**8 October 1997**                                    **Profs. S. Lee and A. Kirillov**

# MASSACHUSETTS INSTITUTE OF TECHNOLOGY

## Hour Exam I for Course 18.06: <u>Linear Algebra</u>

Recitation Instructor:                          Your Name: **SOLUTIONS**


Recitation Time:                               Lecturer:


Grading

1. **40**

2. **16**

3. **16**

4. **28**

**TOTAL: 100**


Do all your work on these pages.
No calculators or notes.
Please work carefully, and check your intermediate results whenever possible.
Point values (total of 100) are marked on the left margin.

**1.** Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 3 & 6 & 10 \end{bmatrix}$.

[16]    **1a.** Give an LU-factorization of $A$.

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix} ; \ U = \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 3 & 6 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U.$$

$l_{21}:$ row $\underline{2}$ $-(2) \times$ row $\underline{1}$
$l_{31}:$ row $\underline{3}$ $-(3) \times$ row $\underline{1}$
$l_{41}:$ row $\underline{4}$ $-(3) \times$ row $\underline{1}$

$l_{32}:$ row $\underline{3}$ $-(1) \times$ row $\underline{2}$
$l_{42}:$ row $\underline{4}$ $-(1) \times$ row $\underline{2}$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix}.$$

[8]     **1b.** Give a basis for the column space of $A$.

Pivot columns in $A$ (column 1, column 3):
$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 10 \\ 10 \end{bmatrix}$$

[8]     **1c.** Give a basis for the nullspace of $A$.

Special solution(s) to $Ux = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ : $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ .

[8]     **1d.** Give the complete solution to $Ax = \begin{bmatrix} 3 \\ 4 \\ 7 \\ 7 \end{bmatrix}$. Recall that $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 3 & 6 & 10 \end{bmatrix}$.

$$x_{\text{complete}} = \begin{bmatrix} 9 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 3 & | & 3 \\ 2 & 4 & 7 & | & 4 \\ 3 & 6 & 10 & | & 7 \\ 3 & 6 & 10 & | & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & | & 3 \\ 0 & 0 & 1 & | & -2 \\ 0 & 0 & 1 & | & -2 \\ 0 & 0 & 1 & | & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & | & 3 \\ 0 & 0 & 1 & | & -2 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & | & 9 \\ 0 & 0 & \boxed{1} & | & -2 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

$\underbrace{\hphantom{xxxxx}}_{Ax=b}$ $\underbrace{\hphantom{xxxxx}}_{Rx=d}$

The pivot variables are $x_1$ and $x_3$; the free variable is $x_2$.

$$x_1 \begin{bmatrix} \boxed{1} \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ \boxed{1} \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 9 \\ -2 \\ 0 \\ 0 \end{bmatrix}}_{d} \text{ gives } x_{\text{particular}} = \begin{bmatrix} 9 \\ 0 \\ -2 \end{bmatrix}.$$

The complete solution is the particular solution plus all linear combinations of the special solution(s).

[8]     **2a.** If possible, give a matrix $A$ which has $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ as a basis for the column space,

and $\begin{bmatrix} 0 \\ 3 \\ 2 \\ -1 \end{bmatrix}$ as a basis for its row space. If not possible, give your reason.

$$A = \begin{bmatrix} 0 & 3 & 2 & -1 \\ 0 & 6 & 4 & -2 \\ 0 & 3 & 2 & -1 \end{bmatrix}, \text{ or any nonzero multiple of it.}$$

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 2 & -1 \\ 0 & 6 & 4 & -2 \\ 0 & 3 & 2 & -1 \end{bmatrix}.$$

[8]     **2b.** Are the vectors $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ a basis for the vector space $\mathbf{R}^3$?

(More than 'yes' or 'no' is needed for full credit.)
Show your work, then briefly explain your answer.

YES.

$$\underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}}_{A} \rightarrow \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} \boxed{1} & 1 & 3 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{-1} \end{bmatrix}}_{r=3 \text{ pivot columns}}.$$

The pivot columns in $A$ are a <u>basis</u> for the column space of $A$.
The column space of $A$ is a $r = 3$-dimensional subspace of $\mathbf{R}^3$
(i.e., <u>all of $\mathbf{R}^3$</u>).

[16]　　　**3.** Given that $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} A = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{P}$, find $A^{-1}$.

$$A^{-1} = \begin{bmatrix} 1 & 0 & 8 \\ 1 & 2 & 3 \\ 2 & 5 & 3 \end{bmatrix}.$$

Premultiply both sides of the original equation by $P^{-1} = P^{T}$.

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}}_{A^{-1}} A = I.$$

**4.** Suppose $x = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ is the only solution to $Ax = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{bmatrix}$.

[12]   **4a.** Fill in each (blank) with a number.
The columns of $A$ span a ____(blank)____ -dimensional subspace of the vector space $\mathbf{R}^{(\text{blank})}$.

---

The columns of $A$ span a 3-dimensional subspace of the vector space $\mathbf{R}^5$.

---

$A \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{bmatrix}$ shows that $A$ is a $m = 5$ by $n = 3$ matrix.

The nullspace of $A$ has dimension $\underbrace{(n - r)} = 0$ since $x = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ is unique.

$A$ has rank $r = n = 3$.
The column space is a subspace of dimension $r = 3$ in $\mathbf{R}^5$.

[16]   **4b.** After applying elementary row operations to $A$, the <u>reduced</u> row echelon form will
be $R = $ _____(give the matrix)_____ .

$R = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Each pivot column in the <u>reduced</u> row echelon form $R$ has 1 as a pivot, with
zeros below and above it.   From 4a, recall that $R$ has $r = 3$ pivot columns.

# 18.06   Professor Strang    Quiz 1    February 26, 1997

1. (36= 4 times 9 points) $A$ is a 3 by 4 matrix and $b$ is a column vector in $\mathbf{R}^3$:

$$A = \begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 7 & 6 & 8 \\ 3 & 9 & 6 & 7 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 7 \\ 7 \end{bmatrix}$$

(a) Reduce $Ax = b$ to echelon form $Ux = c$ and find one solution $x_p$ (if a solution exists).

(b) Find all solutions to this system $Ax = b$ (if solutions exist). Describe this set of solutions geometrically. Is it a subspace?

(c) What is the column space of this matrix $A$? Change the entry 7 in the lower right corner to a different number that gives a smaller column space for the new matrix (call it $M$). The new entry is _____.

(d) Give a right side $b$ so that your new $Mx = b$ has a solution and a right side $b$ so that $Mx = b$ has no solution.

2. (27= 3 times 9 points) Suppose $A$ is a *square invertible* $n$ by $n$ matrix.

(a) What is its column space and what is its nullspace?

(b) Suppose $A$ can be factored into $A = LU$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 7 & 3 & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

Describe the first elimination step in reducing $A$ to $U$. How do you know that $U$ is also invertible?

(c) Find a specific 3 by 3 invertible matrix $A$ that can *not* be factored into this $LU$ form. What factorization is still possible for your example? (You don't have to find the factors.) How do you know your $A$ is invertible?

3. $A$ is an $m$ by $n$ matrix of rank $r$. Suppose $Ax = b$ has *no solution* for some right sides $b$ and *infinitely many solutions* for some other right sides $b$.

  (a) (9) Decide whether the nullspace of $A$ contains only the zero vector and why.

  (b) (9 points) Decide whether the column space of $A$ is all of $\mathbf{R}^m$ and why.

  (c) (10) For this $A$, find all true relations between the numbers $r$, $m$, and $n$.

  (d) (9 points) Can there be a right side $b$ for which $Ax = b$ has exactly one solution? Why or why not?

**18.06**        **Professor Edelman**        **Quiz 1**        **October 1, 1998**

**Your name is** _____.

## Please circle your recitation:

| | | | | | | |
|---|---|---|---|---|---|---|
| 1) | M2 | 2-131 | Darren Crowdy | crowdy@math | 2-335 | 3-7905 |
| 2) | M2 | 2-132 | Yue Lei | yuelei@math | 2-586 | 3-4102 |
| 3) | M3 | 2-131 | Darren Crowdy | crowdy@math | 2-335 | 3-7905 |
| 4) | T10 | 2-131 | Sergiu Moroianu | bebe@math | 2-491 | 3-4091 |
| 5) | T10 | 2-132 | Gabrielle Stoy | stoy@math | 2-235 | 3-4984 |
| 6) | T11 | 2-131 | Sergiu Moroianu | bebe@math | 2-491 | 3-4091 |
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| 8) | T12 | 2-132 | Anda Degeratu | anda@math | 2-229 | 3-1589 |
| 9) | T12 | 2-131 | Edward Goldstein | egold@math | 2-092 | 3-6228 |
| 10) | T1 | 2-131 | Anda Degeratu | anda@math | 2-229 | 3-1589 |
| 11) | T2 | 2-132 | Yue Lei | yuelei@math | 2-586 | 3-4102 |

**1. (35 pts)**  Find the row reduced echelon forms R of all the matrices below:

**(a.)** The $3 \times 4$ matrix of all ones.

**(b.)** A general $m \times n$ matrix of all ones.

**(c.)** The $3 \times 4$ matrix with $a_{ij} = i + j - 1$.

**(d.)** $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{pmatrix}$ .

(e.) $A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$ .

**2. (20 pts)** Sketch the image of the square figure to the left below after applying the map $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$ .

You may use the "graph paper" to its right. Please label the axes clearly.
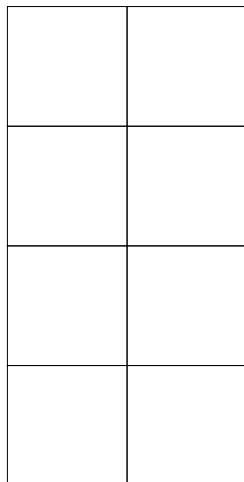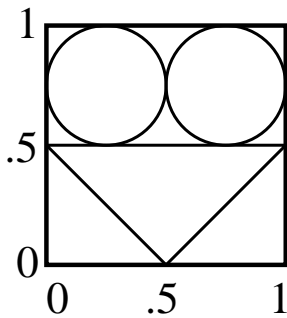
**3. (30 pts)** Please briefly but clearly explain your answers.

**(a.)** Are the set of invertible $2 \times 2$ matrices in $M$ a subspace?

**(b.)** Are the set of singular $2 \times 2$ matrices in $M$ a subspace?

**(c.)** Consider the matrices in $M$ whose nullspace contains $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Is this a subspace?

**4. (15 pts)** Find $L$ and $U$ for the nonsymmetric matrix $A = \begin{pmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{pmatrix}$. (Assume nothing is accidentally zero.)

**18**.06        Professor Strang        Quiz 1        March 6, 1998

**Your name is:** _____

**Please circle your recitation:**

**1.**   Suppose the complete solution to the equation

$$Ax = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} \quad \text{is} \quad x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

**(6) (a)**   What is the dimension of the row space of $A$?

**(12) (b)**   What is the matrix $A$?

**(6) (c)**   Describe exactly all the vectors $b$ for which $Ax = b$ can be solved. (Don't just say that $b$ must be in the column space.)

**ANSWER BELOW AND ON THE NEXT PAGE**

**2.** Suppose the matrix $A$ is this product $BC$ (not $L$ times $U$!):

$$A = BC = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 6 & 6 \end{bmatrix}$$

**(16) (a)** Find bases for the row space and the column space of $A$.

**(8) (b)** Find a basis for the space of all solutions to $Ax = 0$.

**(8) (c)** All these answers will be different if you correctly change one entry in the first factor $B$. Tell me the new matrix $B$.

**3.**    **(12) (a)**   Find the row-reduced echelon form $R$ of $A$ and also the inverse matrix $E^{-1}$ that produces $A = E^{-1}R$.

$$A = \begin{bmatrix} 1 & 0 & 3 & 3 \\ 2 & 0 & 6 & 6 \\ 1 & 1 & 3 & 3 \end{bmatrix} . \qquad \text{Find } R \text{ and } E^{-1}.$$

**(9) (b)**   Separate that multiplication $E^{-1}R$ into columns of $E^{-1}$ times rows of $R$. This allows you to write $A$ as the sum of *two rank-one matrices*. What are those two matrices?

3

**4.**    **(16) (a)**   Suppose $A$ is an $m$ by $n$ matrix of rank $r$. Describe exactly the matrix $Z$ (its shape and all its entries) that comes from transposing the row echelon form of $R'$ (prime means transpose):

$$Z = \text{rref}(\text{rref}(A)')'.$$

**(7) (b)**   Compare $Z$ in Problem 4a with the matrix $ZZ$ that comes from starting with the transpose of $A$ (and not transposing at the end):

$$ZZ = \text{rref}(\text{rref}(A')').$$

Explain in one sentence why $ZZ$ is or is not equal to $Z$.

1. (a) The nullspace has dimension 2. Therefore $3 - r = 2$ and $r = 1$.

   (b) The first column of $A$ comes from knowing the particular solution. The other columns come from knowing the two special solutions in the nullspace:

   $$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

   (c) The vector $b$ must be a multiple of $\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$.

2. (a) One basis for the row space is

   $$\begin{bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 6 \\ 6 \end{bmatrix}.$$

   (b) One basis for the column space (since columns 1 and 3 have pivots) is

   $$B \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix} \quad \text{and} \quad B \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 14 \end{bmatrix}.$$

   (c) All answers are different (because the rank is different) when $b_{33} = 0$:

   $$B = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

3. (a) The row-reduced form is

   $$R = \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   This form was reached by a product of elementary matrices, including a permutation:

   $$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}.$$

   The matrix $E^{-1}$ that recovers $A$ from $R$ is

   $$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(b) The third row of $R$ is zero! So the two column-row multiplications are from columns 1 and 2 of $E^{-1}$ and rows 1 and 2 of $R$:

$$
\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 3 \\ 2 & 0 & 6 & 6 \\ 1 & 0 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = A\,.
$$

4. (a) The matrix $Z$ is $m$ by $n$. All its entries are zero except for $r$ ones at the start of the main diagonal. If $A$ is 3 by 4 of rank $r = 2$, then

$$
Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\,.
$$

(b) The matrix $ZZ$ is the same as $Z$, because $A$ and $A^T$ always have the same rank.

**18**.06      Professor Strang      Quiz 1      March 10, 1999

**Your name is:** _____      **Grading**   1

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**Please circle your recitation:**

1) Mon    2–3    2-131  S. Kleiman      5)  Tues  12–1  2-131  S. Kleiman
2) Mon    3–4    2-131  S. Hollander    6)  Tues   1–2  2-131  S. Kleiman
3) Tues  11–12  2-132  S. Howson       7)  Tues   2–3  2-132  S. Howson
4) Tues  12–1   2-132  S. Howson

**1  (32 pts.)**    The 3 by 3 matrix $A$ is

$$
A = \begin{bmatrix} c & c & 1 \\ c & c & 2 \\ 3 & 6 & 9 \end{bmatrix} .
$$

(a)  Which values of $c$ lead to each of these possibilities?

1. $A = LU$: three pivots without row exchanges

2. $PA = LU$: three pivots after row exchanges

3. $A$ is singular: less than three pivots.      *(Continued)*

(b) For each $c$, what is the rank of $A$?

(c) For each $c$, describe exactly the nullspace of $A$.

(d) For each $c$, give a basis for the column space of $A$.

**2 (21 pts.)**   *A is m by n. Suppose Ax = b has at least one solution for every b.*

(a) The rank of $A$ is _____ .

(b) Describe all vectors in the nullspace of $A^T$.

(c) The equation $A^T y = c$ has (0 or 1)(1 or $\infty$)(0 or $\infty$)(1) solution for every $c$.

**3 (16 pts.)**  Suppose $u, v, w$ are a basis for a subspace of $R^4$, and these are the columns of a matrix $A$.

    (a) How do you know that $A^T y = 0$ has a solution $y \neq 0$?

    (b) How do you know that $Ax = 0$ has only the solution $x = 0$?

**4 (31 pts.)**     (a) To find the first column of $A^{-1}$ (3 by 3), what system $Ax = b$ would you solve?

(b) Find the first column of $A^{-1}$ (if it exists) for

$$A = \begin{bmatrix} a & 3 & 2 \\ 1 & 3 & 0 \\ 1 & b & 0 \end{bmatrix}.$$

(c) For each $a$ and $b$, find the rank of this matrix $A$ and say why.

(d) For each $a$ and $b$, find a basis for the column space of $A$.

**1** (a)  1. (no $c$)

2. (all $c \neq 0$)

3. $c = 0$

(b) rank 3 $c \neq 0$

rank 2 $c = 0$

(c) $N(A) = \{0\}$ if $c \neq 0$

$N(A) = $ all multiples of $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ if $c = 0$.

(d) $c \neq 0$ Give any basis for $R^3$

$c = 0$ one basis is $\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix}$

**2** (a) $m$

(b) Only the zero vector.

(c) (0 or 1) solutions.

**3** The matrix $A$ is 4 by 3. $A^T$ is 3 by 4.

(a) Every system $A^T y = 0$ with more unknowns than equations has a nonzero solution. (By the way, $y$ will be a vector *perpendicular* to the 3-dimensional hyperplane.)

(b) $A$ has independent columns, since $u, v, w$ form a basis.

**4** (a) Solve $Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for the first column of $A^{-1}$.

(b) $\begin{bmatrix} a & 3 & 2 \\ 1 & 3 & 0 \\ 1 & b & 0 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ gives $x = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$ by inspection.

(c) If $b = 3$ then rank$(A) = 2$ (Two equal rows, regardless of $a$)

If $b \neq 3$ then rank$(A) = 3$ (Three independent rows, regardless of $a$)

(d) If $b = 3$ then one basis is $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

If $b \neq 3$ then choose any basis for $R^3$.

**18**.06          Quiz 1          October 8, 1999          Closed Book

**Your name is:** _____          **Grading**  1

                                                                2

**Please circle your recitation:**                              3

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**1  (25 pts.)**   Suppose that row operations (elimination) reduce the matrices $A$ and $B$ to the same row echelon form

$$R = \begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Which of the four subspaces are sure to be the same for $A$ and $B$? ( $C(A) = C(B)$? $N(A) = N(B)$? $C(A^T) = C(B^T)$? $N(A^T) = N(B^T)$?)

(b) Each time the subspaces in part (a) are the same for $A$ and $B$, find a basis for that subspace.

(c) True or False ($A$ is any matrix and $x, y$ are two vectors): If $Ax$ and $Ay$ are linearly independent then $x$ and $y$ are linearly independent.

**2 (25 pts.)** Suppose

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 7 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 4 & 5 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(a) Find a basis for the nullspace of $A$.

(b) Find a basis for the column space of $A$.

(c) Give the complete solution to

$$Ax = \begin{bmatrix} 3 \\ 3 \\ 21 \end{bmatrix}.$$

2

**3 (25 pts.)**  Suppose $A$ is a 3 x 5 matrix and the solutions to $A^T y = 0$ are spanned by the vectors

$$y = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

(a) What is the rank of this $A$?

(b) For all $A$, why does the rank of $A$ equal the rank of the block matrix

$$B = \begin{bmatrix} A & A \\ A & A \end{bmatrix}?$$

(c) If the rank of a matrix $A$ equals the number of rows ($r = m$), what do we know about the equation $Ax = b$?

3

**4 (25 pts.)**    Suppose $A$ is a 4 by 3 matrix, and the complete solution to

$$Ax = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 1 \end{bmatrix} \quad \text{is} \quad x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

(a) What is the third column of $A$?

(b) What is the second column of $A$?

(c) Give all known information about the first column of $A$.

**1**  (a) $N(A) = N(B)$ and $C(A^T) = C(B^T)$

(b) $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix}$ for the row space; $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ -5 \\ 1 \end{bmatrix}$ for the nullspace.

(c) **True**

   Reason: Whenever a combination $cx + dy = 0$, multiply by $A$ to see that $c(Ax) + d(Ay) = 0$.

**2**  (a) $\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ (The first matrix is invertible so it has no effect on the nullspace)

(b) The pivot columns are $1, 2, 4$ (and the first matrix has an effect!) $\begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 28 \end{bmatrix}$.

(c) $x = x_p + x_n = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

**3**  (a) Those vectors $y$ are dependent, they span a space $N(A^T)$ that has dimension 2. So $m - r = 2$ and $m = 3$ and $r = 1$.

(b) The second block of rows copies the first so no increase in the rank. Same for the second block of columns. So those extra blocks leave the rank unchanged.

(c) If $r = m$ then $Ax = b$ has a solution (one or more) for *every* right side $b$.

**4**  (a)–(b) The particular solution says that column $2 +$ column $3 =$ right side $b$. The nullspace solution says that $2(\text{column } 2) + \text{column } 3 = 0$.

   Therefore column $2 = -b$ and column $3 = 2b$.

(c) Since the nullspace is one-dimensional, the 3 by 4 matrix $A$ has rank 2. Therefore we know that the first column of $A$ is *not* a multiple of $b$.

18.06     Professor Strang     Quiz 1     October 2, 2000

**Your name is:** _____

**Please circle your recitation:**

| | | | | | | |
|---|---|---|---|---|---|---|
| 1) | M2 | 2-131 | Holm | 2-181 | 3-3665 | tsh@math |
| 2) | M2 | 2-132 | Dumitriu | 2-333 | 3-7826 | dumitriu@math |
| 3) | M3 | 2-131 | Holm | 2-181 | 3-3665 | tsh@math |
| 4) | T10 | 2-132 | Ardila | 2-333 | 3-7826 | fardila@math |
| 5) | T10 | 2-131 | Czyz | 2-342 | 3-7578 | czyz@math |
| 6) | T11 | 2-131 | Bauer | 2-229 | 3-1589 | bauer@math |
| 7) | T11 | 2-132 | Ardila | 2-333 | 3-7826 | fardila@math |
| 8) | T12 | 2-132 | Czyz | 2-342 | 3-7578 | czyz@math |
| 9) | T12 | 2-131 | Bauer | 2-229 | 3-1589 | bauer@math |
| 10) | T1 | 2-132 | Ingerman | 2-372 | 3-4344 | ingerman@math |
| 11) | T1 | 2-131 | Nave | 2-251 | 3-4097 | nave@math |
| 12) | T2 | 2-132 | Ingerman | 2-372 | 3-4344 | ingerman@math |
| 13) | T2 | 1-150 | Nave | 2-251 | 3-4097 | nave@math |

**1 (30 pts.)** Suppose the matrix $A$ has reduced row echelon form $R$:

$$A = \begin{bmatrix} 1 & 2 & 1 & b \\ 2 & a & 1 & 8 \\ (\text{row} & 3) & & \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) What can you say immediately about row 3 of $A$?

(b) What are the numbers $a$ and $b$?

(c) Describe *all solutions* of $Rx = 0$. Circle the spaces that are the same for $A$ as for $R$: (row space)(column space)(null space).

**2 (30 pts.)**    (a) Find the number $c$ that makes this matrix singular (not invertible):

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 6 \\ 2 & 6 & c \end{bmatrix}$$

(b) If $c = 20$ what are the column space $C(A)$ and the nullspace $N(A)$? Describe them in this specific case (not just repeat their definitions). Also describe $C(A^{-1})$ and $N(A^{-1})$ for the inverse matrix!

(c) With $c = 20$ factor the matrix into $A = LU$ (lower triangular $L$ and upper triangular $U$).

**3 (40 pts.)**   Suppose $A$ is an $m$ by $n$ matrix of rank $r$.

(a) If $Ax = b$ has a solution for every right side $b$, what is the column space of $A$?

(b) In part (a), what are all equations or inequalities that must hold between the numbers $m$, $n$, and $r$.

(c) Give a specific example of a 3 by 2 matrix $A$ of rank 1 with first row $[2\ 5]$. Describe the column space $C(A)$ and the nullspace $N(A)$ completely.

(d) Suppose the right side $b$ is the same as the first column in your example (part c). Find the complete solution to $Ax = b$.

# Math 18.06 Exam 1 Solutions

**1 (30 pts.)**   (a) Because row 3 of $R$ is all zeros, row 3 of $A$ must be a linear combination of rows 1 and 2 of A. The three rows of $A$ are linearly dependent.

(b) After one step of elimination we have

$$\begin{bmatrix} 1 & 2 & 1 & b \\ 0 & a-4 & -1 & 8-2b \\ & (\text{row} & 3) & \end{bmatrix}$$

Looking at $R$ we see that the second column of $A$ is not a pivot column, so $a = 4$. Continuing with elimination, we get to

$$\begin{bmatrix} 1 & 2 & 0 & 8-b \\ 0 & 0 & 1 & 2b-8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Comparing this to $R$ we see that $b = 5$

(c) Setting the free variables $x_2$ and $x_4$ to 1 and 0, and vice versa, and solving $Rx = 0$, we get the nullspace solution

$$x = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

The row space and the nullspace are always the same for $A$ and $R$.

**2 (30 pts.)**   (a) After elimination, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & c-8 \end{bmatrix}$$

So this matrix will not be invertible when $c = 8$

(b) When $c$ is not equal to 8, the matrix is invertible, its rank is 3. So its nullspace is just the zero vector, and its columnspace is all of $\mathbb{R}^3$. The same logic and answers apply to $A^{-1}$.

(c) Using our multipliers from elimination,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 12 \end{bmatrix}$$

**3 (40 pts.)**  (a) There must be a pivot in every row, so $r = m$ and the column space of $A$ is all of $\mathbb{R}^m$

(b) We always have $r \leq n$. From (a) we know $r = m$. From these we deduce also that $m \leq n$

(c) Just use a multiple of $[2,5]$ for the other rows also. For example

$$A = \begin{bmatrix} 2 & 5 \\ 4 & 10 \\ 0 & 0 \end{bmatrix}$$

The column space will be the line in $\mathbb{R}^3$ consisting of all multiples of your first column. The nullspace will be the line in $\mathbb{R}^2$ consisting of all multiples of the nullspace solution $\begin{bmatrix} -5/2 \\ 1 \end{bmatrix}$

(d) Adding the particular solution $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to the nullspace solution from (c) we get the complete solution

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -5/2 \\ 1 \end{bmatrix}$$

# 18.06　　Exam 1　　March 1, 2000　　Closed Book

**Your name is:** _____

**Please circle your recitation:**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1) | M 2 | 2–131 | P. Clifford | | 2) | M 3 | 2–131 | P. Clifford |
| 3) | T 11 | 2–132 | T. de Piro | | 4) | T 12 | 2–132 | T. de Piro |
| 5) | T 1 | 2–131 | T. Bohman | | 6) | T 1 | 2–132 | T. Pietraho |
| 7) | T 2 | 2–132 | T. Pietraho | | 8) | T 2 | 2–131 | T. Bohman |

**Note: Make sure your exam has 5 problems.**

| Problem | | Points possible |
|---|---|---|
| 1 | _____ | 20 |
| 2 | _____ | 20 |
| 3 | _____ | 20 |
| 4 | _____ | 20 |
| 5 | _____ | 20 |
| Total | _____ | 100 |

**1 (20 pts)**   Suppose the $3 \times 3$ matrix $A$ has row $1 +$ row $2 =$ row 3.

(a) Explain why $Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ cannot have a solution.

(b) Which right hand side vector $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ might allow a solution $Ax = b$?

(Give the best answer you can, based on the information provided.)

(c) Why is $A$ not invertible?

**2 (20 pts)** Let

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 0 \\ 0 & 3 & 2 \end{bmatrix}.$$

(a) Factor $A$ as $A = LU$, where $L$ is lower triangular and $U$ is upper triangular.

(b) Find a basis for the column space of $A$.

(c) What is the rank of $A$?

**3 (20 pts)** Suppose

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 7 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 4 & 5 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

(a) What is the rank of $A$?

(b) Find a basis for the nullspace of $A$.

(c) Find the complete solution to

$$Ax = \begin{bmatrix} 10 \\ 15 \\ 85 \end{bmatrix}.$$

**4  (20 pts)**   Let $M$ be the vector space of all $2 \times 2$ matrices and let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

(a) Give a basis for $M$.

(b) Describe a subspace of $M$ which contains $A$ and does not contain $B$.

(c) True (give a reason) or False (give a counterexample): If a subspace of $M$ contains $A$ and $B$, it must contain the identity matrix $I$.

(d) Describe a subspace of $M$ which contains no diagonal matrices except for the zero matrix.

**5  (20 pts)**  If $A^2 = 0$, the zero matrix, explain why $A$ is not invertible.

**Problem 1** (a) After forming the augmented matrix and doing row reduction, the third row becomes $[\,0\ 0\ 0\ -1\,]$, which corresponds to the equation $0 = -1$, so there is no solution.
(b) The same argument shows that in order for $Ax = b$ to have a solution, $b$ must satisfy $b_3 = b_1 + b_2$.
(c) If $A$ were invertible, there would always be a solution $Ax = b$.

**Problem 2** (a)

$$
A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix}.
$$

(b) From $U$ in part (a) we see that every column is a pivot column. The pivot columns from $A$ are a basis for the column space: $(2, 2, 0)$, $(2, 5, 3)$, $(1, 0, 2)$. Since the rank is three, the column space is all of $\mathbf{R}^3$, so another basis would be the standard basis $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. In fact, any three independent vectors in $\mathbf{R}^3$ will do.
(c) The rank is three because there are three pivots.

**Problem 3** (a) This is an $LU$ factorization. The $U$ is the echelon form of $A$, so you can see that there are three pivots, so the rank of $A$ is three.
(b) A basis for $N(A)$ consists of the special solutions. These are $(-1, -2, 1, 0, 0)$ and $(-1, 1, 0, -1, 1)$.
(c) A particular solution is $(-30, -15, 0, 10, 0)$ so the complete solution is

$$
\begin{bmatrix} -30 \\ -15 \\ 0 \\ 10 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.
$$

**Problem 4** (a) One basis would be $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.
(b) The subspace consisting of all multiples of $A$ is a subspace which contains $A$ but not $B$.
(c) True: If a subspace $V$ contains $A$ and $B$, then it contains $A - B = I$.
(d) Same answer as (b) will work.

**Problem 5** There are many different proofs. One is to say that if $A^2 = 0$ then obviously $A^2$ is not invertible. Therefore $A$ isn't invertible, because the product of invertible matrices is invertible. I.e., if $A$ were invertible, then $A^2$ would be invertible.

18.06    Strang, Edelman, Huhtanen    Quiz 1    October 1, 2001

**Your name is:**    _____

**Please circle your recitation:**

**Recitations**

| # | Time | Room | Instructor | Office | Phone | Email @math |
|---|------|------|-----------|--------|-------|-------------|
| Lect. 1 | MWF 12 | 4-270 | M Huhtanen | 2-335 | 3-7905 | huhtanen |
| Lect. 2 | MWF 1 | 4-370 | A Edelman | 2-380 | 3-7770 | edelman |
| Rec. 1 | M 2 | 2-131 | D. Sheppard | 2-342 | 3-7578 | sheppard |
| 2 | M 2 | 2-132 | M. Huhtanen | 2-335 | 3-7905 | huhtanen |
| 3 | M 3 | 2-131 | D. Sheppard | 2-342 | 3-7578 | sheppard |
| 4 | T 10 | 2-132 | A. Lachowska | 2-180 | 3-4350 | anechka |
| 5 | T 10 | 2-131 | S. Kleiman | 2-278 | 3-4996 | kleiman |
| 6 | T 11 | 2-131 | M. Honsen | 2-490 | 3-4094 | honsen |
| 7 | T 11 | 2-132 | A. Lachowska | 2-180 | 3-4350 | anechka |
| 8 | T 12 | 2-131 | M. Honsen | 2-490 | 3-4094 | honsen |
| 9 | T 1 | 2-132 | A. Lachowska | 2-180 | 3-4350 | anechka |
| 10 | T 1 | 2-131 | S. Kleiman | 2-278 | 3-4996 | kleiman |
| 11 | T 2 | 2-132 | F. Latour | 2-090 | 3-6293 | flatour |

**1 (32 pts.)**   Suppose $A$ is the tridiagonal matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

(a) Carry out the row elimination to find the upper triangular factor $U$. (10)

(b) What matrix $L$ yields $A = LU$? (6)

(c) Solve $Ax = b$ with

$$b = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 0 \end{bmatrix}.$$

All components of the solution $x$ happen to be 0's or 1's. What linear combination of the columns of $A$ produces $b$? (10)

(d) If you change the entry $A_{4,4} = 0$ in the right lower-corner of $A$ to $A_{4,4} = \underline{\quad}$ the matrix becomes singular. (Hint: look at pivots) (6)

2

**2 (36 pts.)**   (a) Suppose $A^n = 0$. Show that $(I - A)^{-1} = I + A + A^2 + \cdots + A^{n-1}$. (10)

(b) Assume $A$ and $B$ are commuting matrices (that is, $AB = BA$). If they both are also nonsingular, show that $A^{-1}$ and $B^{-1}$ commute. (10)

(c) Which are true and which false. (Give a good reason!!!)

Let $A$ be an $m$-by-$n$ matrix. Then $Ax = 0$ has always a non-zero solution if

($i$) $\text{rank}(A) < m$ (5)

($ii$) $\text{rank}(A) < n$ (5)

($iii$) $m = n$ and $A^2 = 0$ (6)

3

**3 (32 pts.)** Suppose after elimination on a matrix $A$ we reach its row reduced echelon form

$$R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Find the null space matrix of $A$. (10)

(b) What is the null space of $A^T$? (6)

(c) What is the rank of 2-by-9 block matrix $[\ A\ A\ A\ ]$? (6)

(d) Find a complete solution to $Rx = d$ with $d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. (10)

# FALL 2001, QUIZ 1

## PROBLEM1

a). "As well as possible" means that we find a (the) vector $\vec{x}$ such that the vector

$$\vec{e} = \vec{b} - A\vec{x}$$

has the least possible length.

b). The fact the first ($n$-th) entry in $A^T e$ is zero means that $e$ is orthogonal to the first ($n$-th) row of $A^T$ or, in other words, $e$ is orthogonal to the each column of $A$. Another way to put it is that $e$ is orthogonal to the column space of $A$.

## PROBLEM 2

a) We have

$$A_0 = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}; \quad A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K_0 = A_0^T A_0 = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$K = A^T A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

b).

$$A^T b = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

And so

$$x = \left( A^T A \right)^{-1} A^T b = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

c). The equation

$$A_0^T A_0 \bar{x} = A_0^T b$$

cannot be solved because $A_0$ is not full rank. Since the third column of $A_0$ is a linear combination of the first two (it's minus their sum) it doesn't "contribute" to the column space and, therefore, the column space is the same.

What is the best $\bar{x}$? We can find it by "solving" $A_0^T A_0 \bar{x} = A_0^T b$, for in deriving this equation we never relied on $A_0$'s being full rank. While it cannot be solved uniquely, it can be solved to "within" one degree of freedom Note that

$$\begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix}$$

is a solution. All other solutions can be found by adding any vector from the null space of $A_0^T A_0$ to this one. The null space of $A_0^T A_0$ is clearly

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and so

$$\bar{x} = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix} + C \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

d). Write down G. For the triangle, all nodes are interconnected so

$$G = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and

$$K_0 + G = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

For the 4-node structure, all nodes are connected except 1 and 4. So

$$G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Also,

$$A_0 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

and

$$A_0^T A_0 + G = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The resulting matrix simply tells us how many edges share a given node.

# PROBLEM 3

This is actually a very short problem. Label the edges 1, 2, and 3, starting with the left slanting one and going CW. Let the coordinate system be the standard right-handed Cartesian system. Order degrees of freedom as in (Left Horizontal, Left Vertical, Right Horizontal, Right Vertical).

a). It is clear that the structure can rotate with respect to the bottom pivots. The single word "rotation" would have commanded full credit on part a). Instantaneously, the motion is perpendicular to slanted bars which can be described by the following vector

$$u = \begin{bmatrix} \sin\alpha \\ \cos\alpha \\ \sin\alpha \\ \cos\alpha \end{bmatrix}$$

b). The incidence matrix is

$$A = \begin{bmatrix} -\cos\alpha & \sin\alpha & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & -\cos\alpha & \sin\alpha \end{bmatrix}$$

Now, confirming our intuition in a) we study $Au$:

$$Au = \begin{bmatrix} -\cos\alpha & \sin\alpha & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -\cos\alpha & \sin\alpha \end{bmatrix} \begin{bmatrix} \sin\alpha \\ \cos\alpha \\ \sin\alpha \\ \cos\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

# 18.06    Midterm Exam 1,   Spring, 2001

Name _____                Optional Code _____

Recitation Instructor _____        Email Address _____

Recitation Time _____

This midterm is closed book and closed notes. No calculators, laptops, cell phones or other electronic devices may be used during the exam.

tronic devices may be used during the exam.

There are 3 problems. Good luck.

1. (20pts.) Find a general formula for the solutions of the following linear system of equations,

$$
\begin{array}{rrrrr}
-x_1 & +3x_2 & & +2x_4 & = & 1 \\
& 4x_2 & -12x_2 & +2x_3 & -4x_4 & = & -4 \\
-7x_1 & +21x_2 & +2x_3 & +18x_4 & = & 7
\end{array}
$$

2. (40pts.) Let $A = \begin{pmatrix} 1 & 1 & b \\ a & b & b-a \\ 1 & 1 & 0 \end{pmatrix}$.

   (a) For $a = 2$ and $b = 1$, find the inverse of $A$.

   (b) For which values of $a$ and $b$ is the matrix $A$ not invertible, i.e. it has less than three pivots?

   (c) For what values of $a$ and $b$ is the rank of $A$ equal to 3 ? For what values is it equal to 2, equal to 1 ?

   (d) For $a = b = 2$, describe the nullspace of A.

3. (40pts.) Let $A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$.

   (a) For what vectors $\mathbf{b} = (b_1, b_2, b_3)^T$ does the linear system $A\mathbf{x} = \mathbf{b}$ have a solution?

   (b) Prove that the column space of $A$ is made up of those vectors $(x, y, z)^T \in \mathbb{R}^3$ that satisfy $x + y + z = 0$.

   (c) Prove that the vectors $(x, y, z)^T \in \mathbb{R}^3$ that satisfy $x + y + z = c$ form a subspace of $\mathbb{R}^3$ if and only if $c = 0$.

# 18.06 Midterm Exam 1, Spring, 2001

Name _____          Optional Code _____

Recitation Instructor _____          Email Address _____

Recitation Time _____

This midterm is closed book and closed notes. No calculators, laptops, cell phones or other electronic devices may be used during the exam.

There are 3 problems. Good luck.

1. (20pts.) Find a general formula for the solutions of the following linear system of equations,

$$
\begin{array}{rrrrcr}
-x_1 & +3x_2 & & +2x_4 & = & 1 \\
& 4x_2 & -12x_2 & +2x_3 & -4x_4 & = & -4 \\
-7x_1 & +21x_2 & +2x_3 & +18x_4 & = & 7
\end{array}
$$

- The augmented matrix is

$$
\left( \begin{array}{rrrr|r}
-1 & 3 & 0 & 2 & 1 \\
4 & -12 & 2 & -4 & -4 \\
-7 & 21 & 2 & 18 & 7
\end{array} \right).
$$

The corresponding row reduced matrix is

$$
\left( \begin{array}{rrrr|r}
-1 & 3 & 0 & 2 & 1 \\
0 & 0 & 2 & 4 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array} \right).
$$

If we let $x_4 = b$ and $x_2 = a$, then $x_3 = -2b$ and $x_1 = -1 + 3a + 2b$. The general solution thus takes the form

$$
\mathbf{x} = \left( \begin{array}{r} -1 \\ 0 \\ 0 \\ 0 \end{array} \right) + a \left( \begin{array}{r} 3 \\ 1 \\ 0 \\ 0 \end{array} \right) + b \left( \begin{array}{r} 2 \\ 0 \\ -2 \\ 1 \end{array} \right).
$$

2. (40pts.) Let $A = \begin{pmatrix} 1 & 1 & b \\ a & b & b-a \\ 1 & 1 & 0 \end{pmatrix}$.

(a) For $a = 2$ and $b = 1$, find the inverse of $A$.

(b) For which values of $a$ and $b$ is the matrix $A$ not invertible, i.e. it has less than three pivots?

(c) For what values of $a$ and $b$ is the rank of $A$ equal to 3 ? For what values is it equal to 2, equal to 1 ?

(d) For $a = b = 2$, describe the nullspace of A.

•

(a) The augmented matrix is
$$\left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right).$$

We perform row operations to obtain
$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & -1 & -1 & 3 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array} \right).$$

The matrix on the right is the inverse of $A$.

(b) If we do row operations on the matrix
$$\begin{pmatrix} 1 & 1 & b \\ a & b & b-a \\ 1 & 1 & 0 \end{pmatrix}$$

we obtain
$$\begin{pmatrix} 1 & 1 & b \\ 0 & b-a & b-a-ab \\ 0 & 0 & -b \end{pmatrix}.$$

There are less than three pivot columns if $a = b$ or $b = 0$.

(c) $\mathrm{rk}A = 3$ if $a \neq b \neq 0$. $\mathrm{rk}A = 2$ if $b = 0$ or $a = b \neq 0$. $\mathrm{rk}A = 1$ if $b = a = 0$.

(d) For $a = b = 2$ the row reduced matrix is
$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{pmatrix}.$$

2

Hence, $x_3 = 0$. If we let $x_2 = a$, then $x_1 = -a$. The general solution takes the form

$$\mathbf{x} = a \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

3. (40pts.) Let $A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$.

(a) For what vectors $\mathbf{b} = (b_1, b_2, b_3)^T$ does the linear system $A\mathbf{x} = \mathbf{b}$ have a solution?

(b) Prove that the column space of $A$ is made up of those vectors $(x, y, z)^T \in \mathbb{R}^3$ that satisfy $x + y + z = 0$.

(c) Prove that the vectors $(x, y, z)^T \in \mathbb{R}^3$ that satisfy $x + y + z = c$ form a subspace of $\mathbb{R}^3$ if and only if $c = 0$.

•

(a) The augmented matrix is

$$\left( \begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ -1 & 1 & 0 & b_2 \\ 0 & -1 & 1 & b_3 \end{array} \right).$$

Performing row operations we obtain

$$\left( \begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ 0 & 1 & -1 & b_1 + b_2 \\ 0 & 0 & 0 & b_1 + b_2 + b_3 \end{array} \right).$$

Hence, the vectors $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a solution must satisfy $b_1 + b_2 + b_3 = 0$.

(b) See part (a).

(c) Suppose that the vectors $\mathbf{x} \in \mathbb{R}^3$, which satisfy $x + y + z = c$ form a subspace. Since $\mathbf{0}$ must be in that subspace, and $0 + 0 + 0 = 0$, it follows that $c = 0$.

Now suppose that $c = 0$. Since $0 + 0 + 0 = 0$, $\mathbf{0}$ is in the space. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be in the subspace, and let $a$ and $b$ be real numbers. Consider $a\mathbf{x}_1 + b\mathbf{x}_2$,

$$(ax_1 + bx_2) + (ay_1 + by_2) + (az_1 + bz_2) = a(x_1 + y_1 + z_1) + b(x_2 + y_2 + z_2) = a \cdot 0 + b \cdot 0 = 0.$$

So $a\mathbf{x}_1 + b\mathbf{x}_2$ is also in the space, and therefore this is a subspace of $\mathbb{R}^3$.

4

# 18.06   Professor Strang/Ingerman   Quiz 1    September 27, 2002

**Your name is:** _____

**Please circle your recitation:**

|      |     |       |               |       |        |         |
|------|-----|-------|---------------|-------|--------|---------|
| 1)   | M2  | 2-131 | P.-O. Persson | 2-088 | 2-1194 | persson |
| 2)   | M2  | 2-132 | I. Pavlovsky  | 2-487 | 3-4083 | igorvp  |
| 3)   | M3  | 2-131 | I. Pavlovsky  | 2-487 | 3-4083 | igorvp  |
| 4)   | T10 | 2-132 | W. Luo        | 2-492 | 3-4093 | luowei  |
| 5)   | T10 | 2-131 | C. Boulet     | 2-333 | 3-7826 | cilanne |
| 6)   | T11 | 2-131 | C. Boulet     | 2-333 | 3-7826 | cilanne |
| 7)   | T11 | 2-132 | X. Wang       | 2-244 | 8-8164 | xwang   |
| 8)   | T12 | 2-132 | P. Clifford   | 2-489 | 3-4086 | peter   |
| 9)   | T1  | 2-132 | X. Wang       | 2-244 | 8-8164 | xwang   |
| 10)  | T1  | 2-131 | P. Clifford   | 2-489 | 3-4086 | peter   |
| 11)  | T2  | 2-132 | X. Wang       | 2-244 | 8-8164 | xwang   |

**1 (30 pts.)**   Start with the vectors

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

(a) Find two other vectors $\mathbf{w}$ and $\mathbf{z}$ whose linear combinations fill the same plane $P$ as the linear combinations of $\mathbf{u}$ and $\mathbf{v}$.

(b) Find a 3 by 3 matrix $M$ whose *column space* is that same plane $P$.

(c) Describe all vectors $\mathbf{x}$ in the nullspace ($M\mathbf{x} = \mathbf{0}$) of your matrix $M$.

**2 (30 pts.)**  (a) By elimination put $A$ into its upper triangular form $U$. Which are the pivot columns and free columns?

$$A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 8 & 5 & 2 \\ 1 & 5 & 3 & 1 \end{bmatrix}$$

(b) Describe specifically the vectors in the nullspace of $A$. One way is to find the "special solutions" (*how many??*) to $A\boldsymbol{x} = \boldsymbol{0}$ by setting the free variables to 1 or 0.

(c) Does $A\boldsymbol{x} = \boldsymbol{b}$ have a solution for the right side $\boldsymbol{b} = (3, 8, 5)$? If it does, find one particular solution and then the complete solution to this system $A\boldsymbol{x} = \boldsymbol{b}$.

**3 (40 pts.)**   (a) Apply row elimination to $A$ and find the pivots and the upper triangular $U$. Factor this "Pascal matrix" into $L$ times $U$.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

(b) How do $L$ and $U$ and the pivots confirm that $A$ is invertible?

(c) If you change the entry "20" to what number (??) then $A$ will become singular.

(d) What permutation matrix $P$ will multiply $A$ so that the rows of $PA$ are in reverse order (rows $1, 2, 3, 4$ of $A$ become rows $4, 3, 2, 1$ of $PA$)? What matrix multiplication would put the *columns* in reverse order?

# Course 18.06, Fall 2002: Quiz 1, Solutions

**1** (a) For example

$$
\mathbf{w} = \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} \qquad \text{or} \qquad \mathbf{w} = \mathbf{u} + \mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}, \mathbf{z} = 3\mathbf{u} - \mathbf{v} = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}
$$

(b), (c) For example

$$
M = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{5}{2} & 0 \\ 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{5}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$x_3$ free variable. Let $x_3 = 1$ then $x_1 = x_2 = 0$. Nullspace is all vectors

$$
\lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

**2** (a)

$$
A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 8 & 5 & 2 \\ 1 & 5 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U
$$

Pivot columns 1 and 2, free columns 3 and 4.

(b)

$$
N(A) = \text{linear combination of } \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

(c)

$$
\text{Particular } x_p = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{Complete } x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

**3** (a)

$$
A = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 \\ & & 1 & 3 \\ & & & 1 \end{bmatrix}
$$

(b) $U$ has 4 nonzero entries on the diagonal
$\Rightarrow A$ has 4 nonzero pivots
$\Rightarrow$ Gauss-Jordan will work
$\Rightarrow A^{-1}$ exists

(c) If the last diagonal entry of $U$ was zero $\Rightarrow A_{44} = 1 + 9 + 9 = 19$.

(d)

$$
P = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \Rightarrow PA \text{ has reversed rows \& } AP \text{ has reversed columns.}
$$

# 18.06 Exam 1 #1 Solutions

1 a)

$$\vec{v} \cdot \vec{x} = 0 \Rightarrow x_1 + 2x_2 + x_3 = 0$$
$$\vec{w} \cdot \vec{x} = 0 \Rightarrow 2x_1 + 4x_2 + 3x_3 = 0$$

So the set to be found is the nullspace of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \end{bmatrix}$. The row echelon form

of $A$ is $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. The second variable, $x_2$, is free and the vector $(-2, 1, 0)$ is a basis of the nullspace.

b) Since the set in a) is the nullspace of the matrix $A$, it is a vector space. Generally to prove a set satisfying some property, say $P$, is a vector space, one needs to show:
   (1) If $\vec{x}$ satisfies property $P$, then $c\vec{x}$ also satisfies property $P$, for any $c \in \mathbb{R}$.
   (2) If $\vec{x}, \vec{y}$ satisfy property $P$, then $\vec{x} + \vec{y}$ also satisfies property $P$.

2 a)

$$A = \begin{bmatrix} -2 & 0 & 3 \\ -4 & 3 & -2 \\ 8 & 9 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & 3 \\ 0 & 3 & -8 \\ 0 & 9 & 23 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & 3 \\ 0 & 3 & -8 \\ 0 & 0 & 47 \end{bmatrix}$$

So

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 3 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} -2 & 0 & 3 \\ 0 & 3 & -8 \\ 0 & 0 & 47 \end{bmatrix}$$

b) To solve $A\vec{x} = LU\vec{x} = \vec{b}$, it is equivalent to solve the two equations $L\vec{y} = \vec{b}$ and $U\vec{x} = \vec{y}$.

$$L\vec{y} = \vec{b} \Rightarrow \begin{cases} y_1 = 3 \\ 2y_1 + y_2 = -1 \\ -4y_1 + 3y_2 + y_3 = 13 \end{cases} \Rightarrow \vec{y} = \begin{bmatrix} 3 \\ -7 \\ 46 \end{bmatrix}.$$

$$U\vec{x} = \vec{y} \Rightarrow \vec{y} = \begin{bmatrix} \frac{-3}{94} \\ \frac{13}{47} \\ \frac{46}{47} \end{bmatrix}.$$

3 a) Denote $B = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 5 & 2 & 6 \end{bmatrix}, P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, Then

$$BA = P \Rightarrow P^{-1}BA = I \Rightarrow P^{-1}B = A^{-1}$$

Because $P$ is a permutation matrix, $P^{-1} = P^T$. So

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 5 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 7 \\ 5 & 2 & 6 \\ 1 & 2 & 4 \end{bmatrix}.$$

b) i $B, D$ have full column rank, so the nullspace of each is the zero vector. Now

$$BD\vec{x} = 0 \Rightarrow D\vec{x} \in N(B) = \{0\} \Rightarrow D\vec{x} = 0 \Rightarrow \vec{x} = 0.$$

Hence N(BD)=0.

ii This time only $B$ has full column rank, that is, $N(B) = \{0\}$.

$$BD\vec{x} = 0 \Rightarrow D\vec{x} \in N(B) = \{0\} \Rightarrow D\vec{x} = 0 \Rightarrow \vec{x} \in N(D).$$

So $N(BD) \subseteq N(D)$. On the other hand,

$$D\vec{x} = 0 \Rightarrow BD\vec{x} = B0 = 0 \Rightarrow x \in N(BD) \Rightarrow N(D) \subseteq N(BD).$$

So $N(D) = N(BD)$, which is all we can say about $N(BD)$ without further assumptions on $D$.

iii $r < n$, implies $B$ is not of full column rank and the nullspace of $B$ contains an infinite number of vectors. $r < m$ implies the row echelon form of $B$ has zero rows, so the equation $B\vec{x} = \vec{b}$ has no solutions for some $\vec{b}$. Furthermore, if there is a solution to $B\vec{x} = \vec{b}$, say $\vec{x}_p$, then there are infinitely many solutions since $\vec{x}_p + \vec{x}_n$ is a solution for any $\vec{x}_n$ in $N(B)$. The answer to the question is 0 or infinitely many.

4 a) Apply row operations on $A$ and get the following matrix

$$R = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & c-3 & -3 \\ 0 & 0 & 2(c-3) & -8 \\ 0 & 0 & 0 & d-8 \end{bmatrix}.$$

- No values of $c, d$ will make the rank of $A$ equal to 2.
- if $c \neq 3, d \neq 8$, $R$ is the row echelon form of $A$ and $A$ has rank 4.
- Any other combination of $c, d$ will give rank 3, that is, the rank is 3 if $c = 3$ or $d = 8$.

b) substituting $c = 3, d = 8$ in the matrix $R$, one finds that the third column gives a free variable, and null space of $A$ is spanned by $(-3, 0, 1, 0)$. Use the **augmented matrix** $\left[A \mid \vec{b}\right]$ (NOT $\left[R \mid \vec{b}\right]$) to find a particular solution of the equation $A\vec{x} = \vec{b}$, which is $(-1/2, 1/4, 0, 1/4)$. So the complete solution of the equation is $(-1/2, 1/4, 0, /1/4) + x_3(-3, 0, 1, 0)$.

# 18.06    Fall 2003    Quiz 1    October 1, 2003

**Your name is:**

**Please circle your recitation:**

| | |
|---|---|
| 1. M2 S. Harvey | 7. T11 N. Ganter |
| 2. M2 D. Ingerman | 8. T12 N. Ganter |
| 3. M3 S. Harvey | 9. T12 S. Francisco |
| 4. T10 B. Sutton | 10. T1 K. Cheung |
| 5. T10 C. Taylor | 11. T1 B. Tenner |
| 6. T11 K. Cheung | 12. T2 K. Cheung |

**Grading:**

| Question | Points | Maximum |
|---|---|---|
| Name + rec | | 5 |
| 1 | | 25 |
| 2 | | 15 |
| 3 | | 5 |
| 4 | | 35 |
| 5 | | 15 |
| Extra credit: | | (10) |
| **Total:** | | 100 |

**Remarks:**
Do all your work on these pages.
No calculators or notes.
**Putting your name and recitation name correctly is worth 5 points.**
The exam is worth a total of 100 points.

1

1.   a) (15 points) Find an LU-decomposition of the $3 \times 3$ matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$

b) (10 points) Solve $Ax = b$ where

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

2. (15 points) Let $A$ be an unknown $3 \times 3$ matrix, and let

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Consider the augmented matrix $B = [A \,|\, P\,]$. After performing row operations on $B$ we get the following matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -3 & -4 \\ 0 & 1 & 0 & -1 & 2 & 2 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right].$$

What is $A^{-1}$?

3. (5 points) Find a matrix $A$ such that

$$A \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x - y \\ x + y + 2w \end{bmatrix}.$$

4. All of the questions below refer to the following matrix $A$

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

    a) (5 points) What is the rank of $A$?

    b) (5 points) Do all pairs of columns span the column space, $C(A)$, of $A$? If yes, explain. If no, give a pair of columns that do not span the column space.

c) (10 points) Find a basis for the nullspace $N(A)$ of $A$.

d) (5 points) Does there exist a vector $b \in R^2$ such that $Ax = b$ has no solution?

e) (10 points) Find all solutions of

$$Ax = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Express your solution in the form

$$x = x_{particular} + c_1 x_1 + c_2 x_2$$

where $x_1, x_2$ are special solutions.

5.  a) (6 points) How many $3 \times 3$ permutation matrices are there (including $I$)?

    b) (9 points) Is there a $3 \times 3$ permutation matrix $P$, besides $P = I$, such that $P^3 = I$? If yes, give one such $P$. If no, explain why.

6. **Extra Credit (10 points)** The matrix in question 1 is a Pascal matrix. Find an LU-decomposition of the $6 \times 6$ Pascal matrix

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 3 & 6 & 10 & 15 & 21 \\
1 & 4 & 10 & 20 & 35 & 56 \\
1 & 5 & 15 & 35 & 70 & 126 \\
1 & 6 & 21 & 56 & 126 & 252
\end{bmatrix}
$$

Note: you don't need to write the entire matrix again, just explain how to get the LU-decomposition.

# 18.06    Fall 2003    Quiz 1    October 1, 2003

**Your name is:**

**Please circle your recitation:**

1. M2 S. Harvey
2. M2 D. Ingerman
3. M3 S. Harvey
4. T10 B. Sutton
5. T10 C. Taylor
6. T11 K. Cheung

7. T11 N. Ganter
8. T12 N. Ganter
9. T12 S. Francisco
10. T1 K. Cheung
11. T1 B. Tenner
12. T2 K. Cheung

**Grading:**

| Question | Points | Maximum |
|----------|--------|---------|
| Name + rec |      | 5 |
| 1 |      | 25 |
| 2 |      | 15 |
| 3 |      | 5 |
| 4 |      | 35 |
| 5 |      | 15 |
| Extra credit: |      | (10) |
| **Total:** |      | 100 |

**Remarks:**
Do all your work on these pages.
No calculators or notes.
**Putting your name and recitation name correctly is worth 5 points**.
The exam is worth a total of 100 points.

1

1.   a) (15 points) Find an LU-decomposition of the $3 \times 3$ matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$

Solution:

$$E_{31}E_{21}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

$$U = E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L = (E_{32}E_{31}E_{21})^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Therefore we have,

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

b) (10 points) Solve $Ax = b$ where

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Solution:

From 1(a) we have $A = LU$. Let $c = Ux$ and solve for $Lc = b$ using back substitution to get

$$c = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Now, solve for $Ux = c$ using back substitution to get

$$x = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}.$$

2. (15 points) Let $A$ be an unknown $3 \times 3$ matrix, and let

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Consider the augmented matrix $B = [\,A\,|\,P\,]$. After performing row operations on $B$ we get the following matrix

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -3 & -4 \\ 0 & 1 & 0 & -1 & 2 & 2 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array} \right].$$

What is $A^{-1}$?

Solution:

By performing 2 more row operations on $B$ we get the following augmented matrix

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & -3 \\ 0 & 1 & 0 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array} \right] = [\,I\,|\,A^{-1}P\,].$$

Since $P^{-1} = P$, we have

$$\begin{aligned} A^{-1} &= \begin{bmatrix} 2 & -3 & -3 \\ -1 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 2 & -3 \\ 2 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

4

3. (5 points) Find a matrix $A$ such that

$$A \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x - y \\ x + y + 2w \end{bmatrix}.$$

Solution:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

4. All of the questions below refer to the following matrix $A$

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

a) (5 points) What is the rank of $A$?

Solution:

The rank of $A$ is equal to the number of pivots which is 2.

b) (5 points) Do all pairs of columns span the column space, $C(A)$, of $A$? If yes, explain. If no, give a pair of columns that do not span the column space.

Solution:

No! The column space of A is all of $\mathbb{R}^2$. However, the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ are linearly dependent and hence only span a one-dimensional subspace of $\mathbb{R}^2$.

c) (10 points) Find a basis for the nullspace $N(A)$ of $A$.

Solution:

Let $x_2 = 1$ and $x_4 = 0$. We solve for the pivot variables: $x_1 = -2$ and $x_3 = 0$.

Let $x_2 = 0$ and $x_4 = 1$. We solve for the pivot variables: $x_1 = -1$ and $x_3 = -2$.

A basis for the nullspace is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

d) (5 points) Does there exist a vector $b \in R^2$ such that $Ax = b$ has no solution?

Solution:

No! One possible solution to $Ax = b$ is $x = \begin{bmatrix} b_1 \\ 0 \\ b_2 \\ 0 \end{bmatrix}$.

7

e) (10 points) Find all solutions of

$$Ax = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Express your solution in the form

$$x = x_{particular} + c_1 x_1 + c_2 x_2$$

where $x_1, x_2$ are special solutions.

Solution:

$$x = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

5.  a) (6 points) How many $3 \times 3$ permutation matrices are there (including $I$)?

   Solution: 3!=6

   b) (9 points) Is there a $3 \times 3$ permutation matrix $P$, besides $P = I$, such that $P^3 = I$? If yes, give one such $P$. If no, explain why.

   Solution: Yes,

   $$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

6. **Extra Credit (10 points)** The matrix in question 1 is a Pascal matrix. Find an LU-decomposition of the $6 \times 6$ Pascal matrix

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 3 & 6 & 10 & 15 & 21 \\
1 & 4 & 10 & 20 & 35 & 56 \\
1 & 5 & 15 & 35 & 70 & 126 \\
1 & 6 & 21 & 56 & 126 & 252
\end{bmatrix}
$$

Note: you don't need to write the entire matrix again, just explain how to get the LU-decomposition.

Solution:

Let

$$
U =
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 1 & 3 & 6 & 10 \\
0 & 0 & 0 & 1 & 4 & 10 \\
0 & 0 & 0 & 0 & 1 & 5 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

and $L = U^T$ then $A = LU$.

**Your name is:** _____

**Please circle your recitation:**

# 1 (30 pts.)

(a) Compute the following matrix product

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -5 & -4 & -3 & -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$$

No explanation is necessary.

(b) Let $U$ be the matrix below. Reduce $U$ to a reduced row echelon matrix by row operations (upward elimination). Find the "special solutions" to $Ux = 0$. Also give an expression for the general solution to $Ux = 0$.

$$U = \begin{pmatrix} 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 7 & 5 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

2

**2 (35 pts.)**

(a) Let $A$ and $b$ be as below. For any real number $t$, and any real number $s$: Find the complete solution to the equation $Ax = b$ using the algorithm described in class and in the book. (It depends on $t$ and $s$.)

$$A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & t \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 \\ 0 \\ 0 \\ s \end{pmatrix}$$

(b) First part: For which $t$ are the columns of the matrix $A$ linearly dependent? Second part: Consider $b$ and the first three columns of $A$. For which $s$ are these linearly dependent?

**3 (35 pts.)** The elimination algorithm explained in the course (with "row swapping after Gaussian elimination") was applied to the matrix $A$. Suppose it yields the following equality:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 11 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 10 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

(a) Which row operations do the four elimination matrices in the product correspond to? Please write them down in words in the order in which they were performed on $A$. Why is the upper left hand corner of $A$ zero? (This is the $(1,1)$ entry of $A$.)

(b) The equation implies that $A$ factors as $A = LPUR$. Here $R$ is the matrix on the right hand side of the $=$ sign. The matrices $U$, $P$, and $L$ are invertible $4 \times 4$ matrices. The matrix $U$ is upper triangular. The matrix $P$ is a permutation matrix. And $L$ is lower triangular. Find $U$, $P$, and $L$, and explain how you got them.

**1  (30 pts.)**

(a) (10 pts)

$$
\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -5 & -4 & -3 & -2 & -1 \end{pmatrix}
\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}
=
\begin{pmatrix} 15 & 40 \\ -15 & -20 \end{pmatrix}
$$

(b) (20 pts)

(10 pts) Computing $R$:

$$
U = \begin{pmatrix} 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 7 & 5 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}
\rightarrow
\begin{pmatrix} 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}
$$

$$
\rightarrow
\begin{pmatrix} 1 & 1 & 0 & -9 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}
\rightarrow
\begin{pmatrix} 1 & 1 & 0 & -9 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
= R
$$

(8 pts)  Special solutions to $Ux = 0$:

$$
x_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
, \quad
x_2 = \begin{pmatrix} 9 \\ 0 \\ -7 \\ 1 \\ 0 \end{pmatrix}
$$

(2 pts) General solutions to $Ux = 0$:

$$
x = a \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
+ b \begin{pmatrix} 9 \\ 0 \\ -7 \\ 1 \\ 0 \end{pmatrix}
, \ a, b \in \mathbb{R}
$$

**2 (35 pts.)**

(a) (25 pts) Find the complete solution to the equation $Ax = b$ using the algorithm described in class and in the book.

$$(A \mid b) = \begin{pmatrix} 1 & 0 & 0 & 4 & | & 2 \\ 1 & 0 & 1 & 0 & | & 0 \\ 1 & 1 & 0 & 0 & | & 0 \\ 1 & 2 & 3 & t & | & s \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 4 & | & 2 \\ 0 & 0 & 1 & -4 & | & -2 \\ 0 & 1 & 0 & -4 & | & -2 \\ 0 & 2 & 3 & t-4 & | & s-2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 4 & | & 2 \\ 0 & 1 & 0 & -4 & | & -2 \\ 0 & 0 & 1 & -4 & | & -2 \\ 0 & 0 & 3 & t+4 & | & s+2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 4 & | & 2 \\ 0 & 1 & 0 & -4 & | & -2 \\ 0 & 0 & 1 & -4 & | & -2 \\ 0 & 0 & 0 & t+16 & | & s+8 \end{pmatrix}$$

Therefore, when $t = -16$ and $s \neq -8$ there are no solutions.

When $t \neq -16$ there is a unique solution:

$$x = \begin{pmatrix} 2 - 4\frac{s+8}{t+16} \\ -2 + 4\frac{s+8}{t+16} \\ -2 + 4\frac{s+8}{t+16} \\ \frac{s+8}{t+16} \end{pmatrix}.$$

When $t = -16$ and $s = -8$, there are infinitely many solutions:

$$x = \begin{pmatrix} 2 - 4a \\ -2 + 4a \\ -2 + 4a \\ a \end{pmatrix}, \quad \text{for all } a \in \mathbb{R}.$$

2

(b) First part (5 pts)

From the previous computation, column vectors of $A$ are linearly independent when $t \neq -16$ (all pivots are nonzero).

When $t = -16$, column vectors are linearly dependent:

$$4 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 4 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} - 4 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix} - 1 \cdot \begin{pmatrix} 4 \\ 0 \\ 0 \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Second part (5 pts)

When $s \neq -8$ column vectors are linearly independent. Indeed, after swapping the last column of $A$ with $b$, in the computation in $(a)$ all pivots are nonzero.

When $s = -8$, column vectors are linearly dependent:

$$2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 2 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} - 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix} - 1 \cdot \begin{pmatrix} 4 \\ 0 \\ 0 \\ -8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

3

## 3  (35 pts.)

(a) Row operations (15 pts) :

- add 10 times row 2 to row 3

- add 11 times row 3 to row 4

- swap rows 1 and 2

- add 4 times row 4 to row 1

"Why the upper left corner of $A$ is zero" question. (5 pts)

*Answer:* The upper right corner is zero since otherwise we would never have needed to swap rows 1 and 2.

(b) (15 pts):

The matrix $U$ corresponds to the upward elimination. So we get

$$
U = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

The matrix $P$ corresponds to the permutation of the first two rows. So we have

$$
P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{T} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

The matrix $L$ corresponds the first two elimination steps. So we have

$$
L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 10 & 1 & 0 \\ 0 & 110 & 11 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -10 & 1 & 0 \\ 0 & 0 & -11 & 1 \end{bmatrix}.
$$

4

# 18.06    Fall 2004    Quiz 1    October 13, 2004

**Your name is:**

**Please circle your recitation:**

1. M2 A. Brooke-Taylor

2. M2 F. Liu

3. M3 A. Brooke-Taylor

4. T10 K. Cheung

5. T10 Y. Rubinstein

6. T11 K. Cheung

7. T11 V. Angeltveit

8. T12 V. Angeltveit

9. T12 F. Rochon

10. T1 L. Williams

11. T1 K. Cheung

12. T2 T. Gerhardt

**Grading:**

| Question | Points | Maximum |
|----------|--------|---------|
| Name + rec | | 5 |
| 1 | | 15 |
| 2 | | 55 |
| 3 | | 25 |
| **Total:** | | 100 |

**Remarks:**
Do all your work on these pages.
No calculators or notes.
**Putting your name and recitation section correctly is worth 5 points**.
The exam is worth a total of 100 points.

1. Let
$$A = \begin{bmatrix} 2 & 2 & 2 \\ 4 & 3 & 1 \\ -2 & -1 & 4 \end{bmatrix}.$$

(a) Compute an $LDU$ factorization of $A$ if one exists.

(b) Give all solutions to $Ax = b$ where $b = \begin{bmatrix} 2 \\ -3 \\ 11 \end{bmatrix}$.

2. **One of the entries of $A$ has been modified as there was a mistake.** (Many of the subquestions are independent and can be answered in any order.) By performing row eliminations (and possibly permutations) on the following $4 \times 8$ matrix $A$

$$
\begin{bmatrix}
1 & 2 & 0 & 3 & -1 & 1 & 1 & -2 \\
-3 & -6 & 2 & -7 & 7 & 0 & -6 & 3 \\
1 & 2 & 2 & 5 & 3 & 3 & -1 & 0 \\
2 & 4 & 0 & 6 & -2 & 1 & 3 & 0
\end{bmatrix}
$$

we got the following matrix $B$:

$$
\begin{bmatrix}
1 & 2 & 0 & 3 & -1 & 0 & 2 & 0 \\
0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

(a) What is the rank of $A$?

(b) What are the dimensions of the 4 fundamental subspaces?

(c) How many solutions does $Ax = b$ have? Does it depend on $b$? Justify

(d) Are the rows of $A$ linearly independent? Why?

(e) Do columns 4, 5, 6 and 7 of $A$ form a basis of $R^4$? Why?

(f) Give a basis of $N(A)$.

(g) Give a basis of $N(A^T)$.

(h) (You do not need to do any calculations to answer this question.) What is the reduced row echelon form for $A^T$? Explain.

(i) (Again calculations are not necessary for this part.) Let $B = EA$. Is $E$ invertible? If so, what is the inverse of $E$?

3. For each of these statements, say whether the claim is true or false and give a brief justification.

    (a) **True/False:** The set of $3 \times 3$ non-invertible matrices forms a subspace of the set of all $3 \times 3$ matrices.

    (b) **True/False:** If the system $Ax = b$ has no solution then $A$ does not have full row rank.

(c) **True/False:** There exist $n \times n$ matrices $A$ and $B$ such that $B$ is not invertible but $AB$ is invertible.

(d) **True/False:** For any permutation matrix $P$, we have that $P^2 = I$.

# 18.06    Fall 2004    Quiz 1    October 13, 2004

**Your name is:**

**Please circle your recitation:**

1. M2 A. Brooke-Taylor
2. M2 F. Liu
3. M3 A. Brooke-Taylor
4. T10 K. Cheung
5. T10 Y. Rubinstein
6. T11 K. Cheung

7. T11 V. Angeltveit
8. T12 V. Angeltveit
9. T12 F. Rochon
10. T1 L. Williams
11. T1 K. Cheung
12. T2 T. Gerhardt

**Grading:**

| Question | Points | Maximum |
|:--------:|:------:|:-------:|
| Name + rec | | 5 |
| 1 | | 15 |
| 2 | | 55 |
| 3 | | 25 |
| **Total:** | | 100 |

**Remarks:**

Do all your work on these pages.

No calculators or notes.

**Putting your name and recitation section correctly is worth 5 points**.

The exam is worth a total of 100 points.

1. Let
$$A = \begin{bmatrix} 2 & 2 & 2 \\ 4 & 3 & 1 \\ -2 & -1 & 4 \end{bmatrix}.$$

(a) Compute an $LDU$ factorization of $A$ if one exists.

   **Solution:**

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{E_{31}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} \underbrace{\begin{bmatrix} 2 & 2 & 2 \\ 4 & 3 & 1 \\ -2 & -1 & 4 \end{bmatrix}}_{A} = \begin{bmatrix} 2 & 2 & 2 \\ 0 & -1 & -3 \\ 0 & 1 & 6 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{E_{32}} \begin{bmatrix} 2 & 2 & 2 \\ 0 & -1 & -3 \\ 0 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{bmatrix}$$

So for $A = LU$ decomposition, we have from this that

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 2 & 2 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{bmatrix}$$

But this is not the $U$ we want for $A = LDU$ decomposition; for that, we factor out the pivot values of the old $U$ and put them in $D$. In this way we get

$$\underbrace{\begin{bmatrix} 2 & 2 & 2 \\ 4 & 3 & 1 \\ -2 & -1 & 4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}}_{U}.$$

(b) Give all solutions to $Ax = b$ where $b = \begin{bmatrix} 2 \\ -3 \\ 11 \end{bmatrix}$.

**Solution:** The quick way to do this is by forward and backward substitution, using the result of the previous part. We want $Ax = b$, that is, $LDUx = b$. Setting $DUx = c$, we have to first solve $Lc = b$ for $c$, and then $DUx = c$ for $x$.

Now, $Lc = b$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 11 \end{bmatrix}$$

so clearly we must have $c_1 = 2$, so $c_2 = -7$, and so $c_3 = 6$. With that, $DUx = c$ becomes

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ 6 \end{bmatrix},$$

and hence we get that $x_3 = 2$, so $x_2 = 1$, so finally $x_1 = -2$. Hence, the one and only solution to $Ax = b$ is $x = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$.

2. **One of the entries of $A$ was modified as there was a mistake.** (Many of the subquestions are independent and can be answered in any order.) By performing row eliminations (and possibly permutations) on the following $4 \times 8$ matrix $A$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & -1 & 1 & 1 & -2 \\ -3 & -6 & 2 & -7 & 7 & 0 & -6 & 3 \\ 1 & 2 & 2 & 5 & 3 & 3 & -1 & 0 \\ 2 & 4 & 0 & 6 & -2 & 1 & 3 & 0 \end{bmatrix}$$

we got the following matrix $B$:

$$\begin{bmatrix} 1 & 2 & 0 & 3 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) What is the rank of $A$?

**Solution:**

4. There are 4 pivots in the reduced row echelon form of $A$.

(b) What are the dimensions of the 4 fundamental subspaces?
**Solution:**

$$\dim(C(A)) = \dim(R(A)) = \operatorname{rank}(A) = 4$$
$$\dim(N(A)) = n - \operatorname{rank}(A) = 8 - 4 = 4$$
$$\dim(N(A^T)) = m - \operatorname{rank}(A) = 4 - 4 = 0$$

(c) How many solutions does $Ax = b$ have? Does it depend on $b$? Justify

**Solution:**

$Ax = b$ will have infinitely many solutions for any $b$. There is no row of 0's in the reduced row echelon form to cause there to be no solutions for the "wrong" $b$. There are infinitely many solutions since the nullspace, being 4-dimensional, has infinitely many elements.

(d) Are the rows of $A$ linearly independent? Why?

**Solution:**

Yes. The reduced row echelon form of $A$ has linearly independent rows, and row operations preserve the row space.

(e) Do columns 4, 5, 6 and 7 of $A$ form a basis of $R^4$? Why?

**Solution:**

No. Columns 4, 5, 6 and 7 in $B$ are dependent, and row operations preserve linear dependence and independence of columns. Hence, columns 4, 5, 6 and 7 of $A$ are dependent.

(f) Give a basis of $N(A)$.

**Solution:**We get it as usual from the reduced row echelon form, $B$.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

gives the four equations

$$x_1 = -2x_2 - 3x_4 + x_5 - 2x_7$$
$$x_3 = -x_4 - 2x_5$$
$$x_6 = x_7$$
$$x_8 = 0$$

From these we get the 4 special solutions corresponding to the 4 free variables $x_2$, $x_4$, $x_5$ and $x_7$. The special solutions are a basis for the nullspace. Hence, our basis is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(g) Give a basis of $N(A^T)$.

**Solution:**

We saw that $\dim(N(A^T)) = 0$. Hence, a basis for $N(A^T)$ must contain no vectors, that is, it must be the empty set $\{ \}$, often denoted by $\emptyset$.

(h) (You do not need to do any calculations to answer this question.) What is the reduced row echelon form for $A^T$? Explain.

**Solution:**

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$A^T$ is a $8 \times 4$ matrix with 4 independent columns (since $A$ has 4 independent rows). Thus, every column in the reduced row echelon form must contain a pivot. Hence, the given matrix is the only possible reduced row echelon form of $A^T$.

(i) (Again calculations are not necessary for this part.) Let $B = EA$. Is $E$ invertible? If so, what is the inverse of $E$?

**Solution:**

Yes; $E$ is just the product of the elimination matrices (including possibly permutation matrices) which are applied to $A$ to get $B$. Consider columns 1, 3, 6 and 8 of $A$ - those which become the pivot columns of $B$. The matrix $E$ is what performs this change on the columns. Hence,

$$E \begin{bmatrix} 1 & 0 & 1 & -2 \\ -3 & 2 & 0 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which of course means that this matrix is $E^{-1}$.

3. For each of these statements, say whether the claim is true or false and give a brief justification.

  (a) **True/False:** The set of $3 \times 3$ non-invertible matrices forms a subspace of the set of all $3 \times 3$ matrices.

    **Solution:**

    False. Consider for example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

    The matrix on the right hand side is invertible, but the two on the left hand side are not.

  (b) **True/False:** If the system $Ax = b$ has no solution then $A$ does not have full row rank.

    **Solution:**

    True. For $Ax = b$ to have no solution we must have a row of 0's in the reduced row echelon form. Hence, the number of pivots will be less than the number of rows, and so the matrix $A$ does not have full rank.

(c) **True/False:** There exist $n \times n$ matrices $A$ and $B$ such that $B$ is not invertible but $AB$ is invertible.

**Solution:**

False. Suppose $AB$ is invertible, and consider $C = (AB)^{-1}A$. Then

$$CB = (AB)^{-1}AB = I,$$

so $C$ is an inverse for $B$.

(d) **True/False:** For any permutation matrix $P$, we have that $P^2 = I$.

**Solution:**

False. Consider the permutation matrix

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then

$$P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \neq I.$$

**Your PRINTED name is:** _____

Grading

1

2

3

_____

**Please circle your recitation:**

| | | | | | | |
|---|---|---|---|---|---|---|
| 1) | M 2 | 2-131 | P. Lee | 2-087 | 2-1193 | lee |
| 2) | M 2 | 2-132 | T. Lawson | 4-182 | 8-6895 | tlawson |
| 4) | T 10 | 2-132 | P.-O. Persson | 2-363A | 3-4989 | persson |
| 5) | T 11 | 2-131 | P.-O. Persson | 2-363A | 3-4989 | persson |
| 6) | T 11 | 2-132 | P. Pylyavskyy | 2-333 | 3-7826 | pasha |
| 7) | T 12 | 2-132 | T. Lawson | 4-182 | 8-6895 | tlawson |
| 8) | T 12 | 2-131 | P. Pylyavskyy | 2-333 | 3-7826 | pasha |
| 9) | T 1 | 2-132 | A. Chan | 2-588 | 3-4110 | alicec |
| 10) | T 1 | 2-131 | D. Chebikin | 2-333 | 3-7826 | chebikin |
| 11) | T 2 | 2-132 | A. Chan | 2-588 | 3-4110 | alicec |
| 12) | T 3 | 2-132 | T. Lawson | 4-182 | 8-6895 | tlawson |

**1  (30 pts.)**   Suppose $A$ is $m$ by $n$ with **linearly dependent columns**. Complete with as much true information as possible:

(a) The rank of $A$ is _____

_____

_____ .

(b) The nullspace of $A$ contains _____

_____

_____ .

(c) (more words needed) The equation $A^{\mathrm{T}}y = b$ has no solution for some right hand sides $b$ because _____

_____

_____

_____

_____ .

2

**2 (40 pts.)**    Suppose $A$ is this 3 by 4 matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

(a) A specific basis for the column space of $A$ is _____ .

(b) For which vectors $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ does $Ax = b$ have a solution? Give

conditions on $b_1, b_2, b_3$.

(c) There is no 4 by 3 matrix $B$ for which $AB = I$ (3 by 3). Give a good
reason (is this because $A$ is rectangular?).

(d) Find the complete solution to $Ax = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

3

XX

**3 (30 pts.)**  (a) Find a basis for the vector space of all real 3 by 3 symmetric matrices.

(b) Suppose $A$ is a square invertible matrix. You permute its rows by a permutation $P$ to get a new matrix $B$. How do you know that $B$ is also invertible?

(c) "If 2 matrices have the same shape and the same nullspace, then they have the same column space." **If this is true**, give a reason why. **If this is not true**, find 2 matrices to show it's false.

XXX

**Grading**

Your **PRINTED** name is:   **SOLUTIONS**

**1**

**2**

**3**

**Please circle your recitation:**

| | | | | | | |
|---|---|---|---|---|---|---|
| 1) | M 2 | 2-131 | P. Lee | 2-087 | 2-1193 | lee |
| 2) | M 2 | 2-132 | T. Lawson | 4-182 | 8-6895 | tlawson |
| 4) | T 10 | 2-132 | P.-O. Persson | 2-363A | 3-4989 | persson |
| 5) | T 11 | 2-131 | P.-O. Persson | 2-363A | 3-4989 | persson |
| 6) | T 11 | 2-132 | P. Pylyavskyy | 2-333 | 3-7826 | pasha |
| 7) | T 12 | 2-132 | T. Lawson | 4-182 | 8-6895 | tlawson |
| 8) | T 12 | 2-131 | P. Pylyavskyy | 2-333 | 3-7826 | pasha |
| 9) | T 1 | 2-132 | A. Chan | 2-588 | 3-4110 | alicec |
| 10) | T 1 | 2-131 | D. Chebikin | 2-333 | 3-7826 | chebikin |
| 11) | T 2 | 2-132 | A. Chan | 2-588 | 3-4110 | alicec |
| 12) | T 3 | 2-132 | T. Lawson | 4-182 | 8-6895 | tlawson |

**1  (30 pts.)**  Suppose $A$ is $m$ by $n$ with **linearly dependent columns**. Complete with as much true information as possible:

(a) The rank of $A$ is ...

at most $n - 1$ (and at most $m$, which is a stronger statement if $m < n - 1$).

(b) The nullspace of $A$ contains ...

at least one non-zero vector. (The dimension of the nullspace is $n$ minus the column rank of $A$, i.e., at least 1.)

(c) (more words needed) The equation $A^{\mathrm{T}} y = b$ has no solution for some right hand sides $b$ because ...

the rows of the matrix $A^{\mathrm{T}}$, which are the same as the columns of $A$, are linearly dependent, so $A^{\mathrm{T}}$ is not full row-rank. Thus the reduced row echelon form of $A^{\mathrm{T}}$ contains a row of all zeroes, so the components of $b$ must satisfy a certain linear relation in order for $A^{\mathrm{T}} y = b$ to have a solution.

**2 (40 pts.)**  Suppose $A$ is this 3 by 4 matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

(a) A specific basis for the column space of $A$ is _____ .

(b) For which vectors $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ does $Ax = b$ have a solution? Give conditions on $b_1, b_2, b_3$.

(c) There is no 4 by 3 matrix $B$ for which $AB = I$ (3 by 3). Give a good reason (is this because $A$ is rectangular?).

(d) Find the complete solution to $Ax = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

**Solution**

(a) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ , $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ is a basis for the column space of $A$. So is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ , $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ .

(b) Row reducing the augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$, we get

$$\begin{bmatrix} 1 & 2 & 3 & 4 & b_1 \\ 0 & -1 & -2 & -3 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix} .$$

The linear equations corresponding to the top two rows can be satisfied regardless of the values of $b_1$ and $b_2$, and the bottom row of all zeroes imposes the condition $b_3 - 2b_2 + b_1 = 0$. Hence $Ax = b$ has a solution if and only if $b_3 - 2b_2 + b_1 = 0$.

3

(c) This is because $A$ is not full row-rank, as shown in part (b). If $AB = I$ then we could solve $Ab_1 = $ row 1 of $I$, $Ab_2 = $ row 2 of $I$, $Ab_3 = $ row 3 of $I$, and every equation $Ax = b$. Actually the solution would be $x = Bb$. But in part (b) we saw that $Ax = b$ has no solution for some $b$.

(d) We perform the row reduction

$$
\left[\begin{array}{cccc|c}
1 & 2 & 3 & 4 & 1 \\
2 & 3 & 4 & 5 & 0 \\
3 & 4 & 5 & 6 & -1
\end{array}\right]
\longrightarrow
\left[\begin{array}{cccc|c}
1 & 2 & 3 & 4 & 1 \\
0 & -1 & -2 & -3 & -2 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
\longrightarrow
\left[\begin{array}{cccc|c}
1 & 0 & -1 & -2 & -3 \\
0 & 1 & 2 & 3 & 2 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

Then $x_p = \begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution, and $s_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $s_4 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ are

**special solutions** forming a basis of the nullspace of $A$. Hence the general solution is

$$
x = x_p + x_n = x_p + cs_3 + ds_4 \,.
$$

4

**3 (30 pts.)** (a) Find a basis for the vector space of all real 3 by 3 symmetric matrices.

(b) Suppose $A$ is a square invertible matrix. You permute its rows by a permutation $P$ to get a new matrix $B$. How do you know that $B$ is also invertible?

(c) "If 2 matrices have the same shape and the same nullspace, then they have the same column space." **If this is true**, give a reason why. **If this is not true**, find 2 matrices to show it's false.

**Solution**

(a) The most natural basis is

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.
$$

(b) $A$ being invertible means that $A$ has full rank. Permuting the rows has no effect on the rank, so $B$ has full rank as well, and is thus invertible. (Another argument: the permutation matrix $P$ is invertible, and so $B^{-1} = (PA)^{-1} = A^{-1}P^{-1}$.)

(c) The statement is false. Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ have the same nullspace (the line spanned by the vector $(1, -1)$), but their column spaces differ (for $A$, it's the line spanned by the vector $(1,1)$, and for $B$, it's the line spanned by the vector $(1, 2)$).

5

**Remark**  (now on the web page)

The real 3 by 3 matrices form a vector space $M$. The symmetric matrices in $M$ form a subspace $S$. If you add two symmetric matrices, or multiply by real numbers, the result is still a symmetric matrix. **Problem: Find a basis for $S$.**

When I asked this question on an exam, I realized that a key point needs to be emphasized: **The basis "vectors" for $S$ must lie in the subspace.** They are 3 by 3 symmetric matrices! Then there are two requirements:

1. The basis vectors must be linearly independent.

2. Their combinations must produce every vector (matrix) in $S$.

Here is one possible basis (all symmetric) for this example:

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad S_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad S_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad S_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Since this basis contains 6 vectors, the **dimension of $S$ is 6.**

# 18.06    Professor Strang    Quiz 1    October 4, 2006

**Your PRINTED name is:** _____

**Grading**

**1**

**2**

**3**

**4**

_____

**Please circle your recitation:**

1)  T 10  2-131  K. Meszaros  2-333  3-7826  `karola`

2)  T 10  2-132  A. Barakat   2-172  3-4470  `barakat`

3)  T 11  2-132  A. Barakat   2-172  3-4470  `barakat`

4)  T 11  2-131  A. Osorno    2-229  3-1589  `aosorno`

5)  T 12  2-132  A. Edelman   2-343  3-7770  `edelman`

6)  T 12  2-131  K. Meszaros  2-333  3-7826  `karola`

7)   T 1  2-132  A. Edelman   2-343  3-7770  `edelman`

8)   T 2  2-132  J. Burns     2-333  3-7826  `burns`

9)   T 3  2-132  A. Osorno    2-229  3-1589  `aosorno`

**1 (24 pts.)**  This question is about an $m$ by $n$ matrix $A$ for which

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ has } \textbf{no solutions} \quad \text{and} \quad Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ has } \textbf{exactly one solution.}$$

(a) Give all possible information about $m$ and $n$ and the rank $r$ of $A$.

(b) Find all solutions to $Ax = 0$ and **explain your answer**.

(c) Write down an example of a matrix $A$ that fits the description in part (a).

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**2 (24 pts.)**    The 3 by 3 matrix $A$ reduces to the identity matrix $I$ by the following three
row operations (in order):

$$E_{21}: \quad \text{Subtract } 4\,(\text{row 1}) \text{ from row 2.}$$
$$E_{31}: \quad \text{Subtract } 3\,(\text{row 1}) \text{ from row 3.}$$
$$E_{23}: \quad \text{Subtract row 3 from row 2.}$$

(a) Write the inverse matrix $A^{-1}$ in terms of the $E$'s. **Then compute $A^{-1}$.**

(b) What is the original matrix $A$?

(c) What is the lower triangular factor $L$ in $A = LU$?

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**3 (28 pts.)** This 3 by 4 matrix depends on $c$:

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 3 & c & 2 & 8 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

(a) *For each $c$* find a basis for the column space of $A$.

(b) *For each $c$* find a basis for the nullspace of $A$.

(c) *For each $c$* find the complete solution $x$ to $Ax = \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix}$.

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**4 (24 pts.)**     (a) If $A$ is a 3 by 5 matrix, what information do you have about the nullspace of $A$?

(b) Suppose row operations on $A$ lead to this matrix $R = \texttt{rref}(A)$:

$$R = \begin{bmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Write all known information about the columns of $A$.

(c) In the vector space $M$ of all 3 by 3 matrices (you could call this a matrix space), what subspace $S$ is spanned by all possible row reduced echelon forms $R$?

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# 18.06    Professor Strang    Quiz 1    October 4, 2006

**Your PRINTED name is:**    __**SOLUTIONS**__

_____

**Please circle your recitation:**

| | | | | | | |
|---|---|---|---|---|---|---|
| 1) | T 10 | 2-131 | K. Meszaros | 2-333 | 3-7826 | karola |
| 2) | T 10 | 2-132 | A. Barakat | 2-172 | 3-4470 | barakat |
| 3) | T 11 | 2-132 | A. Barakat | 2-172 | 3-4470 | barakat |
| 4) | T 11 | 2-131 | A. Osorno | 2-229 | 3-1589 | aosorno |
| 5) | T 12 | 2-132 | A. Edelman | 2-343 | 3-7770 | edelman |
| 6) | T 12 | 2-131 | K. Meszaros | 2-333 | 3-7826 | karola |
| 7) | T 1 | 2-132 | A. Edelman | 2-343 | 3-7770 | edelman |
| 8) | T 2 | 2-132 | J. Burns | 2-333 | 3-7826 | burns |
| 9) | T 3 | 2-132 | A. Osorno | 2-229 | 3-1589 | aosorno |

**1 (24 pts.)** This question is about an $m$ by $n$ matrix $A$ for which

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ has \textbf{no solutions}} \quad \text{and} \quad Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ has \textbf{exactly one solution}}.$$

(a) Give all possible information about $m$ and $n$ and the rank $r$ of $A$.

(b) Find all solutions to $Ax = 0$ and **explain your answer**.

(c) Write down an example of a matrix $A$ that fits the description in part (a).

*Solution.*

(a) $Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has *one* solution $\implies N(A) = \{0\}$ so $r = n$. (Also, $m = 3$ since $Ax \in \mathbb{R}^3$.)

$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has no solution $\implies C(A) \neq \mathbb{R}^3$, so $r < m$.

There are two possibilities: $\begin{matrix} m = 3 \\ r = n = 1 \end{matrix}$ and $\begin{matrix} m = 3 \\ r = n = 2 \end{matrix}$.

(b) Since $N(A) = \{0\}$ (because $Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has **1** solution), there is a unique solution to

$Ax = 0$, which is clearly $x = 0$. (Can be either $x = \begin{bmatrix} 0 \end{bmatrix}$ or $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ depending on if $n = 1$ or $n = 2$.)

(c) $A$ could be $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (many more possibilities).

2

**2 (24 pts.)** The 3 by 3 matrix $A$ reduces to the identity matrix $I$ by the following three row operations (in order):

$$E_{21}: \quad \text{Subtract } 4\,(\text{row 1}) \text{ from row 2.}$$
$$E_{31}: \quad \text{Subtract } 3\,(\text{row 1}) \text{ from row 3.}$$
$$E_{23}: \quad \text{Subtract row 3 from row 2.}$$

(a) Write the inverse matrix $A^{-1}$ in terms of the $E$'s. **Then compute $A^{-1}$.**

(b) What is the original matrix $A$?

(c) What is the lower triangular factor $L$ in $A = LU$?

*Solution.*

(a) Apply the three operations to $I$, i.e. $A^{-1} = E_{23}E_{31}E_{21}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix} = A^{-1}$$

(b) Apply the inverse operations in reverse order to $I$, i.e. $A = E_{21}^{-1}E_{31}^{-1}E_{23}^{-1}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} = A$$

$$\text{Check} \quad \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) $L = \begin{bmatrix} 1 & & \\ 4 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ 3 & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$

**3 (28 pts.)**   This 3 by 4 matrix depends on $c$:

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 3 & c & 2 & 8 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

(a) *For each $c$ find a basis for the column space of $A$.*

(b) *For each $c$ find a basis for the nullspace of $A$.*

(c) *For each $c$ find the complete solution $x$ to $Ax = \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix}$.*

*Solution.*

(a)  Elimination gives $\begin{bmatrix} \boxed{1} & 1 & 2 & 4 \\ 0 & c-3 & -4 & -4 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ so there are two cases:

If $c \neq 3$, $c-3$ is a pivot and $U = \begin{bmatrix} \boxed{1} & 1 & 2 & 4 \\ 0 & \boxed{c-3} & -4 & -4 \\ 0 & 0 & \boxed{2} & 2 \end{bmatrix} \longrightarrow R = \begin{bmatrix} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}$

so a basis for $C(A)$ is the first three columns of $A$: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}.$

If $c = 3$, $c-3 = 0$ and $U = \begin{bmatrix} \boxed{1} & 1 & 2 & 4 \\ 0 & 0 & \boxed{-4} & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow R = \begin{bmatrix} \boxed{1} & 1 & 0 & 2 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

so take the first and third columns of $A$ as a basis for $C(A)$: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}.$

4

(b) If $c \neq 3$, the special solutions give $N(A) = \left\{ x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

If $c = 3$, the special solutions give $N(A) = \left\{ x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

(c) By inspection, $x_p = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is one particular solution (other correct answers)

for $c \neq 3$, the complete solution is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

for $c = 3$, the complete solution is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

5

**4 (24 pts.)**   (a) If $A$ is a 3 by 5 matrix, what information do you have about the nullspace of $A$?

(b) Suppose row operations on $A$ lead to this matrix $R = \mathtt{rref}(A)$:

$$R = \begin{bmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Write all known information about the columns of $A$.

(c) In the vector space $M$ of all 3 by 3 matrices (you could call this a matrix space), what subspace $S$ is spanned by all possible row reduced echelon forms $R$?

*Solution.*

(a) $N(A)$ has dimension *at least* 2 (and at most 5).

(b) (**7pts**)  Columns 1, 4, 5 of $A$ form a basis for $C(A)$.

($\approx \mathbf{1pt}$)  Column 2 is $4 \times$ (Column 1); Column 3 is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(c) $A = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \right\}$, the set of upper triangular matrices.

(A basis of six echelon forms is

$$\left\{ \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & \end{bmatrix}, \begin{bmatrix} 1 & & \\ & & 1 \\ & & \end{bmatrix}, \begin{bmatrix} 1 & & \\ & & \\ & & \end{bmatrix}, \begin{bmatrix} 1 & 1 & \\ & & \\ & & \end{bmatrix}, \begin{bmatrix} 1 & & 1 \\ & & \\ & & \end{bmatrix} \right\}.)$$

18.06     Professor Strang     Quiz 1     February 28, 2005

**Your PRINTED name is:** _____

**Please circle your recitation:**

1)   M 2   2-131   A. Chan     2-588   3-4110   alicec

2)   M 3   2-131   A. Chan     2-588   3-4110   alicec

3)   M 3   2-132   D. Testa    2-586   3-4102   damiano

4)   T 10  2-132   C.I. Kim    2-273   3-4380   ikim

5)   T 11  2-132   C.I. Kim    2-273   3-4380   ikim

6)   T 12  2-132   W.L. Gan    2-101   3-3299   wlgan

7)   T 1   2-131   C.I. Kim    2-273   3-4380   ikim

8)   T 1   2-132   W.L. Gan    2-101   3-3299   wlgan

9)   T 2   2-132   W.L. Gan    2-101   3-3299   wlgan

**1 (26 pts.)**   Suppose $A$ is reduced by the usual row operations to

$$R = \begin{bmatrix} 1 & 4 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find the complete solution (if a solution exists) to this system involving the original $A$:

$$Ax = \text{ sum of the columns of } A.$$

**2 (18 pts.)**   Suppose the 4 by 4 matrices $A$ and $B$ have the *same column space*. They may not have the same columns!

(a) Are they sure to have the same number of pivots? YES NO WHY?

(b) Are they sure to have the same nullspace? YES NO WHY?

(c) If $A$ is invertible, are you sure that $B$ is invertible? YES NO WHY?

**3 (40 pts.)** (a) Reduce $A$ to an upper triangular matrix $U$ and carry out the same elimination steps on the right side $b$:

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 & b_1 \\ 3 & 5 & 1 & b_2 \\ -3 & 3 & 2 & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} U & c \end{bmatrix}$$

Factor the 3 by 3 matrix $A$ into $LU =$ (lower triangular)(upper triangular).

(b) If you change the last entry in $A$ from 2 to _____ (what number gives $A_{\mathbf{new}}$?) then $A_{\mathbf{new}}$ becomes singular. *Describe its column space exactly.*

(c) In that singular case from part (b), what condition(s) on $b_1, b_2, b_3$ allow the system $A_{\mathbf{new}}x = b$ to be solved?

(d) *Write down the complete solution to* $A_{\mathbf{new}}x = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}$ (the first column).

**4 (16 pts.)**   Suppose the columns of a 7 by 4 matrix $A$ are linearly independent.

(a) After row operations reduce $A$ to $U$ or $R$, how many rows will be all zero (or is it impossible to tell)?

(b) What is the row space of $A$? Explain why this equation will surely be solvable:

$$A^{\mathrm{T}}y = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

18.06    Professor Strang    Quiz 1    February 28, 2005

**Your PRINTED name is:**    **SOLUTIONS**

**1  (26 pts.)**    Suppose $A$ is reduced by the usual row operations to

$$
R = \begin{bmatrix} 1 & 4 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
$$

Find the complete solution (if a solution exists) to this system involving the original $A$:

$$Ax = \text{ sum of the columns of } A.$$

**Solution**

The complete solution $x = x_{\text{particular}} + x_{\text{nullspace}}$ has

$$
x_{\text{particular}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad x_{\text{nullspace}} = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.
$$

The free variables $x_2$ and $x_4$ can take any values.

The two special solutions came from the nullspace of $R$ = nullspace of $A$.

The particular solution of 1's gives $Ax$ = sum of the columns of $A$.

*Note:*   This also gives $Rx$ = sum of columns of $R$.

1

**2 (18 pts.)**  Suppose the 4 by 4 matrices $A$ and $B$ have the *same column space*. They may not have the same columns!

    (a) Are they sure to have the same number of pivots?  YES  NO  WHY?

    (b) Are they sure to have the same nullspace?  YES  NO  WHY?

    (c) If $A$ is invertible, are you sure that $B$ is invertible?  YES  NO  WHY?

**Solution**

(a) YES.  Number of pivots $=$ **rank** $=$ dimension of the column space.

This is the same for $A$ and $B$.

(b) NO.  The nullspace is not determined by the column space (unless we know that the matrix is symmetric.) Example with same column spaces but different nullspaces:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(c) YES.  If $A$ is invertible, its column space is the whole space $\mathbf{R}^4$. Since $B$ has the same column space, $B$ is also invertible.

**3 (40 pts.)** (a) Reduce $A$ to an upper triangular matrix $U$ and carry out the same elimination steps on the right side $b$:

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 & b_1 \\ 3 & 5 & 1 & b_2 \\ -3 & 3 & 2 & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} U & c \end{bmatrix}$$

Factor the 3 by 3 matrix $A$ into $LU = $ (lower triangular)(upper triangular).

(b) If you change the last entry in $A$ from 2 to _____ (what number gives $A_{\mathbf{new}}$?) then $A_{\mathbf{new}}$ becomes singular. *Describe its column space exactly.*

(c) In that singular case from part (b), what condition(s) on $b_1, b_2, b_3$ allow the system $A_{\mathbf{new}}x = b$ to be solved?

(d) *Write down the complete solution to* $A_{\mathbf{new}}x = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}$ (the first column).

**Solution**

(a) $\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 & b_1 \\ 3 & 5 & 1 & b_2 \\ -3 & 3 & 2 & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 & b_1 \\ 0 & 2 & 0 & b_2 - b_1 \\ 0 & 0 & 3 & b_3 - 3b_2 + 4b_1 \end{bmatrix}$

Here $A = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 5 & 1 \\ -3 & 3 & 2 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(b) If you change $A_{33}$ from 2 to $-1$, the third pivot is reduced by 3 and $A_{\text{new}}$ becomes singular. Its column space is the *plane in* $\mathbf{R}^3$ containing all combinations of the first columns $(3, 3, -3)$ and $(3, 5, 3)$.

(c) We need $b_3 - 3b_2 + 4b_1 = 0$ on the right side (since the left side is now a row of zeros).

(d) $A_{\text{new}}$ gives
$$
\begin{bmatrix}
3 & 3 & 1 & 3 \\
3 & 5 & 1 & 3 \\
-3 & 3 & -1 & -3
\end{bmatrix}
\longrightarrow
\begin{bmatrix}
3 & 3 & 1 & 3 \\
0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

Certainly $x_{\text{particular}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Also $x_{\text{nullspace}} = x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$.

The complete solution is $x_{\text{particular}} +$ any vector in the nullspace.

**4 (16 pts.)**  Suppose the columns of a 7 by 4 matrix $A$ are linearly independent.

(a) After row operations reduce $A$ to $U$ or $R$, how many rows will be all zero (or is it impossible to tell)?

(b) What is the row space of $A$? Explain why this equation will surely be solvable:
$$
A^{\mathrm{T}} y = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
$$

**Solution**

(a) The rank is 4, so there will be $7 - 4 = 3$ rows of zeros in $U$ and $R$.

(b) The row space of $A$ will be all of $\mathbf{R}^4$ (since the rank is 4). Then every vector $c$ in $\mathbf{R}^4$ is a combination of the rows of $A$, which means that $A^{\mathrm{T}} y = c$ is solvable for every right side $c$.

**Your PRINTED name is: SOLUTIONS**

**Grading**

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**Please circle your recitation:**

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| (4) | T 11 | 2-132 | K. Meszaros |
| (5) | T 12 | 2-132 | K. Meszaros |
| (6) | T 1 | 2-132 | Jerin Gu |
| (7) | T 2 | 2-132 | Jerin Gu |

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**Total:**

**Problem 1** *(20 points)*

Are the following sets of vectors in $\mathbb{R}^3$ vector subspaces? Explain your answer.

(a) vectors $(x, y, z)^T$ such that $2x - 2y + z = 0$     **YES**     NO

It is given by a linear equation equal to 0. You can also think about it as the nullspace of the matrix $\begin{pmatrix} 2 & -2 & 1 \end{pmatrix}$.

(b) vectors $(x, y, z)^T$ such that $x^2 - y^2 + z = 0$     YES     **NO**

The vector $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is in the set, but if you multiply by $-1$, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ is not.

(c) vectors $(x, y, z)^T$ such that $2x - 2y + z = 1$     YES     **NO**

It is given by a linear equation <u>not</u> set equal to 0. In particular, it doesn't contain the 0 vector.

(d) vectors $(x, y, z)^T$ such that $x = y$ AND $x = 2z$     **YES**     NO

It is the intersection of two planes! We can think about this set as the nullspace of the matrix $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -2 \end{pmatrix}$.

(e) vectors $(x, y, z)^T$ such that $x = y$ OR $x = 2z$     YES     **NO**

It is the union of two planes! Take for example $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ which is not in the set.

**Problem 2** *(20 points)*

Let $A$ be a $4 \times 3$ matrix with linearly independent columns.

(a) What are the dimensions of the four fundamental subspaces $C(A)$, $N(A)$, $C(A^T)$, $N(A^T)$?

(b) Describe explicitly the nullspace $N(A)$ and the row space $C(A^T)$ of $A$.

(c) Suppose that $B$ is a $4 \times 3$ matrix such that the matrices $A$ and $B$ have exactly the same column spaces $C(A) = C(B)$ and the same nullspaces $N(A) = N(B)$.

Are you sure that in this case $A = B$?     YES     NO

Prove that $A = B$ or give a counterexample where $A \neq B$.

*Solution 2*

(a) The columns are linearly independent, so the rank of the matrix is 3. Then $dimC(A) = 3$, $dimC(A^T) = 3$, $dimN(A) = 0$, $dimN(A^T) = 3$.

(b) Since $C(A^T)$ is a 3-dimensional subspace of $\mathbb{R}^3$, it is all of $\mathbb{R}^3$.

(c) The answer is **NO**, for example
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

**Problem 3** *(20 points)*

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 5 \\ 1 & 3 & 5 & 9 \end{pmatrix}$$

(a) What is the rank of $A$?

(b) Find a matrix $B$ such that the column space $C(A)$ of $A$ equals the nullspace $N(B)$ of $B$.

(c) Which of the following vectors belong(s) to the column space $C(A)$:

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \\ 4 \\ 8 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \quad ?$$

*Solution 3*

We will eliminate the augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & b_1 \\ 1 & 2 & 3 & 5 & b_2 \\ 1 & 3 & 5 & 9 & b_3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 & b_1 \\ 0 & 1 & 2 & 4 & b_2 - b_1 \\ 0 & 2 & 4 & 8 & b_3 - b_1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 & b_1 \\ 0 & 1 & 2 & 4 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{pmatrix}$$

(a) The rank is 2.

(b) We see from the last row of the reduced matrix that the condition for a vector to be in the column space is $b_1 - 2b - 2 + b_3 = 0$. Thus $C(A)$ is $N(B)$ for

$$B = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}.$$

(c) The last two vectors can't belong to the column space because they are in $\mathbb{R}^4$. From the condition mentioned in part (b), we see that $\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ is in the column space, but $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is not.

**Problem 4** *(20 points)*

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 4 & k \end{pmatrix}$$

(a) For which values of $k$ will the system $A\mathbf{x} = (2, 3, 7)^T$ have a unique solution?

(b) For which values of $k$ will it have an infinite number of solutions?

(c) For $k = 4$, find the LU-decomposition of $A$.

(d) For all values of $k$, find the complete solution to the system $A\mathbf{x} = (2, 3, 7)^T$.
(You might need to consider several cases.)

*Solution 4*

We will eliminate the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 3 \\ 3 & 4 & k & 7 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & k-3 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k-5 & 0 \end{pmatrix}$$

(a) and (b) We see from this that no matter what $k$ is there is always at least one solution (there is only a potentially 0 row in the eliminated matrix, and we get a 0 in the augmented vector). We could have seen that by inspection from the original matrix, since

$$\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}.$$

For $k \neq 5$, the matrix has rank 3, so there is a unique solution. For $k = 5$ the matrix has rank 2, so there are infinitely many solutions.

(c) $L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}$ using the multipliers, and $U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$ from the elimination above.

(d) As noted above, for $k \neq 5$ there is a unique solution, given by $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. We can get this from the eliminated matrix, or as mentioned above, by inspection.

For $k = 5$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is a particular solution; the general soltuion is given by adding vectors in the nullspace:

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

.

**Problem 5** *(20 points)*

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 & 0 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 2 & -1 & 0 & 0 \\ 2 & 4 & 0 & 4 & 4 \end{pmatrix}$$

(a) Find a basis of the column space $C(A)$.

(b) Find a basis of the nullspace $N(A)$.

(c) Find linear conditions on $b_1, b_2, b_3, b_4$ that guarantee that the system $A\mathbf{x} = (b_1, b_2, b_3, b_4)^T$ has a solution.

(d) Find the complete solution for the system $A\mathbf{x} = (0, 1, 0, 2)^T$.

*Solution 5*

You could find the following answers by eliminating, but also, by inspection.

(a) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\}$

(b) $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$

(c) $b_3 - b_1 = 0$ and $b_4 - 2b_2 = 0$.

(d) The general solution is:

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$$

18.06      Professor Edelman      Quiz 1      March 6, 2006

**Grading**

**Your PRINTED name is:** _____

**1**

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| 4) | T 10 | 2-132 | F. Liu | 2-333 | 3-7826 | fuliu |
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| 7) | T 1 | 2-131 | F. Liu | 2-333 | 3-7826 | fuliu |
| 8) | T 1 | 2-132 | A. Osorno | 2-229 | 3-1589 | aosorno |
| 9) | T 2 | 2-132 | A. Osorno | 2-229 | 3-1589 | aosorno |

**1 (30 pts.)** Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix}$$

(a) By eliminating, prove that the column space consists of those vectors $(x, y, z)^{\mathrm{T}}$ with $2x - y = 0$.

(b) By eliminating, prove that the row space consists of those vectors $(x, y, z)^{\mathrm{T}}$ with $y + z = 2x$.

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**2 (40 pts.)** For each of the eight cases below, exhibit a $3 \times 3$ matrix $A$ with rank $r$ and the specified condition or argue convincingly that it is impossible.

(a) $r = 0$, $\mathrm{col}(A) = \mathrm{row}(A)$

(b) $r = 1$, $\mathrm{col}(A) = \mathrm{row}(A)$

(c) $r = 2$, $\mathrm{col}(A) = \mathrm{row}(A)$

(d) $r = 3$, $\mathrm{col}(A) = \mathrm{row}(A)$

(e) $r = 0$, $\mathrm{col}(A) \neq \mathrm{row}(A)$

(f) $r = 1$, $\mathrm{col}(A) \neq \mathrm{row}(A)$

(g) $r = 2$, $\mathrm{col}(A) \neq \mathrm{row}(A)$

(h) $r = 3$, $\mathrm{col}(A) \neq \mathrm{row}(A)$

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**3 (30 pts.)** Let

$$
A = \begin{bmatrix} 1 & a & 0 & d & 0 & g \\ 0 & b & 1 & e & 0 & h \\ 0 & c & 0 & f & 1 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}
$$

(a) Find the complete solution to $Ax = v$, if $s = 1$.

(b) Find the complete solution to $Ax = v$, if $s = 0$.

(*Hint:* Best if you don't work too hard!)

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Grading

Your **PRINTED** name is:     **SOLUTIONS**

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1)   M 2   2-131   A. Kasimov   2-339   3-1715   kasimov

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4)   T 10   2-132   F. Liu       2-333   3-7826   fuliu

5)   T 11   2-132   P. Shor      2-369   3-4362   shor

6)   T 12   2-132   P. Shor      2-369   3-4362   shor

7)   T 1   2-131   F. Liu        2-333   3-7826   fuliu

8)   T 1   2-132   A. Osorno     2-229   3-1589   aosorno

9)   T 2   2-132   A. Osorno     2-229   3-1589   aosorno

**1 (30 pts.)** Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix}$$

(a) By eliminating, prove that the column space consists of those vectors $(x, y, z)^{\mathrm{T}}$ with $2x - y = 0$.

(b) By eliminating, prove that the row space consists of those vectors $(x, y, z)^{\mathrm{T}}$ with $y + z = 2x$.

*Solution.*

(a) $\begin{bmatrix} 1 & 1 & 1 & x \\ 2 & 2 & 2 & y \\ 3 & 2 & 4 & z \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & x \\ 0 & 0 & 0 & y - 2x \\ 0 & -1 & 1 & z - 3x \end{bmatrix}$

$\implies y - 2x = 0 \iff 2x - y = 0.$

(b) $CS(A^{\mathrm{T}}) = RS(A)$

$\begin{bmatrix} 1 & 2 & 3 & x \\ 1 & 2 & 2 & y \\ 1 & 2 & 4 & z \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & x \\ 0 & 0 & -1 & y - x \\ 0 & 0 & 1 & z - x \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & x \\ 0 & 0 & -1 & y - x \\ 0 & 0 & 0 & z - x + y - x \end{bmatrix}$

$\implies z + y - 2x = 0 \iff z + y = 2x.$

**2 (40 pts.)** For each of the eight cases below, exhibit a $3 \times 3$ matrix $A$ with rank $r$ and the specified condition or argue convincingly that it is impossible.

(a) $r = 0$, $\text{col}(A) = \text{row}(A)$          (e) $r = 0$, $\text{col}(A) \neq \text{row}(A)$

(b) $r = 1$, $\text{col}(A) = \text{row}(A)$          (f) $r = 1$, $\text{col}(A) \neq \text{row}(A)$

(c) $r = 2$, $\text{col}(A) = \text{row}(A)$          (g) $r = 2$, $\text{col}(A) \neq \text{row}(A)$

(d) $r = 3$, $\text{col}(A) = \text{row}(A)$          (h) $r = 3$, $\text{col}(A) \neq \text{row}(A)$

*Solution.*

(a)   $3 \times 3$   0 matrix

(e)   Not possible.

$$r = 0 \implies CS(A) = 0 = RS(A)$$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d)   $3 \times 3$   identity matrix

(h)   Not possible.

$$r = 3 \implies CS(A) = \mathbb{R}^3 = RS(A).$$

3

**3 (30 pts.)** Let

$$A = \begin{bmatrix} 1 & a & 0 & d & 0 & g \\ 0 & b & 1 & e & 0 & h \\ 0 & c & 0 & f & 1 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}$$

(a) Find the complete solution to $Ax = v$, if $s = 1$.

(b) Find the complete solution to $Ax = v$, if $s = 0$.

(*Hint:* Best if you don't work too hard!)

*Solution.*

(a) No solution.

The last row of $A$ is all zeros. If $s = 1$, we get $0 = 1$.

(b) Let $\bar{x} = (x_1, x_2, x_3, x_4, x_5, x_6)$ and let $x_2, x_4, x_6$ be free variables.

$$x_{\text{particular}} = \begin{bmatrix} p \\ 0 \\ q \\ 0 \\ r \\ 0 \end{bmatrix}$$

4

To find the special solutions:

$$
\text{let } \quad
\begin{aligned}
x_2 &= -1 \\[4pt]
x_4 &= \phantom{-}0 \\[4pt]
x_6 &= \phantom{-}0
\end{aligned}
\qquad
\begin{bmatrix} a \\ -1 \\ b \\ 0 \\ c \\ 0 \end{bmatrix},
\qquad
\begin{aligned}
x_2 &= \phantom{-}0 \\[4pt]
x_4 &= -1 \\[4pt]
x_6 &= \phantom{-}0
\end{aligned}
\qquad
\begin{bmatrix} d \\ 0 \\ e \\ -1 \\ f \\ 0 \end{bmatrix},
\qquad
\begin{aligned}
x_2 &= \phantom{-}0 \\[4pt]
x_4 &= \phantom{-}0 \\[4pt]
x_6 &= -1
\end{aligned}
\qquad
\begin{bmatrix} g \\ 0 \\ h \\ 0 \\ i \\ -1 \end{bmatrix}
$$

The complete solution:

$$
\overline{x} =
\begin{bmatrix} p \\ 0 \\ q \\ 0 \\ r \\ 0 \end{bmatrix}
+ x_2
\begin{bmatrix} a \\ -1 \\ b \\ 0 \\ c \\ 0 \end{bmatrix}
+ x_4
\begin{bmatrix} d \\ 0 \\ e \\ -1 \\ f \\ 0 \end{bmatrix}
+ x_6
\begin{bmatrix} g \\ 0 \\ h \\ 0 \\ i \\ -1 \end{bmatrix}
$$

## SOLUTIONS

**1  (20 pts.)**  Find all solutions to the linear system

$$x + 2y + z - 2w = 5$$

$$2x + 4y + z + w = 9$$

$$3x + 6y + 2z - w = 14$$

Solution:

We perform elimination on the augmented matrix:

$$
\begin{pmatrix}
1 & 2 & 1 & -2 & 5 \\
2 & 4 & 1 & 1 & 9 \\
3 & 6 & 2 & -1 & 14
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 1 & -2 & 5 \\
0 & 0 & -1 & 5 & -1 \\
0 & 0 & -1 & 5 & -1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 1 & -2 & 5 \\
0 & 0 & -1 & 5 & -1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

So $y$ and $w$ are free variables. Thus special solutions to $A\mathbf{x} = 0$ are given by setting $y = 1, w = 0$ and $y = 0, w = 1$ respectively, i.e.

$$
\mathbf{s}_1 =
\begin{pmatrix}
-2 \\
1 \\
0 \\
0
\end{pmatrix},
\qquad
\mathbf{s}_2 =
\begin{pmatrix}
-3 \\
0 \\
5 \\
1
\end{pmatrix}.
$$

Moreover, a particular solution to the system is given by setting $y = w = 0$, i.e.

$$
\mathbf{x}_p =
\begin{pmatrix}
4 \\
0 \\
1 \\
0
\end{pmatrix}.
$$

We could have read these special and particular solutions off even more easily by performing one more elimination step to get the row-reduced echelon matrix:

$$\rightarrow \begin{pmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = R.$$

Notice that the last column gives the values of the pivot variables for the particular solution, and the free columns give the values of the pivot variables in the special solutions (multiplied by $-1$), as was shown in class.

We conclude that the general solutions to this system are given by

$$\mathbf{x} = \mathbf{x}_p + c_1\mathbf{s}_1 + c_2\mathbf{s}_2 = \begin{pmatrix} -2c_1 - 3c_2 + 4 \\ c_1 \\ 5c_2 + 1 \\ c_2 \end{pmatrix},$$

where $c_1$ and $c_2$ are arbitrary constants.

**2 (30 pts.)** In class, we learned how to do "downwards" elimination to put a matrix $A$ in upper-triangular (or echelon) form $U$: not counting row swaps, we subtracted multiples of pivot rows from subsequent rows to put zeros below the pivots, corresponding to multiplying $A$ by elimination matrices.

Instead, we could do elimination "leftwards" by subtracting multiples of pivot columns from leftwards columns, again to get an upper-triangular matrix $U$. For example, let:

$$A = \begin{pmatrix} 7 & 6 & 4 \\ 6 & 3 & 12 \\ 2 & 0 & 1 \end{pmatrix}$$

We could subtract twice the third column from the first column to eliminate the 2, so that we get zeros to the left of the "pivot" 1 at the lower right.

(i) Continue this "leftwards" elimination to obtain an upper-triangular matrix $U$ from the $A$ above, and write $U$ in terms of $A$ multiplied by a sequence of matrices corresponding to each leftwards-elimination step.

(ii) Suppose we followed this process for an arbitrary $A$ (not necessarily square or invertible) to get an echelon matrix $U$. Which of the column space and null space, if any, are the same between $A$ and $U$, and why?

(iii) Is the $U$ that we get by leftwards elimination always the same as the $U$ we get from ordinary downwards elimination? Why or why not?

Solution:

**(i)** The "leftwards" elimination procedure is

$$A = \begin{pmatrix} 7 & 6 & 4 \\ 6 & 3 & 12 \\ 2 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 6 & 4 \\ -18 & 3 & 12 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 35 & 6 & 4 \\ 0 & 3 & 12 \\ 0 & 0 & 1 \end{pmatrix} = U,$$

3

where the first step sent $(\text{col1}) \to (\text{col1}) - 2(\text{col3})$ and the second step sent $(\text{col1}) \to (\text{col1}) + 6(\text{col2})$. Since these operations are linear combinations of the columns, they correspond to multiplying on the *right* by elimination matrices:

$$U = \begin{pmatrix} 35 & 6 & 4 \\ 0 & 3 & 12 \\ 0 & 0 & 1 \end{pmatrix} = A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(One way to get these elimination matrices, as usual, is to do the corresponding operation on the $3 \times 3$ identity matrix.)

**(ii)** As above, $U = AE$ (the elimination matrices multiply on the *right*). It follows that the column space of $A$ is the same as the column space of $U$, but the null spaces are different. Informally, by multiplying $E$ on the *right*, we modify the *input* vectors of $A$ (changing the null space), but the output vectors are still made of columns of $A$ (preserving the column space). To be more careful, we need the fact that $E$ is invertible (as elimination always is); otherwise, $C(AE)$ could be a smaller subspace of $C(A)$.

More precisely, since $U = AE$ above, where $E$ are the elimination matrices, any $\mathbf{x} = U\mathbf{y} = A(E\mathbf{y})$, so any $\mathbf{x}$ in $C(U)$ is in $C(A)$. Also vice-versa, since $A = UE^{-1}$. So $C(U) = C(A)$. However, if $\mathbf{x}$ is in the null space $N(U)$ (i.e. $U\mathbf{x} = \mathbf{0} = AE\mathbf{x}$), this only means $E\mathbf{x}$ is in $N(A)$, not $\mathbf{x}$. So the null spaces are different in general (but have the same dimension).

[Compare to the case of ordinary elimination, which preserves $N(A)$ but changes $C(A)$. Left elimination is equivalent to "upwards" elimination on $A^T$—this preserves the row space of $A^T$, meaning that the column space of $A$ is preserved etc.]

**(iii)** The are *not* the same $U$ in general (although of course there are special cases where they are the same, such as when $A$ is upper-triangular to start with). There are several ways to see this.

The simplest way is to give any counterexample: e.g., apply downwards elimination to $A$ above and you will get a different result. For example, downward elimination never

changes the upper-left corner (7), but the upper-left corner was changed (to 35) by leftwards elimination above.

Abstractly, we know from class that downwards elimination always preserves the null space, whereas we just saw that upwards elimination does not. So, they cannot be the same.

It is *not* sufficient to simply say that left-elimination does different sorts of operations than down-elimination—there are lots of problems where you can do a different sequence of operations and still get the same result. (For example, we could use left elimination to find $A^{-1}$, and of course there is only one possible $A^{-1}$ if it exists at all.)

**3 (20 pts.)**   Determine whether the following statements are true or false, and explain your reasoning.

(♣) If $A^2 = A$, then $A = 0$ or $A = I$.

(◇) Ignoring row swaps, any invertible matrix $A$ has a "UL" factorization (as an alternative to LU factorization): $A$ can be written as $A = UL$ where $U$ and $L$ are some upper and lower triangular matrices, respectively.

(♠) All the $2 \times 2$ matrices that commute with $A = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$ (i.e. all $2 \times 2$ matrices $B$ with $AB = BA$) form a vector space (with the usual formulas for addition of matrices and multiplication of matrices by numbers).

(♡) There is no $3 \times 3$ matrix whose column space equals its nullspace.

---

**Solution**

(♣) False.

Counterexample: if $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $A^2 = A$ but $A \neq I$ and $A \neq 0$.

Another counterexample was given in Pset 2 Problem 8 (a). Note that if we assume $A$ is invertible, then the only solution is $A = I$ (multiply both sides of $A^2 = A$ by $A^{-1}$), but this assumption is not warranted here.

(◇) True.

Instead of "downwards" elimination we can also do "upwards" elimination to put $A$ into lower-triangular form $L$ (possibly with row swaps). In this procedure the corresponding elimination matrices are upper-triangular, but still multiply on the left (since they are still row operations), so we get a UL factorization.

Alternatively, the "leftwards" elimination of problem 2 also leads to a UL factorization, because the (lower-triangular!) elimination matrices multiply on the *right* to give $U = AL^{-1}$.

Technically, however, this process may require *column* swaps if zeros are encountered in pivot positions.

(♠) True.

We need to know that linear combinations of vectors stay in the vectors space. If $B$ is a matrix where $AB = BA$, then clearly $A(cB) = c(AB) = c(BA) = (cB)A$ for any $c$. If $B'$ is another matrix where $AB' = B'A$, then $A(B + B') = AB + AB' = BA + B'A = (B + B')A$. (The other properties of a vector space, associativity etcetera, need not be shown since they are automatic for the usual addition and multiplication operations.)

(♡) True.

Suppose the rank of $A$ is $r$, then the dimension of column space is $r$, and the dimension of null space is $3-r$. Obviously no matter $r = 0, 1, 2, 3$, we always have $r \neq 3-r$. (Equivalently, $r = 3 - r$ would imply a fractional rank $r = 3/2$!) This shows that the two spaces are not the same, since they must have different dimensions.

**4 (30 pts.)** The following information is known about an $m \times n$ matrix $A$:

$$A\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, A\begin{pmatrix} 0 \\ 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, A\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}, A\begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

($\alpha$) Show that the vectors $\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}$ form a basis of $\mathbb{R}^4$.

($\beta$) Give a matrix $C$ and an invertible matrix $B$ such that $A = CB^{-1}$. (You *don't* have to evaluate $B^{-1}$ or find $A$ explicitly. Just say what $B$ and $C$ are and use them to reason about $A$ in the subsequent parts.)

($\gamma$) Find a basis for the null space of $A^T$.

($\delta$) What are $m$, $n$, and the rank $r$ of $A$?

---

Solution:

($\alpha$) We are in $\mathbb{R}^4$, which is four-dimensional, so any four linearly independent vectors forms a basis as shown in class. Thus, we just need to show that these four vectors are linearly independent, which is equivalent to showing that the $4 \times 4$ matrix whose columns are these vectors has full column rank (null space $= \{\mathbf{0}\}$). Proceeding by elimination:

$$B = \begin{pmatrix} 1 & 0 & 2 & 3 \\ -2 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 4 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & -1 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & -7 & -15 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & -1 \end{pmatrix} = U.$$

Thus, there are four pivots, and hence it has full column rank as desired.

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**(β)** The provided equations multiply $A$ by four vectors to get four vectors, which by definition of matrix multiplication (recall the column picture) can be combined into a single equation where $A$ is multiplied by a matrix with four columns to yield a matrix with four columns:

$$A \begin{pmatrix} 1 & 0 & 2 & 3 \\ -2 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 5 & 1 \\ 4 & 0 & 10 & 2 \end{pmatrix}.$$

Thus if we take

$$C = \begin{pmatrix} 2 & 0 & 5 & 1 \\ 4 & 0 & 10 & 2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 & 2 & 3 \\ -2 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \end{pmatrix}$$

we have $A = CB^{-1}$. Since $B$ is precisely the matrix of the basis vectors from part $(\alpha)$, its invertibility follows from above (it is $4 \times 4$ and has 4 pivots).

**(γ)** Since $A = CB^{-1}$, we have

$$A^T = (B^{-1})^T C^T = (B^T)^{-1} C^T.$$

(As in class, because $B$ is invertible, $B^T$ is too.) Just as for elimination (multiplying on the left by an invertible elimination matrix), the null space is preserved when $C^T \to (B^T)^{-1} C^T$. [You need not prove this, because the proof is the same as in class. Recall that if $C^T \mathbf{x} = \mathbf{0}$ then $A^T \mathbf{x} = \mathbf{0}$ from above, and vice versa if we multiply both sides by $B^T$.] That means we just need to find the null space of $C^T$ by elimination:

$$\begin{pmatrix} 2 & 4 \\ 0 & 0 \\ 5 & 10 \\ 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 4 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

in which there is only one free variable, so there is one special solution (the basis of the null space)

$$\mathbf{s}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$

or any multiple thereof. (You can also find this special solution by inspection, without elimination.)

($\delta$) Since $A$ times a 4-vector is a 2-vector, we must have $m = 2$ and $n = 4$. Equivalently, from part ($\beta$) we saw that $A$ was a $2 \times 4$ matrix multiplied by a $4 \times 4$ matrix, giving a $2 \times 4$ matrix. Moreover, from above the dimension of $N(A^T)$ is 1, but this must equal $m - r$, so we obtain $r = 1$.

# 18.06   Professor Edelman   Quiz 1   September 29, 2008

**Your PRINTED name is:** _____

**Please circle your recitation:**

| | | | | | | |
|---|---|---|---|---|---|---|
| 1) | T 10 | 2-131 | J.Yu | 2-348 | 4-2597 | jyu |
| 2) | T 10 | 2-132 | J. Aristoff | 2-492 | 3-4093 | jeffa |
| 3) | T 10 | 2-255 | Su Ho Oh | 2-333 | 3-7826 | suho |
| 4) | T 11 | 2-131 | J. Yu | 2-348 | 4-2597 | jyu |
| 5) | T 11 | 2-132 | J. Pascaleff | 2-492 | 3-4093 | jpascale |
| 6) | T 12 | 2-132 | J. Pascaleff | 2-492 | 3-4093 | jpascale |
| 7) | T 12 | 2-131 | K. Jung | 2-331 | 3-5029 | kmjung |
| 8) | T 1 | 2-131 | K. Jung | 2-331 | 3-5029 | kmjung |
| 9) | T 1 | 2-136 | V. Sohinger | 2-310 | 4-1231 | vedran |
| 10) | T 1 | 2-147 | M Frankland | 2-090 | 3-6293 | franklan |
| 11) | T 2 | 2-131 | J. French | 2-489 | 3-4086 | jfrench |
| 12) | T 2 | 2-147 | M. Frankland | 2-090 | 3-6293 | franklan |
| 13) | T 2 | 4-159 | C. Dodd | 2-492 | 3-4093 | cdodd |
| 14) | T 3 | 2-131 | J. French | 2-489 | 3-4086 | jfrench |
| 15) | T 3 | 4-159 | C. Dodd | 2-492 | 3-4093 | cdodd |

**1 (18 pts.)**   Consider the equation $Ax = b$:

$$\begin{bmatrix} 1 & 0 \\ 4 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} .$$

(a) Put the equation into echelon form $Rx = d$.

(b) For which $b$ are there solutions?

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**2 (24 pts.)** The matrix $A$ has two special solutions:

$$x_1 = \begin{bmatrix} c \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix} d \\ 0 \\ 1 \end{bmatrix}.$$

(a) Describe all the possibilities for the number of columns of $A$.

(b) Describe all the possibilities for the number of rows of $A$.

(c) Describe all the possibilities for the rank of $A$.

Briefly explain your answers.

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**3 (30 pts.)**  Let $A$ be any matrix and $R$ its row reduced echelon form. Answer True or False to the statements below and briefly explain. (Note, if there are any counterexamples to a statement below you must choose false for that statement.)

(a) If $x$ is a solution to $Ax = b$ then $x$ must be a solution to $Rx = b$.

(b) If $x$ is a solution to $Ax = 0$ then $x$ must be a solution to $Rx = 0$.

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**4 (28 pts.)**   A Sudoko puzzle solution such as the example on the last page is a $9x9$ matrix $A$ that among other properties has the numbers 1 through 9 once in every row and in every column.

Hint 1: There is no need to compute at all to solve this problem, and familiarty with Sudoko puzzles are unlikely to help or hurt.

Hint 2: $1+2+3+...+9=45$.

(a) All such matrices $A$ can be written as

$$A = P_1 + 2P_2 + 3P_3 + \ldots + 8P_8 + 9P_9,$$

where the matrices $P_1, \ldots, P_9$ are what kind of matrices? (Looking for what we consider the best possible one word answer. Square would be correct, but would not be acceptable.)

(b) Let $e$ be the $9 \times 1$ vector of nine 1's. What is the rank of the 9x3 matrix whose columns are $e$, $Ae$, and $A^T e$ for any such matrix $A$. Explain your answer.

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## QUIZ 1 ANSWERS

**1.** $\begin{pmatrix} 1 & 0 \\ 4 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$

**a).** (12 points) We reduce: $\begin{pmatrix} 1 & 0 & b_1 \\ 4 & 1 & b_2 \\ 2 & -1 & b_3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 - 4b_1 \\ 0 & -1 & b_3 - 2b_1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 - 4b_1 \\ 0 & 0 & b_3 + b_2 - 6b_1 \end{pmatrix}.$

So the equation becomes:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 - 4b_1 \\ b_3 + b_2 - 6b_1 \end{pmatrix}.$$

**b)** (6 points) Only when $b_3 + b_2 - 6b_1 = 0$.

**2. a)** (8 points) Solutions to $A$ are length 3 column vectors, so $A$ has three columns.

**b)** (8 points) Any number. In fact consider $\begin{pmatrix} 1 & -c & -d \end{pmatrix}$. This kills both of the given vectors. Then, feel free to add any number of rows of zero's below it.

**c)** (8 points) We assume that the matrix is not zero as we are given that these are the only special solutions. To find the rank, we note that each row of $A$ must be in the subspace of $\mathbb{R}^3$ which is orthogonal to $\begin{pmatrix} c & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} d & 0 & 1 \end{pmatrix}$, but since these guys span a plane, this subspace is a line. Thus the rows are all linearly dependant, and so the rank is one.

**3. a)** (20 points) False. To get $A$ to its reduced form, you need to multiply on the left by some elimination matrix $E$; so $Rx = Eb$ is correct, but there is no reason for $Eb - b$ to be in the nullspace of $A$(you can use your favorite non-triangular invertible 2 by 2 matrix to find a counterexample).

**b)** (10 points) True. It is certainly the case that $E0 = 0$.

**4. a)** (10 points) Permutation.

**b)** (18 points) Well, $Pe = e$ for any permutation matrix, because each row has eight zero's and one one. So by part a), $Ae = e + 2e + ... + 9e = 45e$. But since the transpose of a permutation matrix is also a permutation matrix, $A^t e = 45e$ also. But $rank \begin{pmatrix} e & 45e & 45e \end{pmatrix} = 1$.

# 18.06    Professor Strang    Quiz 1    March 3, 2008

**Grading**

1

**Your PRINTED name is:** _____

2

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**Please circle your recitation:** _____

| | | | | | | |
|----|------|--------|--------------|-------|--------|---------|
| 1) | M 2 | 2-131 | A. Ritter | 2-085 | 2-1192 | afr |
| 2) | M 2 | 4-149 | A. Tievsky | 2-492 | 3-4093 | tievsky |
| 3) | M 3 | 2-131 | A. Ritter | 2-085 | 2-1192 | afr |
| 4) | M 3 | 2-132 | A. Tievsky | 2-492 | 3-4093 | tievsky |
| 5) | T 11 | 2-132 | J. Yin | 2-333 | 3-7826 | jbyin |
| 6) | T 11 | 8-205 | A. Pires | 2-251 | 3-7566 | arita |
| 7) | T 12 | 2-132 | J. Yin | 2-333 | 3-7826 | jbyin |
| 8) | T 12 | 8-205 | A. Pires | 2-251 | 3-7566 | arita |
| 9) | T 12 | 26-142 | P. Buchak | 2-093 | 3-1198 | pmb |
| 10) | T 1 | 2-132 | B. Lehmann | 2-089 | 3-1195 | lehmann |
| 11) | T 1 | 26-142 | P. Buchak | 2-093 | 3-1198 | pmb |
| 12) | T 1 | 26-168 | P. McNamara | 2-314 | 4-1459 | petermc |
| 13) | T 2 | 2-132 | B. Lehmann | 2-089 | 2-1195 | lehmann |
| 14) | T 2 | 26-168 | P. McNamara | 2-314 | 4-1459 | petermc |

**1 (18 pts.)** Start with an invertible 3 by 3 matrix $A$. Construct $B$ by subtracting 4 times row 1 of $A$ from row 3. **How do you find $B^{-1}$ from $A^{-1}$?** You can answer in matrix notation, but *you must also answer in words*—what happens to the columns and rows?

Solution (18 points)

We can find $B$ by multiplying $A$ on the left by an appropriate matrix $E_{31}$ (remember, row operations correspond to multiplication on the left, column operations correspond to multiplication on the right). Here, we need

$$
E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \tag{1}
$$

Since $B = E_{31}A$, we know $B^{-1} = (E_{31}A)^{-1} = A^{-1}E_{31}^{-1}$. Thus, $B^{-1}$ can be found by doing some column operations on $A^{-1}$.

What operations specifically? We know

$$
E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \tag{2}
$$

(this is the standard pattern for an $E$ matrix). This represents adding 4 times column 3 to column 1.

**2 (24 pts.)** Elimination on $A$ leads to $U$:

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{leads to} \quad Ux = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(a) Factor the first matrix $A$ into $A = LU$ and also into $A = LDL^{\mathrm{T}}$.

(b) Find the inverse of $A$ by Gauss-Jordan elimination on $AA^{-1} = I$ or by inverting $L$ and $D$ and $L^{\mathrm{T}}$.

(c) If $D$ is diagonal, show that $LDL^{\mathrm{T}}$ is a symmetric matrix for every matrix $L$ (square or rectangular).

$\boxed{\text{Solution}}$ (9+9+6 points)

a) Since $A$ is symmetric, we have an $LDL^T$ decomposition, and we find this first. Since

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix} \tag{3}$$

we must have

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \tag{4}$$

and so

$$L^T = D^{-1}U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

Of course, this means that

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \tag{6}$$

which we could have calculated directly.

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b) If we did it using Gauss-Jordan elimination:

$$
\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 1 & 3 & 7 & 0 & 0 & 1 \end{bmatrix}
\rightsquigarrow
\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & 2 & 6 & -1 & 0 & 1 \end{bmatrix}
$$

$$
\rightsquigarrow
\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & 0 & 4 & 0 & -1 & 1 \end{bmatrix}
$$

(This is $L^{-1}$ on the right hand side, so we can multiply and solve. We may also continue.)

$$
\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & 0 & 4 & 0 & -1 & 1 \end{bmatrix}
\rightsquigarrow
\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & -1/4 & 1/4 \end{bmatrix}
$$

$$
\rightsquigarrow
\begin{bmatrix} 1 & 1 & 0 & 1 & 1/4 & -1/4 \\ 0 & 1 & 0 & -1/2 & 3/4 & -1/4 \\ 0 & 0 & 1 & 0 & -1/4 & 1/4 \end{bmatrix}
$$

$$
\rightsquigarrow
\begin{bmatrix} 1 & 0 & 0 & 3/2 & -1/2 & 0 \\ 0 & 1 & 0 & -1/2 & 3/4 & -1/4 \\ 0 & 0 & 1 & 0 & -1/4 & 1/4 \end{bmatrix}
$$

c) We show a matrix is symmetric by showing that it is equal to its transpose. If $D$ is diagonal, then of course $D = D^T$. Thus:

$$
\begin{aligned}
(LDL^T)^T &= (L^T)^T D^T L^T \\
&= LDL^T
\end{aligned}
$$

**3 (30 pts.)** Suppose the nonzero vectors $a_1, a_2, a_3$ point in different directions in $\mathbb{R}^3$ but

$$3a_1 + 2a_2 + a_3 = \text{zero vector}.$$

The matrix $A$ has those vectors $a_1, a_2, a_3$ in its columns.

(a) Describe the nullspace of $A$ (all $x$ with $Ax = 0$).

(b) Which are the pivot columns of $A$?

(c) I want to show that *all* 3 by 3 matrices with

$$(*) \qquad 3(\text{column } 1) + 2(\text{column } 2) + (\text{column } 3) = \text{zero vector}$$

form a **subspace** $S$ of the space $M$ of 3 by 3 matrices. Now the zero matrix is certainly included.

Suppose $B$ and $C$ are matrices whose columns have this property $(*)$. To show that we have a subspace, we have to prove that every linear combination of $B$ and $C$ ___(finish sentence)___ .

Go ahead and prove that.

---

$\boxed{\text{Solution}}$ (10+10+10 points)

a) First, the problem gives us one vector in the nullspace: the equation

$$3a_1 + 2a_2 + a_3 = \text{zero vector} \tag{7}$$

is the same as saying that $Ax = 0$, where

$$x = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \tag{8}$$

Thus, the nullspace $N(A)$ contains all scalar multiples of this vector $(3, 2, 1)$. Technically, we need to argue that there is nothing else in the nullspace. You must recognize that the matrix has rank 2 either here or in part b to get full credit.

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b) Since we have a linear relationship between the columns, column 3 can not be a pivot column, so the rank is at most 2. The problem also tell us that the three $a_i$ vectors point in different directions. This means that they can't all be on the same line; that is, the column space must be at least a plane. Thus the the rank is exactly 2.

Having rank 2 means that there are 2 pivot columns and one free column. Since column 3 is a linear combination of columns before it, it must be a free column. So columns 1 and 2 are pivots, and column 3 is free.

c) To show that we have a subspace, we have to prove that every linear combination of $B$ and $C$ ___also has property (*)___ .

Suppose that $B$ and $C$ have property (*). Consider the matrix $D = t_1 B + t_2 C$ for any two numbers $t_1, t_2$. For ease of notation, I'll denote the $i$th column of a matrix $A$ by $\text{col}_i(A)$. We have

$$\text{col}_i(D) = t_1 \text{col}_i(B) + t_2 \text{col}_i(C) \tag{9}$$

We can now check that $D$ also satisfies (*):

$$
\begin{aligned}
3\text{col}_1(D) + 2\text{col}_2(D) + \text{col}_3(D) &= 3\left(t_1\text{col}_1(B) + t_2\text{col}_1(C)\right) \\
&\quad +2\left(t_1\text{col}_2(B) + t_2\text{col}_2(C)\right) \\
&\quad + \left(t_1\text{col}_3(B) + t_2\text{col}_3(C)\right) \\
&= t_1\left(3\text{col}_1(B) + 2\text{col}_2(B) + \text{col}_3(B)\right) \\
&\quad +t_2\left(3\text{col}_1(C) + 2\text{col}_2(C) + \text{col}_3(C)\right) \\
&= 0
\end{aligned}
$$

**4 (28 pts.)** Start with this 2 by 4 matrix:

$$A = \begin{bmatrix} 2 & 3 & 1 & -1 \\ 6 & 9 & 3 & -2 \end{bmatrix}$$

(a) Find all special solutions to $Ax = 0$ and **describe the nullspace** of $A$.

(b) Find the complete solution—meaning all solutions $(x_1, x_2, x_3, x_4)$—to

$$Ax = \begin{bmatrix} 2x_1 + 3x_2 + x_3 - x_4 \\ 6x_1 + 9x_2 + 3x_3 - 2x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(c) When an $m$ by $n$ matrix $A$ has rank $r = m$, the system $Ax = b$ can be solved for which $b$ (best answer)? How many special solutions to $Ax = 0$?

---

Solution (10+8+10 points)

a) We find the special solutions by reducing $A$. I'll go all the way to row reduced echelon form:

$$\begin{bmatrix} 2 & 3 & 1 & -1 \\ 6 & 9 & 3 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 3 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 3/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, in turn we set each free variable to 1 and the rest to 0, and solve $Ux = 0$. The special solutions are

$$s_1 = \begin{bmatrix} -3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

$$s_2 = \begin{bmatrix} -1/2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{11}$$

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The nullspace is all linear combinations of these two vectors; it will be a plane in $\mathbb{R}^4$.

b) One way to solve for a particular solution is just to look at the set-up: our $b$ is the negative of the 4th column, so $(0, 0, 0, -1)$ will work.

We could also solve for a particular solution using an augmented matrix:

$$
\begin{bmatrix}
2 & 3 & 1 & -1 & 1 \\
6 & 9 & 3 & -2 & 2
\end{bmatrix}
\rightsquigarrow
\begin{bmatrix}
2 & 3 & 1 & -1 & 1 \\
0 & 0 & 0 & 1 & -1
\end{bmatrix}
$$

We can now back substitute and solve. We can pick the free variables to be whatever we like; we may as well set them to be 0. Then the bottom equation gives us $x_4 = -1$, and the top then gives $x_1 = 0$. This is the same vector as above.

The complete solution is particular solution plus the nullspace:

$$
x_c =
\begin{bmatrix}
0 \\
0 \\
0 \\
-1
\end{bmatrix}
+ c_1
\begin{bmatrix}
-3/2 \\
1 \\
0 \\
0
\end{bmatrix}
+ c_2
\begin{bmatrix}
-1/2 \\
0 \\
1 \\
0
\end{bmatrix}
\tag{12}
$$

c) When an $m$ by $n$ matrix has rank $m$, the dimension of the column space ($=$ rank) is the same as the dimension of the ambient space. Thus, every vector is in the column space; the equation $Ax = b$ can be solved for every $b$. The number of special solutions will be the number of rows minus the rank (note that $n \geq m$ since the rank is no greater than $n$). Thus, we find that there are $n - m$ special solutions.

# 18.06 Spring 2009 Exam 1 Practice

## General comments

Exam 1 covers the first 8 lectures of 18.06:

1. The Geometry of Linear Equations

2. Elimination and Matrix Operations (elimination, pivots, etcetera; different viewpoints of $AB$ and $Ax$ and $x^T A$, e.g. as linear combinations of rows or columns)

3. Elimination Matrices and Matrix Inverses (row operations = multiplying on left by elimination matrices, Gauss-Jordan elimination and what happens when you repeat the elimination steps on $I$)

4. $A = LU$ Factorization (for example, the relationship between $L$ and the elimination steps, and solving problems with $A$ in terms of the triangular matrices $L$ and $U$)

5. Permutations, Dot Products, and Transposes (relationship between dot products and transposes, $(AB)^T = B^T A^T$, permutation matrices, etcetera)

6. Vector Spaces and Subspaces (for example, the column space and nullspace, what is and isn't a subspace in general, and other vector spaces/subspaces e.g. using matrices and functions)

7. Solving $Ax = 0$ (the nullspace), echelon form $U$, row-reduced echelon form $R$ (rank, free variables, pivot variables, special solutions, etcetera)

8. Solving $Ax = 0$ for nonsquare $A$ (particular solutions, relationship of rank/nullspace/columnspace to existence and uniqueness of solutions)

If there is one central technique in all of these lectures, it is **elimination**. You should know elimination forwards and backwards. Literally: we might give you the final steps and ask you to work backwards, or ask you what properties of $A$ you can infer from certain results in elimination. Know how elimination relates to nullspaces and column spaces: elimination doesn't change the nullspace, which is why we can solve $Rx = 0$ to get the nullspace, while it does change the column space...but you can check that $b$ is in the column space of $A$ by elimination (if elimination produces a zero row from $A$, the same steps should produce a zero row from $b$ if $b$ is in the column space). Understand *why* elimination works, not just *how*. Know how/why elimination corresponds to matrix operations (elimination matrices and $L$).

One common mistake that I've warned you about before is: *never compute the inverse of a matrix*, unless you are *specifically asked to*. If you find yourself calculating $A^{-1}$ in order to compute $x = A^{-1}b$, you should instead solve $Ax = b$ for $x$ by elimination & backsubstitution. Computing the inverse matrix explicitly is a lot more work, and more error prone...and fails completely if $A$ is singular or nonsquare.

## Some practice problems

The 18.06 web site has exams from previous terms that you can download, with solutions. I've listed a few practice exam problems that I like below, but there are plenty more to choose from. Note, however that there will be *no questions asking explicitly about linear independence, basis, dimension, or the row space or left nullspace.* Reviewing the homework and solutions is always a good idea, too. The exam will consist of 3 or 4 questions (perhaps with several parts each), and you will have one hour.

1. $A$ is a $4 \times 4$ matrix with rank 2, and $Ax = b$ for some $b$ has three solutions $x = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 2 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 5 \\ 2 \\ 1 \\ 4 \end{pmatrix}$.

    Give the nullspace $N(A)$.

2. If we do a sequence of column operations (adding multiples of one column to another column) on a square matrix $A$ and obtain the identity matrix $I$, then what do we get if we do the same sequence of column operations on $A^{-1}$? (Express your answer in terms of $A$ and/or $A^{-1}$.)

3. If $A$ is $5 \times 3$, $B$ is $4 \times 5$, and $C(A) = N(B)$, then what is $BA$?

4. If $A$ and $B$ are matrices of the same size and $C(A) = C(B)$, does $C(A+B) = C(A)$? If not, give a counter-example.

5. (From spring 2007, exam 1 problem 1.) Are the following sets of vectors in $\mathbb{R}^3$ subspaces? Explain your answers.

    (a) vectors $(x, y, z)^T$ such that $2x - 2y + z = 0$
    (b) vectors $(x, y, z)^T$ such that $x^2 - y^2 + z = 0$
    (c) vectors $(x, y, z)^T$ such that $2x - 2y + z = 1$
    (d) vectors $(x, y, z)^T$ such that $x = y$ **and** $x = 2z$
    (e) vectors $(x, y, z)^T$ such that $x = y$ **or** $x = 2z$

6. (From spring 2007, exam 1 problem 3.) Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 5 \\ 1 & 3 & 5 & 9 \end{pmatrix}$.

    (a) What is the rank of $A$?
    (b) Find a matrix $B$ such that the column space $C(A)$ of $A$ equals the nullspace $N(B)$ of $B$.
    (c) Which of the following vectors belong to the column space $C(A)$? $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$.

7. (From spring 2007, exam 1 problem 4.) Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 4 & k \end{pmatrix}$.

    (a) For which values of $k$ will the system $Ax = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$ have a unique solution?
    (b) For which values of $k$ will the system from (a) have an infinite number of solutions?
    (c) For $k = 4$, find the $LU$ decomposition of $A$.
    (d) For all values of $k$, find the complete solution to $Ax = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$. (You might have to consider several cases.)

8. (From fall 2006 exam 1, problem 4.)

   (a) If $A$ is a 3-by-5 matrix, what information do you have about the nullspace of $A$?

   (b) In the vector space $M$ of all $3 \times 3$ matrices, what subspace is spanned by all possible row-reduced echelon forms $R$?

9. (From spring 2006 exam 1, problem 3.) [Hint: best if you don't work too hard on this problem!] Let

$$A = \begin{pmatrix} 1 & a & 0 & d & 0 & g \\ 0 & b & 1 & e & 0 & h \\ 0 & c & 0 & f & 1 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}.$$

   (a) Find the complete solution to $Ax = v$ if $s = 1$.

   (b) Find the complete solution to $Ax = v$ if $s = 0$.

10. (From spring 2005 exam 1, problem 1.) Suppose $A$ is reduced by the usual row operations to

$$R = \begin{pmatrix} 1 & 4 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

   Find the complete solution (if any exists) to this system involving the original $A$:

$$Ax = \text{sum of the columns of } A.$$

11. (From spring 2005 exam 1, problem 2.) Suppose the $4 \times 4$ matrices $A$ and $B$ have the *same column space*. They may not have the same columns!

   (a) Are they certain to have the same number of pivots? YES or NO. Explain.

   (b) Are they certain to have the same nullspace? YES or NO. Explain.

   (c) If $A$ is invertible, are you sure that $B$ is invertible? YES or NO. Explain.

12. (From spring 2005 exam 1, problem 3.)

   (a) Reduce $A$ to an upper-triangular matrix $U$ and carry out the same elimination steps on the right side $b$:

$$(A\,b) = \begin{pmatrix} 3 & 3 & 1 & b_1 \\ 3 & 5 & 1 & b_2 \\ -3 & 3 & 2 & b_3 \end{pmatrix} \rightarrow (U\,c).$$

   Factor the $3 \times 3$ matrix $A$ into $LU$ (lower triangular times upper triangular).

   (b) If you change the last (lower-right) entry in $A$ from 2 to _____ to get a new matrix $A_{new}$, then $A_{new}$ becomes singular. *Fill in the blank, and describe its column space exactly.*

   (c) In that singular case from (b), what conditions on $b_1$, $b_2$, and $b_3$ allow $A_{new}x = b$ to be solved?

   (d) Write down the complete solution to $A_{new}x = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$ (the first column of $A_{new}$).

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# Solutions

The solutions for all problems from previous exams are posted on the 18.06 web page. Solutions to the first four problems are:

1. The differences between the solutions must be in the nullspace. We have three solutions, hence two differences:
$$\begin{pmatrix} 2 \\ -1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -2 \end{pmatrix} \text{ and } \begin{pmatrix} 5 \\ 2 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -2 \\ 0 \end{pmatrix}.$$ The rank of $A$ is 2 and it has 4 columns, so we only need two independent nullspace vectors to span the nullspace. Hence the nullspace is the span of these two difference vectors (which clearly aren't multiples of one another).

2. A sequence of column operations corresponds to multiplying $A$ on the right by some matrix $E$, like in the problem sets. But if $AE = I$, then $E$ must be $A^{-1}$. Doing the same operations on $A^{-1}$ gives $A^{-1}E = A^{-1}A^{-1} = A^{-2}$.

3. $BA$ is a $4 \times 3$ matrix. Since $C(A) = N(B)$, then $BAx$ for any $x$ gives $B$ multiplied by something in $N(B)$, which gives zero. Since $BAx = 0$ for any $x$, we must have $BA = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

4. No. A simple example is $B = -A$ for any nonzero $A$. $C(-A) = C(A)$ (it's the same columns, just multiplied by $-1$), but $A + (-A) = 0$ and the column space of the zero matrix is just $\{0\} \neq C(A)$.

# 18.06 Quiz 1 Solution

Hold on Monday, 2 March 2009 at 11am in Walker Gym.

Total: 60 points.

**Problem 1:** Your classmate, Nyarlathotep, performed the usual elimination steps to convert $A$ to echelon form $U$, obtaining:

$$U = \begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & 2 & 2 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Find a set of vectors spanning the nullspace $N(A)$.

(b) If $U\vec{y} = \begin{pmatrix} 9 \\ -12 \\ 0 \end{pmatrix}$, find the complete solution $\vec{y}$ (i.e. describe all possible solutions $\vec{y}$).

(c) Nyarla gave you a matrix

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix}$$

and told you that $A = LU$. Describe the complete sequence of elimination steps that Nyarla performed, assuming that she did elimination in the usual way starting with the first column and eliminating downwards. That is, Nyarla first subtracted _____ times the first row from the second row, then subtracted _____ times the first row from the third row, then subtracted _____. (Be careful about signs: *adding* a multiple of a row is the same as subtracting a *negative* multiple of that row.)

(d) If $A\vec{x} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$, then $U\vec{x} = $ _____.

---

Solution (20 points = 5+5+5+5)

(a) The pivots are in the first two columns of $U$, so $x_3$ and $x_4$ are the free variables. Setting $x_3 = 1, x_4 = 0$, we get (from the second row of $U\vec{x} = 0$) $x_2 = -1$

1

and (from the first row) $x_1 = 1 - 4x_2 = 5$; setting $x_3 = 0, x_4 = 1$, we get (from the second row) $x_2 = 3$ and (from the first row)$x_1 = -3 - 4x_2 = -15$. Hence, $N(A)$ is spanned by two special solutions as follows.

$$N(A) = x_3 \begin{pmatrix} 5 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -15 \\ 3 \\ 0 \\ 1 \end{pmatrix} \quad \text{for all } x_3, x_4 \in \mathbb{R}.$$

(b) First, we need to find a particular solution. For this, we may set the free variables to $y_3 = y_4 = 0$. Thus, (from the second row of $U\vec{y} = b$) $y_2 = -6$ and (from the first row) $y_1 = 9 - 4y_2 = 33$. Hence, all the solution to the equations are given by the sum of the particular solution and any vector in the nullspace (all linear combinations of the special solutions):

$$\vec{y} = y_3 \begin{pmatrix} 5 \\ -1 \\ 1 \\ 0 \end{pmatrix} + y_4 \begin{pmatrix} -15 \\ 3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 33 \\ -6 \\ 0 \\ 0 \end{pmatrix} \quad \text{for all } y_3, y_4 \in \mathbb{R}$$

(c) Nyarla first subtracted **2** times the first row from the second row, then subtracted $-1$ time s the first row from the third row, then subtracted **3 times the second row from the third row**.

There are a couple of ways to solve this problem. The easiest is to remember that the $L$ matrix, the product of the inverses of the elimination matrices, is simply composed of the multipliers for each of the elimination steps below each column. Under the first column of $L$ we have 2 and $-1$, and these are thus the multiples of the first row that get subtracted from rows 2 and 3. Under the second column of $L$ we have a 3, and this is the multiple of the second row that gets subtracted from the third row.

The other way to solve it is to just multiply $L$ by $U$ to get $A = LU$, and re-do the elimination process. Obviously, this is a bit more work, but is not too bad.

(d) Applying the same elimination operations in (c) to $A\vec{x}$ should give $U\vec{x}$. So, we have

$$\begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

2

Alternatively, we can just solve $U\vec{x}$ from $A\vec{x}$ as follows. Let $\vec{v} = U\vec{x}$. Then $L\vec{v} = UL\vec{x} = A\vec{x} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$. Thus, we can solve from the top as follows. $v_1 = 0$, $v_2 = 2 - 2v_1 = 2$, and $v_3 = 6 - 3v_2 + v_1 = 0$. Hence, $U\vec{x} = \vec{v} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$.

REMARK: Some students realized that $U\vec{x} = L^{-1}(A\vec{x})$. But several of these students did not get $L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -3 & 1 \end{pmatrix}$ correctly. Be careful that the inverse of $\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}$ is $not$ $\begin{pmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ -l_{31} & -l_{32} & 1 \end{pmatrix}$; the lower left entry should be $l_{21}l_{32} - l_{31}$. (Only for elimination matrices, which have nonzero entries below only a single diagonal, can you always invert just by flipping signs.) More generally, if you find yourself inverting a matrix, you should realize that there is probably an easier way to do it: to multiply $\vec{v} = L^{-1}(A\vec{x})$, it is easier to solve $L\vec{v} = A\vec{x}$ for $\vec{v}$ by elimination (especially since $L$ is triangular, so you can just do forward substitution as above).

**Problem 2:** Which of the following (if any) are subspaces? For any that are *not* a subspace, give an example of how they violate a property of subspaces.

(I) Given some $3 \times 5$ matrix $A$ with full row rank, the set of all solutions to

$$A\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(II) All vectors $\vec{x}$ with $\vec{x}^T \vec{y} = 0$ *and* $\vec{x}^T \vec{z} = 0$ for some given vectors $\vec{y}$ and $\vec{z}$.

(III) All $3 \times 5$ matrices with $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in their column space.

(IV) All $5 \times 3$ matrices with $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in their nullspace.

(V) All vectors $\vec{x}$ with $\|\vec{x} - \vec{y}\| = \|\vec{y}\|$ for some given fixed vector $\vec{y} \neq 0$.

---

Solution (20 points = 4+4+4+4+4)

(I) No. This is not a vector space because $\vec{x} = 0$ is not in this subspace.

(II) Yes. (This is actually just the left nullspace of the matrix whose columns are $\vec{y}$ and $\vec{z}$.)

(III) No. For example, the zero matrix is not in this subset.

(IV) Yes. If the nullspaces of $A_1$ and $A_2$ contain $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, then any linear combination of these matrices does too:

$$(\alpha_1 A_1 + \alpha_2 A_2) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \alpha_1 A_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \alpha_2 A_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0; \text{ for all } \alpha_1, \alpha_2.$$

(V) No. For example, $2\vec{y}$ satisfies the condition (because $\|2\vec{y} - \vec{y}\| = \|\vec{y}\|$) but $\vec{y}$ does not satisfies the condition (because $\|\vec{y} - \vec{y}\| = 0 \neq \|\vec{y}\|$). This violates the fact that a subspace is preserved under multiplication by scalars.

4

REMARK: A common problem we saw in the grading is that some students do not know how to express a counterexample. A counterexample is simply a single specific element of the set that violates a specific property of subspaces, or a specific element that should be in the set but isn't (as in the case of the sets missing $\vec{0}$ above). One such example is all that is needed to disqualify a set as a subspace; no further abstract argument is necessary. If you were asked to find an "example" and you find yourself writing a long, abstract essay, you are probably making a mistake!

**Problem 3:** $A$ is a matrix with a nullspace $N(A)$ spanned by the following three vectors:

$$\begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ -1 \\ 3 \\ 1 \end{pmatrix}.$$

($\alpha$) Give a matrix $B$ such that its column space $C(B)$ is the same as $N(A)$. (There is more than one correct answer.) [Thus, any vector $\vec{y}$ in the nullspace of $A$ satisfies $B\vec{u} = \vec{y}$ for some $\vec{u}$.]

($\beta$) Give a different possible answer to ($\alpha$): another $B$ with $C(B) = N(A)$.

($\gamma$) For some vector $\vec{b}$, you are told that a particular solution to $A\vec{x} = \vec{b}$ is

$$\vec{x}_p = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Now, your classmate Zarkon tells you that a second solution is:

$$\vec{x}_Z = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix},$$

while your other classmate Hastur tells you "No, Zarkon's solution can't be right, but here's a second solution that is correct:"

$$\vec{x}_H = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix}.$$

Is Zarkon's solution correct, or Hastur's solution, or are both correct? (Hint: what should be true of $\vec{x} - \vec{x}_p$ if $\vec{x}$ is a valid solution?)

---

**Solution** (20 points = 5+5+10) ($\alpha$) Since the nullspace is spanned by the given three vectors, we may simply take $B$ to consist of the three vectors as columns, i.e.,

$$B = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ -1 & 1 & 3 \\ 3 & 4 & 1 \end{pmatrix}.$$

$B$ need not be square (many students insisted on square solutions).

($\beta$) For example, we may simply add a zero column to $B$:

$$B = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & -1 & 0 \\ -1 & 1 & 3 & 0 \\ 3 & 4 & 1 & 0 \end{pmatrix}.$$

Or, we could interchange two columns. Or we could multiply one of the columns by $-1$. For example:

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -3 \\ 3 & 4 & -1 \end{pmatrix}.$$

Or we could replace one of the columns by a linear combination of that column with the other two columns (any invertible column operation). Or we could replace $B$ by $-B$ or $2B$. There are many possible solutions. In any case, the solution shouldn't require any significant calculation!

($\gamma$) Since any solution $\vec{x}$ to the equation $A\vec{x} = \vec{b}$ is of the form $\vec{x}_p + \vec{n}$ for some vector $\vec{n}$ in the nullspace, the vector $\vec{x} - \vec{x}_p$ must lie in the nullspace $N(A)$. Thus, we want to look at:

$$\vec{x}_Z - \vec{x}_p = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -4 \end{pmatrix}, \quad \vec{x}_H - \vec{x}_p = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -3 \end{pmatrix}.$$

To determine whether a vector $\vec{y}$ lies in the nullspace $N(A)$, we can just check whether it is in the column space of $B$, i.e. check whether $B\vec{z} = \vec{y}$ has a solution. As we learned in class, we can check this just by doing elimination: if elimination produces a zero row in $B$, it should produce a zero row in the right-hand side. In terms of $B$ from part ($\alpha$) augmented by the right-hand side, this gives:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & -1 & -1 \\ -1 & 1 & 3 & 0 \\ 3 & 4 & 1 & a \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 4 & 4 & a \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & a+4 \end{pmatrix}$$

We can get a solution if and only if $a = -4$. So Zarkon is correct.

REMARK: Several students apparently just stared at the nullspace vectors and found a linear combination that gave $\vec{x}_Z - \vec{x}_p$:

$$\vec{x}_Z - \vec{x}_p = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -4 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 4 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ -1 \\ 3 \\ 1 \end{pmatrix}.$$

Then they stared at Hastur's solution, couldn't find such a combination, and concluded that it was not a solution. This conclusion is correct *in this case*, and was awarded full marks because you were not asked to justify your solution. However, doing elimination is much more systematic and reliable, and ensures that there isn't a linear combination that you simply missed. Use elimination next time!

REMARK: Some students saw the zero components of $\vec{x}_Z - \vec{x}_p$, didn't see any corresponding zero components in the given nullspace vectors, and concluded that $\vec{x}_Z - \vec{x}_p$ was not in the nullspace. This is wrong: the key point is that $\vec{x}_Z - \vec{x}_p$ can be any vector in the nullspace, which means any linear combination of the given nullspace vectors. There are plenty of ways to combine nonzero vectors to get vectors with zero components!

**Problem 1.** Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$.

(a) Find the factorization $A = LU$.

(b) Find the inverse of $A$.

(c) For which values of $c$ is the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & c \end{pmatrix}$ invertible?

$\boxed{\text{Solution}}$

(a) We row reduce $A$ by subtracting row 1 from row 2 ($E_{12}$) and then add row 2 to row 3 ($E_{23}$) to find the upper triangular matrix

$$U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since we can reverse this process and subtract row 2 from row 3 in $U$, followed by adding row 1 to row 2 to obtain $A$, we see that the lower triangular matrix is the product:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Hence we find

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Note that $A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$. We explicitly compute $U^{-1}$ and find:

$$
\left(\begin{array}{ccc|ccc}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & -1 & -2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{array}\right)
\rightarrow
\left(\begin{array}{ccc|ccc}
1 & 2 & 0 & 1 & 0 & -3 \\
0 & -1 & 0 & 0 & 1 & 2 \\
0 & 0 & 1 & 0 & 0 & 1
\end{array}\right)
\rightarrow
\left(\begin{array}{ccc|ccc}
1 & 0 & 0 & 1 & 2 & 1 \\
0 & 1 & 0 & 0 & -1 & -2 \\
0 & 0 & 1 & 0 & 0 & 1
\end{array}\right).
$$

Similary, we compute

$$
L^{-1} = \begin{pmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
-1 & 1 & 1
\end{pmatrix}.
$$

Therefore

$$
A^{-1} = \begin{pmatrix}
1 & 2 & 1 \\
0 & -1 & -2 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
-1 & 1 & 1
\end{pmatrix}
$$

$$
= \begin{pmatrix}
-2 & 3 & 1 \\
3 & -3 & -2 \\
-1 & 1 & 1
\end{pmatrix}.
$$

(c) We row reduce to find :

$$
\begin{pmatrix}
1 & 2 & 3 \\
1 & 1 & 1 \\
0 & 1 & c
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 3 \\
0 & -1 & -2 \\
0 & 0 & c-2
\end{pmatrix}.
$$

Note that $A$ is invertible if and only if it has 3 nonzero pivots. Thus $A$ is invertible when $c \neq 2$.

**Problem 2.** Which of the following are subspaces? Explain why.

(a) All vectors $\mathbf{x}$ in $\mathbb{R}^3$ such that $\mathbf{x}^T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0$.

(b) All vectors $(x, y)^T$ in $\mathbb{R}^2$ such that $x^2 - y^2 = 0$.

(c) All vectors $(x, y)^T$ in $\mathbb{R}^2$ such that $x + y = 2$.

(d) All vectors $\mathbf{x}$ in $\mathbb{R}^3$ which are in the column space AND in the nullspace of the matrix
$$\begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix}.$$

(e) All vectors $\mathbf{x}$ in $\mathbb{R}^3$ which are in the column space OR in the nullspace (or in both) of the matrix
$$\begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix}.$$

Solution

(a) Yes. This equation describes the left nullspace of the matrix $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Since the left nullspace

is a vector space, we are done.

(b) No. Consider the vectors $(1, 1)^T$ and $(1, -1)^T$, both of which satisfy this equation. The sum $(1, 1)^T + (1, -1)^T = (2, 0)^T$ does not satisfy the equation since $2^2 - 0 = 4$.

(c) No. This set does not contain $(0, 0)^T$.

(d) Yes. Note that this column space of this matrix is the span of the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. The

nullspace is spanned by the vectors: $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Since the sum of the two nullspace

basis vectors is the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, the intersection is the column space, which is a vector space.

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(e) Yes. We have already seen in part (d) that the column space is a subspace of the nullspace. Thus the vectors that are in the column space or the nullspace are just the columns in the nullspace, which is a vector space.

**Problem 3.** Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 & 2 \\ -1 & -2 & 0 & 0 & -1 \\ 1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

(a) Find the complete solution of the equation $A\mathbf{x} = \mathbf{0}$.

(b) Find the complete solution of the equation $A\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

(c) Find all vectors $\mathbf{b}$ such that the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(d) Find a matrix $B$ such that $N(A) = C(B)$.

(e) Find bases of the four fundamental subspaces for the matrix $A$.

Solution

In preparation for the next problems, lets first row reduce this matrix with an arbitrary vector augmented.

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 2 & b_1 \\ -1 & -2 & 0 & 0 & -1 & b_2 \\ 1 & 2 & 0 & 0 & 1 & b_3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 & 2 & 2 & b_1 \\ 0 & 0 & 1 & 2 & 1 & b_1 + b_2 \\ 0 & 0 & -1 & -2 & -1 & b_3 - b_1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 1 & 2 & 2 & b_1 \\ 0 & 0 & 1 & 2 & 1 & b_1 + b_2 \\ 0 & 0 & 0 & 0 & 0 & b_2 + b_3 \end{pmatrix}$$

(a) We are finding the nullspace, or the solution when $b_1 = b_2 = b_3 = 0$. Since this matrix has 2 pivots and 3 free columns, so a general solution is just any linear combination of the nullspace basis vectors:

$$c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

(b) To find a general solution, we set $b_1 = 2$, $b_2 = 1$ and $b_3 = -1$ in our augmented matrix row reduction performed above. We find a particular solution to

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix},$$

which is a vector with zeros in the free variables, so we solve directly and find our particular solution:

$$\mathbf{x}_p = \begin{pmatrix} -1 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}.$$

Thus a general solution is just this particular solution plus the general solution for a nullspace vector given in part (a):

$$\begin{pmatrix} -1 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

(c) Finding all vectors $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ has a solution is asking for condition on $\mathbf{b}$ so that it is in the column space. From our original computation, we see that we must have

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$b_2 + b_3 = 0$, so an arbitrary vector in the column space looks like $\begin{pmatrix} b_1 \\ b_2 \\ -b_2 \end{pmatrix}$, where $b_1$ and $b_2$

are any real numbers. In otherwords, the column space is spanned by vectors of the form:

$$b_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

(d) To find a matrix $B$ such that $N(A) = C(B)$, we can simply take a matrix whose column vectors are a basis for the nullspace of $A$. In otherwords, the matrix:

$$B = \begin{pmatrix} -2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(e) Observe that the three vectors in part (a) form a basis for the nullspace of $A$, the two vectors in part (c) form a basis for the column space of (a). Thus all that is left is to find basis vectors for the row space, which we can take to be the two independent row vectors corresponding to the pivot rows of $A$: $\begin{pmatrix} 1 & 2 & 1 & 2 & 2 \end{pmatrix}^T$ and $\begin{pmatrix} 0 & 0 & 1 & 2 & 1 \end{pmatrix}^T$. A basis for the left null space is given by the vector that is in the nullspace of the matrix whose rows are the basis vectors for the column space of $A$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Thus a basis for the left nullspace is given by the vector $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

**Problem 4.** Let $A$ be an $m$ by $n$ matrix. Let $B$ be an $n$ by $m$ matrix. Suppose that $AB = I_m$ is the $m$ by $m$ identity matrix.

1. Let $r = \operatorname{rank}(A)$ denote the rank of the matrix $A$. Choose one answer and be sure to justify it.

    (a) $r \geq m$

    (b) $r \leq m$

    (c) $r = m$

    (d) $r > n$

2. Is $m \leq n$ or is $n \leq m$? Why?

Solution

1. The rank of $A$ is equal to the dimension of the column space $C(A)$. Now the column space of $A$ is the subspace of $\mathbb{R}^m$ that can be written as $A\mathbf{x}$, where $\mathbf{x}$ is a vector in $\mathbb{R}^n$. I claim that $C(A) = \mathbb{R}^m$. This follows because the column space $AB$ is the column space of the identity matrix is all of $\mathbb{R}^m$. So in particular, any vector $\mathbf{v}$ in $\mathbb{R}^m$ can be written as $AB\mathbf{v} = I_m\mathbf{v} = \mathbf{v}$. Thus any vector $\mathbf{v}$ in $\mathbb{R}^m$ is in the column space of $A$ because it can be written as $\mathbf{v} = A\mathbf{x}$ by simply setting $\mathbf{x} = B\mathbf{v}$. Therefore we have seen that the dimension of the column space is $m$, and thus the answer is (c).

2. We know that the rank of $A$ must be less than or equal to the smallest dimension of $A$. Since $r = m$, it must be the case that $m \leq n$.

**18.06 Fall 2010 Exam 1 solutions**
**1 (30 pts.)**
Start with the 3 by 4 matrix:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \end{bmatrix}.$$

(a) (10 pts.) What are all the special solutions to $Ax = 0$, and describe the nullspace of $A$.

**Solution:** We can do row operations to get the matrix $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The free columns are 1, 3, and 4. We get each special solution by setting one of $x_1, x_3, x_4$ to 1 and the other two to 0. The answers are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix},$$

and the nullspace of $A$ is the span of these three vectors.

(b) (10 pts.) What is the rank of $A$, and describe the column space of $A$.

**Solution:** $A$ has 1 pivot, so it has rank 1. Each column is a multiple of the second column, so the column space is spanned by $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

(c) (5 pts.) Find all solutions to $Ax = \begin{bmatrix} 0 \\ 6 \\ 12 \end{bmatrix}$.

**Solution:** A particular solution to this problem is $x_p = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 0 \end{bmatrix}$. We can add any vector in the nullspace of $A$ to get another solution, and this gives all solutions, so the general form is

$$\begin{bmatrix} 0 \\ 6 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix},$$

where $c_1, c_2, c_3$ are some numbers.

(d) (5 pts.) Can $A$ be written as $A = uv^T$ for some vectors $u$ and $v$? If so what are these vectors, or if not, why not?

**Solution:** Yes, we know that a matrix can be written as $uv^T$ if and only if it has rank 1 (or 0). Since the second column spans the column space, we take it to be $u$. Then the entries of $v$ say which multiples the other columns are of $u$. So the answer

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is

$$u = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad v^T = (0, 1, 2, 3).$$

**2 (30 pts.)**
Consider the matrix

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix},$$

where $ps = rq$ and $pr \neq 0$.

(a) (5 pts.) Describe simply and clearly the column space of $A$.
  **Solution:** The first column is nonzero since $p \neq 0$ and $r \neq 0$. The second column is a multiple of the first column. To see this, multiply the first column by $q/p$ and use the assumption that $ps = rq$. So the column space is spanned by $\begin{bmatrix} p \\ r \end{bmatrix}$.

(b) (10 pts.) Write as simply as possible the special solution(s) to $Ax = 0$, if any.
  **Solution:** Subtract $r/p$ times the first row from the second row to transform $A$ into the matrix $\begin{bmatrix} p & q \\ 0 & 0 \end{bmatrix}$, where again we used that $ps = rq$ to simplify. So column 2 is the only free column because $p \neq 0$. Setting $x_2 = 1$ gives the special solution $\begin{bmatrix} -q/p \\ 1 \end{bmatrix}$.

(c) (5 pts.) What are all the solutions to $Ax = 0$?
  **Solution:** All multiples of the vector $\begin{bmatrix} -q/p \\ 1 \end{bmatrix}$.

(d) (10 pts.) Write $A$ as simply as possible in row reduced echelon form.
  **Solution:** We did one row operation in (b). To finish, divide the first column by $p$ to get

$$\begin{bmatrix} 1 & q/p \\ 0 & 0 \end{bmatrix}.$$

**3 (20 pts.)**
(In the questions below, you can choose any $n$ that works for an example, or prove that for all $n$, there is no example.)

(a) (10 pts.) Can you find independent vectors $v, w, x$ and $y$ in some space $R^n$ and where $A = vw^T + xy^T$ is invertible? or prove that no such example exists?
  **Solution:** No. Matrices of the form $vw^T$ always have rank $\leq 1$. The sum of two matrices of rank $\leq 1$ has rank $\leq 2$. So if $A$ is invertible we need $n \leq 2$. But we also have the requirement that $v, w, x, y$ are independent vectors, which forces $n \geq 4$. So we can't pick any $n$ to make both inequalities happen.

(b) (10 pts.) Can you find vectors $v, w, x$ and $y$ that span some space $R^n$ and where $A = vw^T + xy^T$ is invertible? or prove that no such example exists?
  **Solution:** Yes. There are many possibilities. The easiest is to take $n = 1$. Then $v, w, x, y$ are just numbers and $A$ being invertible means it is nonzero. So we could take $v = w = 1$ and $x = y = 0$ for example.

**4 (20 pts.)**

Write an informal computer program to calculate $xx^Tx$, for any $n \times 1$ column vector $x$. The program should only use about $2n$ operations and no more than about $2n$ numbers in memory. You can write the program in MATLAB or your favorite language. It is not important that you remember exact syntax, but it is important that your operations are clear and unambiguous.

**Solution:** The naive way to do this is to first multiply $xx^T$ and then multiply by $x$. But $xx^T$ is a $n \times n$ matrix so doesn't meet the requirement of having roughly $2n$ numbers in memory (or the $2n$ operations requirement).

The point is that matrix multiplication is associative, so we can instead do $x(x^Tx)$, i.e., calculate $x^Tx$ which is a single number, and then multiply $x$ by this number. If the entries of $x$ are $x_1, \ldots, x_n$, then the final answer would be

$$\begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}, \quad \text{where } c = x_1^2 + x_2^2 + \cdots + x_n^2.$$

So a program that does this should roughly look like this:

```
c := 0;
for i from 1 to n do c := c + x[i]*x[i];
for i from 1 to n do answer[i] := x[i]*c;
return answer;
```