18.06 Quiz 1 March 1, 2010 Professor Strang

Your PRINTED name is: ____________________ **1.**

Your recitation number or instructor is ____________________ **2.**

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1. Forward elimination changes $A\mathbf{x} = \mathbf{b}$ to a row reduced $R\mathbf{x} = \mathbf{d}$: the complete solution is

$$\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \mathbf{c_1} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \mathbf{c_2} \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

(a) **(14 points)** What is the 3 by 3 reduced row echelon matrix R and what is $\mathbf{d}$?

Solution: First, since R is in reduced row echelon form, we must have

$$\boxed{\mathbf{d} = \begin{bmatrix} 4 & 0 & 0 \end{bmatrix}^T}$$

The other two vectors provide special solutions for R, showing that R has rank 1: again, since it is in reduced row echelon form, the bottom two rows must be all 0, and the top row is $\begin{bmatrix} 1 & -2 & -5 \end{bmatrix}^T$, i.e. $\boxed{R = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$.

(b) **(10 points)** If the process of elimination subtracted 3 times row 1 from row 2 and then 5 times row 1 from row 3, what matrix connects R and $\mathbf{d}$ to the original A and $\mathbf{b}$? Use this matrix to find A and $\mathbf{b}$.

Solution: The matrix connecting R and $\mathbf{d}$ to the original A and $\mathbf{b}$ is

$$E = E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}}$$

That is, $R = EA$ and $E\mathbf{b} = \mathbf{d}$. Thus, $A = E^{-1}R$ and $\mathbf{b} = E^{-1}\mathbf{d}$, giving

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & -2 & -5 \\ 3 & -6 & -15 \\ 5 & -10 & -25 \end{bmatrix}}$$

$$\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 12 \\ 20 \end{bmatrix}}$$

2. Suppose A is the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix}.$$

(a) **(16 points)** Find all special solutions to $Ax = 0$ and describe in words the whole nullspace of A.

Solution: First, by row reduction

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the special solutions are

$$\boxed{s_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, s_2 = \begin{bmatrix} 0 \\ -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}}$$

Thus, $N(A)$ is a plane in $\mathbb{R}^4$ given by all linear combinations of the special solutions.

(b) **(10 points)** Describe the column space of this particular matrix A. "All combinations of the four columns" is not a sufficient answer.

Solution: $C(A)$ is a plane in $\mathbb{R}^3$ given by all combinations of the pivot columns, namely

$$\boxed{c_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix}}$$

(c) **(10 points)** What is the reduced row echelon form $R^* = \text{rref}(B)$ when B is the 6 by 8 block matrix

$$B = \begin{bmatrix} A & A \\ A & A \end{bmatrix} \text{ using the same } A?$$

Solution: Note that B immediately reduces to

$$B = \begin{bmatrix} A & A \\ 0 & 0 \end{bmatrix}$$

We reduced A above: the row reduced echelon form of of B is thus

$$\boxed{B = \begin{bmatrix} rref(A) & rref(A) \\ 0 & 0 \end{bmatrix}, rref(A) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

3. **(16 points) Circle the words** that correctly complete the following sentence:

(a) Suppose a 3 by 5 matrix A has rank $r = 3$. Then the equation $Ax = b$

(always / sometimes but not always)

has (a unique solution / many solutions / no solution).

Solution: the equation $Ax = b$ $\boxed{\text{always}}$ has $\boxed{\text{many solutions}}$.

(b) What is the column space of A? Describe the nullspace of A.

Solution: $\boxed{\text{The column space is a 3-dimensional space inside a 3-dimensional space}}$, i.e. it contains all the vectors, and $\boxed{\text{the nullspace has dimension } 5 - 3 = 2 > 0 \text{ inside } \mathbb{R}^5.}$

4. Suppose that A is the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 5 \\ 2 & 4 \end{bmatrix}.$$

(a) **(10 points)** Explain in words how knowing all solutions to $A\mathbf{x} = \mathbf{b}$ decides if a given vector $\mathbf{b}$ is in the column space of A.

Solution: The column space of A contains all linear combinations of the columns of A, which are precisely vectors of the form $A\mathbf{x}$ for an arbitrary vector $\mathbf{x}$. Thus, $\boxed{A\mathbf{x} = \mathbf{b} \text{ has a solution if and only if } \mathbf{b} \text{ is in the column space of } A}$.

(b) **(14 points)** Is the vector $\mathbf{b} = \begin{bmatrix} 8 \\ 28 \\ 14 \end{bmatrix}$ in the column space of A?

Solution: $\boxed{\text{Yes}}$. Reducing the matrix combining A and $\mathbf{b}$ gives

$$\left[\begin{array}{cc|c} 2 & 1 & 8 \\ 6 & 5 & 28 \\ 2 & 4 & 14 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & 8 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & 8 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{b}$, and $\mathbf{b}$ is in the column space of A.

18.06 Professor Edelman Quiz 1 October 5, 2011

Grading

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Your PRINTED name is: ____________________

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2	T 10	2-132	Niels Moeller	2-588	3-4110	`moller`
3	T 10	2-146	Kestutis Cesnavicius	2-089	2-1195	`kestutis`
4	T 11	2-132	Niels Moeller	2-588	3-4110	`moller`
5	T 12	2-132	Yan Zhang	2-487	3-4083	`yanzhang`
6	T 1	2-132	Taedong Yun	2-342	3-7578	`tedyun`

1 (30 pts.)

Let $A = \begin{bmatrix} 0 & 0 \\ 6 & 9 \\ 2 & 3 \end{bmatrix}$.

(a) (6 pts.) Circle the best answer: The column space of A is a

a. point $\boxed{\text{b. line}}$ c. plane d. three dimensional space. Explain very briefly.

$\boxed{\text{The matrix has rank 1, seen by inspection or by elimination.}}$

(b) (6 pts.) True or $\boxed{\text{False}}$: The row space of A is a vector subspace of R^3, i.e., consists of a collection of vectors with three components that are closed under all linear combinations. Explain very briefly.

$\boxed{\text{The rowspace is a subspace of } R^2\text{, collections of vectors with } \textbf{two} \text{ components.}}$

(c) (6 pts.) Circle the best answer:

a. Matrix A has full column rank.

b. Matrix A has full row rank.

$\boxed{\text{c. Matrix } A \text{ has neither full column rank nor full row rank.}}$

Explain very briefly.

$\boxed{\text{The rank is both smaller than than the number of rows and the number of columns.}}$

(d) (12 pts.) Let $b = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$. Find the complete solution to $Ax = b$.

The second column of A is free.

The particular solution is then $x_p = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$. There is one special solution $\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$.

The complete solution consists of all vectors of the form

$$\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}.$$

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2 (23 pts.)

A is a square 3×3 matrix whose LU decomposition exists with no row exchanges. Carefully provide a count of the exact number of operations required to compute the three parameters l_{12}, l_{13} and l_{23} of L and the six parameters of U. The questions below count first all the divisions, then all the multiplications, and then all the subtractions that occur.

Avoid any unnecessary operations. (Operations on the elements being eliminated are unnecessary since that element's ultimate fate is known to be 0.)

Note: Other answers maybe ok with explanation.

(a) (5 pts.) We recall that computation of each multiplier l_{ij} requires one division. The exact number of divisions in the LU decomposition of our 3×3 A is $\boxed{3}$.

Each of the three multipliers requires one division to compute.

(b) (9 pts.) We recall that multipliers l_{ij} multiply the jth row but only to the right of column i. The exact number of multiplications in the entire LU decomposition of our 3×3 A is

$5=2^2+1^2$

The first row(column) is the pivot row(column) for the first two eliminations. First l_{21} scales A_{12} and A_{13} as a byproduct of the elimination of the (2,1) entry. Then l_{31} also scales A_{12} and A_{13}. By the time we have computed l_{32}, it multiplies the value now in A_{23}.

In general, we phrased the question in a way that counted for the fact that no computation need occur in the first row or first column. The total number of multiplications in general would then be $(n-1)^2 + \ldots + 1$.

(c) (9 pts.) We recall that after l_{ij} does its job of multipying row j to the right of column i, we subtract row j from row i but only to the right of column i. The exact number of subtractions in the entire LU decomposition of our 3×3 A is $\boxed{5}$.

$\boxed{\text{5=}2^2 + 1\text{. The subtraction count is always the same as the multiplication count.}}$

3 (27 pts.)

A is a matrix which has two special solutions to $Ax = 0$. All other solutions, we recall, are linear combinations of the two special solutions. The two special solutions are

$$\begin{bmatrix} 3 \\ 1 \\ 4 \\ 0 \\ 5 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}.$$

(a) (9 pts.) What is r =rank(A)? What is the dimension of the column space $C(A)$? What is the dimension of the nullspace $N(A)$? Very briefly explain your three numbers.

$r = 3 =$ (5 columns)-(2 special solutions)
dim($C(A)$)=$r = 3$
dim($N(A)$)=2=number of special solutions

(b) (9 pts.) B is a matrix that is the same as A except that its second row is (row 2 of A)-(row 1 of A). What is a basis for the nullspace $N(B)$?

$\boxed{\text{The two special solutions above, since } N(B) = N(A).}$

(c) (9 pts.) C is a matrix that is the same as A except that its second column is (column 2 of A)-(column 1 of A). What is a basis for the nullspace $N(C)$? (Hint: If M is invertible, it may be useful to know that if y is in $N(C)$, then $M^{-1}y$ is in $N(CM)$.)

$C = AM$, where M is the matrix given in the hint. We recognize matrices such as M. The inverse of M looks the same as M except the $(2,1)$ entry is $+1$ not -1. M^{-1} takes a vector and replaces the first component with the sum of the first two components. Thus the new basis consists of the multiplication of M^{-1} times the previous basis vectors: $\begin{bmatrix} 4 \\ 1 \\ 4 \\ 0 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}$.

4 (20 pts.)

The next question concerns M_4, the 16 dimensional space of 4x4 real matrices.

(a) (10 pts.) True or $\boxed{\text{False}}$. The twenty-four 4 x 4 permutation matrices are independent members of M_4? Explain briefly.

At most 16 members of this space can be independent because it is a 16 dimensional vector

(b) (10 pts.) True or $\boxed{\text{False.}}$ The twenty-four 4x4 permutation matrices span M_4? (Hint: is any row sum possible?) Explain briefly.

To span the space, every matrix must be a linear combination of permulation matrices. This is impossible. Why? Because every row of a permutation matrix sums to 1, so every row of a linear combination of permutation matrices is the same. Try it! Thus a matrix with different row sums can not be a linear combination of permutation matrices.

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18.06 Quiz 1 Professor Strang February 28, 2011

Your PRINTED name is ______________________ 1.

Your Recitation Instructor (and time) is ______________________ 2.

Instructors: (Pires)(Hezari)(Sheridan)(Yoo) 3.

1. (a) By elimination find the **rank** of A and the pivot columns of A (in its column space):

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 6 & 3 & 9 \\ 2 & 4 & 2 & 9 \end{bmatrix}.$$

(b) Find the special solutions to $Ax = 0$ and then find **all solutions** to $Ax = 0$.

(c) For which number b_3 does $Ax = \begin{bmatrix} 3 \\ 9 \\ b_3 \end{bmatrix}$ have a solution?

Write the **complete solution** x (the general solution) with that value of b_3.

(a) $\begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 6 & 3 & 9 \\ 2 & 4 & 2 & 9 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-3R_1+R_2} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{3}R_2+R_3} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$r = \text{rank}(A) = 2$, pivot columns are $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 9 \\ 9 \end{bmatrix}$

(b) Special Solutions: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. $N(A) = \left\{ c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

all Solutions $c_1, c_2 \in \mathbb{R}$

(c) $\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 3 \\ 3 & 6 & 3 & 9 & 9 \\ 2 & 4 & 2 & 9 & b_3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 3 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & b_3-6 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 3 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & b_3-6 \end{array}\right]$

Hence to have a Solution we need $b_3 - 6 = 0 \Rightarrow \boxed{b_3 = 6}$

For this value of b_3, a particular solution is given by $x_p = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

complete solution: $x_c = x_p + x_n = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

nullsolution

2. Suppose A is a 3 by 5 matrix and the equation $Ax = b$ has a solution for every b. What are $(a)(b)(c)(d)$? (If you don't have enough information to answer, tell as much about the answer as you can.)

(a) Column space of A : Since $AX = b \Rightarrow b = x_1 A_1 + \cdots + x_n A_n$ where $A_1, \ldots A_n$ are the columns of A, every b in $\mathbb{R}^3$ is in $C(A)$. So $C(A) = \mathbb{R}^3$.

(b) Nullspace of A Since $C(A) = \mathbb{R}^3$, we have $r = 3$ and therefore # free variables = # special solutions $= n - r = 5 - 3 = 2$. Hence $N(A)$ is a plane in $\mathbb{R}^5$.

(c) Rank of A

By (a) $C(A) = \mathbb{R}^3$ therefore $r = 3$.
We can also argue by saying that $r = m$ or we have a constraint on b.

(d) Rank of the 6 by 5 matrix $B = \begin{bmatrix} A \\ A \end{bmatrix}$.

We can use elimination to obtain

$$\begin{bmatrix} A \\ A \end{bmatrix} \longrightarrow \begin{bmatrix} A \\ 0 \end{bmatrix}.$$

But $\begin{bmatrix} A \\ 0 \end{bmatrix}$ has rank 3. Therefore $\operatorname{rank}(B) = 3$.

3. (a) When an odd permutation matrix P_1 multiplies an even permutation matrix P_2, the product P_1P_2 is odd (EXPLAIN WHY).

P_1 applies an odd number of row exchanges to I and P_2 applies an even number. Hence P_1P_2 applies odd + even = odd number of row exchanges.

(b) If the columns of B are vectors in the nullspace of A, then AB is 0 matrix (EXPLAIN WHY). Let $B = [B_1 | \cdots | B_K]$, where $B_1, \ldots, B_K$ are the columns of B. Since each B_i is in $N(A)$ we have $AB_i = 0$.

Then $AB = [AB_1 | \cdots\cdots | AB_K] = [0 | \cdots | 0] = 0$

(c) If $c = 0$, factor this matrix into $A = LU$ (lower triangular times upper triangular):

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & c \end{bmatrix}.$$

(d) That matrix A is invertible unless $c =$ 21.

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & c \end{bmatrix} \xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \\ 0 & 6 & c-3 \end{bmatrix} \xrightarrow{-3R_2+R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & c-21 \end{bmatrix}$

Hence A is invertible unless $c - 21 = 0 \Rightarrow \boxed{c = 21}$

(a) when $c = 0$ we get $A = LU$

where $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

18.06 Professor Edelman Quiz 1 October 3, 2012

Grading

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Your PRINTED name is: ____________________

Please circle your recitation:

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4	T 11	2-132	Rosalie Belanger-Rioux	2-331	3-5029	robr
5	T 12	2-132	Geoffroy Horel	2-490	3-4094	ghorel
6	T 1	2-132	Tiankai Liu	2-491	3-4091	tiankai
7	T 2	2-132	Tiankai Liu	2-491	3-4091	tiankai

1 (22 pts.)

Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 3 & 4 \end{pmatrix}$ and $M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 3 & 4 \end{pmatrix}$.

a) (5 pts.) Which are the pivot columns and which are the free columns of A ?

We subtract twice the first row from the second and three times the first row from the third to find the row echelon form:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We see that the first column is free and the second and third columns are pivots.

b) (5 pts.) Which are the pivot columns and which are the free columns of M ?

We subtract twice the first row from the second and three times the first row from the third to find the row echelon form:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We see that all three columns are pivot columns and thus there are no free columns.

c) (6 pts.) For which $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ are there solutions to $Ax = b$? For those b, write down the complete solution.

We form the augmented matrix:

$$\begin{pmatrix} 0 & 1 & 1 & b_1 \\ 0 & 2 & 2 & b_2 \\ 0 & 3 & 4 & b_3 \end{pmatrix}$$

As before, we subtract twice the first row from the second and three times the first row from the third:

$$\begin{pmatrix} 0 & 1 & 1 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 1 & b_3 - 3b_1 \end{pmatrix}.$$

We now find the reduced echelon form by subtracting the third row from the first:

$$\begin{pmatrix} 0 & 1 & 0 & 4b_1 - b_3 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 1 & b_3 - 3b_1 \end{pmatrix}.$$

The second row gives the equation $0 = b_2 - 2b_1$ so for $Ax = b$ to have solutions, we must have $b_2 = 2b_1$.

The third row gives $x_3 = b_3 - 3b_1$. The first row gives $x_2 = 4b_1 - b_3$.

The nullspace is 1-dimensional and by inspection we see that it contains $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, so the general solution to $Ax = b$ is

$$x = \begin{pmatrix} c \\ 4b_1 - b_3 \\ b_3 - 3b_1 \end{pmatrix},$$

where c may be any real number.

d) (6 pts.) For which $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ are there solutions to $Mx = b$? For those b, write down the complete solution.

Since M is square has no free columns, for any b there will be a solution to $Mx = b$. We form the augmented matrix:

$$\begin{pmatrix} 0 & 1 & 1 & b_1 \\ 1 & 2 & 2 & b_2 \\ 0 & 3 & 4 & b_3 \end{pmatrix}$$

As before, we subtract twice the first row from the second and three times the first row from the third:

$$\begin{pmatrix} 0 & 1 & 1 & b_1 \\ 1 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 1 & b_3 - 3b_1 \end{pmatrix}.$$

We now find the reduced echelon form by subtracting the third row from the first:

$$\begin{pmatrix} 0 & 1 & 0 & 4b_1 - b_3 \\ 1 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 1 & b_3 - 3b_1 \end{pmatrix}.$$

The third equation gives $x_3 = b_3 - 3b_1$, the first equation gives $x_1 = 4b_1 - b_3$, and the second equation gives $x_2 = b_2 - 2b_1$.

2 (24 pts.)

Consider the vector space of polynomials of the form $p(x) = ax^3 + bx^2 + cx + d$. Are the following subspaces? Explain briefly in a way that we are sure you understand subspaces.Note: We have written down the answers in detail to be pedagogical, but you didn't need to write so much.

a) (6 pts.) Those $p(x)$ for which $p(1) = 0$.

This is a subspace, because any linear combination of polynomials of degree at most 3 with a root at 1 is still a polynomial with degree at most 3 and a root at 1 : let $p(x) = fp_1(x)+gp_2(x)$ where f, g are any real number, then $p(1) = fp_1(1) + gp_2(1) = f \cdot 0 + g \cdot 0 = 0$, as required.

b) (6 pts.) Those $p(x)$ for which $p(0) = 1$.

This is not a vector space. One of many reasons is I can add two polynomials that have value 1 at 0. Then they will have value 2 at 0. Hence we have exhibited a linear combination of polynomials that are in the set which is not itself in the set. Hence this cannot be a subspace.

c) (6 pts.) Those $p(x)$ for which $a + b = c + d$.

This is a subspace. Consider two polynomials in the set, say $p_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$, $i = 1, 2$. Then, since they are in the set, we know $a_i + b_i = c_i + d_i$ for $i = 1, 2$. Now take any linear combination of those polynomials, say $p(x) = fp_1(x) + gp_2(x)$. Writing $p(x)$ all out and rearranging terms, we get $p(x) = ax^3 + bx^2 + cx + d$, where $a = a_1 + a_2$, $b = b_1 + b_2$, etc. Clearly now, using $a_i + b_i = c_i + d_i$ for $i = 1, 2$. we have $a + b = c + d$. Hence $p(x)$, or in fact any linear combination of members of the set, is still in the set: we have a subspace.

d) (6 pts.) Those $p(x)$ for which $a^2 + b^2 = c^2 + d^2$.

This is not a subspace. You can guess from what we wrote down for c) that here, things might go wrong. For example, take $p_1(x) = x^3 + x$ and $p_2(x) = -x^3 + x$. Both of them are in the set, but if you add them up you get $p(x) = p_1(x) + p_2(x) = 2x$, which is not in the set.

3 (27 pts.)

a) (9 pts.) Find an LU decomposition of the matrix $A = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$, where we assume $a \neq 0$. L is unit lower triangular (1's on the diagonal) and U is upper triangular.

Use elimination. The upper-left entry a is our pivot, and we want to eliminate the c just below it, so subtract c/a times row 1 from row 2, to get

$$U = \begin{pmatrix} a & b \\ 0 & -bc/a \end{pmatrix}.$$

Our elimination matrix was

$$E_{21} = \begin{pmatrix} 1 & 0 \\ -c/a & 1 \end{pmatrix},$$

so

$$L = E_{21}{}^{-1} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix}.$$

We have $A = LU$ for the above L and U.

b) (9 pts.) Find a "PU" decomposition of the matrix $A = \begin{pmatrix} 0 & a & b \\ c & d & e \\ 0 & 0 & f \end{pmatrix}$, where P is a permutation matrix, and U is upper triangular.

The matrix A fails to be upper triangular because of the c in the first column, so we swap rows 1 and 2. Then $A = PU$, where

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} c & d & e \\ 0 & a & b \\ 0 & 0 & f \end{pmatrix}.$$

(Note that in this case $PA = U$ also, because $P = P^{-1}$. However, the problem asked for an $A = PU$ decomposition, so just writing $PA = U$ is not enough.)

c) (9 pts.) Find an "X'X" decomposition of the matrix $A = \begin{pmatrix} a^2 + b^2 + c^2 & ad + be + cf \\ ad + be + cf & d^2 + e^2 + f^2 \end{pmatrix}$. The matrix X that you need to find satisfies $A = X^T X$, and need not be a square matrix.

By inspection, $A = X^T X$ where X is the 3×2 matrix

$$X = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}.$$

Many other matrices X would work, but why make things unnecessarily complicated?

4 (27 pts.)

Either construct a matrix A or argue that it is impossible, where the nullspace of A is exactly the multiples of $(1, 1, 1, 1)$ and the dimensions (number of rows, number of columns) of A are

a) (9 pts.) 2 $\times$4

This is impossible. Indeed, a 2 by 4 matrix has rank at most 2 (the rank is the dimension of the row space). We know that $dimN(A) + dimC(A) = 4$, therefore, the dimension of the null space is at least 2, but the problem requires the null space to have dimension 1.

b) (9 pts.) 3×4

This one is possible. Since the null space has dimension 1, the rank has to be 3. If we find a rank 3 matrix having $(1, 1, 1, 1)$ in its null space then it's going to answer the problem. The following matrix works :

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

The ranks is 3 because there are 3 pivot columns and $(1, 1, 1, 1)$ is in the null space because the sum of the coefficients in each row is 0.

c) (9 pts.) 4×4

Here we need a rank 3 matrix with $(1, 1, 1, 1)$ in its null space. One possible answer is :

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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18.06 Quiz 1 Professor Strang March 9, 2012

Please PRINT your name ______________________________ **1.**

2.

Please Circle your Recitation: **3.**

4.

r1	T	11	4-159	Ailsa Keating
r2	T	11	36-153	Rune Haugseng
r3	T	12	4-159	Jennifer Park
r4	T	12	36-153	Rune Haugseng
r5	T	1	4-153	Dimiter Ostrev
r6	T	1	4-159	Uhi Rinn Suh
r7	T	1	66-144	Ailsa Keating
r8	T	2	66-144	Niels Martin Moller
r9	T	2	4-153	Dimiter Ostrev
r10	ESG			Gabrielle Stoy

1. **(36 pts.)** Suppose the 4 by 4 matrix A (with 2 by 2 blocks) is already reduced to its rref form

$$A = \begin{bmatrix} I & 3I \\ 0 & 0 \end{bmatrix}.$$

(a) Find a basis for the column space $C(A)$.

(b) Describe all possible bases for $C(A)$.

(c) Find a basis (special solutions are good) for the nullspace $N(A)$.

(d) Find the complete solution x to the 4 by 4 system

$$Ax = \begin{bmatrix} 5 \\ 4 \\ 0 \\ 0 \end{bmatrix}.$$

Solution.

(a) The column space is spanned by the vectors $(1,0,0,0),(0,1,0,0),(3,0,0,0),(0,3,0,0)$. We then put them in a matrix and do a Gaussian elimination to find independent vectors. This tells us that the basis for the column space is $\{(1,0,0,0),(0,1,0,0)\}$

(b) The column space can be described by

$$C(A) = \{(x,y,0,0) \mid x,y \in \mathbb{R}\},$$

so the basis of $C(A)$ is the set of any two independent vectors $(x_1,x_2,0,0)$ and $(x_3,x_4,0,0)$. This means that the matrix

$$A = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}$$

has full rank (in other words $x_1x_4 - x_2x_3 \neq 0$ must hold).

(c) We observe that $(3,0,-1,0)$ and $(0,3,0,-1)$ are two independent vectors belonging to the null space. Since the column space has dimension 2, the null space has dimension $4-2=2$, so any basis of $N(A)$ has two elements. Hence, $\{(3,0,-1,0),(0,3,0,-1)\}$ is a basis for $N(A)$.

(d) We start by looking for $x_{\text{particular}}$ via elimination. Note that the matrix is already in a reduced row echelon form:

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 5 \\ 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So $x_{\text{particular}} = (5,4,0,0)$. Then the complete solution is given by

$$\begin{aligned} x &= x_{\text{particular}} + x_{\text{nullspace}} \\ &= (5,4,0,0) + (3a,3b,-a,-b) \\ &= (5+3a, 4+3b, -a, -b) \end{aligned}$$

for any $a,b \in \mathbb{R}$.

$\square$

2. **(16 pts.)** Suppose the matrix A is m by n of rank r, and the matrix B is M by N of rank R. Suppose the column space $C(A)$ is contained in (possibly equal to) the column space $C(B)$. (This means that every vector in $C(A)$ is also in $C(B)$.) What relations must hold between m and M, n and N, and r and R?

It might be good to write down an example of A and B where all the columns are different.

Solution. The column space of A is contained in $\mathbb{R}^m$, and the column space of B is contained in $\mathbb{R}^M$. If $C(A) \subseteq C(B)$, this means they are contained in the same Euclidean space, so $M = m$. The dimension of the column space is the rank of the matrix, so if $C(A) \subseteq C(B)$, this means $\dim C(A) \leq \dim C(B)$, hence $r \leq R$. There are no relations between N and n; $n = N$ if $A = B$, $n \leq N$ if $B = [A\ A]$, and $n \geq N$ if $A = [B\ B]$.

□

3. (a) **(16 pts.)** Suppose three matrices satisfy $AB = C$. If the columns of B are dependent, show that the columns of C are dependent.

(b) **(12 pts.)** If A is 5 by 3 and B is 3 by 5, show using part (a) or otherwise that $AB = I$ is impossible.

Solution. (a) The columns of B being dependent means by definition that there is a vector $\mathbf{x} \neq 0$ such that $B\mathbf{x} = 0$. But then we also have

$$C\mathbf{x} = (AB)\mathbf{x} = A(B\mathbf{x}) = A(\mathbf{0}) = \mathbf{0},$$

which means that the same $\mathbf{x} \neq 0$ works to show that the columns of C are dependent.

(b) The columns of B are dependent, since these are five vectors in $\mathbb{R}^3$, and $5 > 3$. Thus, by part (a), the columns of AB must be dependent. However, columns of I are independent, so AB can never equal I. [Note: Switching the order matters here. One can indeed find a 3×5 matrix A, and a 5×3 matrix B such that $AB = I$ is the 3×3 identity - hence any "proof" that is insensitive to the order of A and B must be flawed].

□

4. **(20 pts.)** Apply row elimination to reduce this invertible matrix from A to I. Then write A^{-1} as a product of three (or more) simple matrices coming from that elimination. Multiply these matrices to find A^{-1}.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix}.$$

Solution. Swapping rows 1 and 2 corresponds to

$$P := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Subtracting 4 times row 1 from row 3 corresponds to

$$E_{31} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}.$$

Subtracting row 3 from row 2 corresponds to

$$E_{23} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Putting them together, we get

$$E_{23}E_{31}PA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = I.$$

Hence, $A^{-1} = E_{23}E_{31}P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & -4 & 1 \end{pmatrix}.$

□

Please PRINT your name ______________________ **1.**

2.

Please Circle your Recitation: **3.**

r1	T	10	36-156	Russell Hewett
r2	T	11	36-153	Russell Hewett
r3	T	11	24-407	John Lesieutre
r4	T	12	36-153	Stephen Curran
r5	T	12	24-407	John Lesieutre
r6	T	1	36-153	Stephen Curran
r7	T	1	36-144	Vinoth Nandakumar
r8	T	1	24-307	Aaron Potechin
r9	T	2	24-307	Aaron Potechin
r10	T	2	36-144	Vinoth Nandakumar
r11	T	3	36-144	Jennifer Park

(1.) **(30 pts.)** For a 3 by 3 matrix A, suppose all three multipliers are $l_{21} = l_{31} = l_{32} = 3$. Each l_{ij} multiplies pivot row j when it is subtracted from row i.

(a) Assuming no row exchanges, what is A, if elimination reaches

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & g \end{bmatrix}?$$

(b) In case $g = 0$, the three columns of A must be dependent. Find the nullspace (a vector space) of A.

(c) In case $g \neq 0$, what is the column space of U? What is the column space of the original matrix A? How do you know?

(2.) **(40 pts.)** A is a 2 by 4 matrix with exactly two special solutions to $Ax = 0$:

$$x = s_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad x = s_2 = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

(a) Find the reduced row echelon form R of A.

(b) What is the column space of A?

(c) What is the complete solution to $Rx = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$?

(d) Find a combination of columns 2, 3, 4 that equals the zero vector. (Not OK to use $0\,(\text{col}\,2) + 0\,(\text{col}\,3) + 0\,(\text{col}\,4) = 0$. The problem is to show that these 3 columns are dependent.)

(3.) **(30 pts.)** Suppose A is the 2 by 3 matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}.$$

(a) Find all 3 by 2 matrices X with

$$AX = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

What is a basis for that space of matrices?

(b) Find one 3 by 2 matrix X with

$$AX = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(c) Find the complete solution of this matrix equation: *all* 3 by 2 matrices X with

$$AX = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

18.06 Professor Edelman Quiz 1 October 4, 2013

Grading

1

2

3

4

5

Your PRINTED name is: ____________________

Please circle your recitation:

1	T 9	2-132	Dan Harris	E17-401G	3-7775	dmh
2	T 10	2-132	Dan Harris	E17-401G	3-7775	dmh
3	T 10	2-146	Tanya Khovanova	E18-420	4-1459	tanya
4	T 11	2-132	Tanya Khovanova	E18-420	4-1459	tanya
5	T 12	2-132	Saul Glasman	E18-301H	3-4091	sglasman
6	T 1	2-132	Alex Dubbs	32-G580	3-6770	dubbs
7	T 2	2-132	Alex Dubbs	32-G580	3-6770	dubbs

1 (30 pts.)

Consider the matrix $A = \begin{bmatrix} 2 & 3 & 5 \\ 2 & 4 & 5 \\ -2 & 0 & -5 \end{bmatrix}$ and the general right hand side $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

a) (18 pts.) Reduce A to an upper triangular matrix U and carry out the same steps on the right side b by working with the augmented matrix $[A\ b]$. Factor the 3 by 3 matrix A into LU =(lower triangular)(upper triangular).

b) (6 pts.) Describe the column space of A exactly through a condition on b.

c) (6 pts.) What are the special solutions to $Ax = 0$?

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2 (25 pts.)

Consider the matrix $A = \begin{bmatrix} 0.451 & 0.3 & 0 & 0.2 & 1 & -.1 \\ 0.673 & 0.7 & 1 & 0.5 & 1 & -.3 \end{bmatrix}$. (Big Hint: The questions asked here can all be readily done with mental arithmetic if you reorder your world view.)

a) (5 pts.) What is the column space of A? (Explain briefly.)

c) (20 pts.) Write down four independent solutions to $Ax = 0$.

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3 (15 pts.)

a) (4 pts.) Complete these sentences appropriately for a 3×3 matrix A.

If the column space is a plane, the nullspace is a ____________________.

If the column space is a line, the nullspace is a ____________________.

If the column space is all of R^3, the nullspace ____________________.

If the column space is the zero vector, the nullspace ____________________.

b) (11 pts.) Find a 7×7 matrix A whose column space equals its nullspace, or argue briefly it can not exist. (Hint: part 3a might provide a clue.)

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4 (15 pts.)

The vector space S consists of 2×2 matrices whose entries are linear functions of the symbol x. For example, $\begin{bmatrix} x & 2-x \\ 1+x & 4+10x \end{bmatrix}$ is one member of S, and the general form of a member of S is

$$A = \begin{bmatrix} a+bx & e+fx \\ c+dx & g+hx \end{bmatrix}.$$

Write down a basis for S.

5 (15 pts.)

An elimination step (a multiple of one row subtracted from another row) may be written in Julia as

```
A[j,:]-=m*A[i,:]
```

where we assume $i \neq j$.

The same row operation in matrix form is expressed in linear algebra by "replace A with EA," where E is the matrix formed from the identity with $-m$ in the (j, i) entry.

If Gauss-Jordan is performed on an n by n non-singular matrix A, augmented with I, provide an exact count in terms of n of the general number of required elimination steps. (Hint: we are counting in units of row operations, not elemental operations; the exact answer has the form $an^2 + bn + c$). We want the exact answer and a very short reason why.)

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18.06 F13 EXAM 1

SOLUTIONS

1.

a) $$\left(\begin{array}{ccc|c} 2 & 3 & 5 & b_1 \\ 2 & 4 & 5 & b_2 \\ -2 & 0 & -5 & b_3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 2 & 3 & 5 & b_1 \\ 0 & 1 & 0 & b_2 - b_1 \\ 0 & 3 & 0 & b_3 + b_1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 2 & 3 & 5 & b_1 \\ 0 & 1 & 0 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 + b_1 - 3(b_2 - b_1) \end{array}\right)$$

$$U = \begin{pmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix}$$

b) From bottom row of augmented matrix,

$$b_3 + b_1 - 3(b_2 - b_1) = 4b_1 - 3b_2 + b_3 = 0.$$

c) Free column of U is on right, so put a 1 in bottom entry

$$\begin{pmatrix} -5/2 \\ 0 \\ 1 \end{pmatrix}$$

---

2. a) A contains columns $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ which are independent

$\therefore$ column space $= \mathbb{R}^2$

b) First subtract top row from bottom row so that we have columns $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0.451 & 0.3 & 0 & 0.2 & 1 & -0.1 \\ 0.222 & 0.4 & 1 & 0.3 & 0 & -0.2 \end{pmatrix}$$

Four independent solutions:

$$\begin{pmatrix} 1 \\ 0 \\ -0.222 \\ 0 \\ -0.451 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ -0.4 \\ 0 \\ -0.3 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ -0.3 \\ 1 \\ -0.2 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0.2 \\ 0 \\ 0.1 \\ 1 \end{pmatrix}$$

(corresponding to the free columns 1, 2, 4, 6).

---

3. a) If the column space is a plane, the nullspace is a line.

If the column space is a line, the nullspace is a plane.

If the column space is $\mathbb{R}^3$, the nullspace is zero.

If the column space is zero, the nullspace is $\mathbb{R}^3$.

b) $\dim(\text{column space}) + \dim(\text{nullspace}) = 7$

If column space = nullspace then they both have dimension $7/2$, but dimension cannot be a fraction.

So the matrix A cannot exist.

4.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$$

---

5. Gauss-Jordan for nonsingular square matrix:
In each of the n columns, we use the pivot to reduce each of the other $n-1$ entries to 0.
Therefore we use $n(n-1)$ operations.

SOLUTIONS TO EXAM 1

Problem 1 (30 pts)

(a) Since the multipliers are all 3, the row operations we had goes:

- subtract three times row 1 to row 2;
- subtract three times row 1 to row 3;
- subtract three times row 2 to row 3,

where the end step gives us

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & g \end{bmatrix}.$$

So now we just need to reverse the row operations. Reversing the last step means adding three times row 2 to row 3:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 6 & g+3 \end{bmatrix}.$$

Reversing the second step means adding three times row 1 to row 3:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 3 & 9 & g+6 \end{bmatrix},$$

And reversing the first step means adding three times row 1 to row 2:

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 9 & g+6 \end{bmatrix}.$$

(b) To find the nullspace of A, we first note that there are three columns in A, and that the rank of A is 2 in the case when $g = 2$, so the dimension of $N(A)$ is one. So we need to find just one vector that are in $N(A)$; then this vector will be a basis for $N(A)$. By using Gaussian elimination to $Ax = 0$, we get $Ux = 0$, from which we deduce that $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ is a solution.

Therefore, $N(A) = c \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$.

(c) If $g \neq 0$, then we observe that $C(U) = \mathbb{R}^3$. Since row rank is equal to the column rank, the column rank (i.e. the dimension of the column space) or A is also 3. Since the column space of A is contained in $\mathbb{R}^3$, which is 3-dimensional, it must be the whole space as well. So $C(A) = \mathbb{R}^3$.

Problem 2 (40 pts)

(a) Any row of A is orthogonal to the two special solutions given in the problem. That is, any row

$$r = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

satisfies $r \cdot s_1 = r \cdot s_2 = 0$. This is just a system of two linear equations, so we need to solve the equation

$$\begin{bmatrix} 3 & 1 & 0 & 0 \\ 6 & 0 & 2 & 1 \end{bmatrix} r = 0$$

whose complete solution is given by

$$c \begin{bmatrix} 1 \\ -3 \\ 0 \\ -6 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \end{bmatrix},$$

from which we get the reduced row echelon form of A given by

$$\begin{bmatrix} 1 & -3 & 0 & -6 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

(b) R has two pivots, and therefore A has two pivots and $r(A) = 2$. Two independent columns in $\mathbb{R}^2$ span $\mathbb{R}^2$, so $C(A) = \mathbb{R}^2$.

Partial credit was given if the student referred back to A for the column space and if they gave $\mathbb{R}^2$ with incomplete reasoning. Most point were lost if they just gave $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as the basis without indicating where they came from (or from reading them off of R, not A). There were lots of right answer, with wrong (or no) reasons.

(c) The free variables are x_2, x_4 so the particular solution is

$$x_p = \begin{bmatrix} 3 \\ 0 \\ 6 \\ 0 \end{bmatrix}.$$

The complete solution is

$$x_c = x_p + c_1 \cdot s_1 + c_2 \cdot s_2 = \begin{bmatrix} 3 \\ 0 \\ 6 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

(d) We have

$$2 \cdot \begin{bmatrix} -3 \\ 0 \end{bmatrix} - 2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -6 \\ -2 \end{bmatrix}.$$

One can find these coefficients by inspection, or combining the special solutions:

$$2s_1 - s_2 = \begin{bmatrix} 0 \\ 2 \\ -2 \\ -1 \end{bmatrix}.$$

Problem 3 (30 pts)

(a) Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be a column of X. Then x, y and z satisfy

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

We apply elimination to get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

from which we deduce that $y = 0$, and $x = -z$. So each column of X is a multiple of the vector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Since there are two columns of X, X can be written as

$$\begin{bmatrix} a & b \\ 0 & 0 \\ -a & -b \end{bmatrix}.$$

The basis for this space of matrices is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

(b) We first solve

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Elimination gives We apply elimination to get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

From which we see that we can take $y = -1/2, x = 3/2, z = 0$. This will be the first column of X.

We now solve

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Elimination gives We apply elimination to get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

And we can take $y = 1/2, x = -1/2, z = 0$. This is the second column of X.

So one possible solution for X is

$$\begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \\ 0 & 0 \end{bmatrix}.$$

(c) The set of complete solutions is given by

$$X_{\text{particular}} + X_{\text{special}} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \\ 0 & 0 \end{bmatrix} + a \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

18.06 **EXAM 1** October 1, 2014

Your PRINTED name is: ____________________

Please circle your recitation:

R01	T 9	E17-136	Darij Grinberg
R02	T 10	E17-136	Darij Grinberg
R03	T 10	24-307	Carlos Sauer
R04	T 11	24-307	Carlos Sauer
R05	T 12	E17-136	Tanya Khovanova
R06	T 1	E17-139	Michael Andrews
R07	T 2	E17-139	Tanya Khovanova

Grading

1	
2	
3	
4	
Total:	

Each problem is 25 points, and each of its five parts (a)–(e) is 5 points.

In all problems, write all details of your solutions. Just giving an answer is not enough to get a full credit. Explain how you obtained the answer.

Problem 1. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 7 & 10 \end{pmatrix}$. (a) Find the $A = LU$ factorization of the matrix A.

(b) Solve the system $A\mathbf{x} = (3, 10, 20)^T$.

Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 7 & k \end{pmatrix}$ (obtained by replacing the bottom right entry of A by parameter k).

(c) For which values of k is the matrix B singular?

(d) Find all values of k for which the system $B\mathbf{x} = (1, 2, 3)^T$ has infinitely many solutions. (You don't need to solve the system in this part.)

(e) Find all values of k for which the system $B\mathbf{x} = (10, 1, 2014)^T$ has exactly one solution. (You don't need to solve the system in this part.)

Problem 2. Which of the following sets of vectors are vector subspaces of $\mathbb{R}^3$? Explain your answer?

(a) All vectors $(x, y, z)^T$ such that $10\,x + y + 2014\,z = 0$.

(b) All vectors $(x, y, z)^T$ such that $x + y + z \leq 2014$.

(c) All vectors $(x, y, z)^T$ such that $x + y + z = 0$ AND $x + 2y + 3z = 0$.

(d) All vectors $(x, y, z)^T$ such that $x + y + z = 0$ OR $x + 2y + 3z = 0$.

(e) All vectors $(b_1, b_2, b_3)^T$ such that the system $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{x} = (b_1, b_2, b_3)^T$ has a solution.
(You don't need to solve the system in this part.)

Problem 3. Let $A = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 3 \\ -1 & -2 & 0 & 2 & 3 \end{pmatrix}$.

(a) Find the complete solution of $A\mathbf{x} = \mathbf{0}$.

(b) Find the complete solution of $A\mathbf{x} = (1, 2, 0)^T$.

(c) Find the linear condition(s) on b_1, b_2, b_3 that guarantee that the system $A\mathbf{x} = (b_1, b_2, b_3)^T$ has a solution.

(d) Find the rank of A and dimensions of the four fundamental subspaces of A.

(e) Find bases of the four fundamental subspaces of A.

Problem 4. Which of the following statements are true? Explain your answer.

(a) Matrices A and $R = RREF(A)$ always have the same column space $C(A) = C(R)$.

(b) Matrices A and $R = RREF(A)$ always have the same row space $C(A^T) = C(R^T)$.

(c) If A is an $m \times n$ matrix with linearly independent columns, then $m \geq n$.

(d) If A is an $m \times n$ matrix of rank $r = m$, then the left nullspace $N(A^T)$ contains only the zero vector $\mathbf{0}$.

(e) If two $m \times n$ matrices A and B have exactly the same 4 fundamental subspaces

$$C(A) = C(B), \quad N(A) = N(B), \quad C(A^T) = C(B^T), \quad N(A^T) = N(B^T),$$

then $A = B$. (Prove that $A = B$ or give a counterexample where $A \neq B$.)

If needed, you can use this extra sheet for your calculations.

If needed, you can use this extra sheet for your calculations.

Exam Solutions

Problem 1

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 7 & 10 \end{pmatrix}$.

(a) Find the $A = LU$ factorization of the matrix A.

(b) Solve the system $Ax = (3, 10, 20)^T$.

Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 7 & k \end{pmatrix}$ (obtained by replacing the bottom right entry by the parameter k).

(c) For which values of k is the matrix B singular?

(d) Find all values of k for which the system $Bx = (1, 2, 3)^T$ has infinitely many solutions.

(e) Find all values of k for which the system $Bx = (10, 1, 2014)^T$ has exactly one solution.

Answers:

Let's put the matrix $B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 7 & k \end{pmatrix}$ into LU form.

We get $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 4 & k-3 \end{pmatrix}$ by substracting 2 lots of row 1 from row 2 and 3 lots of row 1 from row 3.

We get $U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & k-7 \end{pmatrix}$ by subtracting 2 lots of row 2 from row 3. We see $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$.

(a) By setting $k = 10$ we obtain $A = LU$ where $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$.

(b) We spot that $(1, 1, 1)^T$ is a solution. L and U are invertible since their diagonals are nonzero and so A is invertible. Thus, this is the only solution.

Alternatively, we can solve $Lc = (3, 10, 20)^T$ via substitution to give $c = (3, 4, 3)$ and $Ux = c = (3, 4, 3)^T$ via substitution to give $x = (1, 1, 1)^T$.

(c) B is singular if and only if U is singular. U is singular if and only if the last row is zero, i.e. if $k = 7$.

(d) This equation always has a solution since the first column is $(1, 2, 3)^T$. When $k \neq 7$ the matrix is invertible and there is only one solution. When $k = 7$ the vector $(0, 1, -1)^T$ is in the nullspace, so there are infinitely many solutions: in fact, we can see that the solutions are $(1, 0, 0)^T + c(0, 1, -1)$ for $c \in \mathbb{R}$.

(e) When $k \neq 7$ the matrix B is invertible so there exists exactly one solution.

Problem 2

Which are of the following sets of vectors are vector subspaces of $\mathbb{R}^3$? Explain your answer.

(a) All vectors $(x, y, z)^T$ such that $10x + y + 2014z = 0$

(b) All vectors $(x, y, z)^T$ such that $x + y + z \leq 2014$.

(c) All vectors $(x, y, z)^T$ such that $x + y + z = 0$ AND $x + 2y + 3z = 0$.

(d) All vectors $(x, y, z)^T$ such that $x + y + z = 0$ OR $x + 2y + 3z = 0$.

(e) All vectors $(b_1, b_2, b_3)^T$ such that $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} x = (b_1, b_2, b_3)^T$ has a solution.

Answers:

(a) Yes: this is the null space of the 3×1 matrix $\begin{pmatrix} 10 & 1 & 2014 \end{pmatrix}$.

(b) No: $(1, 0, 0)$ is in the set under consideration but $2015(1, 0, 0)$ is not.

(c) Yes: this is the null space of the 3×2 matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$.

(d) No: $(-1, 0, 1)$ and $(1, 1, -1)$ are in the set under consideration but their sum $(0, 1, 0)$ is not.

(e) Yes: this is the column space of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.

Problem 3

Let $A = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 3 \\ -1 & -2 & 0 & 2 & 3 \end{pmatrix}$

(a) Find the complete solution of $Ax = 0$.

(b) Find the complete solution of $Ax = (1,2,0)^T$.

(c) Find the linear condition(s) on b_1, b_2, b_3 that guarantee that the system $Ax = (b_1, b_2, b_3)^T$ has a solution.

(d) Find the rank of A and dimensions of the four subspaces of A.

(e) Find bases of the four fundamental subspaces of A.

Answers:

Let's put the matrix $\begin{pmatrix} 1 & 2 & 1 & 0 & 0 & b_1 \\ 1 & 2 & 2 & 2 & 3 & b_2 \\ -1 & -2 & 0 & 2 & 3 & b_3 \end{pmatrix}$ into RREF.

We get $\begin{pmatrix} 1 & 2 & 1 & 0 & 0 & b_1 \\ 0 & 0 & 1 & 2 & 3 & b_2 - b_1 \\ 0 & 0 & 1 & 2 & 3 & b_1 + b_3 \end{pmatrix}$ followed by $\begin{pmatrix} 1 & 2 & 0 & -2 & -3 & 2b_1 - b_1 \\ 0 & 0 & 1 & 2 & 3 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & 0 & 2b_1 - b_2 + b_3 \end{pmatrix}$.

(a) We read of the special vectors as

$$x_1 = (-2,1,0,0,0)^T,\ x_2 = (2,0,-2,1,0)^T \text{ and } x_3 = (3,0,-3,0,1)^T.$$

The complete solution to $Ax = 0$ is $x = c_1x_1 + c_2x_2 + c_3x_3$ for $c_1, c_2, c_3 \in \mathbb{R}$.

(b) Let $x_p = (0,0,1,0,0)^T$. Then $Ax_p = (1,2,0)^T$ so that the complete solution to $Ax = (1,2,0)^T$ is $x = x_p + c_1x_1 + c_2x_2 + c_3x_3$ for $c_1, c_2, c_3 \in \mathbb{R}$.

(c) We read of the linear combination of b_1, b_2 and b_3 in the zero row above: $2b_1 - b_2 + b_3 = 0$ is the condition that guarantees $Ax = (b_1, b_2, b_3)^T$ has a solution.

(d) We see there are two pivot columns in RREF(A) and so the rank of A is 2. Thus $\dim C(A) = \dim C(A^T) = 2$, $\dim N(A) = 5 - 2 = 3$, and $\dim N(A^T) = 3 - 2 = 1$.

(e) To give a basis for $C(A)$ we read of the the columns corresponding to pivot columns in RREF: $\{(1,1,-1)^T, (1,2,0)^T\}$. We already computed a basis for $N(A)$ in a): $\{x_1, x_2, x_3\}$. To give a basis for $C(A^T)$ we find two independent rows:

$$\{(1,2,1,0,0)^T, (1,2,2,2,3)^T\}.$$

To give a basis for $N(A^T)$ we read off the coefficients of the relation in b): $\{(2,-1,1)^T\}$.

Problem 4

Which of the following statements are true? Explain your answer.

(a) Matrices A and $R = RREF(A)$ always have the same column space $C(A) = C(R)$.

(b) Matrices A and $R = RREF(A)$ always have the same row space $C(A^T) = C(R^T)$.

(c) If A is an $m \times n$ matrix with linearly independent columns, then $m \geq n$.

(d) If A is an $m \times n$ matrix of rank $r = m$, the the left nullspace $N(A^T)$ contains only the zero vector 0.

(e) If two $m \times n$ matrices A and B have the same 4 fundamental spaces

$$C(A) = C(B),\ N(A) = N(B),\ C(A^T) = C(B^T),\ N(A^T) = N(B^T),$$

then $A = B$.

Answers:

(a) No. The matrices $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $R = REEF(A) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ have different column spaces.

(b) Yes. Row operations do not alter the row space.

(c) Yes. If A has linearly independent columns, then the column space has dimension n. But the column space is a subspace of $\mathbb{R}^m$, which has dimension m. Thus $n \leq m$.

(d) Yes. $r = m$ tells us that the rows are independent, so that $N(A^T) = 0$. Alternatively, we can note that $\dim N(A^T) = m - r = 0$

(e) No. I and $2I$ have the same fundamental spaces.

18.06 Exam I **Professor Strang** **March 7, 2014**

Your PRINTED Name is: ______________________________

Please circle your section:

R01	T	10	36-144	Qiang Guang
R02	T	10	35-310	Adrian Vladu
R03	T	11	36-144	Qiang Guang
R04	T	11	4-149	Goncalo Tabuada
R05	T	11	E17-136	Oren Mangoubi
R06	T	12	36-144	Benjamin Iriarte Giraldo
R07	T	12	4-149	Goncalo Tabuada
R08	T	12	36-112	Adrian Vladu
R09	T	1	36-144	Jui-En (Ryan) Chang
R10	T	1	36-153	Benjamin Iriarte Giraldo
R11	T	1	36-155	Tanya Khovanova
R12	T	2	36-144	Jui-En (Ryan) Chang
R13	T	2	36-155	Tanya Khovanova
R14	T	3	36-144	Xuwen Zhu

Grading 1:

2:

3:

____

Question 3.(c) is a 1 point question. All other questions are worth 11 points each.

1. Suppose the blocks in A are 3 by 3 (so A is 6 by 6), and $F = ones(3)$ is the all-ones matrix:

$$A = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

(a) Find a basis for the nullspace $N(A)$.

(b) Find a basis for the left nullspace $N(A^T)$.

(c) Exactly which matrices have dimension of nullspace of A equal to dimension of nullspace of A^T ?

2. **(a)** What value of q gives A a different rank compared to all other values of q?

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & -1 & 3 & 4 \\ 4 & 3 & 9 & q \end{bmatrix}$$

(b) With that special value of q, what are the conditions on b_1, b_2, b_3 for $Ax = b$ to have a solution?

(c) If those conditions are satisfied by b_1, b_2, b_3, what are all the solutions x (the complete solution to $Ax = b$ with that special value of q) ?

3. Suppose the nullspace of A (5 by 4 matrix) is spanned by v and w, which are special solutions to $Ax = 0$:

$$v = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad w = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

(a) What is the row reduced echelon form $R = rref(A)$? We don't have to know A.

(b) Which vectors in R^4 can be rows of A? How many of the 5 rows of A will be independent?

(c) ***One point question:*** What is the dimension of the matrix space containing all 5 by 4 matrices A that have those vectors v and w in their nullspace?

(d) If C is any 4 by 7 matrix of rank $r = 4$, find the column space of C. Explain clearly why $Cx = b$ always has infinitely many solutions.

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18.06 Exam I   Professor Strang   March 7, 2014

Solutions

Question 3.(c) is a 1 point question. All other questions are worth 11 points each.

1. Suppose the blocks in A are 3 by 3 (so A is 6 by 6), and $F = ones(3)$ is the all-ones matrix:

$$A = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

(a) Find a basis for the nullspace $N(A)$.

(b) Find a basis for the left nullspace $N(A^T)$.

(c) Exactly which matrices have dimension of nullspace of A equal to dimension of nullspace of A^T ?

Solution.

(a) Matrix A is already in its $rref$. There are three pivots. Therefore, $r = \dim C(A) = \dim R(A) = 3$. Hence, $\dim N(A) = 6 - r = 3$. The special basis is $[-1, -1, -1, 1, 0, 0]'$, $[-1, -1, -1, 0, 1, 0]'$, and $[-1, -1, -1, 0, 0, 1]'$.

(b) The dimension of the left nullspace is: $\dim N(A^T) = 6 - r = 3$, with a basis: $[0, 0, 0, 1, 0, 0]'$, $[0, 0, 0, 0, 1, 0]'$, and $[0, 0, 0, 0, 0, 1]'$.

(c) The dimension of the nullspace of A is $n - r$, the dimension of the left nullspace is $m - r$. They are equal when $n - r = m - r$. That is, when $m = n$. The dimensions are equal when the matrix is square.

2. **(a)** What value of q gives A a different rank compared to all other values of q?

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & -1 & 3 & 4 \\ 4 & 3 & 9 & q \end{bmatrix}$$

(b) With that special value of q, what are the conditions on b_1, b_2, b_3 for $Ax = b$ to have a solution?

(c) If those conditions are satisfied by b_1, b_2, b_3, what are all the solutions x (the complete solution to $Ax = b$ with that special value of q) ?

Solution.

(a) Start the elimination: replace row 2 (r_2) with $r_1 - 2r_2$, then replace r_3 with $r_3 - 4r_1$ to get:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -5 & -3 & 2 \\ 0 & -5 & -3 & q-4 \end{bmatrix}.$$

Continue by replacing the third row with the difference of the third minus the second row to get the triangular matrix U:

$$U = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -5 & -3 & 2 \\ 0 & 0 & 0 & q-6 \end{bmatrix}.$$

When $q = 6$, U has two pivots, and the rank of A is 2. Otherwise, the rank of A is 3.

(b) Repeat the elimination steps on $b = (b_1, b_2, b_3)$ to get $(b_1, b_2 - 2b_1, b_3 - b_2 - 2b_1)$. The condition is for the last coordinate to be zero: $b_3 - b_2 - 2b_1 = 0$.

(c) There are two pivots and two free variables: x_3 and x_4. The nullspace is 2-dimensional with special solutions: $[-9/5, -3/5, 1, 0]'$ and $[-9/5, 2/5, 0, 1]'$. We can get a particular solution when we assign free variables to be zero, and solve for the pivot variables:

$$\left[\begin{array}{ccccc} x_1 & + & 2x_2 & = & b_1 \\ & & -5x_2 & = & b_2 - 2b_1 \end{array}\right].$$

The result is: $[b_1/5{+}2b_2/5, -b_2/5{+}2b_1/5, 0, 0]'$. The complete solution is: $[b_1/5{+}2b_2/5, -b_2/5{+}2b_1/5, 0, 0]'{+}c[-9/5, -3/5, 1, 0]'{+}d[-9/5, 2/5, 0, 1]'$.

3. Suppose the nullspace of A (5 by 4 matrix) is spanned by v and w, which are special solutions to $Ax = 0$:

$$v = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad w = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

(a) What is the row reduced echelon form $R = rref(A)$? We don't have to know A.

(b) Which vectors in R^4 can be rows of A? How many of the 5 rows of A will be independent?

(c) ***One point question:*** What is the dimension of the matrix space containing all 5 by 4 matrices A that have those vectors v and w in their nullspace?

(d) If C is any 4 by 7 matrix of rank $r = 4$, find the column space of C. Explain clearly why $Cx = b$ always has infinitely many solutions.

Solution.

(a) It is clear that x_2 and x_4 are free variables. We can reconstruct a part of R right away:

$$R = \begin{bmatrix} 1 & a & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

To find a, b, and c we can use the fact that $r_1 \cdot v = r_1 \cdot w = r_2 \cdot v = r_2 \cdot w = 0$. We get $a = -4$, $b = -1$, $c = -2$. Thus,

$$R = \begin{bmatrix} 1 & -4 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) The dimension of the nullspace is 2 and is equal to $n - r$. So $r = 4 - 2 = 2$, Therefore, the dimension of the row space is 2. Thus, exactly two rows in A are independent. The row space of A is the same as the row space of R. So any row in A must be a linear combination of the first two rows of R: $c[1, -4, 0, -1] + d[0, 0, 1, -2]$.

(c) Each row can be any vector in a 2-dimensional space orthogonal to v and w. There are five rows. So the dimension of the space of all such matrices is 10.

(d) The rank of the matrix is equal to the dimension of the column space. Thus the dimension of the column space is 4. Therefore, the column space spans all of R^4. A basis of the column space is $[1, 0, 0, 0]'$, $[0, 1, 0, 0]'$, $[0, 0, 1, 0]'$ and $[0, 0, 0, 1]'$. Hence, for any vector b there exists a solution. In addition, the dimension of the nullspace is 3. Therefore, there are infinitely many solutions.

18.06 Exam I Professor Strang March 2, 2015

Your PRINTED Name is: ________________________________

Please CIRCLE your section:

R01	T10	26-302	Dmitry Vaintrob
R02	T10	26-322	Francesco Lin
R03	T11	26-302	Dmitry Vaintrob
R04	T11	26-322	Francesco Lin
R05	T11	26-328	Laszlo Lovasz
R06	T12	36-144	Michael Andrews
R07	T12	26-302	Netanel Blaier
R08	T12	26-328	Laszlo Lovasz
R09	T1pm	26-302	Sungyoon Kim
R10	T1pm	36-144	Tanya Khovanova
R11	T1pm	26-322	Jay Shah
R12	T2pm	36-144	Tanya Khovanova
R13	T2pm	26-322	Jay Shah
R14	T3pm	26-322	Carlos Sauer
ESG			Gabrielle Stoy

Grading 1:

2:

3:

____

1. (36 points) Start with the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & -2 & 4 & 0 \\ 3 & -3 & 7 & 0 \end{bmatrix}$$

(a) Find a basis for the column space $\mathbf{C}(A)$.

(b) Find a basis for the null space $\mathbf{N}(A)$.

(c) Find a basis for the row space $\mathbf{C}(A^T)$.

(d) Write the complete solution to $Ax = b$.

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & -2 & 4 & 0 \\ 3 & -3 & 7 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

2. (32 points)

(a) Suppose the matrices A and B have the same column space. Give an example where A and B have different nullspaces — or say why this is impossible.

(b) Again A and B have the same column space. Give an example where A and B have different ranks r — or say why this is impossible.

(c) CIRCLE TRUE or FALSE:
If B is a square matrix then $\mathbf{C}(B) = \mathbf{C}(B^T)$.

(d) If the columns of a 5 by 3 matrix M are linearly independent and x in $\mathbf{R}^3$ is not the zero vector, then you know that Mx is ____________.
I am looking for an answer that uses independence of columns and $x \neq 0$.

3. (32 points)

(a) Find a 3 by 3 matrix A whose column space is the plane $x+y+z=0$ in $\mathbf{R}^3$. (This means: $\mathbf{C}(A)$ consists of all column vectors (x, y, z) with $x+y+z=0$.)

(b) How do you know that a 3 by 3 matrix A with that column space is not invertible?

(c) Does there exist a matrix B whose column space is spanned by $(1, 2, 3)$ and $(1, 0, 1)$ and whose nullspace is spanned by $(1, 2, 3, 6)$? If so, construct B. If not, explain why not.

(d) Is this set of matrices a vector space or not? All 3 by 3 matrices with the column vector $(1, 1, 1)$ in their column space. Yes or No with a reason.

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18.06 Exam I **Professor Strang** **March 2, 2015**

Solutions

1. (36 points) Start with the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & -2 & 4 & 0 \\ 3 & -3 & 7 & 0 \end{bmatrix}$$

(a) Find a basis for the column space $\mathbf{C}(A)$.

(b) Find a basis for the null space $\mathbf{N}(A)$.

(c) Find a basis for the row space $\mathbf{C}(A^T)$.

(d) Write the complete solution to $Ax = b$.

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & -2 & 4 & 0 \\ 3 & -3 & 7 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Solution. We can notice that the second row is 2 times the first row. This means that rows are dependent and the rank of the matrix is less than 3. In addition, we see that not all the rows are multiples of the same row, that means the rank of the matrix is more than 1. Therefore, it must be equal to 2. It follows that $\dim \mathbf{C}(A) = \dim \mathbf{C}(A^T) = 2$. The dimension of the null space is $n - r = 4 - 2 = 2$.

We can also find the dimensions by calculating U. As the second row is proportional to the first one, we need to swap the second and the third row and have a $PA = LU$ decomposition:

$$U = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution (1a). Any two independent columns of A would form a basis. That means columns 1 and 3, or columns 2 and 3. The default answer is to pick the pivot columns: that is pick columns that are not linearly dependent on other columns to the left of them. In this case, these are columns 1 and 3: $(1, 2, 3)$ and $(2, 4, 7)$.

Solution (1b). The fact that the last column is zero means $(0, 0, 0, 1)$ is in the null space. The fact that the first column plus the second is zero means $(1, -1, 0, 0)$ is in the nullspace. Or, we can use U and the default basis for the null space is $(-1, 1, 0, 0)$ and $(0, 0, 0, 1)$.

Solution (1c). Similarly to 1a), we can pick any two independent rows of A as the basis. That means row 1 and row 3, or row 2 and row 3. Or, we can pick all the non-zero rows in the U matrix. The most common bases would be either $(1, -1, 2, 0)$ and $(3, -3, 7, 0)$, or $(1, -1, 2, 0)$ and $(0, 0, 1, 0)$.

Solution (1d). Using the the row swap and the elimination on b, the augmented matrix U is:

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The particular solution is $(-1, 0, 1, 0)$ and the complete solution: $(-1, 0, 1, 0) + a(-1, 1, 0, 0) + b(0, 0, 0, 1)$.

2. (32 points)

(a) Suppose the matrices A and B have the same column space. Give an example where A and B have different nullspaces — or say why this is impossible.

Solution. The simplest way to provide an example is to add dependent columns to matrix A. For example, matrices $[1]$ and $[1\ 0]$ have the same column space and different null spaces.

(b) Again A and B have the same column space. Give an example where A and B have different ranks r — or say why this is impossible.

Solution. The rank is the dimension of the column space. That means the rank is the same for both matrices.

(c) CIRCLE TRUE or FALSE:
If B is a square matrix then $\mathbf{C}(B) = \mathbf{C}(B^T)$.

Solution. False. Consider $B = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Then the column space of B is $(1, 0)^T$ and the column space of B^T is $(0, 1)^T$.

(d) If the columns of a 5 by 3 matrix M are linearly independent and x in $\mathbf{R}^3$ is not the zero vector, then you know that Mx is ________.
I am looking for an answer that uses independence of columns and $x \neq 0$.

Solution. Mx is a non zero vector in $\mathbf{R}^5$.

3. (32 points)

(a) Find a 3 by 3 matrix A whose column space is the plane $x+y+z=0$ in $\mathbf{R}^3$. (This means: $\mathbf{C}(A)$ consists of all column vectors (x, y, z) with $x+y+z=0$.)

Solution. The column space must have dimension 2. That means that any two independent vectors in the plane will do. For example,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

(b) How do you know that a 3 by 3 matrix A with that column space is not invertible?

Solution. The plane is 2-dimensional, so the rank of the matrix is 2, which is less than the size of the matrix.

(c) Does there exist a matrix B whose column space is spanned by $(1,2,3)$ and $(1,0,1)$ and whose nullspace is spanned by $(1,2,3,6)$? If so, construct B. If not, explain why not.

Solution. Such matrix does not exist. The dimensions of such a matrix must be 3 by 4 ($m=3$ and $n=4$). The dimension of the column space is 2, because the given vectors are independent. That means the dimension of the nullspace must be $4-2=2$. The null space cannot be spanned by 1 vector.

(d) Is this set of matrices a vector space or not? All 3 by 3 matrices with the column vector $(1,1,1)$ in their column space. YES or NO with a reason.

Solution. Consider matrices

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

They contain $(1,1,1)$ in their column space, but there sum does not:

$$A+B=\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}.$$

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