\\1\end{pmatrix}}{\| \quad " \quad \|}$

$= \begin{pmatrix}-2\\0\\2\end{pmatrix} \cdot \frac{1}{\sqrt{8}} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}$

An orthogonal basis is given by $\left\{ \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix} \right\}$.

(b) Must find v s.t. $a_1^T v = 0$ and $a_2^T v = 0$ or similarly

$q_1^T v = 0$ and $q_2^T v = 0$ (since q_1 and q_2 are a basis for $C(A)$!)

This is the same as solving:

$\begin{pmatrix} - & q_1^T & - \\ - & q_2^T & - \end{pmatrix} v = 0$ or $\begin{pmatrix} \frac{1}{\sqrt{3}}(1 & 1 & 1) \\ \frac{1}{\sqrt{2}}(1 & 0 & -1) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{matrix}0\\0\\0\end{matrix}$

$\Rightarrow$ $v_1 - v_3 = 0$, $v_1 + v_2 + v_3 = 0$ So $v_1 = v_3$, $v_2 = -2v_1$ and $v = \begin{pmatrix}1\\-2\\1\end{pmatrix}$

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(c) $v \in N(A^T)$. Why? Let Q be the matrix with columns given by q_1 and q_2 from part (a).

Then $C(A) = C(Q) \implies N(A^T) = N(Q^T)$

And from part (b), we saw that v was a solution to $Q^T v = 0$. Since $C(Q)$ is 2-dim $N(Q^T)$ is 1-dimensional, so is actually spanned by v.

(d) Take Q to be Q from part (c).

$$Q^T Q = \begin{pmatrix} - & q_1^T & - \\ - & q_2^T & - \end{pmatrix} \begin{pmatrix} | & | \\ q_1 & q_2 \\ | & | \end{pmatrix} = \begin{pmatrix} \underset{\|}{\overset{1}{\overset{\|}{q_1^T q_1}}} & \overset{0}{\overset{\|}{q_1^T q_2}} \\ \underset{0}{\underset{\|}{q_2^T q_1}} & \underset{1}{\underset{\|}{q_2^T q_2}} \end{pmatrix} = I.$$

Problem 2. Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \\ 0 & -1 \end{pmatrix}$, and let $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$.

(a) What is the projection of $\mathbf{b}$ onto the column space of A?

(b) Give an orthogonal basis for each of the four fundamental subspaces of A.

(c) Use least squares approximation to solve $A\mathbf{x} = \mathbf{b}$.

(a) Want to find $p = A\hat{x}$ (i.e. in $C(A)$) closest to b.
Or equivalently $A^T(b - A\hat{x}) = 0 \Rightarrow A^TA\hat{x} = A^Tb$
Since columns of A are independent $\hat{x} = (A^TA)^{-1}A^Tb$
So the projection $A\hat{x}$ is given by:

$A\hat{x} = p = A(A^TA)^{-1}A^Tb$:

$$A^TA = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 6 \end{pmatrix}$$

So $(A^TA)^{-1} = \begin{pmatrix} 1/5 & 0 \\ 0 & 1/6 \end{pmatrix}$

$$p = \begin{pmatrix} 2 & -1 \\ 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1/5 & 0 \\ 0 & 1/6 \end{pmatrix}\underbrace{\begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}}_{\begin{pmatrix} 0 \\ -6 \end{pmatrix}}$$

$$\begin{pmatrix} 1/5 & 0 \\ 0 & 1/6 \end{pmatrix}\begin{pmatrix} 0 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}$$

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(b) From part (a): $A^T A = \begin{pmatrix} 5 & 0 \\ 0 & 6 \end{pmatrix}$ shows that the columns of A are already orthogonal. They are clearly independant as A has 2 pivots, so

$C(A)$ has basis: $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \right\}$

Since $\dim C(A) = \dim C(A^T) = 2$, the rowspace is all of $\mathbb{R}^2$, so we can take any basis

or $C(A^T)$ has a basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

(These are clearly orthogonal.)

$N(A) = \vec{0}$ since A has full rank.

$N(A^T)$ is spanned by a vector v s.t. $A^T v = 0$, so $\left\{ \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \right\}$ works.

(c) Want to solve $A^T A \hat{x} = A^T b$ or $\hat{x} = (A^T A)^{-1} A^T b$.
From part (a) $\hat{x} = \begin{pmatrix} 1/5 & 0 \\ 0 & 1/6 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

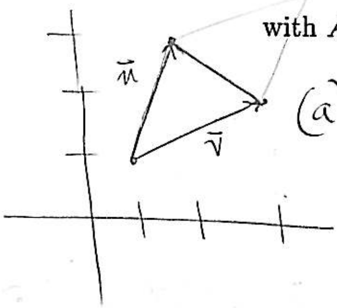
Problem 3.

(a) Find the area of the triangle on the plane $\mathbb{R}^2$ with the vertices $(1,1)$, $(2,3)$, $(3,2)$.

(b) Calculate the determinant of the 4 by 4 matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

(c) Find the inverse of the matrix A from part (b). Check your answer by multiplying it with A.



(a) Take $\vec{u} = (2,3) - (1,1) = (1,2)$

$\vec{v} = (3,2) - (1,1) = (2,1)$.

Then the Area of triangle $= \frac{1}{2}$ area of parallelogram.

So Area $= \frac{1}{2}\left|\det\begin{pmatrix} \vec{u} & \vec{v} \end{pmatrix}\right| = \frac{1}{2}\left|\det\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\right| = \frac{1}{2}|1-4| = \boxed{\frac{3}{2}}$

(b) $\det A = 1 \cdot \det\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} - (-1)\det\begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ → 0

$= 1 - (1+1) = \boxed{-1}$

row 2 and 3 are dependant, so det of this matrix $= 0$

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There is more than one way to do this, but the fewest operations are required by the following:

(c):

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

row 2 $\to$ row 2 + row 1
row 3 $\to$ row 3 + row 4

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

row 1 $\to$ row 1 − row 3
row 4 $\to$ row 4 − row 2

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right)$$

multiply row 2 + 3 by (−1) and switch them.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix}$$

Check $A^{-1}A = I$:

$$\begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

18.06 Spring 2009 Exam 2 Practice

General comments

Exam 2 covers the first 18 lectures of 18.06. It does *not* cover determinants (lectures 19 and 20). There will also be *no* questions on graphs and networks. The topics covered are (very briefly summarized):

1. All of the topics from exam 1.

2. Linear independence [key point: the columns of a matrix A are independent if $N(A) = \{0\}$], bases (an independent set of vectors that spans a space), and dimension of subspaces (the number of vectors in *any* basis).

3. The four fundamental subspaces (key points: their dimensions for a given rank r and $m \times n$ matrix A, their relationship to the solutions [if any] of $Ax = b$, their orthogonal complements, and how/why we can find bases for them via the elimination process).

4. What happens to the four subspaces as we do matrix operations, especially elimination steps and more generally how the subspaces of AB compare to those of A and B. The fact (important for projection and least-squares!) that $A^T A$ has the same rank as A, the same null space as A, and the same column space as A^T, and why (we proved this in class and another way in homework).

5. Orthogonal complements $S^\perp$ for subspaces S, especially (but not only) the four fundamental subspaces.

6. Orthogonal projections: given a matrix A, the projection of b onto $C(A)$ is $p = A\hat{x}$ where $\hat{x}$ solves $A^T A\hat{x} = A^T b$ [always solvable since $C(A^T A) = C(A^T)$]. If A has full column rank, then $A^T A$ is invertible and we can write the projection matrix $P = A(A^T A)^{-1}A^T$ (so that $A\hat{x} = Pb$, but it is *much* quicker to solve $A^T A\hat{x} = A^T b$ by elimination than to compute P in general). $e = b - A\hat{x}$ is in $C(A)^\perp = N(A^T)$, and $I - P$ is the projection matrix onto $N(A^T)$.

7. Least-squares: $\hat{x}$ minimizes $\|Ax - b\|^2$ over all x, and is the *least-squares* solution. That is, $p = A\hat{x}$ is the *closest* point to b in $C(A)$. Application to least-square curve fitting, minimizing the sum of the squares of the errors.

8. Orthonormal bases, forming the columns of a matrix Q with $Q^T Q = I$. The projection matrix onto $C(Q)$ is just QQ^T, and $\hat{x} = Q^T b$. Obtaining Q from A (i.e., an orthonormal basis from any basis) by Gram-Schmidt, and the correspondence of this process to $A = QR$ factorization where $R = Q^T A$ is invertible and upper-triangular. Using $A = QR$ to solve equations (either $Ax = b$ or $A^T A\hat{x} = A^T b$). Q is an *orthogonal matrix* only if it is square, in which case $Q^T = Q^{-1}$.

9. Dot products of functions, and hence Gram-Schmidt, orthonormal bases (e.g. Fourier series or orthogonal polynomials), orthogonal projection, and least-squares for functions.

As usual, the exam questions may turn these concepts around a bit, e.g. giving the answer and asking you to work backwards towards the question, or ask about the same concept in a slightly changed context. We want to know that you have really internalized these concepts, not just memorizing an algorithm but knowing *why* the method works and where it came from.

Some practice problems

The 18.06 web site has exams from previous terms that you can download, with solutions. I've listed a few practice exam problems that I like below, but there are plenty more to choose from. (Note: exam 2 in several previous terms asked about determinants; we *won't* have any determinant questions until exam 3.) The exam will consist of 3 or 4 questions (perhaps with several parts each), and you will have one hour. You can find the solutions to these problems on the 18.06 web site (in the section for old exams/psets). On the last page I give practice problems for orthogonal functions and orthogonal projections of functions.

1. (Fall 2002 exam 2.) **(a)** Choose c and the last column of Q so that you have an orthogonal matrix:

$$Q = c \begin{bmatrix} 1 & -1 & -1 & ? \\ -1 & 1 & -1 & ? \\ -1 & -1 & -1 & ? \\ -1 & -1 & 1 & ? \end{bmatrix}.$$

   (b) Project $b = (1,1,1,1)^T$ onto the first column of Q. Then project b onto the plane spanned by the first two columns. **(c)** Suppose the last column of this matrix (where the ?'s are) were changed to $(1,1,1,1)^T$. Call this new matrix A. If Gram-Schmidt is applied to the 4 columns of A, what would be the 4 outputs q_1, q_2, q_3, q_4? (Don't do a lot of calculations...please!)

2. (Fall 2008 exam 2.) [The parts of this question are independent and can be done in any order.] **(a)** P is the projection matrix onto $C(A)$, where A has independent columns. Q is a square orthogonal matrix with the same number of rows as A. In its simplest form, in terms of P and Q, what is the projection matrix onto the column space of QA? **(b)** The vectors a, b, and c are independent. The matrix P is the projection matrix onto the span of a and b. Suppose we apply Gram-Schmidt onto the vectors a, b,and c to produce orthonormal vectors q_1,q_2,and q_3. Write the unit vector q_3 in simplest form in terms of P and c only. **(c)** The vectors a, b, and c are independent, and the matrix A has these three vectors as its columns. You are given the QR decomposition of A, where Q is orthogonal and R is 3×3 upper-triangular as usual. Write $\|c\|$ in terms of only the elements of R, in simplest form.

3. (Fall 2008 exam 2.) Suppose we have obtained from measurements n data points (t_i, b_i) and you are asked to find a best least-squares fit function of the form $y = C + Dt + E(1-t)$. Are C, D, and E uniquely determined? Write down a solvable system of equations that gives a solution to the least-squares problem.

4. (Fall 2008 exam 2.) **(a)** If A is invertible, must the column space of A^{-1} be the same as the column space of A? **(b)** If A is square, must the column space of A^2 be the same as the column space of A?

5. (Fall 2005 exam 1.) Suppose A is $m \times n$ with *linearly dependent column*s. Complete with as much true information as possible: **(a)** The rank of A is .....? **(b)** The nullspace of A contains .....? **(c)** The equation $A^T y = b$ has no solution for some right-hand sides b because ......? (more words needed)

6. (Fall 2005 exam 1.) Suppose A is the 3×4 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}.$$

   (a) A basis for $C(A)$ is .....? **(b)** For which vectors $b = (b_1, b_2, b_3)^T$ does $Ax = b$ have a solution? (Give specific conditions on $b_{1,2,3}$.) **(c)** Explain why there is no 4×3 matrix B for which $AB = I$ (3×3). Give a good reason (the mere fact that A is rectangular is *not* sufficient).

7. (Spring 2005 exam 1.) Suppose the columns of a 7×4 matrix A are linearly independent. **(a)** After row operations reduce A to U or R, how many rows will be all zero (or is it impossible to tell)? **(b)** Assume that no row swaps were required for elimination. What is the row space of A? Explain why this equation will surely be solvable: $A^T y = (1,0,0,0)^T$.

8. (Fall 2005 exam 2.) The matrix Q has orthonormal columns q_1, q_2, q_3:

$$Q = \begin{bmatrix} 0.1 & 0.5 & a \\ 0.7 & 0.5 & b \\ 0.1 & -0.5 & c \\ 0.7 & -0.5 & d \end{bmatrix}.$$

(a) What equations must be satisfied by the numbers a, b, c, d? Is there a unique choice for those (real) numbers, apart from multiplying them all by -1? **(c)** Suppose Gram-Schmidt starts with those same first two columns and with the third column $a_3 = (1,1,1,1)^T$. What third column would it choose for q_3. (You can leave a square root as $\sqrt{\cdots}$ if you want to.)

9. (Fall 2005 exam 2.) Our measurements at times $t = 1,2,3$ are $b = 1,4,b_3$. We want to fit those points by the nearest line $C+Dt$, using least-squares. **(a)** Which value for b_3 will put the three points on a straight line? Give C and D for this line. Will least squares choose that line if the third measurement is $b_3 = 9$? (Yes or no.) **(b)** What is the lienar system $Ax = b$ that would be solved exactly for $x = (C,D)$ if the three points do lie on a line? Compute the projection matrix P onto the column space of A. You can use the 2×2 inverse formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. **(c)** What is the rank of that projection matrix P? How is the column space of P related to the column space of A? (You can answer this part without your answer from **b**.) **(d)** Suppose $b_3 = 1$. Write down the equation for the best least-squares solution $\hat{x}$, and show that the best straight line is horizontal in this case.

10. (Fall 2006 exam 2.) Suppose we take measurements at the 21 equally spaced times $t = -10,-9,\ldots,9,10$. All measurements are $b_i = 0$ except that $b_{11} = 1$ at the middle time $t = 0$. **(a)** Using least squares, what are the best C and D to fit those 21 points by a straight line $C+Dt$? **(b)** You are projecting the vector b onto what subspace? (*Give a basis.*) Find a nonzero vector perpendicular to that subspace.

11. (Fall 2006 exam 2.) The Gram-Schmidt methods produces orthonormal vectors q_1, q_2, q_3 from independent vectors a_1, a_2, a_3 in $\mathbb{R}^5$. Put those vectors into the columns of 5×3 matrices Q and A, respectively. **(a)** Give formulas using Q and A for the projection matrices P_Q and P_A onto the column spaces of Q and A, respectively. **(b)** Does P_Q equal P_A, and why or why not? What is $P_Q Q$? **(c)** Suppose a_4 is a new vector, and a_1, a_2, a_3, a_4 are independent. Which of the following (if any) is the new Gram-Schmidt vector q_4? **1:** $\frac{P_Q a_4}{\|P_Q a_4\|}$. **2:** $\frac{a_4 - \frac{a_4^T a_1}{a_1^T a_1} a_1 - \frac{a_4^T a_2}{a_2^T a_2} a_2 - \frac{a_4^T a_3}{a_3^T a_3} a_3}{\|\cdots\text{same vector}\cdots\|}$. **3.** $\frac{a_4 - P_A a_4}{\|a_4 - P_A a_4\|}$.

12. (Spring 2004 exam 2.) We are given two vectors a and b in $\mathbb{R}^4$, $a = (2,5,2,4)^T$ and $b = (1,2,1,0)^T$. **(a)** Find the projection p of the vector b onto the line through a. Check(!) that the error $e = b - p$ is perpendicular to....what? **(b)** The subspace S of all vectors in $\mathbb{R}^4$ that are perpendicular to this a is 3-dimensional. Compute the projection q of b onto this perpendicular subspace S. (It doesn't need a big computation!)

13. (Spring 2004 exam 2.) Suppose that q_1, q_2, and q_3 are 3 orthonormal vectors in $\mathbb{R}^n$. They go into the columns of an $n\times 3$ matrix Q. **(a)** What inequality ($\le$ or $\ge$) do you know for n? Is there any condition on n required in order to have $Q^T Q = I$? Is there any condition on n required to have $QQ^T = I$? **(b)** Give a nice matrix formula involving b and Q for the projection p of a vector b onto the column space of Q. Complete the sentence: p is the closest vector .... **(c)** Suppose the projection of b onto that column space is $p = c_1 q_1 + c_2 q_2 + c_3 q_3$. Find a formula for c_1 that only involves b and q_1 (possibly using dot products).

14. (Spring 2005 exam 2.) If the output vectors from Gram-Schmidt are: $q_1 = (\cos\theta, \sin\theta)^T$ and $q_2 = (-\sin\theta, \cos\theta)$ for some θ, describe all possible input vectors a_1 and a_2.

15. (Spring 2005 exam 2.) If a and b are nonzero vectors in $\mathbb{R}^n$, what number x minimizes the squared length $\|b - xa\|^2$?

16. (Spring 2005 exam 2.) Find the projection p of the vector $b = (1,2,6)^T$ onto the plane $x+y+z = 0$ in $\mathbb{R}^3$. (You may want to first find a basis for this 2-dimensional subspace, perhaps even an orthogonal basis.)

17. (Spring 2005 exam 2.) You are given the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Suppose P_1 is the projection matrix onto the 1-dimensional subspace spanned by the first column of A. Suppose P_2 is the projection matrix onto the 2-dimensional column space of A. After thinking a little, compute the product $P_2 P_1$.

None of the first two exams in previous terms covered orthogonal functions—these are in the standard 18.06 syllabus, especially Fourier series, but previously weren't covered until later in the term, after eigenproblems. A couple of problems about orthogonal functions appeared on your last problem set, which you should review, and a couple more practice problems on this topic are:

1. Suppose you are given three functions $a_1(t)$, $a_2(t)$, and $b(t)$ for $0 \le t \le 1$. Define dot products of any two functions $f(t)$ and $g(t)$ by $f(t) \cdot g(t) = \int_0^1 f(t)g(t)dt$ (hence, $\|f(t)\| = \sqrt{\int_0^1 f(t)^2 dt}$). Suppose we want the "best-fit" function $p(t) = Ca_1(t) + Da_2(t)$ that minimizes $\|p(t) - b(t)\|$ over all possible C and D. Give an explicit formula for $p(t)$ in terms of some integrals and other expressions involving $a_1(t)$, $a_2(t)$, and $b(t)$ only.

2. The functions $q_1(t) = \sin(t)/\sqrt{\pi}$, $q_2(t) = \sin(2t)/\sqrt{\pi}$, and $q_3(t) = \cos(t)/\sqrt{\pi}$ are orthonormal if we define dot products of any two functions $f(t)$ and $g(t)$ by $f(t) \cdot g(t) = \int_0^{2\pi} f(t)g(t)dt$. **(a)** Write the function $b(t) = t$ as the sum of two functions, one in the span of q_1, q_2 and q_3 and one perpendicular to q_1, q_2 and q_3. You should write your answer explicitly in terms of integrals *etc.*, but you need not evaluate the integrals (this isn't 18.01). **(b)** If you were to do Gram-Schmidt on the set of four functions q_1, q_2, q_3, b, in that order, what would you get?

Solutions:

1. This is just a least-squares problem. There are a couple of ways to do this, but the way we learned in class is to first find an orthonormal basis by Gram-Schmidt: $q_1(t) = a_1/\|a_1\| = a_1(t)/\sqrt{\int_0^1 a_1(t)^2 dt}$, $q_2(t) = (a_2 - q_1[q_1 \cdot a_2])/\|\cdots\| = [a_2 - q_1 \int_0^1 q_1(t)a_2(t)dt]/\|\cdots\|$. Then $p(t) = q_1(q_1 \cdot b) + q_2(q_2 \cdot b) = q_1(t)\int_0^1 q_1(t')b(t')dt' + q_2(t')\int_0^1 q_2(t')b(t')dt'$.

2. **(a)** We are just writing $b(t) = p(t) + e(t)$, where $p(t)$ is the orthogonal projection and $e(t) = b(t) - p(t)$. Exactly as for vectors, we can write the orthogonal projection as:

$$p(t) = \sum_{i=1}^{3} q_i(q_i \cdot b) = \frac{\sin(t)}{\pi}\int_0^{2\pi} \sin(t')t'dt' + \frac{\sin(2t)}{\pi}\int_0^{2\pi} \sin(2t')t'dt' + \frac{\cos(t)}{\pi}\int_0^{2\pi} \cos(t')t'dt',$$

and thus $e(t) = t - p(t)$ is perpendicular to q_1, q_2, q_3. **(b)** q_1 to q_3 are already orthonormal, so they wouldn't be changed by Gram-Schmidt. When you do Gram-Schmidt on the last function $b(t)$, you would subtract off the projection and then normalize...but this is precisely the function $q_4(t) = e(t)/\|e(t)\| = e(t)/\sqrt{\int_0^{2\pi} e(t')^2 dt'}$.

The key point that I want you to understand is that you just do exactly the same steps as you would for vectors, and the "only" change is that the dot products become some kind of integral (depending on what the function dot product was chosen to be).

1 (30 pts.)

(a) (25 pts.) Compute the determinant (as a function of x) of the 4×4 matrix

$$A = \begin{bmatrix} x & x & x & x \\ x & x & 0 & 0 \\ x & 0 & x & x \\ x & 0 & x & 1 \end{bmatrix}$$

(Note that all the entries x in the matrix represent the same number.)

There are several ways to do this. One way is to use the cofactor formula on the second row, which gives

$$-x * \det \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & x & 1 \end{bmatrix} + x * \det \begin{bmatrix} x & x & x \\ x & x & x \\ x & x & 1 \end{bmatrix}.$$

The second determinant is zero because the first two rows are dependent, so we just need to expend the first. The first column is easy since there is only one nonzero element. Expanding gives $-x * x(x - x^2) = \boxed{x^4 - x^3}$.

(b) (5 pts.) Find all values of x for which A is singular.

A is singular exactly when the determinant is 0. This means $x^4 - x^3 = x^3(1 - x) = 0$, which means x is $\boxed{0 \text{ or } 1}$.

2 (35 pts.)

Let P_1 be the projection matrix onto the line through $(1, 1, 0)$ and P_2 is the projection matrix onto the line through $(0, 0, 1)$.

(a) (15 pts.) Compute $P = P_2P_1$. Note that there is a harder way and an easier way to perform this computation. Either way is valid. (The easier way uses associativity of matrix multiplication. Always a useful trick.)

We can calculate P_1 and P_2 directly via $P = A(A^TA)^{-1}A^T$ to obtain

$$P_2P_1 = B(B^TB)^{-1}B^TA(A^TA)^{-1}A^T,$$

where $B = [0, 0, 1]^T$ and $A = [1, 1, 0]^T$. However, in the middle of that mess is $B^TA = 0 * 1 + 0 * 1 + 1 * 0 = 0$, so the whole thing is the zero matrix.

The intutition behind this is that the two lines we are projecting onto are orthogonal. When we projected onto the first vector, we get some multiple of the first vector. Since this is orthogonal to the second vector, the second projection must get us 0.

(b) (5 pts.) Is $P = P_2P_1$ a projection matrix? (Explain simply.)

A matrix P is a projection matrix exactly when $P^2 = P$ and when P is symmetric, both of which are obviously true for the zero matrix (it is a projection onto the 0-dimensional space that has only the zero vector).

(c) (15 pts.) What are the four fundamental subspaces associated with P?

We have that $\boxed{C(P) = C(P^T) = 0}$, since both have column vectors that are all 0. We also have $\boxed{N(P) = N(P^T) = R^3}$ because every vector in R^3 gets sent to 0 via the zero matrix.

3 (35 pts.)

(a) (10 pts.) Perform Gram Schmidt on the two vectors $u = (1, 1, 1, 1)$ and $t = (t_1, t_2, t_3, t_4)$. The answer should be in the form q_1 and q_2, an orthonormal pair of vectors. You may wish to use the notation $\bar{t}$ for the mean of t, and $\|t - \bar{t}u\|$ for the norm of $t - \bar{t}u$.

For q_1, we take u and normalize by $\|u\| = 2$, so dividing by the norm gives $\boxed{q_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}}$.

For q_2, we need to first subtract out the projection onto u (or q_1). This gives $t - (t^T u)u/(u^u) = t - (1/4)\sum t_i u = t - \bar{t}u$. Then we need to normalize, so the answer is $\boxed{q_2 = \frac{t - \bar{t}u}{\|t - \bar{t}u\|}}$.

(b) (10 pts.) Write a “QR" decomposition of $[u\ t]$, i.e. find a 4×2 matrix Q, and a 2×2 matrix R such $[u\ t] = QR$, where $Q^T Q = I$, and $R_{2,1} = 0$.

We already have Q by putting the two vectors from the previous part together, so

$$Q = \begin{bmatrix} 1/2 & \frac{t_1 - \bar{t}u}{\|t - \bar{t}u\|} \\ 1/2 & \frac{t_2 - \bar{t}u}{\|t - \bar{t}u\|} \\ 1/2 & \frac{t_3 - \bar{t}u}{\|t - \bar{t}u\|} \\ 1/2 & \frac{t_4 - \bar{t}u}{\|t - \bar{t}u\|} \end{bmatrix}.$$

We can get R by either multiplying $Q^T A$ or by eyeballing (since we know $QR = A$, it isn't hard to see, say, that getting all 1's in the first column needs 2 copies of the first column and none of the second, etc.). Multiplying $\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ \frac{t_1 - \bar{t}u}{\|t - \bar{t}u\|} & \frac{t_2 - \bar{t}u}{\|t - \bar{t}u\|} & \frac{t_3 - \bar{t}u}{\|t - \bar{t}u\|} & \frac{t_4 - \bar{t}u}{\|t - \bar{t}u\|} \end{bmatrix} \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \\ 1 & t_4 \end{bmatrix}$

gives $\boxed{R = \begin{bmatrix} 2 & 2\bar{t} \\ 0 & \|t - \bar{t}u\| \end{bmatrix}}$.

(c)(5 pts.) When is R singular?

R is singular when its determinant is 0, so exactly when $\|t - \bar{t}u\| = 0$. This happens exactly when each $t_i = \bar{t}$, which happens when $\boxed{t = au}$ for some constant a (equivalently, $t_1 = t_2 = t_3 = t_4$).

(d)(10 pts) Use the QR decomposition of $A = [u\ t]$ (and not the normal equations with A^TA), to compute the slope of the best fit line $C + Dt$ to the data (t_i, b_i) for $i = 1, 2, 3, 4$. (In other words, compute a simple expression for D.)

The first key step is realizing that $\begin{bmatrix} C \\ D \end{bmatrix} = R^{-1}Q^Tb$. One can derive this, say, via the original formula

$$
\begin{aligned}
x &= (A^TA)^{-1}A^Tb && (1)\\
&= (R^TQ^TQR)^{-1}(R^TQ^T)b && (2)\\
&= R^{-1}I(R^T)^{-1}R^TQ^Tb && (3)\\
&= R^{-1}Q^Tb. && (4)
\end{aligned}
$$

Now, a simplifying observation is that R^{-1} is still upper triangular, with the form $\frac{1}{2\|t-\bar{t}u\|}\begin{bmatrix} blah & blah \\ 0 & 2 \end{bmatrix}$. Since we only care about D, the second coordinate, we just need to multiply the lower right term by the second term of Q^tb, which is q_2^Tb. So the answer is $\boxed{\dfrac{q_2^Tb}{\|t-\bar{t}u\|} = \dfrac{(t-\bar{t}u)^Tb}{\|t-\bar{t}u\|^2}}$.

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